

ON FINITE-TIME HYPERBOLICITY

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ABSTRACT. A solution of a nonautonomous ordinary differential equation is finite-time hyperbolic, i.e. hyperbolic on a compact interval of time, if the linearisation along that solution exhibits a strong exponential dichotomy. As a finite-time variant and strengthening of classical asymptotic facts, it is shown that finite-time hyperbolicity guarantees the existence of stable and unstable manifolds of the appropriate dimensions. Eigenvalues and -vectors are often unsuitable for detecting hyperbolicity. A (dynamic) partition of the extended phase space is used to circumvent this difficulty. It is proved that any solution staying clear of the elliptic and degenerate parts of the partition is finite-time hyperbolic. This extends and unifies earlier partial results.

1. Introduction. Ever since the emergence of dynamical systems theory, hyperbolicity has been recognised as a fundamental concept. Variations and generalisations of hyperbolicity as well as their ramifications continue to play a vital role in modern dynamics [14]. Prominent examples from the more recent past include the formation of the theories of nonuniformly and partially hyperbolic systems, nonuniform exponential dichotomies as well as finite-time dynamics. The development of the latter field is to a considerable extent driven by applications in fluid dynamics and oceanography, where a nonautonomous differential equation may describe the instationary velocity field around an airfoil, within a tornado, or of a stretch of ocean surface. The mathematical models used for these problems are often based on sophisticated measurements or elaborate numerical simulations. Naturally, therefore, these models, as well as any conclusions drawn from them, are valid only over a bounded interval of time. Many classical, that is, asymptotic concepts of dynamics do not apply in such a situation and have to be modified or replaced altogether with appropriate finite-time substitutes. The identification and systematic study of these finite-time concepts, informally referred to as *finite-time dynamics*, has recently experienced a surge in activity, see e.g. [1, 3, 16, 17] and the many references therein. The present article contributes to this development a thorough finite-time analysis of one of dynamics' fundamental notions viz. hyperbolicity and is organised as follows: Based upon a form of (finite-time) hyperbolicity that originates from [7], the main result of Section 2 (Theorem 3) asserts that every hyperbolic solution of a nonautonomous differential equation comes with stable and unstable manifolds. While the statement of the theorem is exactly what could be expected by analogy with classical results, its proof is different as a stronger conclusion (concerning the

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uniform monotone growth and decay of solutions) has to be derived from somewhat weaker assumptions. One possible way of establishing hyperbolicity is to impose conditions on the (typically time-dependent) eigenvalues and -vectors of the associated linearisation. As evidenced by examples, this approach has both its merits and pitfalls. To avoid the latter, a partition of the extended phase space is exploited in Section 3. Early versions of dynamic partitions have been utilised in [9, 10]. In [1], the concept has been extended to arbitrary dimensions, norms, and time-varying vector fields. In this general setting, the main results (Theorem 7 and Corollary 8) show that dynamic partitions, although essentially a Eulerian concept, can be very useful in identifying (Lagrangian) hyperbolic behaviour: Any solution that does not intersect the elliptic and degenerate parts of the partition is, in fact, hyperbolic. This extension of earlier results in [2, 10] turns out to be best possible in several ways.

2. Finite-time hyperbolicity. Consider the nonautonomous ordinary differential equation

$$\dot{x} = f(t, x), \tag{1}$$

where $f : I \times U \rightarrow \mathbb{R}^d$ is C^1 with continuous second derivatives $D_{tx}f, D_{xt}f, D_{xx}f$, and $I \subset \mathbb{R}$ and $U \subset \mathbb{R}^d$ are, respectively, a non-degenerate closed interval and a non-empty open set. The linearisation along a solution $\mu : I \rightarrow U$ of (1) is

$$\dot{y} = D_x f(t, \mu(t))y. \tag{2}$$

In the classical asymptotic theory (i.e. for $I = \mathbb{R}$) a solution μ of (1) is *hyperbolic* if the associated linearisation (2) has an exponential dichotomy, see e.g. [5, 14, 15]. In this article, only the finite-time case will be studied in detail. Thus assume from now on that $I = [t_-, t_+]$ with $-\infty < t_- < t_+ < +\infty$. For the finite-time case, the notion of hyperbolicity must be modified as quantitative or transitive effects have to be taken into account [3]. To allow for sufficient flexibility in quantifying e.g. the growth and decay of solutions of (2), arbitrary norms on \mathbb{R}^d induced by an inner product will be considered, i.e., the family of norms $\|\cdot\|_\Gamma = \sqrt{\langle \cdot, \Gamma \cdot \rangle}$ will be used, where $\Gamma \in \mathbb{R}^{d \times d}$ is any symmetric positive definite matrix, i.e. $\Gamma^\top = \Gamma > 0$; here, as usual, $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$ is the standard inner product on \mathbb{R}^d , and the symbol $\|\cdot\|_\Gamma$ also denotes the norm induced on $\mathbb{R}^{d \times d}$. Quantities depending on Γ have their dependence made explicit by a subscript which is suppressed only if Γ equals $\text{id}_{d \times d}$, the $d \times d$ identity matrix.

To define finite-time hyperbolicity, instead of (2) consider more generally any nonautonomous *linear* equation

$$\dot{y} = A(t)y, \tag{3}$$

where $A : I \rightarrow \mathbb{R}^{d \times d}$ is C^1 . Let $\Phi : I \times I \rightarrow \mathbb{R}^{d \times d}$ denote the associated evolution operator, i.e., $y : t \mapsto \Phi(t, s)\eta$ is, for any $\eta \in \mathbb{R}^d$, the unique solution of (3) satisfying $y(s) = \eta$. A projection-valued function $P : I \rightarrow \mathbb{R}^{d \times d}$ is an *invariant projector* if $P(t)\Phi(t, s) = \Phi(t, s)P(s)$ for all $t, s \in I$. Note that $t \mapsto P(t)$ is continuous for any invariant projector, and $\text{rk } P(t)$ is a constant non-negative integer not larger than d , denoted henceforth as $\text{rk } P$.

Definition 1. Let $\Gamma^\top = \Gamma > 0$. Equation (3) is *hyperbolic* (on I w.r.t. $\|\cdot\|_\Gamma$) if there exists an invariant projector P , together with positive constants α, β , such

that for all $t, s \in I$ and $y \in \mathbb{R}^d$,

$$\|\Phi(t, s)P(s)y\|_\Gamma \leq e^{-\alpha(t-s)}\|P(s)y\|_\Gamma, \quad \forall t \geq s, \tag{4}$$

$$\|\Phi(t, s)(\text{id}_{d \times d} - P(s))y\|_\Gamma \leq e^{\beta(t-s)}\|(\text{id}_{d \times d} - P(s))y\|_\Gamma, \quad \forall t \leq s. \tag{5}$$

A solution μ of (1) is hyperbolic (on I w.r.t. $\|\cdot\|_\Gamma$) if the associated linearisation (2) is hyperbolic.

Remark 2. (i) The estimates (4) and (5) incorporate a finite-time variant of the classical notion of an exponential dichotomy [5, 15]. They are more restrictive than the latter because an arbitrary multiplicative constant on the right-hand side of (4) or (5) would render the concept meaningless.

(ii) In essence, hyperbolicity according to Definition 1 is equivalent to *uniform hyperbolicity* as advocated in [7, Def.1]: With the notation used in that paper, one can simply take $E_\tau^+(h) = \ker P(\tau)$, $E_\tau^-(h) = \text{im } P(\tau)$ for all τ, h .

(iii) Recall that if (3) is *autonomous* (i.e., if A does not depend on t), then it has a (classical) exponential dichotomy if and only if $\Re\lambda \neq 0$ for every eigenvalue λ of A . In this case, there exists a norm $\|\cdot\|_\Gamma$ w.r.t. which (3) is hyperbolic on every compact interval. To see this, let E^s and E^u be the sum of the generalised eigenspaces of A corresponding to eigenvalues with negative and positive real part, respectively. Then $\mathbb{R}^d = E^s \oplus E^u$, and with P_0 denoting the projection onto E^s along E^u define Γ via

$$\begin{aligned} \langle x, \Gamma y \rangle &= \int_0^{+\infty} e^{2\alpha\sigma} [e^{\sigma A} P_0 x, e^{\sigma A} P_0 y] \, d\sigma \\ &\quad + \int_{-\infty}^0 e^{-2\beta\sigma} [e^{\sigma A} (\text{id}_{d \times d} - P_0) x, e^{\sigma A} (\text{id}_{d \times d} - P_0) y] \, d\sigma, \quad \forall x, y \in \mathbb{R}^d, \end{aligned} \tag{6}$$

where $[\cdot, \cdot]$ is any inner product on \mathbb{R}^d , and $\alpha, \beta > 0$ are any numbers with

$$\alpha < \min\{-\Re\lambda : \Re\lambda < 0\}, \quad \beta < \min\{\Re\lambda : \Re\lambda > 0\}.$$

It is readily confirmed that $\Gamma^\top = \Gamma > 0$. Moreover, since $\Phi(t, s) = e^{(t-s)A}$,

$$\begin{aligned} \|\Phi(t, s)P_0 y\|_\Gamma^2 &= \int_0^{+\infty} e^{2\alpha\sigma} [e^{\sigma A} P_0 e^{(t-s)A} P_0 y, e^{\sigma A} P_0 e^{(t-s)A} P_0 y] \, d\sigma \\ &= e^{-2\alpha(t-s)} \int_{t-s}^{+\infty} e^{2\alpha\sigma} [e^{\sigma A} P_0 y, e^{\sigma A} P_0 y] \, d\sigma \leq e^{-2\alpha(t-s)} \|P_0 y\|_\Gamma^2, \end{aligned}$$

for all $t \geq s$ and $y \in \mathbb{R}^d$. Hence (4), and similarly (5), holds with $P(s) \equiv P_0$, and every solution of $\dot{x} = Ax$ is hyperbolic on any I w.r.t. $\|\cdot\|_\Gamma$. Note, however, that the converse is not true in general, i.e., (3) may, for constant A , be hyperbolic on *every* interval I and w.r.t. $\|\cdot\|_\Gamma$ for *all* Γ even though $\Re\lambda = 0$ for some eigenvalue λ of A , see [3, Exp.24].

(iv) In Definition 1, $\text{rk } P$ is uniquely determined whereas, unlike in the classical case, the invariant projector P itself may not be unique, see [3, Exp.27].

