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# More on finite-time hyperbolicity

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**Abstract**

A solution of a nonautonomous ordinary differential equation is finite-time hyperbolic, i.e. hyperbolic on a compact interval of time, if the linearisation along that solution exhibits a strong exponential dichotomy. In analogy to classical asymptotic facts, it is shown that finite-time hyperbolicity is robust, that is, it persists under small perturbations. Eigenvalues and -vectors may be misleading with regards to hyperbolicity. This is demonstrated by means of simple examples.

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Hyperbolicity is widely recognised as a fundamental notion of dynamical systems theory. While extensions and refinements of the classical, that is, asymptotic concept continue to play a vital role in modern dynamics, much attention has recently been drawn to the systematic study of suitable finite-time analogues. This note contributes to *finite-time dynamics* a brief discussion of two practical aspects of the hyperbolicity concept developed and utilised e.g. in [1, 3, 4, 6, 8].

## 1 Hyperbolicity is robust

Consider the nonautonomous ordinary differential equation

$$\dot{x} = f(t, x)\,, \tag{1}$$

where $f : I \times U \to \mathbb{R}^d$ is $C^1$, $I = [t_-, t_+]$ with $-\infty < t_- < t_+ < +\infty$, and $U \subset \mathbb{R}^d$ is a non-empty open set. The linearisation of (1) along any solution $\mu : I \to U$ is

$$\dot{y} = D_x f\big(t, \mu(t)\big) y\,. \tag{2}$$

To quantify growth and decay of solutions of (2), arbitrary inner product norms $\|\cdot\|_\Gamma = \sqrt{\langle\cdot,\Gamma\cdot\rangle}$ are considered, where $\Gamma \in \mathbb{R}^{d\times d}$ is any symmetric positive definite matrix, i.e. $\Gamma^\top = \Gamma > 0$, and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^d$; the symbol $\|\cdot\|_\Gamma$ also denotes the induced norm on $\mathbb{R}^{d\times d}$. Quantities depending on $\Gamma$ have their dependence made explicit by a subscript which is suppressed only if $\Gamma$ equals $\mathrm{id}_{d\times d}$, the $d\times d$ identity matrix.

To define finite-time hyperbolicity, instead of (2) consider more generally any nonautonomous *linear* equation

$$\dot{y} = A(t)y\,, \tag{3}$$

where $A : I \to \mathbb{R}^{d\times d}$ is continuous. Let $\Phi : I\times I \to \mathbb{R}^{d\times d}$ denote the associated evolution operator, i.e., $y : t \mapsto \Phi(t,s)\eta$ is, for any $\eta\in\mathbb{R}^d$, the unique solution of (3) satisfying $y(s) = \eta$. A projection-valued function $P : I \to \mathbb{R}^{d\times d}$ is an *invariant projector* for (3) if $P(t)\Phi(t,s) = \Phi(t,s)P(s)$ for all $t,s\in I$. Note that $t\mapsto P(t)$ is continuous, and $\mathrm{rk}P(t)$ is constant, for any invariant projector.

**Definition 1** *Let $\Gamma^\top = \Gamma > 0$. Equation* (3) *is* hyperbolic (*on $I$ w.r.t. $\|\cdot\|_\Gamma$*) *if there exists an invariant projector $P$ for* (3)*, together with constants $\alpha,\beta>0$, such that for every $y\in\mathbb{R}^d$,*

$$\big\|\Phi(t,s)P(s)y\big\|_\Gamma \le e^{-\alpha(t-s)}\big\|P(s)y\big\|_\Gamma\,,\ \ \forall t\ge s\,, \tag{4}$$

$$\big\|\Phi(t,s)\big(\mathrm{id}_{d\times d}-P(s)\big)y\big\|_\Gamma \le e^{\beta(t-s)}\big\|\big(\mathrm{id}_{d\times d}-P(s)\big)y\big\|_\Gamma\,,\ \ \forall t\le s\,. \tag{5}$$

*A solution $\mu$ of* (1) *is hyperbolic if the associated linearisation* (2) *is hyperbolic.*

