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MULTI-DIMENSIONAL DYNAMICAL SYSTEMS AND BENFORD'S LAW

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Abstract. One-dimensional projections of (at least) almost all orbits of many multi-dimensional dynamical systems are shown to follow Benford's law, i.e. their (base b) mantissa distribution is asymptotically logarithmic, typically for all bases b. As a generalization and unification of known results it is proved that under a (generic) non-resonance condition on $A \in \mathbb{C}^{d\times d}$, for every $z \in \mathbb{C}^d$ real and imaginary part of each non-trivial component of $(A^n z)_{n\in\mathbb{N}_0}$ and $(e^{At}z)_{t\geq 0}$ follow Benford's law. Also, Benford behavior is found to be ubiquitous for several classes of non-linear maps and differential equations. In particular, emergence of the logarithmic mantissa distribution turns out to be generic for complex analytic maps T with $T(0) = 0$, $|T'(0)| < 1$. The results significantly extend known facts obtained by other, e.g. number-theoretical methods, and also generalize recent findings for one-dimensional systems.

1. **Introduction.** Benford's law is the probability distribution for the mantissa with respect to base $b \in \mathbb{N} \setminus \{1\}$ given by $\mathbb{P}(\text{mantissa}_b \leq t) = \log_b t$ for all $t \in [1, b[$; the most well-known special case is that

$$\mathbb{P}\big(\text{first significant digit}_{10} = d\big) = \log_{10}\left(1 + d^{-1}\right), \qquad d = 1, \ldots, 9\,.$$

Although first discovered by Newcomb [16], this logarithmic law for significant digits gained popularity following an article by Benford [4] which contained extensive empirical evidence of the distribution in diverse tables of data. Since Benford's article, numerous examples of data sets following Benford's law have been found in real-life data [11, 17], in stochastic processes [17, 21], in many classical sequences of numbers such as (2^n), $(n!)$ and the Fibonacci numbers [2, 4, 7, 8], and in data from physical experiments and numerical simulations related to dynamical systems [20, 23].

Recently, a fairly complete analysis of Benford's law for (autonomous as well as non-autonomous) dynamical systems on the real line having 0 and $\pm\infty$ as an attractor was presented in [6], where the emergence of the logarithmic mantissa distribution was shown to be a common phenomenon for such systems. Since convergence to any finite limit may be translated to convergence to the origin, focusing on orbits that converge to 0 or $\pm\infty$ is not as restrictive as it may appear. Moreover, it is completely natural in view of [10], where Benford's law has been characterized

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as the only continuous mantissa distribution which is *base-invariant*. It is natural to require that a general pattern of mantissa distribution, if one exists at all, does not depend on the particular choice of the base. Base-invariance, however, implies that for a sequence (x_n) to follow Benford's law for all bases, every weak limit of $\frac{1}{n}\sum_{l=1}^{n}\delta_{x_l}$ on the extended real line $\mathbb{R}\cup\{\pm\infty\}$ necessarily is a convex combination of point-masses at 0 and $\pm\infty$. Thus only at these points can *stable* dynamics generate Benford's distribution with respect to all bases, as often they do.

The present article extends the main results in [6] to multi-dimensional systems, where some important new aspects arise. For example, unlike in the one-dimensional case, zero and infinity can no longer justifiably be assumed attractors in higher dimensions: while some components of the sequence (x_n) in $\mathbb{R}^d$ with $d \geq 2$ may converge to 0, others may converge to $\pm\infty$, and still Benford's law may hold for each component $(x_n^{(j)})$ and for every base. Also, the problem of *resonances*, which does not exist in the one-dimensional setting, is crucial in the multi-dimensional framework. Nevertheless, as in [6], the emergence of Benford's logarithmic distribution turns out to be typical for all the important families of dynamical systems considered below. These results thus complement other explanations of the ubiquity of Benford's law in numerical data.

The organization of this article is as follows. Section 2 contains definitions, the basic relationship between Benford sequences and uniform distribution mod 1, and preliminary results about uniform distribution of one-dimensional sequences constructed from higher-dimensional ones. Section 3 introduces the notion of an (exponentially) resonant spectrum and presents a fairly complete analysis of *linear* autonomous difference and differential equations: if the matrix A does not have a resonant and exponentially resonant spectrum (a property that is generic), then each component of $(A^n x)_{n\in\mathbb{N}_0}$ and $(e^{At}x)_{t\geq 0}$ either follows Benford's law or else is trivial. In Section 4, three different classes of *non-linear* systems are studied: maps with a dominant linear or a dominant polynomial part; and complex analytic maps, interpreted as two-dimensional real maps. In all three cases, the dominant part (the first non-vanishing term in the Taylor expansion in case of a complex analytic map) is shown to typically generate Benford sequences, and a shadowing argument shows that the same is true for the full non-linear system.

2. **Preliminaries.**

Throughout, b will always denote a natural number larger than one (called a *base*). Every positive real number x can be written uniquely as $x = M_b(x)b^l$ with $M_b(x) \in [1, b[$ and the appropriate integer l. The function $M_b : \mathbb{R}^+ \to [1, b[$ is called the (*base* b) *mantissa function*; for convenience $M_b(0) := 0$ for all b. For every real x, the numbers $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the largest integer not larger, and the smallest integer not smaller than x, respectively. The number $\lfloor M_b(x)\rfloor \in \{1,\ldots,b-1\}$ is called the *first significant digit* of x (with respect to base b). For a given base b, $\log_b$ will denote the logarithm with respect to b, where, to avoid cumbersome formulations, $\log_b 0 := 0$ for all b; if used without a subscript, the log symbol denotes the natural logarithm. The cardinality of the finite set A is $\#A$, and $\mathbf{1}_B$ stands for the indicator function of any set $B \subseteq \mathbb{R}$.

Definition 2.1. (i) A sequence $(x_n)_{n\in\mathbb{N}_0}$ of real numbers is a *b-Benford sequence* if

$$\lim_{n\to\infty}\frac{\#\{l \leq n : M_b(|x_l|) \leq t\}}{n} = \log_b t \qquad \text{for all } t \in [1, b[\,,$$

and it is called a *strict Benford sequence* (or simply a *Benford sequence*) if it is a b-Benford sequence for every $b \in \mathbb{N}\setminus\{1\}$.

(ii) A measurable real-valued function $f : [0, +\infty[\to \mathbb{R}$ is a *b-Benford function* if

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{[1,t[} \circ M_b(|f(\tau)|)\, d\tau = \log_b t \qquad \text{for all } t \in [1, b[\,,$$

and it is called a *strict Benford function* (or simply a *Benford function*) if it is a b-Benford function for every $b \in \mathbb{N}\setminus\{1\}$.