In dynamical systems theory a most fundamental implication of hyperbolicity is the existence of (local) stable and unstable manifolds [14, 15]. It will now be shown that finite-time hyperbolicity entails the existence of similar objects, see also FIGURE 1. To this end, denote by $t \mapsto \varphi(t; s, \xi)$ the (unique) solution of (1) with $x(s) = \xi \in U$. If $\mu : I \rightarrow U$ is a solution of (1) then $\varphi(t; s, \xi)$ exists for all $t \in I$, provided that $\|\xi - \mu(s)\|_\Gamma$ is sufficiently small.

Theorem 3. Assume the solution $\mu : I \rightarrow U$ of (1) is hyperbolic w.r.t. $\|\cdot\|_\Gamma$, with invariant projector P and constants $\alpha, \beta > 0$, and let $k = \text{rk } P$. Then, for every $0 < \rho < 1$ there exist C^1 -manifolds $W^s, W^u \subset I \times U$ of dimensions $k + 1$ and $d - k + 1$, respectively, with the following properties:

- (i) W^s and W^u are invariant, i.e., $(s, \xi) \in W^s$ if and only if $(t, \varphi(t; s, \xi)) \in W^s$ for all $t \in I$, and similarly for W^u ;
- (ii) For every $t \in I$ the fibres

$$W^s(t) = \{x \in U : (t, x) \in W^s\}, \quad W^u(t) = \{x \in U : (t, x) \in W^u\},$$

are C^1 -manifolds of dimensions k and $d - k$, respectively, with

$$T_{\mu(t)}W^s(t) = \text{im } P(t), \quad T_{\mu(t)}W^u(t) = \text{ker } P(t),$$

and

$$W^s(t) \cap W^u(t) = \{\mu(t)\}; \tag{7}$$

- (iii) For every $(s, \xi) \in W^s$,

$$\|\varphi(t; s, \xi) - \mu(t)\|_\Gamma \leq e^{-\alpha\rho(t-s)} \|\xi - \mu(s)\|_\Gamma, \quad \forall t \geq s; \tag{8}$$

similarly, for every $(s, \xi) \in W^u$,

$$\|\varphi(t; s, \xi) - \mu(t)\|_\Gamma \leq e^{\beta\rho(t-s)} \|\xi - \mu(s)\|_\Gamma, \quad \forall t \leq s. \tag{9}$$

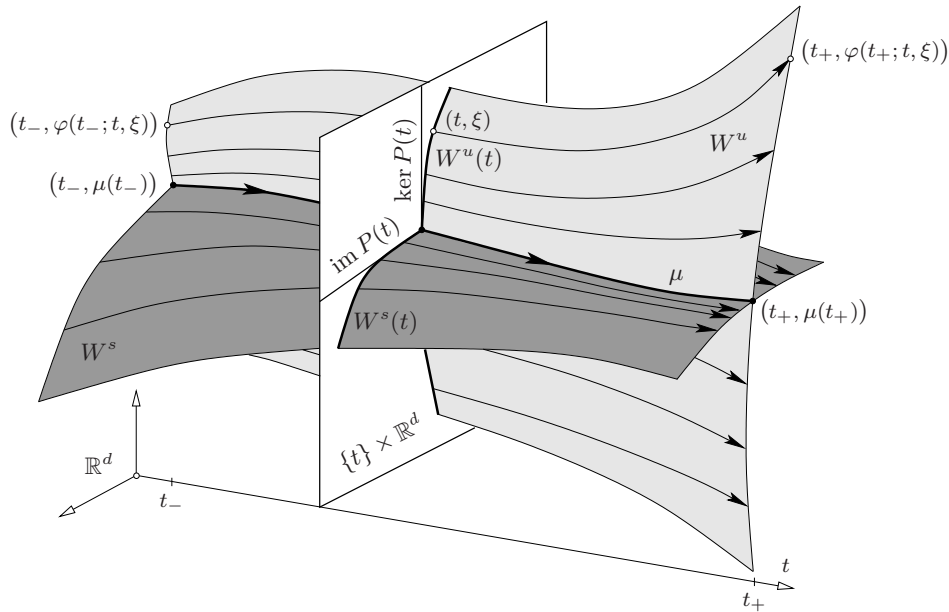


FIGURE 1. Stable and unstable manifolds of a hyperbolic solution.

Proof. For the reader's convenience the argument, essentially a simplified and generalised version of A.2–A.8 in [8], is divided into three steps. Firstly, (1) is extended to $\mathbb{R} \times \mathbb{R}^d$ in a way that makes it more amenable to analysis. Then, using a contraction mapping argument, W^s and W^u are identified as families of solutions to an integral equation equivalent to (1) in its extended form; properties (i) and (ii) follow directly from this. Finally, (iii) is established via a Gronwall-type estimate.

Step 1. Note first that substituting $x = \mu + z$ transforms (1) into

$$\dot{z} = A(t)z + g(t, z), \tag{10}$$

where $A = D_x f(\cdot, \mu) : I \rightarrow \mathbb{R}^{d \times d}$ and g are C^1 , and

$$g(t, 0) = 0, \quad D_z g(t, 0) = 0, \quad \forall t \in I. \tag{11}$$

By assumption the linearisation $\dot{y} = A(t)y$ is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$, with invariant projector $P : I \rightarrow \mathbb{R}^{d \times d}$ and constants $\alpha, \beta > 0$. Given $0 < \varepsilon \leq 1$, let $I_\varepsilon := [t_- - \varepsilon, t_+ + \varepsilon]$ and choose a C^1 -function $\tilde{A} : I_\varepsilon \rightarrow \mathbb{R}^{d \times d}$ with $\tilde{A}(t) = A(t)$ for all $t \in I$. Also pick $\tilde{g} \in C^1(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d)$ such that $\tilde{g}(t, z) = g(t, z)$ for all $t \in I$ and $\|z\|_\Gamma$ sufficiently small, and $\tilde{g}(t, \cdot) = 0$ for all $t \in \mathbb{R} \setminus I_\varepsilon$. By (11), \tilde{g} can in fact be chosen such that, with the appropriate $\delta_\varepsilon > 0$,

$$\|\tilde{g}(t, z_1) - \tilde{g}(t, z_2)\|_\Gamma \leq \varepsilon \|z_1 - z_2\|_\Gamma, \quad \forall t \in \mathbb{R},$$

provided that $\|z_1\|_\Gamma + \|z_2\|_\Gamma < \delta_\varepsilon$. Denote by $\tilde{\Phi}$ the evolution operator associated with $\dot{y} = \tilde{A}(t)y$ on I_ε ; clearly $\tilde{\Phi}(t, s) = \Phi(t, s)$ for all $t, s \in I$. Define

$$\begin{aligned} A_- &:= -(\alpha + \beta)\tilde{\Phi}(t_- - \varepsilon, t_-)P(t_-)\tilde{\Phi}(t_- - \varepsilon, t_-)^{-1} + \beta \text{id}_{d \times d}, \\ A_+ &:= -(\alpha + \beta)\tilde{\Phi}(t_+ + \varepsilon, t_+)P(t_+)\tilde{\Phi}(t_+ + \varepsilon, t_+)^{-1} + \beta \text{id}_{d \times d}, \end{aligned}$$

and let $\tilde{A}(t)$ equal A_- and A_+ whenever $t < t_- - \varepsilon$ and $t > t_+ + \varepsilon$, respectively. With this, the linear equation $\dot{y} = \tilde{A}(t)y$ together with its associated evolution operator, again denoted by $\tilde{\Phi}$, is now defined on the entire real axis and has a (classical) exponential dichotomy: With $P(t) := P(t_- - \varepsilon)$ for all $t \leq t_- - \varepsilon$ and $P(t) := P(t_+ + \varepsilon)$ for all $t \geq t_+ + \varepsilon$, and with some $C_1 \geq 1$ independent of ε , the estimates

$$\|\tilde{\Phi}(t, s)P(s)y\|_\Gamma \leq C_1 e^{-\alpha(t-s)} \|P(s)y\|_\Gamma, \quad \forall t \geq s, \tag{12}$$

$$\|\tilde{\Phi}(t, s)(\text{id}_{d \times d} - P(s))y\|_\Gamma \leq C_1 e^{\beta(t-s)} \|(\text{id}_{d \times d} - P(s))y\|_\Gamma, \quad \forall t \leq s, \tag{13}$$

hold for all $y \in \mathbb{R}^d$, but also

$$\sup_{t \in \mathbb{R}} (\|\tilde{A}(t)\|_\Gamma + \|P(t)\|_\Gamma) \leq C_1.$$

This follows immediately from the fact that $P : \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ is continuous and constant outside I_ε . Note that $\tilde{A} : \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ is C^1 except for the two jump discontinuities at $t_- - \varepsilon, t_+ + \varepsilon$. Also, with some constant $C_2 \geq 1$ that does not depend on ε , the estimate $\|\tilde{\Phi}(t, s)\|_\Gamma \leq C_2$ holds whenever $t, s \in I_\varepsilon$. Let $C = 1 + \max(C_1, C_2)$. With these preparations, consider (10) with A and g replaced by \tilde{A} and \tilde{g} , respectively, but for ease of notation suppress the tilde from now on. For $t \notin I_\varepsilon$ the right-hand side of (10) is continuous and linear (in z). With the appropriate $\delta_0 > 0$, therefore, the solution $z = \varphi(\cdot; s, \zeta)$ of (10) with $z(s) = \zeta$ exists for all $t \in \mathbb{R}$, provided that $\|\zeta\|_\Gamma < \delta_0$; assume w.l.o.g. that $\delta_\varepsilon \leq \delta_0$.