The estimates in Definition 1 incorporate a finite-time variant of the classical notion of an *exponential dichotomy* that is more restrictive than the latter because an arbitrary multiplicative constant on the right-hand side of (4) or (5) would render the concept meaningless. Consequently, to establish the robustness of finite-time hyperbolicity, classical arguments using Gronwall-type estimates (see e.g. [10]) do not apply directly. Instead, the alternative argument presented in Lemma 3 below makes use of [3, Lem.9], restated here as

**Proposition 2** *Equation* (3) *is hyperbolic on $I$ w.r.t. $\|\cdot\|_\Gamma$, with invariant projector $P$ and constants $\alpha,\beta>0$, if and only if, for all $t\in I$ and $y\in\mathbb{R}^d$,*

$$\frac{d}{dt}\|\Phi(t,t_-)P(t_-)y\|_\Gamma \le -\alpha\|\Phi(t,t_-)P(t_-)y\|_\Gamma\,, \tag{6}$$

*as well as*

$$\frac{d}{dt}\big\|\Phi(t,t_-)\big(\mathrm{id}_{d\times d}-P(t_-)\big)y\big\|_\Gamma \ge \beta\big\|\Phi(t,t_-)\big(\mathrm{id}_{d\times d}-P(t_-)\big)y\big\|_\Gamma\,. \tag{7}$$

**Lemma 3** *Let $A,\widetilde{A} : I \to \mathbb{R}^{d\times d}$ be continuous, and assume* (3) *is hyperbolic, with constants $\alpha,\beta>0$. Then, given $0<\widetilde{\alpha}<\alpha$, $0<\widetilde{\beta}<\beta$, there exists $\delta>0$ such that*

$$\dot{y} = \widetilde{A}(t)y \tag{8}$$

*is hyperbolic as well, with constants $\widetilde{\alpha},\widetilde{\beta}$, whenever $\max_{t\in I}\|\widetilde{A}(t)-A(t)\|_\Gamma<\delta$.*

*Proof*. For every continuous $B : I \to \mathbb{R}^{d\times d}$, let $\|B\|_\infty := \max_{t\in I} \|B(t)\|_\Gamma$, and denote by $\Phi$ and $\widetilde{\Phi}$ the evolution operators associated with (3) and (8), respectively. Also, recall the trivial estimate

$$e^{-|t-s|\|A\|_\infty}\|y\|_\Gamma \le \|\Phi(t,s)y\|_\Gamma \le e^{|t-s|\|A\|_\infty}\|y\|_\Gamma\,, \quad \forall t,s\in I\,, \tag{9}$$

and note that $\widetilde{P} : t \mapsto \widetilde{\Phi}(t,t_-)P(t_-)\widetilde{\Phi}(t,t_-)^{-1}$ is an invariant projector for (8). For the latter equation, the variation of constants formula yields

$$\widetilde{\Phi}(t,t_-) - \Phi(t,t_-) = \int_{t_-}^{t} \Phi(t,\tau)\big(\widetilde{A}(\tau) - A(\tau)\big)\widetilde{\Phi}(\tau,t_-)\,\mathrm{d}\tau\,,$$

which together with (9) implies that, for all $t \in I$,

$$\begin{aligned}\big\|\widetilde{\Phi}(t,t_-) - \Phi(t,t_-)\big\|_\Gamma &\le \int_{t_-}^{t} e^{(t-\tau)\|A\|_\infty}\|\widetilde{A}-A\|_\infty e^{(\tau-t_-)\|\widetilde{A}\|_\infty}\mathrm{d}\tau \\ &\le e^{(t-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty \int_{t_-}^{t} e^{(\tau-t_-)\|\widetilde{A}-A\|_\infty}\mathrm{d}\tau \\ &\le e^{(t-t_-)\|A\|_\infty}\left(e^{(t-t_-)\|\widetilde{A}-A\|_\infty} - 1\right) \\ &\le 2(t-t_-)e^{(t-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty\,,\end{aligned}$$

provided that $\|\widetilde{A}-A\|_\infty < \delta_1 := (t_+ - t_-)^{-1}$. Given $y \in \mathbb{R}^d$, define the two $C^1$-functions $\phi, \widetilde{\phi} : I \to \mathbb{R}$ as

$$\phi : t \mapsto \tfrac{1}{2}\|\Phi(t,t_-)P(t_-)y\|_\Gamma^2\,, \quad \widetilde{\phi} : t \mapsto \tfrac{1}{2}\|\widetilde{\Phi}(t,t_-)P(t_-)y\|_\Gamma^2\,.$$