The following correspondence between Benford sequences and uniform distribution modulo one is a standard tool in the context of Benford's law [6, 8] since it allows the powerful classical tools of uniform distribution theory to be applied. The term *(continuously) uniformly distributed modulo one* will henceforth be abbreviated as *(c.) u.d.* mod 1.

Proposition 2.2 ([8]). (i) *A sequence $(x_n)_{n\in\mathbb{N}_0}$ of real numbers is a b-Benford sequence if and only if* $(\log_b |x_n|)_{n\in\mathbb{N}_0}$ *is u.d.* mod 1.

(ii) *A measurable function $f : [0, +\infty[\to \mathbb{R}$ is a b-Benford function if and only if* $(\log_b |f(t)|)_{t\geq 0}$ *is c.u.d.* mod 1.

This article studies Benford properties of recursively defined sequences,

$$x_n := T(x_{n-1})\,, \qquad n = 1, 2, \ldots\,, \tag{2.1}$$

where T denotes a map from $\mathbb{R}^d$ or a part thereof into itself. For ease of notation, no distinction will be made between row and column vectors, e.g. $x \in \mathbb{R}^d$ should be thought of as a column but nevertheless will be written as $x = \big(x^{(1)}, \ldots, x^{(d)}\big)$. In addition to (2.1), the solution of the linear initial value problem

$$\dot{x} = Ax\,, \qquad x(0) = x_0 \in \mathbb{R}^d$$

with $A \in \mathbb{R}^{d\times d}$ will also be analyzed with respect to Benford's law. The corresponding one-dimensional systems ($d = 1$) have been studied in [6], and the results presented below provide natural generalizations of that work. For $n \in \mathbb{N}$ the n-fold composition of T with itself is denoted by T^n, and $T^0 := id$. The sequence generated by (2.1) subject to the initial condition $x_0 = x$ is thus $\big(T^n(x)\big)_{n\in\mathbb{N}_0}$; this sequence will be denoted by $O_T(x)$ and referred to as the *orbit* of x under T. Note that this interpretation of the orbit as a *sequence* differs from the standard terminology in dynamical systems theory (e.g. [13]) where the orbit of x is the mere *set* $\{x_n : n \in \mathbb{N}_0\}$. For any function φ defined on $\mathbb{R}^d$, the symbol $\varphi\big(O_T(x)\big)$ stands for the sequence $\big(\varphi \circ T^n(x)\big)_{n\in\mathbb{N}_0}$; for example, $\|O_T(x)\|$ denotes the sequence of d-dimensional Euclidean norms $(\|x_n\|)_{n\in\mathbb{N}_0}$, and $O_T^{(j)}(x) = \big(x_n^{(j)}\big)_{n\in\mathbb{N}_0}$ denotes the sequence of j-th components of $O_T(x)$.

In light of Proposition 2.2, a study of Benford's law for dynamical systems will make use of results from uniform distribution theory (cf. [9, 14]). The following auxiliary results will be used later; proofs are included for completeness.

Lemma 2.3. *Let the sequence $(y_n)_{n\in\mathbb{N}_0}$ of real numbers be u.d.* mod 1, *and assume that $(x_n)_{n\in\mathbb{N}_0}$ has the following property: for every $\varepsilon > 0$, there exists $J_\varepsilon \subseteq \mathbb{N}$ such that*

$$\underline{\lim}_{n\to\infty} \frac{\#(J_\varepsilon \cap \{1, \ldots, n\})}{n} > 1 - \varepsilon \qquad \text{and} \qquad |x_n - y_n| < \varepsilon \ \text{ for all } n \in J_\varepsilon\,.$$

Then $(x_n)_{n\in\mathbb{N}_0}$ is also u.d. mod 1.

Proof. Fix any non-empty interval $[a,b] \subseteq]0,1[$. For each sufficiently small ε,

$$\#\{l \leq n : x_l \in [a,b]+\mathbb{Z}\} \leq \#(\{l \leq n : y_l \in [a-\varepsilon, b+\varepsilon]+\mathbb{Z}\} \cap J_\varepsilon) + \#(J_\varepsilon^c \cap \{1\ldots,n\}) .$$

Therefore $\overline{\lim}_{n\to\infty} \frac{1}{n}\#\{l \leq n : x_l \in [a,b]+\mathbb{Z}\} \leq b-a+3\varepsilon$, and analogously $\underline{\lim}_{n\to\infty} \frac{1}{n}\#\{l \leq n : x_l \in [a,b]+\mathbb{Z}\} \geq b-a-3\varepsilon$. Since ε was arbitrary, (x_n) is u.d. mod 1. □

The following lemma, a straightforward generalization of a result in [22], guarantees uniform distribution of certain sequences constructed from real-valued functions on the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. The (normalized) Haar measure of this compact Abelian group will be denoted by $\lambda_{\mathbb{T}^d}$.

Lemma 2.4. *Let $f : \mathbb{T}^d \to \mathbb{R}$ be continuous $\lambda_{\mathbb{T}^d}$-almost everywhere, and assume that $(\zeta_n)_{n\in\mathbb{N}_0}$ is uniformly distributed on $\mathbb{T}^{d+1}$. Then $(x_n)_{n\in\mathbb{N}_0}$ defined by*

$$x_n := \zeta_n^{(0)} + f\big(\zeta_n^{(1)}, \ldots, \zeta_n^{(d)}\big)$$

is u.d. mod 1.

Proof. Fix an integer $h \neq 0$, and define a function $F_h : \mathbb{T}^{d+1} \to \mathbb{C}$ by

$$F_h\big(x^{(0)}, \ldots, x^{(d)}\big) := e^{2\pi i h \left(x^{(0)} + f(x^{(1)},\ldots,x^{(d)})\right)} .$$

Since it is bounded and continuous almost everywhere, F_h is Riemann-integrable, and

$$\frac{1}{N}\sum_{l=1}^{N} e^{2\pi i h x_l} = \frac{1}{N}\sum_{l=1}^{N} F_h(\zeta_l) \to \int_{\mathbb{T}^{d+1}} F_h \, d\lambda_{\mathbb{T}^{d+1}} = \int_{\mathbb{T}^1} e^{2\pi i h x^{(0)}} dx^{(0)} \int_{\mathbb{T}^d} e^{2\pi i f} d\lambda_{\mathbb{T}^d} = 0$$

as $N \to \infty$, which shows that (x_n) is u.d. mod 1 (see [14]). □

Remark 2.5. (i) It was noted already in [22] that Lemma 2.4 remains valid if (ζ_n) is asymptotically distributed according to $\lambda_{\mathbb{T}^1} \otimes \mu$, where μ is any probability measure on $\mathbb{T}^d$, and f is continuous μ-almost everywhere.