Step 2. To identify W^s and W^u a contraction mapping argument will be used. To motivate this approach, for any solution z of (10) let

$$z^s(t) := P(t)z(t), \quad z^u(t) := (\text{id}_{d \times d} - P(t))z(t),$$

so that $z = z^s + z^u$, and observe that (10) may, for any $\tau_1, \tau_2 \in \mathbb{R}$, equivalently be written as a pair of integral equations,

$$z^s(t) = \Phi(t, \tau_1)z^s(\tau_1) + \int_{\tau_1}^t \Phi(t, \sigma)P(\sigma)g(\sigma, z) \, d\sigma, \tag{14}$$

$$z^u(t) = \Phi(t, \tau_2)z^u(\tau_2) + \int_{\tau_2}^t \Phi(t, \sigma)(\text{id}_{d \times d} - P(\sigma))g(\sigma, z) \, d\sigma, \tag{15}$$

for all $t \in \mathbb{R}$. If $\|z(t)\|_\Gamma$ is bounded as $t \rightarrow +\infty$ then by (13), for every $t \geq t_-$,

$$\|\Phi(t, \tau_2)z^u(\tau_2)\|_\Gamma \leq C_1 e^{\beta(t-\tau_2)} \|z^u(\tau_2)\|_\Gamma \rightarrow 0 \quad \text{as } \tau_2 \rightarrow +\infty.$$

Thus, for every solution z of (10) that remains bounded in forward time (15) takes the form

$$z^u(t) = - \int_t^{+\infty} \Phi(t, \sigma)(\text{id}_{d \times d} - P(\sigma))g(\sigma, z) \, d\sigma, \quad \forall t \in \mathbb{R}. \tag{16}$$

In view of (14) and (16) let Ω^+ be the linear space

$$\Omega^+ = \{ \omega = (\omega_1, \omega_2) \mid \omega_1, \omega_2 : [t_-, +\infty[\rightarrow \mathbb{R}^d \text{ bounded and continuous,} \\ \omega_1(t) \in \text{im } P(t), \omega_2(t) \in \text{ker } P(t) \quad \forall t \geq t_- \},$$

which is a Banach space when endowed with the norm

$$\|\omega\| := \sup_{t \geq t_-} (\|\omega_1(t)\|_\Gamma + \|\omega_2(t)\|_\Gamma),$$

and denote by Ω_δ^+ the closed δ -ball in Ω^+ centred at $(0, 0)$. For every $\tau \in I$ and $\zeta \in \text{im } P(\tau)$ define the map $\mathcal{F}^{(\tau, \zeta)} : \Omega_\delta^+ \rightarrow \Omega^+$ according to

$$\mathcal{F}^{(\tau, \zeta)} : \omega \mapsto \left(\Phi(t, \tau)\zeta + \int_\tau^t \Phi(t, \sigma)P(\sigma)g(\sigma, \omega_1 + \omega_2) \, d\sigma, \right. \\ \left. - \int_t^{+\infty} \Phi(t, \sigma)(\text{id}_{d \times d} - P(\sigma))g(\sigma, \omega_1 + \omega_2) \, d\sigma \right).$$

Note that $\mathcal{F}^{(\tau, \zeta)}(0, 0) = (\Phi(\cdot, \tau)\zeta, 0)$ so that in particular $\mathcal{F}^{(\tau, 0)}(0, 0) = (0, 0)$ for all $\tau \in I$. Also,

$$\|\mathcal{F}^{(\tau, \zeta)}(\omega) - \mathcal{F}^{(\tau, \widehat{\zeta})}(\omega)\| = \|(\Phi(\cdot, \tau)(\zeta - \widehat{\zeta}), 0)\| \leq C\|\zeta - \widehat{\zeta}\|_\Gamma \tag{17}$$

for all $\tau \in I, \zeta, \widehat{\zeta} \in \text{im } P(\tau)$ and $\omega \in \Omega_\delta^+$. It will now be shown that $\mathcal{F}^{(\tau, \zeta)}$ is in fact a contraction of Ω_δ^+ into itself, provided that δ and $\|\zeta\|_\Gamma$ are sufficiently small. To this end observe that $\|\omega\| \leq \delta_\varepsilon$ implies

$$\|\mathcal{F}^{(\tau, \zeta)}(\omega)\| \leq \sup_{t \geq t_-} \left(\|\Phi(t, \tau)\zeta\|_\Gamma + \left| \int_\tau^t \|\Phi(t, \sigma)P(\sigma)g(\sigma, \omega_1 + \omega_2)\|_\Gamma \, d\sigma \right| + \right. \\ \left. + \int_t^{+\infty} \|\Phi(t, \sigma)(\text{id}_{d \times d} - P(\sigma))g(\sigma, \omega_1 + \omega_2)\|_\Gamma \, d\sigma \right) \\ \leq C\|\zeta\|_\Gamma + C^2\varepsilon\delta_\varepsilon(\alpha^{-1} + t_+ - t_- + \beta^{-1});$$

here the estimates

$$\left| \int_\tau^t \|\Phi(t, \sigma)P(\sigma)g(\sigma, \omega_1 + \omega_2)\|_\Gamma \, d\sigma \right| \leq \int_\tau^t C e^{-\alpha(t-\sigma)} C\varepsilon\delta_\varepsilon \, d\sigma \leq C^2\varepsilon\delta_\varepsilon\alpha^{-1} \quad \text{if } t \geq \tau, \\ \int_t^\tau C^2\varepsilon\delta_\varepsilon \, d\sigma \leq C^2\varepsilon\delta_\varepsilon(t_+ - t_-) \quad \text{if } t < \tau,$$

as well as

$$\int_t^{+\infty} \|\Phi(t, \sigma)(\text{id}_{d \times d} - P(\sigma))g(\sigma, \omega_1 + \omega_2)\|_{\Gamma} d\sigma \leq \int_t^{+\infty} C e^{\beta(t-\sigma)} C \varepsilon \delta_{\varepsilon} d\sigma = C^2 \varepsilon \delta_{\varepsilon} \beta^{-1}$$

have been used. Thus if

$$\varepsilon < \frac{1}{2C^2}(\alpha^{-1} + t_+ - t_- + \beta^{-1})^{-1} =: \varepsilon_1, \quad \|\zeta\|_{\Gamma} < \frac{\delta_{\varepsilon}}{2C} =: \delta'_{\varepsilon}, \tag{18}$$

then $\mathcal{F}^{(\tau, \zeta)}(\Omega_{\delta_{\varepsilon}}^+) \subset \Omega_{\delta_{\varepsilon}}^+$. Moreover, with (18) for all $\omega, \widehat{\omega} \in \Omega_{\delta_{\varepsilon}}^+$,

$$\begin{aligned} \|\mathcal{F}^{(\tau, \zeta)}(\omega) - \mathcal{F}^{(\tau, \zeta)}(\widehat{\omega})\| &\leq \varepsilon C^2(\alpha^{-1} + t_+ - t_- + \beta^{-1}) \|\omega - \widehat{\omega}\| \\ &\leq \frac{1}{2} \|\omega - \widehat{\omega}\|, \end{aligned} \tag{19}$$

showing that $\mathcal{F}^{(\tau, \zeta)}$ is indeed a contraction whenever (18) holds, with a contraction factor that does not depend on τ, ζ . It follows that for $\varepsilon < \varepsilon_1$ and every $\tau \in I$ and $\zeta \in \text{im } P(\tau)$ there exists a unique fixed point $\omega^{(\tau, \zeta)} = (\omega_1^{(\tau, \zeta)}, \omega_2^{(\tau, \zeta)}) \in \Omega_{\delta_{\varepsilon}}^+$ of $\mathcal{F}^{(\tau, \zeta)}$, provided that $\|\zeta\|_{\Gamma} < \delta'_{\varepsilon}$; clearly $\omega^{(\tau, 0)} = (0, 0)$ and $\omega_1^{(\tau, \zeta)}(\tau) = \zeta$ for all τ . For notational convenience, let $\omega^* = \omega^{(t_-, \zeta)}$ and define

$$W^s := \{(t, \omega_1^*(t) + \omega_2^*(t)) : t \in I, \zeta \in \text{im } P(t_-), \|\zeta\|_{\Gamma} < \delta'_{\varepsilon}\} \subset I \times U.$$

Since $\omega_1^* + \omega_2^* : I \rightarrow U$ is a solution of (10), the set W^s thus defined is invariant. Note also that (17) and (19) imply

$$\|\omega^{(\tau, \zeta)} - \omega^{(\tau, \widehat{\zeta})}\| \leq 2C \|\zeta - \widehat{\zeta}\|_{\Gamma}, \tag{20}$$

so that in particular $\|\omega_2^{(\tau, \zeta)}(\tau) - \omega_2^{(\tau, \widehat{\zeta})}(\tau)\|_{\Gamma} \leq 2C \|\zeta - \widehat{\zeta}\|_{\Gamma}$. For every $t \in I$, therefore, $W^s(t)$ is the graph of a Lipschitz function: Every element in $W^s(t)$ can be represented in the form $\zeta + w(t, \zeta)$ where $\zeta \in \text{im } P(t)$, $w(t, \zeta) \in \ker P(t)$, and $w(t, 0) = 0$ as well as $\|w(t, \zeta)\|_{\Gamma} \leq 2C \|\zeta\|_{\Gamma}$. Since $(\tau, \zeta, \omega) \mapsto \mathcal{F}^{(\tau, \zeta)}(\omega)$ is C^1 , so are W^s and w , see for instance [12, Thm.C.7]. Furthermore, from (20) it follows that

$$\begin{aligned} \|\omega_2^{(\tau, \zeta)}(t) - \omega_2^{(\tau, \widehat{\zeta})}(t)\|_{\Gamma} &\leq \int_t^{\infty} \|\Phi(t, \sigma)(\text{id}_{d \times d} - P(\sigma))(g(\sigma, \omega_1^{(\tau, \zeta)} + \omega_2^{(\tau, \zeta)}) - g(\sigma, \omega_1^{(\tau, \widehat{\zeta})} + \omega_2^{(\tau, \widehat{\zeta})}))\|_{\Gamma} d\sigma \\ &\leq C^2 \varepsilon \beta^{-1} \|\omega^{(\tau, \zeta)} - \omega^{(\tau, \widehat{\zeta})}\| \\ &\leq 2C^3 \varepsilon \beta^{-1} \|\zeta - \widehat{\zeta}\|_{\Gamma}, \end{aligned}$$

and hence in particular $\|w(t, \zeta) - w(t, \widehat{\zeta})\|_{\Gamma} \leq 2C^3 \varepsilon \beta^{-1} \|\zeta - \widehat{\zeta}\|_{\Gamma}$ for all $t \in I$ and $\zeta, \widehat{\zeta} \in \text{im } P(t)$ with $\|\zeta\|_{\Gamma} + \|\widehat{\zeta}\|_{\Gamma} \leq \delta_{\varepsilon}$. Under the assumption that

$$\varepsilon < \frac{\beta}{2C^3} =: \varepsilon_2, \tag{21}$$

the map $(t, \zeta) \mapsto (t, \zeta + w(t, \zeta))$ is one-to-one on the set

$$\{(t, \zeta) : t \in I, \zeta \in \text{im } P(t), \|\zeta\|_{\Gamma} \leq \delta_{\varepsilon}\}.$$

Consequently, $W^s \subset I \times U$ is a $(k+1)$ -dimensional C^1 -manifold, and $\dim W^s(t) \equiv k$. Since $D_z g(t, 0) \equiv 0$ also $D_{\zeta} w(t, 0) \equiv 0$ and therefore $T_0 W^s(t) = \text{im } P(t)$ for all $t \in I$. (If f is C^r then $(\tau, \zeta, \omega) \mapsto \mathcal{F}^{(\tau, \zeta)}(\omega)$ is C^r as well, and [12, Thm.C.7] shows that in this case W^s is C^r also.)