For notational convenience, let $\eta = P(t_-)y$. It follows from the estimate

$$\begin{aligned}\big|\dot{\widetilde{\phi}} - \dot{\phi}\big| &= \big|\langle\Gamma\widetilde{A}\widetilde{\Phi}\eta, \widetilde{\Phi}\eta\rangle - \langle\Gamma A\Phi\eta, \Phi\eta\rangle\big| \\ &\le \big|\langle\Gamma(\widetilde{A}-A)\widetilde{\phi}\eta, \widetilde{\phi}\eta\rangle\big| + \big|\langle\Gamma A\widetilde{\Phi}\eta, \widetilde{\Phi}\eta\rangle - \langle\Gamma A\Phi\eta, \Phi\eta\rangle\big| \\ &\le 2\|\widetilde{A}-A\|_\infty\widetilde{\phi} + \|A\|_\infty\|(\widetilde{\Phi}-\Phi)\eta\|_\Gamma(\|\widetilde{\Phi}\eta\|_\Gamma + \|\Phi\eta\|_\Gamma) \\ &\le 2\|\widetilde{A}-A\|_\infty\widetilde{\phi} + \|A\|_\infty(t_+-t_-)e^{(t_+-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty\|\eta\|_\Gamma(\sqrt{8\widetilde{\phi}} + \sqrt{8\phi}) \\ &\le 2\|\widetilde{A}-A\|_\infty\left(\widetilde{\phi} + 2\|A\|_\infty(t_+-t_-)e^{(t_+-t_-)(2\|A\|_\infty + \|\widetilde{A}-A\|_\infty)}(\widetilde{\phi}+\phi)\right)\,,\end{aligned}$$

which is valid whenever $\|\widetilde{A}-A\|_\infty < \delta_1$, that

$$\big|\dot{\widetilde{\phi}}(t) - \dot{\phi}(t)\big| \le C\|\widetilde{A}-A\|_\infty\big(\widetilde{\phi}(t) + \phi(t)\big)\,, \quad \forall t \in I\,,$$

where $C$ depends only on $t_+ - t_- + \|A\|_\infty$. With Proposition 2, therefore,

$$\begin{aligned}\dot{\widetilde{\phi}} &\le \dot{\phi} + C\|\widetilde{A}-A\|_\infty(\widetilde{\phi}+\phi) \;\le\; -2\alpha\phi + C\|\widetilde{A}-A\|_\infty(\widetilde{\phi}+\phi) \\ &\le -2(\alpha - C\|\widetilde{A}-A\|_\infty)\widetilde{\phi} + (2\alpha + C\|\widetilde{A}-A\|_\infty)\big|\widetilde{\phi}-\phi\big|\,,\end{aligned} \tag{10}$$

whenever $\|\widetilde{A}-A\|_\infty < \delta_1$. Under the latter condition, observe that also

$$\begin{aligned}
\big|\widetilde{\phi}-\phi\big| &\le \tfrac{1}{2}\|(\widetilde{\phi}-\phi)\eta\|_\Gamma(\|\widetilde{\phi}\eta\|_\Gamma+\|\phi\eta\|_\Gamma)\\
&\le (t_+-t_-)e^{(t_+-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty\|\eta\|_\Gamma\left(\sqrt{2\widetilde{\phi}}+\sqrt{2\phi}\right)\\
&\le 2(t_+-t_-)e^{(t_+-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty\left(e^{(t_+-t_-)\|\widetilde{A}\|_\infty}\widetilde{\phi}+e^{(t_+-t_-)\|A\|_\infty}\phi\right)\\
&\le 2(t_+-t_-)e^{1+2(t_+-t_-)\|A\|_\infty}\|\widetilde{A}-A\|_\infty(\widetilde{\phi}+\phi)\\
&\le 2C\|\widetilde{A}-A\|_\infty\widetilde{\phi}+C\|\widetilde{A}-A\|_\infty\big|\widetilde{\phi}-\phi\big|\,,
\end{aligned}$$