(ii) Lemma 2.4 reflects the fact that $\lambda_{\mathbb{T}^1} * \nu \,(\mathrm{mod}\ 1) = \lambda_{\mathbb{T}^1}$ for every Borel probability measure ν on $\mathbb{R}$; here $*$ symbolizes the convolution of finite measures on $\mathbb{T}^1$. An interesting probabilistic interpretation of this fact is the following. Let the random variable ξ be uniformly distributed on $[0,1]$, and let η be any(!) real-valued random variable. If ξ and η are *independent*, then $\xi + \eta \,(\mathrm{mod}\ 1)$ is again uniform on $[0,1]$. (Clearly, the assumption of independence is crucial and cannot be dropped.)

For the following corollary, recall that the real numbers $\rho_1, \ldots, \rho_k$ are *rationally independent* (or $\mathbb{Q}$-*independent*, for short) if $\sum_{l=1}^{k} q_l \rho_l = 0$ implies that $(q_1, \ldots, q_k) = (0, \ldots, 0)$, where $q_l \in \mathbb{Q}$ for all l; otherwise $\rho_1, \ldots, \rho_k$ are said to be $\mathbb{Q}$-*dependent*.

Corollary 2.6. *Let $f : \mathbb{T}^d \to \mathbb{R}$ be continuous on a set of full $\lambda_{\mathbb{T}^d}$-measure, and assume that the $d+2$ real numbers $1, \rho_0, \rho_1, \ldots, \rho_d$ are $\mathbb{Q}$-independent. Then $(x_n)_{n\in\mathbb{N}_0}$ defined by*

$$x_n := n\rho_0 + f(n\rho_1, \ldots, n\rho_d)$$

is u.d. mod 1.

Proof. By Weyl's criterion [12, 14] the sequence (ζ_n) with $\zeta_n := (n\rho_0, n\rho_1, \ldots, n\rho_d)$ is uniformly distributed on $\mathbb{T}^{d+1}$, and thus the conclusion follows from Lemma 2.4. □

Corollary 2.7. *Let $f : \mathbb{T}^d \to \mathbb{R}$ be continuous on a set of full $\lambda_{\mathbb{T}^d}$-measure, and let $p \in \mathbb{N}\backslash\{1\}$. For $\lambda_{\mathbb{T}^{d+1}}$-almost every $(\xi_0, \xi_1, \dots, \xi_d) \in \mathbb{T}^{d+1}$, the sequence $(x_n)_{n\in\mathbb{N}_0}$ defined by*

$$x_n := p^n \xi_0 + f(p^n \xi_1, \dots, p^n \xi_d)$$

is u.d. mod 1.

Proof. The homomorphism $x \mapsto px$ of $\mathbb{T}^{d+1}$ is ergodic with respect to $\lambda_{\mathbb{T}^{d+1}}$, see [13]. For almost every point $\xi = (\xi_0, \xi_1, \dots, \xi_d) \in \mathbb{T}^{d+1}$, the sequence (ζ_n) with $\zeta_n := (p^n\xi_0, p^n\xi_1, \dots, p^n\xi_d)$ is thus uniformly distributed on $\mathbb{T}^{d+1}$, and again the conclusion follows from Lemma 2.4. □

The next lemma is a special case of an important permanence principle in the theory of uniform distribution [18, 19].

Lemma 2.8. *Let $(x_n)_{n\in\mathbb{N}_0}$ be u.d.* mod 1. *Then, for every $\alpha \in \mathbb{R}$ and $b \in \mathbb{N}\backslash\{1\}$, the sequence $(x_n + \alpha \log_b n)_{n\in\mathbb{N}_0}$ is also u.d.* mod 1.

Proof. Set $f(x) := \alpha \log_b x$. Then $f \in C^1(\mathbb{R}^+)$, and $\lim_{x\to\infty} x f'(x) = \frac{\alpha}{\log b}$ is finite, and the conclusion follows from [19]. □

The real part, imaginary part, complex conjugate and absolute value (modulus) of a number $z \in \mathbb{C}$ is denoted by $\Re z$, $\Im z$, $\overline{z}$ and $|z|$, respectively. For $z \neq 0$, the argument $\arg z$ is the unique number in $]-\pi, \pi]$ which satisfies $z = |z|e^{i \arg z}$; for ease of notation $\arg 0 := 0$. The unit circle S^1 is interpreted as $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. If A is a real or complex matrix, then $\sigma(A) \subseteq \mathbb{C}$ symbolizes the *spectrum* of A, i.e. the *set* of eigenvalues of A. The symbol $\|A\|$ denotes the matrix norm of A as induced by the Euclidean norm on $\mathbb{C}^d$, that is, $\|A\| = \sqrt{|\mu|}$ where μ is the eigenvalue of $\overline{A}^T A$ with maximal modulus. In the context of the linear-algebraic considerations in the next section, Corollary 2.6 and Lemma 2.8 will be used mainly via

Lemma 2.9. *Assume that $1, \rho_0, \rho_1, \dots, \rho_d$ are $\mathbb{Q}$-independent, and let $(z_n)_{n\in\mathbb{N}_0}$ be a convergent sequence in $\mathbb{C}$; also let $b \in \mathbb{N}\backslash\{1\}$, $\alpha \in \mathbb{R}$ and $c_1, \dots, c_{2d} \in \mathbb{C}$ such that $c_{2l-1} + \overline{c_{2l}} \neq 0$ for at least one $l \in \{1, \dots, d\}$. Then $(x_n)_{n\in\mathbb{N}_0}$ defined by*

$$x_n := n\rho_0 + \alpha \log_b n + \\ + \log_b \big|\Re(c_1 e^{2\pi i n\rho_1} + c_2 e^{-2\pi i n \rho_1} + \ldots + c_{2d-1} e^{2\pi i n \rho_d} + c_{2d} e^{-2\pi i n \rho_d} + z_n)\big|$$

is u.d. mod 1.