The manifold W^u is constructed in a similar manner: Taking the limit $\tau_1 \rightarrow -\infty$ in (14) leads to the definition of a contraction on a closed ball within the Banach space

$$\Omega^- = \{ \omega = (\omega_1, \omega_2) \mid \omega_1, \omega_2 :] - \infty, t_+] \rightarrow \mathbb{R}^d \text{ bounded and continuous,} \\ \omega_1(t) \in \text{im } P(t), \omega_2(t) \in \text{ker } P(t) \ \forall t \leq t_+ \},$$

with the norm

$$\| \omega \| := \sup_{t \leq t_+} (\| \omega_1(t) \|_\Gamma + \| \omega_2(t) \|_\Gamma),$$

and the remaining argument is completely analogous to the one above. Overall, with the choice of $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$ this completes the proof of (i) and (ii) except for the intersection property (7); the latter will follow immediately once (8) and (9) have been established.

Step 3. The argument for W^u again being completely analogous, to prove (iii) it suffices to verify (8). Since $\| \omega_2^*(t) \|_\Gamma \leq 2C \| \omega_1^*(t) \|_\Gamma$ for all $t \in I$,

$$\begin{aligned} \| \omega_1^*(t) \|_\Gamma &\leq \| \Phi(t, s) \omega_1^*(s) \|_\Gamma + \int_s^t \| \Phi(t, \sigma) P(\sigma) g(\sigma, \omega_1^* + \omega_2^*) \|_\Gamma \, d\sigma \\ &\leq e^{-\alpha(t-s)} \| \omega_1^*(s) \|_\Gamma + \int_s^t e^{-\alpha(t-\sigma)} C \varepsilon \| \omega_1^*(\sigma) + \omega_2^*(\sigma) \|_\Gamma \, d\sigma \\ &\leq e^{-\alpha(t-s)} \| \omega_1^*(s) \|_\Gamma + \int_s^t e^{-\alpha(t-\sigma)} C \varepsilon (1 + 2C) \| \omega_1^*(\sigma) \|_\Gamma \, d\sigma \end{aligned}$$

holds for all $t \geq s$. By means of $h(t) := e^{\alpha t} \| \omega_1^*(t) \|_\Gamma$, the latter estimate can be rewritten as

$$h(t) \leq h(s) + \varepsilon C (1 + 2C) \int_s^t h(\sigma) \, d\sigma \leq h(s) + 3\varepsilon C^2 \int_s^t h(\sigma) \, d\sigma,$$

so by Gronwall's Lemma $h(t) \leq h(s) e^{3\varepsilon C^2(t-s)}$. Thus $t \mapsto e^{(\alpha - 3\varepsilon C^2)t} \| \omega_1^*(t) \|_\Gamma$ is non-increasing, and

$$\frac{d}{dt} \| \omega_1^*(t) \|_\Gamma^2 \leq -2(\alpha - 3\varepsilon C^2) \| \omega_1^*(t) \|_\Gamma^2, \quad \forall t \in I.$$

To prove (8) it is enough to confirm that

$$k : t \mapsto e^{2\alpha\rho t} \| \omega_1^*(t) + \omega_2^*(t) \|_\Gamma^2 \quad (t \in I)$$

is non-increasing. To this end, observe that with (21),

$$\begin{aligned} e^{-2\alpha\rho t} \frac{d}{dt} k &= 2\alpha\rho \| \omega_1^* + \omega_2^* \|_\Gamma^2 + \frac{d}{dt} \| \omega_1^* + \omega_2^* \|_\Gamma^2 \\ &\leq 2\alpha\rho (\| \omega_1^* \|_\Gamma + \| \omega_2^* \|_\Gamma)^2 + \frac{d}{dt} \| \omega_1^* \|_\Gamma^2 + 2\langle \dot{\omega}_1^*, \Gamma \omega_2^* \rangle + 2\langle \dot{\omega}_2^*, \Gamma(\omega_1^* + \omega_2^*) \rangle \\ &\leq 2\alpha\rho (1 + 2C^3\varepsilon\beta^{-1})^2 \| \omega_1^* \|_\Gamma^2 - 2(\alpha - 3\varepsilon C^2) \| \omega_1^* \|_\Gamma^2 + \\ &\quad + 2\langle A\omega_1^* + Pg(\cdot, \omega_1^* + \omega_2^*), \Gamma \omega_2^* \rangle + 2\langle A\omega_2^* + Pg(\cdot, \omega_1^* + \omega_2^*), \Gamma(\omega_1^* + \omega_2^*) \rangle \\ &\leq (2\alpha\rho (1 + 6C^3\varepsilon\beta^{-1}) - 2\alpha + 6\varepsilon C^2 + 60C^5\varepsilon\beta^{-1}) \| \omega_1^* \|_\Gamma^2 \\ &= 2(-\alpha(1 - \rho) + 3\varepsilon C^2(1 + 2C\alpha\rho\beta^{-1} + 10C^3\beta^{-1})) \| \omega_1^* \|_\Gamma^2, \end{aligned}$$

from which it follows that indeed $\frac{d}{dt} k \leq 0$, provided that

$$\varepsilon < \frac{\alpha(1 - \rho)}{3C^2(1 + 2C\alpha\rho\beta^{-1} + 10C^3\beta^{-1})} =: \varepsilon_3.$$

Thus choosing $\varepsilon < \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ completes the proof of (iii).

Finally, to verify the intersection property (7) let $\xi \in W^s(t) \cap W^u(t)$ for some $t \in I$; assume w.l.o.g. that $t < t_+$. For all $\tau > 0$ with $t + \tau \leq t_+$,

$$\|\xi - \mu(t)\|_\Gamma \leq e^{-\beta\rho\tau} \|\varphi(t + \tau; t, \xi) - \mu(t + \tau)\|_\Gamma \leq e^{-(\alpha+\beta)\rho\tau} \|\xi - \mu(t)\|_\Gamma,$$

and hence $\xi = \mu(t)$. □

Remark 4. (i) The sets W^s and W^u are, respectively, local stable and unstable manifolds for the hyperbolic solution μ . Unlike their classical counterparts they are not uniquely determined. (In the above proof of Theorem 3 they depend on the choice of the extensions \tilde{A} and \tilde{g} . Moreover, recall that the invariant projector P upon which that proof relies heavily may not be unique either.) It can be shown, however, that an appropriately defined distance between any two invariant manifolds satisfying (7)–(9) is $\mathcal{O}(e^{-\gamma(t_+ - t_-)})$ with some $\gamma > 0$, see [7]. Hence for practical purposes W^s and W^u may be considered unique for all sufficiently long time intervals. In the terminology of [8, 10], $W^s(t)$ and $W^u(t)$ correspond to a k -dimensional repelling and a $(d - k)$ -dimensional attracting *material surface*, respectively.

(ii) In [6], an alternative concept of finite-time (un)stable manifolds is advocated, leading to manifolds that are open subsets of $I \times U$ and contain, respectively, the sets W^s and W^u considered here.

(iii) The fact that $\|\cdot\|_\Gamma$ is induced by an inner product has been used only in Step 3 of the proof. Hence Definition 1 and Theorem 3 remain virtually unchanged for arbitrary norms on \mathbb{R}^d ; however, the bounds $e^{-\alpha\rho(t-s)}$ and $e^{\beta\rho(t-s)}$ have to be replaced by the slightly weaker $\rho^{-1}e^{-\alpha(t-s)}$ and $\rho^{-1}e^{\beta(t-s)}$, respectively.

(iv) Intuitively, it may seem desirable to have $\rho = 1$ in (8) and (9). That this is impossible in general, and that consequently the bounds in Theorem 3 are best possible, can be seen already from a simple autonomous example. Concretely, let $H : \mathbb{R} \rightarrow \mathbb{R}$ be the C^∞ -function with

$$H(x) = \begin{cases} e^{-x^{-2}} \sin(x^{-2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

define $h(x) := H(x) + xH'(x)$ so that h is C^∞ with $h(0) = h'(0) = 0$, and consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 - h(x_1) \end{bmatrix}. \tag{22}$$

It is readily confirmed that the equilibrium $\mu = 0$ of (22) is hyperbolic on every interval I and w.r.t. every norm $\|\cdot\|_\Gamma$ on \mathbb{R}^2 , with $\alpha = \beta = 1$, its (classical) stable manifold being

$$\left\{ \begin{bmatrix} x_1 \\ H(x_1) \end{bmatrix} : x_1 \in \mathbb{R} \right\}.$$

More generally, the solution of (22) with $x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_{10}e^{-t} \\ (x_{20} - H(x_{10}))e^t + H(x_{10}e^{-t}) \end{bmatrix}.$$

With this it is easy to check that for instance with the Euclidean norm the derivative $\frac{d}{dt}(e^{2t}\|x(t)\|^2)$ is actually *positive* arbitrarily close to the origin. Thus no matter how

W^s is chosen, (8) cannot possibly hold with $\rho = 1$ in even the tiniest neighborhood of μ . Note, however, that in accordance with Theorem 3(iii), for every $\varepsilon > 0$,

$$\|x(t)\| \leq \|x(s)\|e^{-(1-\varepsilon)(t-s)}, \quad \forall t \geq s,$$

holds for all solutions in W^s sufficiently close to the origin.