which in turn implies that

$$\big|\widetilde{\phi}(t)-\phi(t)\big| \le 4C\|\widetilde{A}-A\|_\infty\widetilde{\phi}(t)\,,\quad \forall t\in I\,, \tag{11}$$

provided that $\|\widetilde{A}-A\|_\infty < \delta_2 := (2C)^{-1} < \delta_1$. Combining (10) and (11) yields

$$\dot{\widetilde{\phi}}(t) \le -2\big(\alpha-2C(1+2\alpha)\|\widetilde{A}-A\|_\infty\big)\widetilde{\phi}(t)\,,\quad \forall t\in I\,,$$

whenever $\|\widetilde{A}-A\|_\infty < \delta_2$. With $\delta := \dfrac{\min(1,\alpha-\widetilde{\alpha})}{2C(1+2\alpha)} > 0$ therefore $\|\widetilde{A}-A\|_\infty < \delta$ implies that $\dot{\widetilde{\phi}}(t) \le -2\widetilde{\alpha}\widetilde{\phi}(t)$ for all $t\in I$. This establishes $(\widetilde{6})$. A completely analogous argument proves $(\widetilde{7})$. Overall, $\|\widetilde{A}-A\|_\infty < \delta$ ensures that (8) is hyperbolic on $I$ w.r.t. $\|\cdot\|_\Gamma$, with invariant projector $\widetilde{P}$ and constants $\widetilde{\alpha},\widetilde{\beta}$. □

**Remark 4** (i) Note that $\delta$ in Lemma 3 depends only on $\alpha-\widetilde{\alpha}$, $\beta-\widetilde{\beta}$, and $t_+-t_-+\|A\|_\infty$. Usually, it is not possible to choose $\widetilde{\alpha}=\alpha$ or $\widetilde{\beta}=\beta$, not even if (3) and (8) are autonomous.

(ii) It was shown in [3, Exp.24] that, perhaps somewhat surprisingly,

$$\dot{y} = \begin{bmatrix} 0 & 1\\ 0 & 0\end{bmatrix} y$$

is hyperbolic for every $I$ and $\Gamma$. Thus, by Lemma 3,

$$\dot{y} = \begin{bmatrix} a_1 & 1\\ a_2 & a_3\end{bmatrix} y \tag{12}$$

is hyperbolic as well, provided that $\max_{t\in I}\sum_{i=1}^3|a_i(t)|$ is sufficiently small for the continuous functions $a_1,a_2,a_3 : I\to\mathbb{R}$. If so, even though the (possibly time-dependent) eigenvalues of (12) may be both positive or negative, the rank of any invariant projector according to Definition 1 equals *one.*

The desired robustness result is an immediate consequence of Lemma 3. It asserts that hyperbolicity according to Definition 1 is robust under variations of the initial data and $C^1$-small perturbations of the right-hand side in (1).

**Theorem 5** *Assume the solution $\mu$ of* (1) *is hyperbolic on $I$ w.r.t. $\|\cdot\|_\Gamma$. Then there exists $\delta > 0$ such that for every $C^1$-function $\widetilde{f} : I \times U \to \mathbb{R}^d$ with*

$$\sup_{t\in I}\Big( \big\|\widetilde{f}\big(t,\mu(t)\big) - f\big(t,\mu(t)\big)\big\|_\Gamma + \big\|D_x\widetilde{f}\big(t,\mu(t)\big) - D_x f\big(t,\mu(t)\big)\big\|_\Gamma \Big) < \delta\,, \quad (13)$$

*every solution $\widetilde{\mu} : I \to U$ of*

$$\dot{x} = \widetilde{f}(t,x) \qquad (14)$$

*is hyperbolic as well, provided that $\|\widetilde{\mu}(t_0) - \mu(t_0)\|_\Gamma < \delta$ for some $t_0 \in I$.*