Proof. Set $z_\infty := \lim_{n\to\infty} z_n$, and define $h : \mathbb{T}^d \to \mathbb{R}$ by

$$h\big(x^{(1)}, \dots, x^{(d)}\big) := \Re\big(c_1 e^{2\pi i x^{(1)}} + c_2 e^{-2\pi i x^{(1)}} + \ldots + c_{2d-1} e^{2\pi i x^{(d)}} + c_{2d} e^{-2\pi i x^{(d)}} + z_\infty\big);$$

by the assumption on the numbers c_l, the smooth function h is not constant. Corollary 2.6 applies to $f := \log_b |h|$ and, by Lemma 2.8,

$$y_n := n\rho_0 + \alpha \log_b n + \log_b |h(n\rho_1, \dots, n\rho_d)|$$

is u.d. mod 1. Given $\varepsilon > 0$, choose $0 < \delta < \frac{1}{2}\min\big(\varepsilon, (\log b)^{-1}\big)$ so small that $\lambda_{\mathbb{T}^d}\big(\{x : |h(x)| > \delta\}\big) > 1 - \varepsilon$. For

$$J_\varepsilon := \big\{n \in \mathbb{N} : |h(n\rho_1, \dots, n\rho_d)| > \delta\,,\ |z_n - z_\infty| < \delta^2 \log b\big\} \subseteq \mathbb{N}\,,$$

$\underline{\lim}_{n\to\infty} \frac{1}{n} \#(J_\varepsilon \cap \{1, \dots, n\}) > 1 - \varepsilon$. Furthermore, for every $n \in J_\varepsilon$

$$|x_n - y_n| = \left|\log_b \frac{|h(n\rho_1, \dots, n\rho_d) + \Re(z_n - z_\infty)|}{|h(n\rho_1, \dots, n\rho_d)|}\right| < \big|\log_b(1 - \delta \log b)\big| < \varepsilon\,,$$

since $|\log(1-x)| < 2x$ for all $0 < x < \frac{1}{2}$. The claim thus follows from Lemma 2.3. $\square$

3. **Linear systems.** This section studies, under the perspective of Benford's law, linear difference and differential equations, $x_{n+1} = Ax_n$ and $\dot{x} = Ax$, respectively. Throughout, $A \in \mathbb{R}^{d\times d}$ denotes a constant real matrix. Since the continuous-time case will be easily grasped once the discrete-time case is analyzed, first consider the discrete initial value problem

$$x_{n+1} = Ax_n \quad (n \in \mathbb{N}_0), \qquad x_0 = x, \tag{3.2}$$

the solution of which is $x_n = A^n x$, so every component of x_n is a weighted sum of entries of A^n. Let $\sigma(A)^+ = \{\lambda_1, \ldots, \lambda_s\} \subseteq \mathbb{C}$ be the "upper half" of the spectrum of A, i.e. $\sigma(A)^+ = \{\lambda \in \sigma(A) : \Im\lambda \geq 0\}$. (The usage of $\sigma(A)^+$ refers to the fact that non-real eigenvalues of real matrices always occur in conjugate pairs.) Without loss of generality, assume that the eigenvalues in $\sigma(A)^+$ are labeled such that $|\lambda_1| \geq \ldots \geq |\lambda_s|$. A universal expression for the j-th component ($j = 1, \ldots, d$) of x_n is

$$x_n^{(j)} = \Re\big(p_1^{(j)}(n)\lambda_1^n + \ldots + p_s^{(j)}(n)\lambda_s^n\big), \qquad n \geq d. \tag{3.3}$$

In (3.3), the $p_l^{(j)}$ denote polynomials with complex coefficients and degrees $k_l^{(j)} < d$. Fix $j \in \{1, \ldots, d\}$ for the following considerations. If $x_n^{(j)}$ does not vanish for all but finitely many n, let $s_j \in \{1, \ldots, s\}$ be the minimal index l such that $p_l^{(j)} \not\equiv 0$. To analyze (3.3) as $n \to \infty$ it is useful to distinguish two cases.

<u>Case 1:</u> $|\lambda_{s_j}| > |\lambda_{s_j+1}|$.

In this case a *dominant* eigenvalue occurs, and (3.3) may be written in the form

$$x_n^{(j)} = |\lambda_{s_j}^n| n^{k_{s_j}^{(j)}} \Re\Big(c_{s_j}^{(j)} \left(\frac{\lambda_{s_j}}{|\lambda_{s_j}|}\right)^n + z_j(n)\Big), \tag{3.4}$$

where $c_{s_j}^{(j)} := \lim_{n\to\infty} n^{-k_{s_j}^{(j)}} p_{s_j}^{(j)}(n) \neq 0$, and $z_j(n) \to 0$ as $n \to \infty$. Therefore

$$\log_b\big|x_n^{(j)}\big| = n\log_b|\lambda_{s_j}| + k_{s_j}^{(j)}\log_b n + \log_b\Big|\Re\big(c_{s_j}^{(j)} e^{in\arg\lambda_{s_j}} + z_j(n)\big)\Big| \tag{3.5}$$

provided that $x_n^{(j)} \neq 0$, and Lemma 2.9 implies that $\big(x_n^{(j)}\big)_{n\in\mathbb{N}_0}$ is a b-Benford sequence if either λ_{s_j} is a real number and $\log_b|\lambda_{s_j}|$ is irrational, or if $\lambda_{s_j} \in \mathbb{C}\backslash\mathbb{R}$, and $1, \log_b|\lambda_{s_j}|, \frac{1}{2\pi}\arg\lambda_{s_j}$ are $\mathbb{Q}$-independent.

<u>Case 2:</u> $|\lambda_{s_j}| = |\lambda_{s_j+1}| = \ldots = |\lambda_{t_j}| > 0$ for some $t_j > s_j$.

Here several *different* eigenvalues with the same modulus occur. Assume without loss of generality that $0 \leq \arg\lambda_{s_j} < \arg\lambda_{s_j+1} < \ldots < \arg\lambda_{t_j} \leq \pi$, and let $k^{(j)}$ be the maximal degree of the polynomials $p_{s_j}^{(j)}, \ldots, p_{t_j}^{(j)}$. From (3.3) it follows that

$$x_n^{(j)} = |\lambda_{s_j}^n| n^{k^{(j)}} \Re\Big(c_{s_j}^{(j)} \left(\frac{\lambda_{s_j}}{|\lambda_{s_j}|}\right)^n + \ldots + c_{t_j}^{(j)} \left(\frac{\lambda_{t_j}}{|\lambda_{t_j}|}\right)^n + z_j(n)\Big); \tag{3.6}$$

again $c_l^{(j)} := \lim_{n\to\infty} n^{-k^{(j)}} p_l^{(j)}(n) \in \mathbb{C}$ for $l = s_j, \ldots, t_j$, where $c_l^{(j)} \neq 0$ for at least one l, and $z_j(n) \to 0$ as $n \to \infty$. Consequently, if $x_n^{(j)} \neq 0$ then

$$\begin{aligned}\log_b\bigl|x_n^{(j)}\bigr| = {} & n \log_b|\lambda_{s_j}| + k^{(j)}\log_b n + \\ & + \log_b\left|\Re\bigl(c_{s_j}^{(j)}\left(\frac{\lambda_{s_j}}{|\lambda_{s_j}|}\right)^n + \ldots + c_{t_j}^{(j)}\left(\frac{\lambda_{t_j}}{|\lambda_{t_j}|}\right)^n + z_j(n)\bigr)\right|, \end{aligned} \tag{3.7}$$

and Lemma 2.9 yields the uniform distribution of $\bigl(\log_b|x_n^{(j)}|\bigr)_{n\in\mathbb{N}_0}$ if the $t_j - s_j + 3$ numbers $1, \log_b|\lambda_{s_j}|, \frac{1}{2\pi}\arg\lambda_{s_j}, \ldots, \frac{1}{2\pi}\arg\lambda_{t_j}$ are $\mathbb{Q}$-independent. If λ_{s_j} is positive then clearly $0 = \frac{1}{2\pi}\arg\lambda_{s_j}$ has to be removed from this list; the same is true for negative λ_{t_j} unless $\lambda_{s_j} > 0$.