(v) It is well-known and -documented that, except for the autonomous case, the (generally t -dependent) eigenvalues of A are largely irrelevant e.g. for the stability of (3), see for instance [18, Ch.6]. On the other hand, if the variation of eigenvalues and -vectors is small enough in an appropriate sense then some insight concerning finite-time behaviour may still be gained from them. In this spirit and for $d = 2$ and $\Gamma = \text{id}_{2 \times 2}$, [7, Thm.1] and [11, Thm.1] present conditions on the eigenvalues and -vectors of A that ensure finite-time hyperbolicity. Practicable as these conditions may be e.g. in a fluid dynamics context, it must be remembered that hyperbolicity according to Definition 1 denominates a very uniform, monotone growth and decay of (some) solutions of (2), and the eigenvalue information may be inconclusive in this regard, even in the autonomous case. For a concrete example consider

$$\dot{x} = A_0x \quad \text{with} \quad A_0 = \begin{bmatrix} -1 & 6 \\ 0 & -7 \end{bmatrix}. \tag{23}$$

While obviously $\lambda < 0$ for every eigenvalue λ of A_0 , it follows from [3, Exp.14] or an elementary calculation that (23) is *not* hyperbolic on I w.r.t. $\|\cdot\|$ whenever

$$t_+ - t_- \geq \frac{1}{3} \log(1 + 2\sqrt{2}) - \frac{1}{6} \log 7 \approx 0.1232,$$

i.e., unless I is fairly short. Remark 2(iii) implies that an appropriately chosen Γ will make (23) hyperbolic: With $\alpha = \frac{1}{2}$, $P_0 = \text{id}_{2 \times 2}$ and $[\cdot, \cdot] = \langle \cdot, \cdot \rangle$, one obtains from (6) that

$$\Gamma = \frac{1}{91} \begin{bmatrix} 91 & 78 \\ 78 & 79 \end{bmatrix},$$

and (23) is hyperbolic on every I w.r.t. $\|\cdot\|_\Gamma$. Alternatively, Theorem 7 below shows that Γ could also be chosen as

$$\Gamma = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

since in this case

$$S_\Gamma = \frac{1}{2}(\Gamma A_0 + A_0^\top \Gamma) = \begin{bmatrix} -1 & -1 \\ -1 & -8 \end{bmatrix}$$

is negative definite.

It is in view of examples like (23) that the present article does not attempt to establish finite-time hyperbolicity by imposing conditions on the spectral data of A . Rather, an altogether different approach will be presented in the following section.

(vi) Hyperbolicity w.r.t. $\|\cdot\|_\Gamma$ is invariant under appropriate changes of coordinates. Concretely, call the matrix $C \in \mathbb{R}^{d \times d}$ Γ -orthogonal if it preserves $\|\cdot\|_\Gamma$, that is, if $C^\top \Gamma C = \Gamma$. As detailed in [10], specific applications may require changes of coordinates as general as

$$x = Q(t)\xi + b(t), \tag{24}$$

where $Q : I \rightarrow \mathbb{R}^{d \times d}$, $b : I \rightarrow \mathbb{R}^d$ are C^1 , and $Q(t)$ is Γ -orthogonal for every $t \in I$. It is readily confirmed that a solution of (1) transformed to ξ -coordinates is hyperbolic w.r.t. $\|\cdot\|_\Gamma$ if and only if the corresponding solution of the original equation is, with invariant projector QPQ^{-1} and the same constants α, β . With regard to (v)

note also that the eigenvalues of the linearisation may change drastically under the transformation (24).

3. Dynamic partition and hyperbolicity of linear systems. Finite-time hyperbolic solutions may be difficult to detect on the sole basis of Definition 1. The main result of this section, Theorem 7 and Corollary 8 below, provides a simple condition guaranteeing finite-time hyperbolicity. This result exploits the *instantaneous* behaviour of (1) as it is encoded in its associated *dynamic partition*. To recall the latter concept, let $\mu : I \rightarrow U$ be a solution of (1) and $y : I \rightarrow \mathbb{R}^d$ any solution of the linearisation (2). Then

$$\frac{d}{dt} \|y(t)\|_{\Gamma}^2 = 2\langle y(t), S_{\Gamma}(t, \mu(t))y(t) \rangle, \quad \forall t \in I,$$

where S_{Γ} , the so-called Γ -strain tensor of (1), is given by

$$S_{\Gamma}(t, x) = \frac{1}{2}(\Gamma D_x f + (D_x f)^{\top} \Gamma);$$

moreover,

$$\frac{d^2}{dt^2} \|y(t)\|_{\Gamma}^2 = 2\langle y(t), M_{\Gamma}(t, \mu(t))y(t) \rangle, \quad \forall t \in I,$$

with the Γ -strain acceleration tensor of (1) defined as

$$M_{\Gamma}(t, x) = D_t S_{\Gamma} + (D_x S_{\Gamma})f + S_{\Gamma}(D_x f) + (D_x f)^{\top} S_{\Gamma}.$$

If $S_{\Gamma}(t, \mu(t))$ is negative or positive definite then neighboring solutions are instantaneously attracted or repelled by μ . If $S_{\Gamma}(t, \mu(t))$ is indefinite then the behaviour of solutions near μ is determined by the sign of $\langle \cdot, M_{\Gamma} \cdot \rangle$ on the Γ -zero-strain set $Z_{\Gamma} := \{y \in \mathbb{R}^d : \langle y, S_{\Gamma} y \rangle = 0\}$. For ease of notation, denote the restriction of the quadratic form $y \mapsto \langle y, M_{\Gamma} y \rangle$ to Z_{Γ} by $M_{Z_{\Gamma}}$ and call this function indefinite, positive definite, and negative definite if it attains, respectively, positive as well as negative values, only positive values, and only negative values on $Z_{\Gamma} \setminus \{0\}$.

Definition 5. Let $\Gamma = \Gamma^{\top} > 0$ and consider the differential equation (1). A point $(t, x) \in I \times U$ is called

- ▷ *attracting* if $S_{\Gamma}(t, x)$ is negative definite;
- ▷ *repelling* if $S_{\Gamma}(t, x)$ is positive definite;
- ▷ *elliptic* if $S_{\Gamma}(t, x)$ is regular but indefinite, and $M_{Z_{\Gamma}}$ is indefinite;
- ▷ *hyperbolic* if $S_{\Gamma}(t, x)$ is regular but indefinite, and $M_{Z_{\Gamma}}$ is positive definite;
- ▷ *quasi-hyperbolic* if $S_{\Gamma}(t, x)$ is regular but indefinite, and $M_{Z_{\Gamma}}$ is negative definite.

The sets of all attracting, repelling, elliptic, hyperbolic, and quasi-hyperbolic points are denoted by $\mathcal{A}_{\Gamma}, \mathcal{R}_{\Gamma}, \mathcal{E}_{\Gamma}, \mathcal{H}_{\Gamma}, \mathcal{Q}_{\Gamma}$, respectively. Points in

$$\mathcal{D}_{\Gamma} := I \times U \setminus (\mathcal{A}_{\Gamma} \cup \mathcal{R}_{\Gamma} \cup \mathcal{E}_{\Gamma} \cup \mathcal{H}_{\Gamma} \cup \mathcal{Q}_{\Gamma})$$

are called *degenerate*.

Remark 6. (i) The open sets $\mathcal{A}_{\Gamma}, \dots, \mathcal{Q}_{\Gamma}$ together with the closed set \mathcal{D}_{Γ} evidently form a partition of $I \times U$, the *dynamic partition* associated with (1). Thus for every $t \in I$ and $\mathcal{T} \in \{\mathcal{A}, \dots, \mathcal{Q}, \mathcal{D}\}$ the t -fibres $\mathcal{T}_{\Gamma}(t) := \{x \in U : (t, x) \in \mathcal{T}\}$ form a partition of U .

(ii) Definition 5 is a slight extension of the EPH partition introduced in [9, 10] for $d \in \{2, 3\}$ and $\Gamma = \text{id}_{d \times d}$. Some basic properties of the *dynamic partition* are discussed in [1, 3, 6]. FIGURE 2 shows typical velocity fields for different parts of the

partition. It is most important to keep in mind that the indicated integral curves (dashed lines) do generally *not* correspond to solutions of (1).

(iii) If (1) describes the Lagrangian motion of a particle within an *incompressible* fluid flow then $\text{trace } S = \text{trace } D_x f = \text{div } f = 0$ and hence $\mathcal{A} = \mathcal{R} = \emptyset$. Furthermore, with $S = [s_{ij}]$,

$$\text{trace } M = \text{trace} \left(\frac{d}{dt} S(t, \mu(t)) \right) + \text{trace} (S D_x f + (D_x f)^\top S) = 2 \sum_{i,j=1}^d s_{ij}^2 \geq 0,$$

which in turn implies that $\langle y, My \rangle \geq 0$ for some non-zero $y \in Z$, and thus $\mathcal{Q} = \emptyset$ as well. Under the assumption of incompressibility and for $\Gamma = \text{id}_{d \times d}$, therefore, in [9, 10] only the elliptic, hyperbolic, and degenerate parts of the dynamic partition play a role. Any point $(t, x) \in I \times U$ for which $S(t, x)$ is regular but indefinite and M_Z is merely positive *semi*-definite is labelled *parabolic* in [9]; according to Definition 5, any such point is degenerate.

(iv) An important aspect of the dynamic partition is its invariance under changes of coordinates (24). For (1) transformed to ξ -coordinates, an argument similar to [1, Lem.2.5] shows that the type of each point (t, ξ) is identical with the type of the corresponding point (t, x) . Thus for any fixed Γ the properties introduced in Definition 5, as well as all statements derived from them, are invariant under any change of coordinates (24); they are Γ -*objective*, cf. [10].

(v) If (1) is linear in x , i.e., if $f(t, x) = A(t)x$ with a C^1 -function $A : I \rightarrow \mathbb{R}^{d \times d}$, then the type of (t, x) is independent of x , or equivalently $\mathcal{T}_\Gamma(t) \in \{\emptyset, \mathbb{R}^d\}$ for every $t \in I$ and $\mathcal{T} \in \{\mathcal{A}, \dots, \mathcal{D}\}$. In this case, it is legitimate to call (1) attracting, repelling, etc. at t if the fibre $\mathcal{A}_\Gamma(t)$, $\mathcal{R}_\Gamma(t)$, etc. equals \mathbb{R}^d .

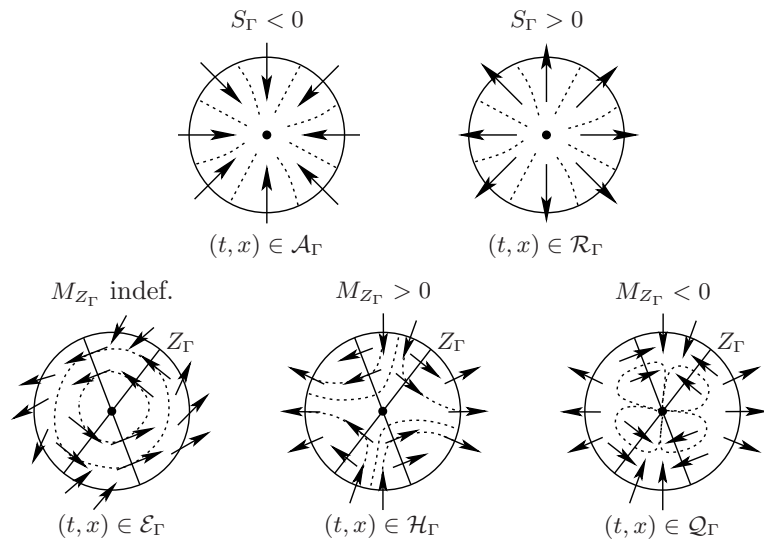


FIGURE 2. Snapshots of velocity fields corresponding to different parts of the dynamic partition: If S_Γ is definite then (t, x) is attracting or repelling (top row); if S_Γ is regular but indefinite then (t, x) is elliptic, hyperbolic or quasi-hyperbolic.