*Proof.* Given $\varepsilon > 0$, choose $\delta_1 > 0$ so small that

$$T_{\delta_1} := \big\{(t,x) : t \in I, \|x - \mu(t)\|_\Gamma \le \delta_1\big\} \subset I \times U$$

and $\|D_x\widetilde{f}(t,x) - D_x\widetilde{f}(t,y)\|_\Gamma < \frac{1}{2}\varepsilon$ whenever $x, y \in T_{\delta_1}$ and $\|x-y\|_\Gamma < \delta_1$. Also, pick $\delta_2 > 0$ small enough to ensure that $\max_{t\in I} \|\widetilde{f}\big(t,\mu(t)\big) - f\big(t,\mu(t)\big)\|_\Gamma < \delta_2$ and $\|x_0 - \mu(t_0)\| < \delta_2$ for some $t_0 \in I$ imply that the solution of (14) with $x(t_0) = x_0$ exists for all $t \in I$ and satisfies $\max_{t\in I} \|x(t) - \mu(t)\|_\Gamma < \delta_1$. With $\delta := \min(\frac{1}{2}\varepsilon, \delta_1, \delta_2)$, it follows from (13) that

$$\begin{aligned} &\big\|D_x\widetilde{f}\big(t,\widetilde{\mu}(t)\big) - D_x f\big(t,\mu(t)\big)\big\|_\Gamma \\ &\qquad \le \big\|D_x\widetilde{f}\big(t,\widetilde{\mu}(t)\big) - D_x\widetilde{f}\big(t,\mu(t)\big)\big\|_\Gamma + \big\|D_x\widetilde{f}\big(t,\mu(t)\big) - D_x f\big(t,\mu(t)\big)\big\|_\Gamma \\ &\qquad \le \tfrac{1}{2}\varepsilon + \delta < \varepsilon\,, \end{aligned}$$

if only $\|\widetilde{\mu}(t_0) - \mu(t_0)\|_\Gamma < \delta$ for some $t_0 \in I$. Since $\varepsilon > 0$ was arbitrary, Lemma 3 applies with $A(t) = D_x f\big(t,\mu(t)\big)$ and $\widetilde{A}(t) = D_x\widetilde{f}\big(t,\widetilde{\mu}(t)\big)$. □

## 2 How (not) to detect hyperbolicity

If the system (3) is *autonomous*, then it has a (classical) exponential dichotomy if and only if no eigenvalue of $A$ lies on the imaginary axis. It thus seems natural to use eigenvalues as a tool to detect hyperbolicity: If the eigenvalues and -vectors vary sufficiently little over time then, hopefully, some insight concerning finite-time behaviour can be gained from them. In this spirit and for $d = 2$ and $\Gamma = \mathrm{id}_{2\times 2}$, [6, Thm.1] and [9, Thm.1] present conditions on the spectral data of $A$ that ensure finite-time hyperbolicity.

Relying on spectral data in a finite-time context does have its pitfalls, though. This fact, already hinted at by Remark 4(ii), is elucidated further through the following simple example which is phrased in the terminology of [7] so as to make it directly accessible to readers of that paper. Specifically, a family $\mathcal{L} = \{\mathcal{L}_t : t \in I\}$ of $C^1$-curves $\mathcal{L}_t : \mathbb{R} \to \mathbb{R}^d$ is referred to as a *material line* of (1) if it is invariant in the sense that, for any $s, t \in I$,

$$x_0 \in \mathcal{L}_s(\mathbb{R}) \quad \text{if and only if} \quad x(t; s, x_0) \in \mathcal{L}_t(\mathbb{R})\,;$$

here $x(\cdot; s, x_0)$ denotes the unique solution of (1) with $x(s) = x_0$. The obvious fluid dynamical interpretation is that, at each time $t$, the set $\mathcal{L}_t(\mathbb{R})$ represents a smooth curve of fluid particles advected by the velocity field $f$. A material line $\mathcal{L}$ is *attracting* if for every solution $\mu$ of (1) with $\mu(t) \in \mathcal{L}_t(\mathbb{R})$ for some (and hence every) $t \in I$, there exists $\alpha > 0$ and a smooth family $X$ of $(d-1)$-dimensional subspaces, invariant under the linearisation (2) along $\mu$, i.e. $\Phi(t,s)X(s) = X(t)$ for all $s, t \in I$, such that $X(t)$ is, for every $t \in I$, transversal to $T_{\mu(t)}\mathcal{L}_t(\mathbb{R})$, and

$$\|\Phi(t,s)x\| \le e^{-\alpha(t-s)}\|x\| , \quad \forall t \ge s, x \in X(s) . \tag{15}$$