The above considerations show that each component $\bigl(x_n^{(j)}\bigr)$ either vanishes eventually, or else is a b-Benford sequence provided that the base b and the potential dominant eigenvalues satisfy a non-resonance condition. In order to conveniently summarize this observation, it is useful to introduce the notion of an (exponentially) b-resonant spectrum.

Definition 3.1. (i) A set $\Gamma \subseteq \mathbb{C}$ is called *b-resonant* if there exists a finite non-empty subset $\Gamma_0 = \{\gamma_1, \ldots, \gamma_r\} \subseteq \Gamma$ with $|\gamma_1| = \ldots = |\gamma_r| =: |\Gamma_0|$ such that 1, $\log_b|\Gamma_0|$ and the elements of $\frac{1}{2\pi}\arg\Gamma_0$ are $\mathbb{Q}$-dependent, where

$$\frac{1}{2\pi}\arg\Gamma_0 := \begin{cases} \bigl\{\frac{1}{2\pi}\arg\gamma_1, \ldots, \frac{1}{2\pi}\arg\gamma_r\bigr\} \setminus \{0, \frac{1}{2}\} & \text{if } \#(\Gamma_0 \cap \mathbb{R}) \leq 1\,, \\ \bigl\{\frac{1}{2\pi}\arg\gamma_1, \ldots, \frac{1}{2\pi}\arg\gamma_r\bigr\} \setminus \{0\} & \text{if } \#(\Gamma_0 \cap \mathbb{R}) = 2\,. \end{cases}$$

(ii) Let $A \in \mathbb{R}^{d\times d}$ be a real matrix with spectrum $\sigma(A) \subseteq \mathbb{C}$. The matrix A has *b-resonant spectrum* if the set $\sigma(A)^+$ is b-resonant.

(iii) The matrix $A \in \mathbb{R}^{d\times d}$ is said to have *exponentially b-resonant spectrum* if e^{At} has b-resonant spectrum for all $t \in [0, \delta]$ for some $\delta > 0$.

Remark 3.2. (i) If $\#\Gamma_0 = 1$, then $\frac{1}{2\pi}\arg\Gamma_0$ is empty if $\gamma_1 \in \mathbb{R}$, and otherwise $\frac{1}{2\pi}\arg\Gamma_0 = \bigl\{\frac{1}{2\pi}\arg\gamma_1\bigr\}$. The definition of the set $\frac{1}{2\pi}\arg\Gamma_0$ therefore reflects the cases distinguished above: while *one* real eigenvalue of maximal modulus, i.e. $\frac{1}{2\pi}\arg\lambda_{s_j} \in \{0, \frac{1}{2}\}$, is perfectly acceptable in (3.5) and (3.7), the presence of *two* real eigenvalues of maximal modulus, necessarily of opposite sign, $\frac{1}{2\pi}\arg\lambda_{s_j} = 0$, $\frac{1}{2\pi}\arg\lambda_{t_j} = \frac{1}{2}$, may invalidate (3.7) and has to be excluded.

(ii) It is easily checked that A has exponentially b-resonant spectrum if and only if there exists a non-empty set $\{\lambda_{j_1}, \ldots, \lambda_{j_r}\} \subseteq \sigma(A)^+$ with $\Re\lambda_{j_1} = \ldots = \Re\lambda_{j_r}$ such that $\Re\lambda_{j_1}/\log b$ and the elements of $\bigl\{\frac{1}{2\pi}\Im\lambda_{j_1}, \ldots, \frac{1}{2\pi}\Im\lambda_{j_r}\bigr\} \setminus \{0\}$ are $\mathbb{Q}$-dependent.

(iii) Non-zero eigenvalues of different absolute value are not b-resonant unless one of their moduli is a rational power of b. An eigenvalue zero, as well as eigenvalues on the unit circle, are resonant, and eigenvalues on the imaginary axis are exponentially resonant. Also, real eigenvalues of opposite sign are resonant. Different real non-zero eigenvalues, however, are not exponentially resonant. Clearly, the (trivial) one-dimensional case is included in Definition 3.1: the real 1×1 matrix $A = (a)$ has b-resonant spectrum if and only if $\log_b|a|$ is irrational, and it has exponentially resonant spectrum precisely if $a = 0$ (cf. [6]).

(iv) Let $\mathcal{M}_b$ denote the set of all $d \times d$-matrices having b-resonant spectrum. It is not difficult to see that $\mathcal{M}_b$, considered as a subset of $\mathbb{R}^{d\times d}$, is a set of first category, and so is $\bigcup_b \mathcal{M}_b$. A *typical* matrix will thus not have b-resonant spectrum for any b. A similar statement holds for exponential resonance.

The observations preceding Definition 3.1, in particular equations (3.5) and (3.7), are the basis for the proof of the following theorem, which generalizes and unifies results for recursive sequences in [7, 15, 22].

Theorem 3.3. *Assume that $A \in \mathbb{R}^{d\times d}$ does not have b-resonant spectrum. Then, for every $x \in \mathbb{R}^d$, each component of $(A^n x)_{n\in\mathbb{N}_0}$ either equals zero for all but finitely many n, or else is a b-Benford sequence; the same is true for $(\|A^n x\|)_{n\in\mathbb{N}_0}$.*

Proof. The assertion about individual components of $x_n = A^n x$ merely summarizes the arguments in Case 1 and 2 considered above. As for the Euclidean norm, $\log_b \|x_n\| = \frac{1}{2}\log_b \sum_{j=1}^d \bigl(x_n^{(j)}\bigr)^2$, it is clear from the explicit formula (3.6) that without loss of generality $|\lambda_{s_j}| = |\lambda_{s_0}|$ and $k^{(j)} = k_0$ for all j. Therefore

$$x_n^{(j)} = |\lambda_{s_0}^n| n^{k_0}\Bigl(f_j\bigl(n\frac{\arg\lambda_{s^*}}{2\pi},\ldots,n\frac{\arg\lambda_{t^*}}{2\pi}\bigr)+\Re z_j(n)\Bigr) \tag{3.8}$$

with $s^* := \min_{j=1}^d s_j$ and $t^* := \max_{j=1}^d t_j$, and the functions $f_j : \mathbb{T}^{t^*-s^*+1} \to \mathbb{R}$ defined by

$$f_j(x_{s^*},\ldots,x_{t^*}) := \Re(c_{s_j}^{(j)} e^{2\pi i x_{s_j}} + \ldots + c_{t_j}^{(j)} e^{2\pi i x_{t_j}}) .$$