As indicated earlier, the main goal of this section is to provide a simple condition which allows finite-time hyperbolicity to be inferred from the instantaneous

information encoded in the dynamic partition. The linear version of this condition which greatly generalises [2, Thm.2.1] is

Theorem 7. *Let $\Gamma = \Gamma^\top > 0$ and assume that $A : I \rightarrow \mathbb{R}^{d \times d}$ is C^1 . If*

$$\mathcal{E}_\Gamma(t) \cup \mathcal{D}_\Gamma(t) = \emptyset, \quad \forall t \in I, \tag{25}$$

then (3) is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$.

Before turning to the proof of Theorem 7, note that an immediate consequence of the latter is the following generalisation of [10, Thm.1].

Corollary 8. *Let $\Gamma^\top = \Gamma > 0$. Assume the solution $\mu : I \rightarrow U$ of (1) satisfies*

$$\mu(t) \notin \mathcal{E}_\Gamma(t) \cup \mathcal{D}_\Gamma(t), \quad \forall t \in I,$$

that is, μ does not intersect the elliptic or degenerate parts of the dynamic partition. Then μ is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$. In particular, therefore, with $k = \text{rk } P$ for one (and hence every) invariant projector for (2) according to Definition 1, the solution μ has a $(k + 1)$ -dimensional stable and a $(d - k + 1)$ -dimensional unstable manifold as described in Theorem 3.

The proof of Theorem 7 relies on a combination of classical perturbation and approximation results with a simple topological observation concerning certain subsets of the (real) Grassmannian $G_{k,d}$. Since no reference is known to the author a proof of this observation is included. To this end, recall that $G_{k,d}$ is, for each $k \in \{0, \dots, d\}$, defined as the set of all k -dimensional subspaces of \mathbb{R}^d . Elements of $G_{k,d}$ will henceforth be labelled X, Y, \dots . Let π_X be the orthogonal projection onto $X \in G_{k,d}$ and define the distance between any two elements X, Y of $G_{k,d}$ as $\|\pi_X - \pi_Y\|$. With this, $G_{k,d}$ becomes a compact metric space, and in fact a smooth manifold of dimension $k(d - k)$, see e.g. [4]. Denote by Δ_k the diagonal matrix whose first k entries equal 1, and whose remaining $d - k$ entries equal -1 , i.e.

$$\Delta_k = \text{diag}[\underbrace{1, \dots, 1}_{k \text{ times}}, \underbrace{-1, \dots, -1}_{d-k \text{ times}}] \in \mathbb{R}^{d \times d}.$$

For every $|\delta| < 1$ define

$$\mathcal{P}_k^\delta := \{X \in G_{k,d} : \langle \Delta_k x, x \rangle > \delta \langle x, x \rangle \quad \forall x \in X \setminus \{0\}\} \subset G_{k,d}.$$

Clearly, \mathcal{P}_k^δ is open and non-empty, and $\mathcal{P}_k^{\delta_1} \supset \mathcal{P}_k^{\delta_2}$ whenever $\delta_1 \leq \delta_2$, see also FIGURE 3 for the cases $d = 2, 3$.

Lemma 9. *For every $0 < k < d$ and $|\delta| < 1$, the sets \mathcal{P}_k^δ and $\partial \mathcal{P}_k^\delta$ are homeomorphic to, respectively, the open unit ball and the unit sphere in $\mathbb{R}^{k(d-k)}$.*

Proof. Let e_1, \dots, e_d represent the standard basis in \mathbb{R}^d . Given $X \in G_{k,d}$, choose a basis b_1, \dots, b_k of X such that

$$b_j = e_j + \sum_{i=1}^{d-k} \beta_{ij} e_{i+k}, \quad \forall j = 1, \dots, k, \tag{26}$$

with the appropriate matrix $B = [\beta_{ij}] \in \mathbb{R}^{(d-k) \times k}$. Such a choice is possible because $\langle \Delta_k x, x \rangle = -\langle x, x \rangle < \delta \langle x, x \rangle$ whenever $x \in \text{span}\{e_{k+1}, \dots, e_d\} \setminus \{0\}$, and in fact the basis b_1, \dots, b_k is determined uniquely by (26). Thus the map

$$p : \begin{cases} \mathcal{P}_k^\delta & \rightarrow \mathbb{R}^{(d-k) \times k} \\ X & \mapsto [\beta_{ij}] \end{cases}$$

is well defined and one-to-one; clearly, it is also continuous. Moreover, for every $x \in X \setminus \{0\}$, say

$$x = \sum_j x_j b_j = \sum_j x_j e_j + \sum_{i,j} \beta_{ij} x_j e_{i+k},$$

the estimate

$$\sum_j x_j^2 - \sum_{j,r} (B^\top B)_{jr} x_j x_r = \langle \Delta_k x, x \rangle > \delta \langle x, x \rangle = \delta \left(\sum_j x_j^2 + \sum_{j,r} (B^\top B)_{jr} x_j x_r \right)$$

implies that $\|B\| < \sqrt{\frac{1-\delta}{1+\delta}}$, hence $p(\mathcal{P}_k^\delta)$ is contained in an open ball of radius $\sqrt{\frac{1-\delta}{1+\delta}}$ in $\mathbb{R}^{(d-k) \times k}$. Conversely, given any $C = [\gamma_{ij}] \in \mathbb{R}^{(d-k) \times k}$ with $\|C\| < \sqrt{\frac{1-\delta}{1+\delta}}$, define

$$X := \text{span} \left\{ e_j + \sum_{i=1}^{d-k} \gamma_{ij} e_{i+k} : j = 1, \dots, k \right\} \in G_{k,d},$$

and observe that, for every $x = \sum_j x_j e_j + \sum_{i,j} \gamma_{ij} x_j e_{i+k} \in X \setminus \{0\}$,

$$\langle x, x \rangle = \sum_j x_j^2 + \sum_{j,r} (C^\top C)_{jr} x_j x_r < \sum_j x_j^2 + \frac{1-\delta}{1+\delta} \sum_j x_j^2 = \frac{2}{1+\delta} \sum_j x_j^2,$$

but also

$$\begin{aligned} \langle \Delta_k x, x \rangle &= \sum_j x_j^2 - \sum_{j,r} (C^\top C)_{jr} x_j x_r > \sum_j x_j^2 - \frac{1-\delta}{1+\delta} \sum_j x_j^2 = \frac{2\delta}{1+\delta} \sum_j x_j^2 \\ &> \delta \langle x, x \rangle, \end{aligned}$$

showing that $X \in \mathcal{P}_k^\delta$ and $p(X) = C$. As $C \mapsto \text{span} \{e_j + \sum_i \gamma_{ij} e_{i+k} : j = 1, \dots, k\}$ is continuous, p maps \mathcal{P}_k^δ homeomorphically onto an open $\sqrt{\frac{1-\delta}{1+\delta}}$ -ball in $\mathbb{R}^{(d-k) \times k}$. This ball clearly is homeomorphic to the open unit ball (w.r.t. the Euclidean norm) in $\mathbb{R}^{k(d-k)}$.

To prove the statement about $\partial \mathcal{P}_k^\delta$ simply note that

$$\begin{aligned} \partial \mathcal{P}_k^\delta &= \left\{ X \in G_{k,d} : \langle \Delta_k x, x \rangle \geq \delta \langle x, x \rangle \quad \forall x \in X \right. \\ &\quad \left. \text{but } \langle \Delta_k x_0, x_0 \rangle = \delta \langle x_0, x_0 \rangle \text{ for some } x_0 \in X \setminus \{0\} \right\}. \end{aligned}$$

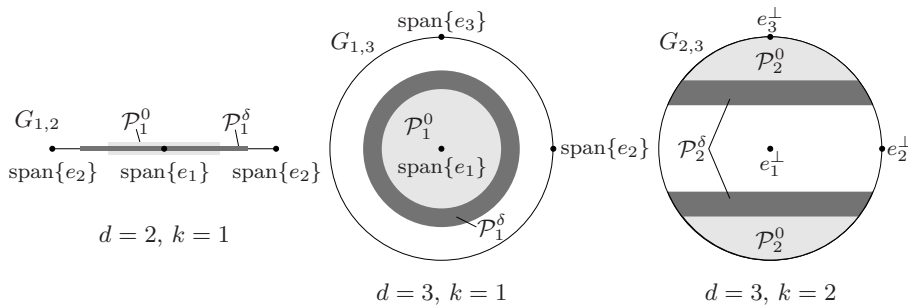


FIGURE 3. Visualising Lemma 9 for $d = 2$ (left) and $d = 3$ with $-1 < \delta < 0$; the Grassmannian $G_{1,2}$ is represented as an interval whose endpoints are identified, whereas $G_{1,3}$ and $G_{2,3}$ are discs with any two diametral points on the periphery identified.

Essentially the same argument as before shows that $\partial\mathcal{P}_k^\delta$ is homeomorphic to a sphere of radius $\sqrt{\frac{1-\delta}{1+\delta}}$ in $\mathbb{R}^{(d-k)\times k}$, and hence to the (Euclidean) unit sphere in $\mathbb{R}^{k(d-k)}$. \square

Corollary 10. *For every $0 < k < d$ and $|\delta| < 1$, the set $\partial\mathcal{P}_k^\delta$ is not a retract of $\overline{\mathcal{P}_k^\delta}$, that is, there does not exist a continuous map $r : \overline{\mathcal{P}_k^\delta} \rightarrow \partial\mathcal{P}_k^\delta$ with $r(X) = X$ for all $X \in \partial\mathcal{P}_k^\delta$.*

Proof of Theorem 7. Note first that by (25) the type of (3) is constant: By the symmetry and continuity of S_Γ and M_Γ , (3) is either attracting, repelling, hyperbolic, or quasi-hyperbolic for all $t \in I$. For the former two cases the claim is easily established. Indeed, if for instance $\mathcal{A}_\Gamma(t) \equiv \mathbb{R}^d$ then, for some $\alpha > 0$,

$$\frac{d}{dt}\|y(t)\|_\Gamma^2 = 2\langle y(t), S_\Gamma y(t) \rangle \leq -2\alpha\|y(t)\|_\Gamma^2, \quad \forall t \in I, \tag{27}$$

and (3) is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$ with $P(t) \equiv \text{id}_{d \times d}$. Similarly, if $\mathcal{R}_\Gamma(t) \equiv \mathbb{R}^d$ then (3) is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$ with $P(t) \equiv 0$. It remains to verify the statement for the hyperbolic and quasi-hyperbolic cases. Since the argument is completely analogous for either case, assume from now on that $\mathcal{H}_\Gamma(t) \equiv \mathbb{R}^d$. Let k denote the number of negative eigenvalues (counted with multiplicity) of S_Γ ; by assumption, k is a constant integer with $0 < k < d$. For the reader's convenience, the proof is divided into two steps. Note beforehand that the evolution operator Φ associated with (3) naturally induces a family of continuous maps on $G_{k,d}$, denoted by the same symbol, that is $\Phi(t, s) : X \mapsto \Phi(t, s)X := \{\Phi(t, s)x : x \in X\}$.