For any $\kappa > 0$, consider now the autonomous linear equation

$$\dot{x} = \begin{bmatrix} -1 & 6 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & \kappa \end{bmatrix} x . \tag{16}$$

Since the $(x_1, x_2)$-plane and the $x_3$-axis are both invariant under the flow generated by (16), corresponding respectively to two negative and one positive eigenvalue, it seems plausible that e.g. the $x_3$-axis is an attracting material line. In fact, Case 1 of [7, Thm.1], asserts that *every* solution of (16) is contained in an attracting material line, and hence (16) allows for *many* attracting material lines. Plausible though this may be, it is actually not true:

**Claim 6** *No material line of* (16) *is attracting.*

To verify this claim, suppose $\mathcal{L}$ was an attracting material line of (16) and $\mu$ a solution in $\mathcal{L}$. Denote by $G_{2,3}$ the set of all two-dimensional subspaces of $\mathbb{R}^3$. It follows from (15) that $\frac{d}{dt}\frac{1}{2}\|\Phi(t,s)x\|^2\big|_{t=s} \le -\alpha\|x\|^2$ for all $x \in X(s)$, where $X(s) \in G_{2,3}$ is transversal to $T_{\mu(s)}\mathcal{L}_s(\mathbb{R})$. Note that $\frac{d}{dt}\frac{1}{2}\|\Phi(t,s)x\|^2\big|_{t=s} = \langle Cx, x\rangle$ with the symmetric matrix

$$C = \begin{bmatrix} -1 & 3 & 0 \\ 3 & -7 & 0 \\ 0 & 0 & \kappa \end{bmatrix} .$$

Thus Claim 6 will follow immediately once it is demonstrated that

$$\max\nolimits_{x \in X, \|x\|=1} \langle Cx, x\rangle \ge 0 , \quad \forall X \in G_{2,3}. \tag{17}$$

To prove (17), first recall the following elementary fact from linear algebra.

**Proposition 7** *Let $X \ne \{0\}$ be a subspace of $\mathbb{R}^d$, and $C, D \in \mathbb{R}^{d\times d}$ symmetric matrices with $D > 0$. Then $\big\{\langle Cx, x\rangle : x \in X, \langle Dx, x\rangle = 1\big\} = [\rho_-, \rho_+]$, where $\rho_+$ and $\rho_-$ denote, respectively, the largest and smallest eigenvalue of $[\langle Cb_i, b_j\rangle][\langle Db_i, b_j\rangle]^{-1} \in \mathbb{R}^{l\times l}$, and $\{b_1, \ldots, b_l\}$ is any basis of $X$.*

Denote by $X_{\vartheta,\varphi} \subset \mathbb{R}^3$ the two-dimensional space

$$X_{\vartheta,\varphi} = \begin{bmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi \\ \sin\vartheta \end{bmatrix}^{\perp} , \quad 0 \le \vartheta \le \tfrac{1}{2}\pi,\, 0 \le \varphi \le 2\pi ;$$

every $X \in G_{2,3}$ equals $X_{\vartheta,\varphi}$ for the appropriate $\vartheta$, $\varphi$. To apply Proposition 7 with $D = \mathrm{id}_{3\times 3}$ and $X = X_{\vartheta,\varphi}$, deduce from a straightforward computation that $[\langle Cb_i, b_j\rangle][\langle Db_i, b_j\rangle]^{-1}$ is similar to $\kappa\,\mathrm{id}_{2\times 2} + E_1 E_2$, where

$$E_1 = -\kappa\,\mathrm{id}_{2\times 2} + \begin{bmatrix} -1 & 3 \\ 3 & -7 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 - \cos^2\vartheta\cos^2\varphi & -\cos^2\vartheta\cos\varphi\sin\varphi \\ -\cos^2\vartheta\cos\varphi\sin\varphi & 1-\cos^2\vartheta\sin^2\varphi \end{bmatrix}.$$