From (3.8) it follows that

$$\log_b \|x_n\| = n\log_b|\lambda_{s_0}| + k_0\log_b n + \frac{1}{2}\log_b\Bigl|\sum_{j=1}^d f_j^2\bigl(n\frac{\arg\lambda_{s^*}}{2\pi},\ldots,n\frac{\arg\lambda_{t^*}}{2\pi}\bigr) + \widetilde{z}_n\Bigr|$$

with $\lim_{n\to\infty}\widetilde{z}_n = 0$. Using Corollary 2.6 and Lemma 2.8, an argument analogous to the proof of Lemma 2.9 shows that $(\log_b\|x_n\|)$ is u.d. mod 1 if $\{\lambda_{s^*},\ldots,\lambda_{t^*}\}$ is not b-resonant. Therefore, if A does not have b-resonant spectrum, then $(\|x_n\|)_{n\in\mathbb{N}_0}$ is either a b-Benford sequence or else vanishes for all but finitely many n. □

Remark 3.4. (i) The arguments leading to Theorem 3.3 did not exploit the specific structure of the matrix A, as reflected for instance through the polynomials $p_l^{(j)}$ in (3.3). A refined analysis of (3.3), which takes into account the specific form of A as well as the initial value x_0, will naturally lead to a refined version of Theorem 3.3. Consider as an example the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1+e & 1-e \\ 1-e & 1+e\end{pmatrix} \qquad \text{with } \sigma(A) = \{1, e\}.$$

Even though A has b-resonant spectrum for every base b, $\bigl((A^n x)^{(j)}\bigr)_{n\in\mathbb{N}_0}$ for $j = 1,2$, and also $(\|A^n x\|)_{n\in\mathbb{N}_0}$, are strict Benford sequences for every $x = (x^{(1)}, x^{(2)})$ with $x^{(1)} \neq x^{(2)}$, i.e., for x not an element of the eigenspace corresponding to the resonant eigenvalue 1. Requiring $\mathbb{Q}$-independence of $1, \log_b|\lambda|, \frac{1}{2\pi}\arg\lambda$ for individual eigenvalues, and considering the position of the corresponding eigenspaces relative to each other as well as relative to the coordinate axes, may thus enable a detailed refinement of Theorem 3.3; the details are left to the reader.

(ii) By Remark 3.2(iv), the emergence of (strict) Benford sequences in (3.2) is a generic phenomenon.

Example 3.5. (i) As a classical example consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1\end{pmatrix}$$

whose characteristic polynomial is $\chi_A(\lambda) = \lambda^2 - \lambda - 1$, with roots $\kappa, -\kappa^{-1}$ where $\kappa = \frac{1+\sqrt{5}}{2}$. Since χ_A is irreducible over $\mathbb{Q}$ and has two roots of different absolute value, it follows that $\log_b \kappa$ is irrational for every base b. Hence $\big((A^n x)^{(j)}\big)_{n\in\mathbb{N}_0}$ for $j = 1, 2$, and $(\|A^n x\|)_{n\in\mathbb{N}_0}$ are strict Benford sequences for all $x \neq 0$. From

$$(A^{n+2}x)^{(2)} = (A^{n+1}x)^{(1)} + (A^{n+1}x)^{(2)} = (A^n x)^{(2)} + (A^{n+1}x)^{(2)}$$

it is immediate to deduce the well-known fact [7] that the sequences $(F_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ of Fibonacci and Lucas numbers are strict Benford sequences, since F_n and L_n equal $(A^n x)^{(2)}$, with $x = (1, 0)$ and $x = (-1, 2)$, respectively.

(ii) As indicated in Remark 3.4(i), the sufficient condition in Theorem 3.3 is not necessary. For instance, if $\frac{1}{2\pi}\arg\lambda_l = \frac{p_l}{2q}$ with $p_l, q \in \mathbb{N}$ and $p_l \leq q$ for all $l = s_j, \ldots, t_j$, then $\log_b\big|x_n^{(j)}\big|$ is u.d. mod 1 provided that $\log_b|\lambda_{s_j}|$ is irrational and

$$\Re\big(c_{s_j}^{(j)} e^{\pi i q^{-1} k p_{s_j}} + \ldots + c_{t_j}^{(j)} e^{\pi i q^{-1} k p_{t_j}}\big) \neq 0 \qquad \text{for all } k = 0, 1, \ldots, 2q-1\,. \tag{3.9}$$

As a simple example, consider the matrix

$$A = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} \qquad \text{with } \sigma(A) = \{\pm ie\}\,,$$

which has b-resonant spectrum for every base b; with the notation of (3.4) one has $c_1^{(1)} = x^{(1)} - ix^{(2)}$, $c_1^{(2)} = ic_1^{(1)}$, $z_{1,2} = 0$ and $k_1^{(1,2)} = 0$. Therefore (3.9) holds for $x^{(1)}x^{(2)} \neq 0$, and $\big((A^n x)^{(j)}\big)_{n\in\mathbb{N}_0}$ is a strict Benford sequence for $j = 1, 2$ and all x with $x^{(1)}x^{(2)} \neq 0$.

(iii) On the other hand, for a matrix A with b-resonant spectrum, $(A^n x)$ may certainly fail to have components which are b-Benford sequences. This may happen even if A is a 2×2 matrix with non-real eigenvalues $\lambda, \overline{\lambda}$, and any two numbers among $1, \log_b|\lambda|, \frac{1}{2\pi}\arg\lambda$ are rationally independent. For example, again let $\kappa = \frac{1+\sqrt{5}}{2}$ and take

$$A = b^{\kappa^2}\begin{pmatrix} \cos 2\pi\kappa & \sin 2\pi\kappa \\ -\sin 2\pi\kappa & \cos 2\pi\kappa \end{pmatrix} \qquad \text{with } \sigma(A) = \big\{b^{\kappa^2} e^{\pm 2\pi i\kappa}\big\}\,.$$

It is readily checked that $\big((A^n x)^{(j)}\big)$ is not a b-Benford sequence for any $x \in \mathbb{R}^2$, whereas $(\|A^n x\|)$ is obviously b-Benford whenever $x \neq 0$.