Step 1. Assume for the time being that A is polynomial in t . (This assumption will be dropped in Step 2 below.) In this case, there exist C^1 (in fact, real-analytic) functions $Q : I \rightarrow \mathbb{R}^{d \times d}$ and $\lambda_1, \dots, \lambda_d : I \rightarrow \mathbb{R}^+$ (some of which may coincide) such that $Q(t)$ is orthogonal for every $t \in I$, and

$$Q(t)^\top S_\Gamma(t)Q(t) = \text{diag}[-\lambda_1(t), \dots, -\lambda_k(t), \lambda_{k+1}(t), \dots, \lambda_d(t)], \quad \forall t \in I.$$

(The analyticity of A is essential here, see [13, Thm.II.6.1 and Exp.II.5.3] for details as well as a C^∞ counter-example.) For every $t \in I$, define the invertible matrix

$$M(t) := \text{diag}[\sqrt{\lambda_1(t)}, \dots, \sqrt{\lambda_d(t)}]Q(t)^\top \Phi(t, t_-),$$

and note that, for any $y \in \mathbb{R}^d$,

$$\frac{d}{dt}\|\Phi(t, t_-)y\|_\Gamma^2 = -2\langle \Delta_k M(t)y, M(t)y \rangle.$$

Suppose $X \in G_{k,d}$ and $\alpha > 0$ could be found such that

$$\frac{d}{dt}\|\Phi(t, t_-)\xi\|_\Gamma^2 \leq -2\alpha\|\Phi(t, t_-)\xi\|_\Gamma^2, \quad \forall t \in I, \xi \in X. \tag{28}$$

Clearly, with $P(s)$ denoting any projection onto $\Phi(s, t_-)X$, this would imply that, for every $y \in \mathbb{R}^d$,

$$\|\Phi(t, s)P(s)y\|_\Gamma \leq e^{-\alpha(t-s)}\|P(s)y\|_\Gamma, \quad \forall t \geq s,$$

and hence identify $(\Phi(t, t_-)X)_{t \in I}$ as a possible family of stable subspaces, as required for hyperbolicity on I w.r.t. $\|\cdot\|_\Gamma$. It will now be shown that X and α satisfying (28) do indeed exist.

Assume to the contrary that for every $X \in G_{k,d}$ there exists $t \in I$ such that $\frac{d}{dt}\|\Phi(t, t_-)\xi\|_\Gamma^2 = 0$ for some $\xi \in X \setminus \{0\}$. In particular, therefore, for every $X \in \mathcal{P}_k^0$

one can find $t \in I$ with $M(t)M(t_-)^{-1}X \in \partial\mathcal{P}_k^0$. Define the forward hitting time τ^+ of $\partial\mathcal{P}_k^0$ as

$$\tau^+ : \begin{cases} \overline{\mathcal{P}_k^0} & \rightarrow I, \\ X & \mapsto \inf\{t \geq t_- : M(t)M(t_-)^{-1}X \in \partial\mathcal{P}_k^0\}. \end{cases}$$

Clearly, $\tau^+(X) = t_-$ for every $X \in \partial\mathcal{P}_k^0$, and being the hitting time of a closed set, the function τ^+ is l.s.c.; also, $M(\tau^+(X))M(t_-)^{-1}X \in \partial\mathcal{P}_k^0$. Given $X \in \mathcal{P}_k^0$, let $\xi \in X \setminus \{0\}$ such that

$$\frac{d}{dt}\|\Phi(t, t_-)M(t_-)^{-1}\xi\|_\Gamma^2|_{t=\tau^+(X)} = 0.$$

Hyperbolicity implies that $\frac{d^2}{dt^2}\|\Phi(t, t_-)M(t_-)^{-1}\xi\|_\Gamma^2|_{t=\tau^+(X)} > 0$, and hence

$$\langle \Delta_k M(t)M(t_-)^{-1}\xi, M(t)M(t_-)^{-1}\xi \rangle < 0$$

whenever $t - \tau^+(X) > 0$ is sufficiently small. For every $\varepsilon > 0$, therefore,

$$\tau^+(Y) < \tau^+(X) + \varepsilon,$$

provided that Y is sufficiently close to X , i.e., the function τ^+ is u.s.c. as well. Consequently, the map

$$T^+ : \begin{cases} \overline{\mathcal{P}_k^0} & \rightarrow \partial\mathcal{P}_K^0, \\ X & \mapsto M(\tau^+(X))M(t_-)^{-1}X, \end{cases}$$

is continuous, and $T^+(X) = X$ for all $X \in \partial\mathcal{P}_K^0$. This contradicts Corollary 10 and shows in turn that there exists $\delta > 0$ and $Y \in \mathcal{P}_k^\delta$ such that $M(t)M(t_-)^{-1}Y \in \mathcal{P}_k^\delta$ holds for all $t \in I$. For every $\xi \in M(t_-)^{-1}Y$ and $t \in I$,

$$\begin{aligned} \frac{d}{dt}\|\Phi(t, t_-)\xi\|_\Gamma^2 &= -2\langle \Delta_k M(t)\xi, M(t)\xi \rangle \leq -2\delta\|M(t)\xi\|^2 \leq -2\delta\lambda_{\min}\|\Phi(t, t_-)\xi\|^2 \\ &\leq -2\delta\lambda_{\min}\gamma_{\max}^{-1}\|\Phi(t, t_-)\xi\|_\Gamma^2, \end{aligned}$$

where $\lambda_{\min} := \min_{t \in I} \min_i \lambda_i(t) > 0$, and $\gamma_{\max} > 0$ is the largest eigenvalue of Γ . Thus (28) holds for $X = M(t_-)^{-1}Y$ and $\alpha = \delta\lambda_{\min}\gamma_{\max}^{-1} > 0$.

A completely analogous argument yields the existence of $Y \in G_{d-k, d}$ and $\beta > 0$ such that

$$\frac{d}{dt}\|\Phi(t, t_+)\eta\|_\Gamma^2 \geq 2\beta\|\Phi(t, t_+)\eta\|_\Gamma^2, \quad \forall t \in I, \eta \in Y,$$

and therefore also

$$\|\Phi(t, s)(\text{id}_{d \times d} - \tilde{P}(s))\|_\Gamma \leq e^{\beta(t-s)}\|\text{id}_{d \times d} - \tilde{P}(s)\|_\Gamma, \quad \forall t \leq s,$$

with $\tilde{P}(s)$ denoting any projection along $\Phi(s, t_+)Y$. Overall, therefore, if $P(t)$ is chosen as the projection onto $\Phi(t, t_-)X$ along $\Phi(t, t_+)Y$, then (3) is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$.

Step 2. To complete the proof, drop the assumption that A be polynomial in t ; hence $A : I \rightarrow \mathbb{R}^{d \times d}$ is merely assumed to be C^1 from now on. Given $\varepsilon > 0$, choose $A_\varepsilon : I \rightarrow \mathbb{R}^{d \times d}$ such that A_ε is polynomial in t and

$$\|\Phi_\varepsilon(t, t_-) - \Phi(t, t_-)\|_\Gamma + \|S_\Gamma^{(\varepsilon)}(t) - S_\Gamma(t)\|_\Gamma + \|M_\Gamma^{(\varepsilon)}(t) - M_\Gamma(t)\|_\Gamma < \varepsilon, \quad \forall t \in I;$$

here Φ_ε , $S_\Gamma^{(\varepsilon)}$, and $M_\Gamma^{(\varepsilon)}$ denote, respectively, the evolution operator, the Γ -strain tensor, and the Γ -strain acceleration tensor associated with $\dot{x} = A_\varepsilon(t)x$. As shown

in Step 1, for all sufficiently small $\varepsilon > 0$, there exists $\alpha_\varepsilon > 0$ and $X_\varepsilon \in G_{k,d}$ such that

$$\frac{d}{dt} \|\Phi_\varepsilon(t, t_-)x\|_\Gamma^2 \leq -2\alpha_\varepsilon \|\Phi_\varepsilon(t, t_-)x\|_\Gamma^2, \quad \forall t \in I, x \in X_\varepsilon. \tag{29}$$

Since $G_{k,d}$ is compact, it is possible to choose (ε_n) with $\varepsilon_n \rightarrow 0$ such that $X_{\varepsilon_n} \rightarrow X_0$ for some $X_0 \in G_{k,d}$. Clearly $\frac{d}{dt} \|\Phi(t, t_-)x\|_\Gamma^2 \leq 0$ for all $t \in I, x \in X_0$. Suppose $\frac{d}{dt} \|\Phi(t, t_-)x_0\|_\Gamma^2|_{t=t_0} = 0$ for some $t_0 \in I, x_0 \in X_0 \setminus \{0\}$. Since (3) is hyperbolic at t_0 , for all $t - t_0 > 0$ sufficiently small,

$$\frac{d}{dt} \|\Phi(t, t_-)x_0\|_\Gamma^2 > 0.$$

Obviously, this contradicts (29) for small ε . (Strictly speaking, for this argument to work in the case $t_0 = t_+$ also, (3) has to be extended slightly to the right.) Consequently, $\frac{d}{dt} \|\Phi(t, t_-)x\|_\Gamma^2 < 0$ for all $t \in I, x \in X_0 \setminus \{0\}$, and hence

$$\frac{d}{dt} \|\Phi(t, t_-)x\|_\Gamma^2 \leq -2\alpha_0 \|\Phi(t, t_-)x\|_\Gamma^2, \quad \forall t \in I, x \in X_0,$$

with some $\alpha_0 > 0$. Since a completely analogous argument again yields the existence of a $(d - k)$ -dimensional unstable space, the proof is complete. \square