It follows that the maximum of $\left\{\langle Cx, x\rangle : x \in X_{\vartheta,\varphi}, \|x\| = 1\right\}$ is $\kappa + \tau$, with $\tau$ denoting the largest zero of the quadratic function

$$p_{\vartheta,\varphi} : t \mapsto t^2 + \Big(2(\kappa+4) - \cos^2\vartheta\big(\kappa+4-3\sqrt{2}\sin(2\varphi+\tfrac{1}{4}\pi)\big)\Big)t + (\kappa^2+8\kappa-2)\sin^2\vartheta\,.$$

If $0 < \kappa \le 3\sqrt{2} - 4$ then $p_{\vartheta,\varphi}(0) \le 0$ and hence $\tau \ge 0$. On the other hand,

$$p_{\vartheta,\varphi}(3\sqrt{2} - 4 - \kappa) = 3\sqrt{2}(3\sqrt{2} - 4 - \kappa)\cos^2\vartheta\big(1 + \sin(2\varphi + \tfrac{1}{4}\pi)\big) \le 0$$

whenever $\kappa > 3\sqrt{2} - 4$, so that $\kappa + \tau \ge 3\sqrt{2} - 4$ in this case. Overall therefore

$$\max\nolimits_{x\in X, \|x\|=1} \langle Cx, x\rangle \ge \min(\kappa, 3\sqrt{2} - 4) > 0\,, \quad \forall X \in G_{2,3}\,.$$

Clearly, this strengthened form of (17) proves Claim 6.

**Remark 8** (i) A straightforward computation confirms that (16) is hyperbolic w.r.t. $\|\cdot\|$ if and only if $t_+ - t_- < \frac{1}{6}\log\frac{9+4\sqrt{2}}{7} \approx 0.1232$. In this case, the rank of any invariant projector for (16) according to Definition 1 equals *one*, and not *two* as might be expected.

(ii) If $A$ is constant and has no eigenvalue on the imaginary axis, then there always exist uncountably many $\Gamma = \Gamma^\top > 0$ such that (3) is hyperbolic w.r.t. $\|\cdot\|_\Gamma$ on *every* compact interval $I$, see [1, Rem.2] and [2, Thm.2.9]. For example, (16) is hyperbolic on every $I$ w.r.t. $\|\cdot\|_\Gamma$, where

$$\Gamma = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, if the definition of attractivity is adapted in that $\|\cdot\|$ in (15) is replaced by $\|\cdot\|_\Gamma$, then every trajectory of (16) is indeed contained in an attracting material line. Not restricting oneself to the Euclidean norm may thus be beneficial even in the most elementary of circumstances.

(ii) The reader may wonder exactly which part of the alleged proof of [7, Thm.1] is problematic. The answer is simple: As the above example shows, linear changes of coordinates do generally not preserve finite-time hyperbolicity, not even if they are time-*independent*. Concretely, $x = My$ with the appropriate non-singular matrix $M$ transforms (16) into $\dot{y} = \mathrm{diag}\,[-1, -7, \kappa]$, for which e.g. every trajectory not contained in the $(y_1, y_2)$-plane, and hence in particular the $y_3$-axis is an attracting material line.

(iv) The usage of time-dependent spectral data to detect finite-time hyperbolicity can be avoided altogether. Based on a *dynamic partition* of the extended phase space, [1, Cor.9] presents a neat condition guaranteeing that a solution $\mu$ of (1) is hyperbolic. The dynamic partition does not involve eigenvalues or -vectors but rather utilises a classification of the points in $I \times U$ according to their qualitative instantaneous behaviour. The interested reader may want to consult [1, 2, 5, 6, 8] where aspects of this useful concept are developed in detail.

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