As a continuous-time analogue of (3.2), consider the initial value problem

$$\dot{x} = Ax\,, \qquad x(0) = x_0\,, \tag{3.10}$$

the solution of which is $x : t \mapsto e^{At}x_0$. The following lemma relates the continuous-time to the discrete-time case.

Lemma 3.6. *If $A \in \mathbb{R}^{d\times d}$ does not have exponentially b-resonant spectrum, then e^{At} has b-resonant spectrum for at most countably many $t > 0$.*

Proof. Let $\sigma(A)^+ = \{\lambda_1, \ldots, \lambda_s\} \subseteq \mathbb{C}$. If A does not have exponentially b-resonant spectrum, then for some sequence (t_n) with $t_n > 0$, $\lim_{n\to\infty} t_n = 0$, the matrices e^{At_n} do not have b-resonant spectrum. This implies that

$$\Re\lambda_{j_1}/\log b \text{ and the elements of } \Big\{\frac{1}{2\pi}\Im\lambda_{j_1}, \ldots, \frac{1}{2\pi}\Im\lambda_{j_r}\Big\} \setminus \{0\} \tag{3.11}$$

are $\mathbb{Q}$-independent whenever $\{\lambda_{j_1}, \ldots, \lambda_{j_r}\} \subseteq \sigma(A)^+$ with $\Re\lambda_{j_1} = \ldots = \Re\lambda_{j_r}$. Assume that e^{At} has b-resonant spectrum for a given $t > 0$. Then for some $r \in \mathbb{N}$ and appropriate integers $q_{-1}, q_0, \ldots, q_r$ not all of which are zero,

$$q_{-1} + q_0\log_b\big|e^{\lambda_{j_1}t}\big| + \frac{q_1}{2\pi}\arg e^{\lambda_{j_1}t} + \ldots + \frac{q_r}{2\pi}\arg e^{\lambda_{j_r}t} = 0 \tag{3.12}$$

and $|e^{\lambda_{j_1}t}| = \ldots = |e^{\lambda_{j_r}t}|$. The $\mathbb{Q}$-independence of the numbers (3.11) implies that (3.12) has at most countably many solutions $t > 0$. $\square$

As in the discrete-time case, there is a simple sufficient condition for the solution of (3.10) to have Benford functions as its components. Again, the emergence of Benford functions from (3.10) turns out to be a generic phenomenon, cf. Remark 3.2(iv).

Theorem 3.7. *Assume that $A \in \mathbb{R}^{d\times d}$ does not have exponentially b-resonant spectrum. Then, for every $x \in \mathbb{R}^d$, each component of $(e^{At}x)_{t\geq 0}$ either equals zero identically, or else is a b-Benford function; the same is true for $(\|e^{At}x\|)_{t\geq 0}$.*

Proof. The argument follows easily from Theorem 3.3, Lemma 3.6 and the well-known fact (see [14]) that $f : [0, +\infty[\to \mathbb{R}$ is c.u.d. mod 1 if $\big(f(nh)\big)_{n\in\mathbb{N}_0}$ is u.d. mod 1 for almost all $h \in [0, \delta]$ for some $\delta > 0$. By Lemma 3.6, choose $\delta > 0$ such that the spectrum of $B = e^{Ah}$ lacks b-resonance for all but countably many $h \in [0, \delta]$. For almost all h and every $x \in \mathbb{R}^d$, the quantities

$$\log_b\big|(e^{At}x)^{(j)}\big|_{t\,=\,nh} = \log_b\big|(B^n x)^{(j)}\big| \quad \text{and} \quad \log\|e^{At}x\|_{t\,=\,nh} = \log_b\|B^n x\|$$

therefore either vanish eventually or else yield u.d. mod 1 sequences. The components of $e^{At}x$ are analytic in t, and so $\big((e^{At}x)^{(j)}\big)_{t\geq 0}$ and $(\|e^{At}x\|)_{t\geq 0}$ either vanish identically, or else are b-Benford functions. $\square$

Remark 3.8. For simplicity, and also because Benford's law by its very nature is a statement about *real* sequences, only real matrices were considered in this section. It is, however, straightforward to formulate Theorems 3.3 and 3.7 for *complex* matrices and for real and imaginary parts of the solutions of (3.2) and (3.10), respectively. Indeed, given $A \in \mathbb{C}^{d\times d}$, denote by $A_{\mathbb{R}}$ its *realification*, i.e. the real matrix

$$A_{\mathbb{R}} := \begin{pmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{pmatrix} \in \mathbb{R}^{2d\times 2d}\,.$$

It is easily seen that $\sigma(A_{\mathbb{R}}) = \sigma(A) \cup \overline{\sigma(A)}$. Also, since $(A^n)_{\mathbb{R}} = (A_{\mathbb{R}})^n$ for all $n \in \mathbb{N}_0$, $\varphi(A)_{\mathbb{R}} = \varphi(A_{\mathbb{R}})$ whenever φ is an entire function with $\varphi(\mathbb{R}) \subseteq \mathbb{R}$; in particular $e^{A_{\mathbb{R}}} = (e^A)_{\mathbb{R}}$. For complex matrices, eigenvalues do not necessarily occur in conjugate pairs, and representation (3.3) for $z_n = A^n z$ takes the form

$$z_n^{(j)} = p_1^{(j)}(n)\lambda_1^n + \ldots + p_s^{(j)}(n)\lambda_s^n\,, \tag{3.13}$$

where $\{\lambda_1, \ldots, \lambda_s\}$ represents the *full* spectrum of A. If $\Im\lambda_l < 0$ then λ_l^n in (3.13) can be replaced by $\overline{\lambda_l}^n e^{2in\arg\lambda_l}$. Lemma 2.9 thus implies that only the "upper half" of the symmetrisized full spectrum $\sigma(A) \cup \overline{\sigma(A)}$ has to be considered in (3.13). Therefore, say that $A \in \mathbb{C}^{d\times d}$ has (exponentially) b-resonant spectrum if and only if the realification $A_{\mathbb{R}}$ has the corresponding property. Using this this tailor-made definition, it is clear that Theorem 3.3 correspondingly holds for $A \in \mathbb{C}^{d\times d}$ and $\Re(A^n z)_{n\in\mathbb{N}_0}$ and $\Im(A^n z)_{n\in\mathbb{N}_0}$ with $z \in \mathbb{C}^d$; similarly, Theorem 3.7 remains valid for $\Re(e^{At}z)_{t\geq 0}$ and $\Im(e^{At}z)_{t\geq 0}$.