Example 11. To compare Theorem 7 with [7, Thm.1] and [11, Thm.1] which are based on estimates on the time-dependent eigenstructure of $D_x f$, consider

$$\dot{x} = \begin{bmatrix} a & 2 \\ 0 & -a \end{bmatrix} x, \tag{30}$$

where $a : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 ; thus

$$S(t) = \begin{bmatrix} a & 1 \\ 1 & -a \end{bmatrix}, \quad M(t) = \begin{bmatrix} \dot{a} + 2a^2 & 2a \\ 2a & -\dot{a} + 4 + 2a^2 \end{bmatrix},$$

and $\mathcal{A}(t) \equiv \mathcal{R}(t) \equiv \emptyset$ because $\det S(t) = -(a^2 + 1) < 0$. The quantities required to apply [7, Thm.1] and [11, Thm.1] are, with the notation used in these papers,

$$\lambda_{\min} = \lambda_{1\min} = \lambda_{2\min} = \min_{t \in I} |a(t)|, \quad \alpha = \frac{\lambda_{\min}}{\sqrt{1 + \lambda_{\min}^2}}, \quad \beta = \max_{t \in I} \frac{|\dot{a}(t)|}{1 + a^2(t)}.$$

(i) For $a(t) = e^t$ it is checked easily that the matrix M is positive definite for all $t \in \mathbb{R}$, and hence $\mathcal{H}(t) \equiv \mathbb{R}^2$. By Theorem 7, (30) is hyperbolic w.r.t. $\|\cdot\|$ on every interval $I \subset \mathbb{R}$; in particular, every solution is hyperbolic and comes with stable and unstable manifolds according to Theorem 3. By contrast, [7, Thm.1(i)] guarantees the existence of (un)stable manifolds for (30) only for

$$2\sqrt{2}\beta < \frac{e^{2t_-}}{\sqrt{1 + e^{2t_-}}}, \tag{31}$$

and finite-time hyperbolicity of (30) on I whenever (31) holds with $2\sqrt{2}$ replaced by $2 + \sqrt{2}$. With

$$2\beta = \begin{cases} (\cosh t_+)^{-1} & \text{if } t_+ < 0, \\ 1 & \text{if } 0 \in I, \\ (\cosh t_-)^{-1} & \text{if } t_- > 0, \end{cases}$$

this gives the restrictions

$$t_- > \frac{1}{2} \log \frac{\sqrt{33} - 1}{2} \approx 0.4319 \quad \text{and} \quad t_- > \frac{1}{2} \log \frac{\sqrt{25 + 16\sqrt{2}} - 1}{2} \approx 0.5410$$

for, respectively, the existence of (un)stable manifolds and finite-time hyperbolicity of (30). Note that the cited results are inconclusive whenever $I \subset \mathbb{R}^-$.

(ii) Choose $a(t) = \tanh(2t)$. In this case, [7, Thm.1] and [11, Thm.1] do not apply at all if $0 \in I$. (As in (i), it is possible to give explicit expressions for the numbers $t_{ii} > t_i > 0$ such that [7, Thm.1(*)] applies for $* = i, ii$ precisely if $[-t_*, t_*] \cap I = \emptyset$; numerically, $t_i \approx 0.7331$ and $t_{ii} \approx 0.7741$.) On the other hand, it follows from the positive definiteness of

$$M(t) = \begin{bmatrix} 2 & 2a(t) \\ 2a(t) & 2 + 2a^2(t) \end{bmatrix}$$

for all $t \in \mathbb{R}$ that $\mathcal{H}(t) \equiv \mathbb{R}^2$, and again (30) is hyperbolic w.r.t. $\|\cdot\|$ on every $I \subset \mathbb{R}$.

(iii) Let $a(t) = 6t$. With $\Gamma = \text{id}_{2 \times 2}$, a straightforward calculation shows that (30) is elliptic at t whenever $|t| < t_0$, and hyperbolic for $|t| > t_0$, where $t_0 = \frac{1}{6}\sqrt{\rho}$ with ρ denoting the real root of $\rho^3 + 2\rho^2 + 7\rho - 3 = 0$; numerically, $t_0 \approx 0.1027$. Therefore Theorem 7 applies only if $[-t_0, t_0] \cap I = \emptyset$. Similarly, [7, Thm.1(*)] applies for $* = i, ii$ whenever $[-t_*, t_*] \cap I = \emptyset$, where $t_i \approx 0.4267$ and $t_{ii} \approx 0.4338$. Thus in this example the time intervals excluded by [7, Thm.1] are significantly larger than the one excluded on the basis of Theorem 7. Choosing Γ appropriately may make the latter disappear altogether. Indeed, with $\Gamma = \text{diag}[3, 1]$,

$$S_\Gamma = 3 \begin{bmatrix} 6t & 1 \\ 1 & -2t \end{bmatrix}, \quad M_\Gamma = 6 \begin{bmatrix} 3 + 36t^2 & 6t \\ 6t & 1 + 12t^2 \end{bmatrix},$$

and $\det S_\Gamma = -9(12t^2 + 1) < 0$ together with $M_\Gamma > 0$ shows that (30) is hyperbolic w.r.t. $\|\cdot\|_\Gamma$ on every interval $I \subset \mathbb{R}$.

Remark 12. (i) Theorem 7 has a partial converse in the attracting or repelling case, as follows immediately from (27): If (3) is hyperbolic on I w.r.t. $\|\cdot\|_\Gamma$ with $\text{rk } P = d$ (or $\text{rk } P = 0$) then $\mathcal{A}_\Gamma(t) \equiv \mathbb{R}^d$ (or $\mathcal{R}_\Gamma(t) \equiv \mathbb{R}^d$). A similar converse does not hold in general for $0 < d < k$. As a simple example consider the autonomous equation

$$\dot{x} = Ax \quad \text{with} \quad A = \begin{bmatrix} -\alpha & -2 \\ 0 & -2 - \alpha \end{bmatrix}, \tag{32}$$

where $\alpha > 0$, and let $\Gamma = \text{id}_{2 \times 2}$. An elementary calculation confirms that $\mathcal{E}(t) \equiv \mathbb{R}^2$ whenever $\alpha < \sqrt{2} - 1$. On the other hand, it follows from [3, Exp.14] that (32) is hyperbolic on I w.r.t. $\|\cdot\|$, provided that

$$t_+ - t_- < T_\alpha \quad \text{with} \quad 2T_\alpha = \log \frac{1 + (1 + \alpha)\sqrt{1 - 2\alpha - \alpha^2}}{2\alpha + \alpha^2} > 0,$$

that is, on sufficiently short time intervals. (Note that $T_\alpha = -\frac{1}{2} \log \alpha + \mathcal{O}(\alpha)$ as $\alpha \searrow 0$.) Put differently, for one (and hence every) solution of (32) to be hyperbolic, the total time spent in the elliptic domain \mathcal{E} must not exceed T_α . Assuming incompressibility and $d = 2$, a similar observation yields a characterisation of Lagrangian elliptic behaviour, see [9, Thm.2]. Even though the latter result does not apply to (32), it seems plausible that quite generally a (Lagrangian) hyperbolic solution can spend only a limited amount of time in the elliptic or degenerate regions.

(ii) Theorem 7 may fail even if (25) is violated only for a single $t \in I$, as can be seen from the simple example $A(t) = \text{diag}[t, -t]$ for which $\mathcal{H}_\Gamma(t) = \mathbb{R}^2$ for all $t \neq 0$, yet (3) is not hyperbolic w.r.t. $\|\cdot\|_\Gamma$ on any interval containing 0.

(iii) In view of Example 11 it may be conjectured that $\mu(t) \in \mathcal{H}(t)$ for all $t \in I$ holds automatically whenever μ meets the conditions of [7, Thm.1(ii)]. At the time

of writing the author does not know of any proof of, or counter-example to, this conjecture.

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REFERENCES

- [1] A. Berger, T. S. Doan and S. Siegmund, *Nonautonomous finite-time dynamics*, Discrete Continuous Dynam. Systems - B, **9** (2008), 463–492.
- [2] A. Berger, T. S. Doan and S. Siegmund, *A remark on finite-time hyperbolicity*, PAMM Proc. Appl. Math. Mech., **8** (2008), 10917–10918.
- [3] A. Berger, T. S. Doan and S. Siegmund, *A definition of spectrum for differential equations on finite time*, J. Differential Equations, **246** (2009), 1098–1118.
- [4] M. Berger and B. Gostiaux, “Differential Geometry: Manifolds, Curves, and Surfaces,” Springer, 1988.
- [5] A. Coppel, “Dichotomies in Stability Theory,” Lecture Notes in Mathematics **629**, Springer, 1978.
- [6] L. H. Duc and S. Siegmund, *Hyperbolicity and invariant manifolds for planar nonautonomous systems on finite time intervals*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **18** (2008), 641–674.
- [7] G. Haller, *Finding finite-time invariant manifolds in two-dimensional velocity fields*, Chaos, **10** (2000), 99–108.
- [8] G. Haller, *Distinguished material surfaces and coherent structures in three-dimensional fluid flows*, Physica D, **149** (2001), 248–277.
- [9] G. Haller, *Lagrangian structures and the rate of strain in a partition of two-dimensional turbulence*, Physics of Fluids, **13** (2001), 3365–3385.
- [10] G. Haller, *An objective definition of a vortex*, J. Fluid Mech., **525** (2005), 1–26.
- [11] G. Haller and G. Yuan, *Lagrangian coherent structures and mixing in two-dimensional turbulence*, Physica D, **147** (2000), 352–370.
- [12] M. C. Irwin, “Smooth Dynamical Systems,” World Scientific, 2001.
- [13] T. Kato, “Perturbation Theory for Linear Operators,” Springer, 1980.
- [14] A. Katok and B. Hasselblatt, “Introduction to the Modern Theory of Dynamical Systems,” Cambridge University Press, 1995.
- [15] K. Palmer, “Shadowing in Dynamical Systems. Theory and Applications,” Kluwer, 2000.
- [16] R. M. Samelson and S. Wiggins, “Lagrangian Transport in Geophysical Jets and Waves. The Dynamical Systems Approach,” Springer, 2006.
- [17] S. C. Shadden, F. Lekien and J. E. Marsden, *Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows*, Physica D, **212** (2005), 271–304.
- [18] F. Verhulst, “Nonlinear Differential Equations and Dynamical Systems,” Springer, 1990.

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