4. **Some non-linear examples.** The analysis of linear systems in the previous section has been fairly complete. This level of completeness should certainly not be expected for the vast class of *non-linear* maps on $\mathbb{R}^d$. As the subsequent results show, there are nevertheless several important families of maps for which a general statement about the emergence of Benford sequences can be made. If the map

under consideration has a dominant term of linear or polynomial growth, then this dominant part is responsible for the generation of Benford sequences in a sense made precise below. As a final class of examples, analytic maps of the complex plane are studied. Contrary to the usual approach in dynamics [3, 13], these maps have to be considered as two-dimensional real systems here. As is the case for linear systems, the emergence of Benford's logarithmic mantissa distribution turns out to be generic for a reasonably chosen, fairly general class of complex analytic maps.

4.1. **Linearly dominated systems.** Let the map T on $\mathbb{R}^d$ be given by

$$T : x \mapsto Ax + f(x)\,, \tag{4.14}$$

where $A \in \mathbb{R}^{d\times d}$, and f denotes a bounded continuous function. In order to have infinity as an attractor for (4.14), assume that all eigenvalues of A have absolute value larger than one. In this case it depends on the linearization $x \mapsto Ax$ of (4.14) whether T generates Benford sequences or not.

Theorem 4.1. *Let T be given by* (4.14)*, and assume that $|\lambda| > 1$ for every $\lambda \in \sigma(A)$. If A does not have b-resonant spectrum, then for sufficiently large $\|x\|$, every unbounded component of $O_T(x)$ is a b-Benford sequence, and so is $\|O_T(x)\|$.*

Proof. As in [6, Thm. 3.1] the following shadowing argument is crucial. Define a continuous map h by setting

$$h(x) := x + \sum_{l=0}^{\infty} A^{-(l+1)} f\circ T^l(x) = \lim_{l\to\infty} A^{-l}T^l(x)\,,$$

and observe that $h\circ T(x) = A\circ h(x)$ for all x; also $\sup_{x\in\mathbb{R}^d}\|h(x)-x\| < \infty$. Thus if $\|x\|$ is sufficiently large, $\overline{\lim}_{n\to\infty}\|A^n h(x) - T^n(x)\| < \infty$ and $\|T^n(x)\| \to \infty$ as $n\to\infty$, in which case $\big(A^n h(x)^{(j)} - T^n(x)^{(j)}\big)$ remains bounded as $n\to\infty$ for all $j = 1,\ldots,d$. Therefore if $O_T^{(j)}(x)$ is unbounded, then so is $\big(A^n h(x)^{(j)}\big)$, and by Theorem 3.3 the latter is a b-Benford sequence. Since $|\lambda| > 1$ for all $\lambda\in\sigma(A)$, it follows from the representations (3.4) and (3.6) that for any $\varepsilon > 0$, there exists a set $J_\varepsilon \subseteq \mathbb{N}$ with lower density at least $1-\varepsilon$ such that $\lim_{n\in J_\varepsilon, n\to\infty}|A^n h(x)^{(j)}| = \infty$. Therefore

$$\Big|\log_b|T^n(x)^{(j)}| - \log_b|A^n h(x)^{(j)}|\Big| = \left|\log_b\left|\frac{T^n(x)^{(j)} - A^n h(x)^{(j)}}{A^n h(x)^{(j)}} + 1\right|\right| < \varepsilon$$

for all sufficiently large $n\in J_\varepsilon$, and thus $O_T^{(j)}(x)$ is also a b-Benford sequence by Lemma 2.3. By virtue of $\|T^n(x)\|\to\infty$ and the elementary estimate $\big|\|A^n h(x)\| - \|T^n(x)\|\big| \le \|A^n h(x) - T^n(x)\|$, the assertion about $\|O_T(x)\|$ follows immediately from Theorem 3.3. $\square$

Remark 4.2. (i) Bounded components of $O_T(x)$ need not be Benford sequences, as the simple two-dimensional example $T : x\mapsto ex + (1-e, 0)$ shows; for $x = (1, x^{(2)})$ one finds $T^n x = \big(1, e^n x^{(2)}\big)$, and the first component is constant, no matter how large $\|x\|$ is.

(ii) Boundedness, on the other hand, does not necessarily rule out Benford behavior. Indeed, for the slightly modified map $T : x\mapsto ex + f(x)$ with $f\big(x^{(1)}, x^{(2)}\big) := \big((e-e^{-1})\min(|x^{(1)}|, 1), 0\big)$, it is readily checked that both components of $O_T(x)$, and also $\|O_T(x)\|$, are strict Benford sequences provided that $x^{(1)} \notin \{-1-e^{-1}, 0\}$ and $x^{(2)} \neq 0$. Notice, however, that $O_T^{(1)}(x)$ is bounded if $x^{(1)} \in\,]-1-e^{-1}, 0[\,$.

Theorem 4.1 has a direct continuous-time counterpart. Consider the initial value problem

$$\dot{x} = Ax + f(x)\,, \qquad x(0) = x_0\,, \tag{4.15}$$

with $A \in \mathbb{R}^{d\times d}$ and f a bounded function which is assumed to be C^1 in order to guarantee (local) existence and uniqueness of the solution of (4.15); this solution will be denoted as $(\varphi_t x_0)_{t\geq 0}$. Again, for $(\varphi_t x_0)_{t\geq 0}$ to go off to infinity for all sufficiently large $\|x_0\|$, assume that $\Re\lambda > 0$ for all $\lambda \in \sigma(A)$.

Theorem 4.3. *Assume that all eigenvalues of $A \in \mathbb{R}^{d\times d}$ have positive real part, and that $f \in C^1$ is bounded. If A does not have exponentially b-resonant spectrum, then the solution $(\varphi_t x_0)_{t\geq 0}$ of (4.15) exists for all $t \geq 0$ and, for all sufficiently large $\|x_0\|$, every unbounded component of $(\varphi_t x_0)_{t\geq 0}$ as well as $(\|\varphi_t x_0\|)_{t\geq 0}$ is a b-Benford function.*

Proof. The argument reduces the continuous-time case to the discrete-time case studied in Theorem 4.1. Since the estimate $\|Ax + f(x)\| \leq \|A\|\,\|x\| + C$ holds with $C := \sup_{x\in\mathbb{R}^d} \|f(x)\|$, the solution of (4.15) is uniquely defined for all $t \geq 0$, see [1]. Implicitly, $\varphi_t x_0$ may be represented via the integral equation

$$\varphi_t x_0 = e^{At}x_0 + \int_0^t e^{A(t-\tau)} f(\varphi_\tau x_0)\, d\tau$$

for all $t \geq 0$, $x_0 \in \mathbb{R}^d$. Therefore, for any positive h and all $n \in \mathbb{N}_0$ and $x_0 \in \mathbb{R}^d$, $\varphi_{nh}x_0 = T_h^n(x_0)$ with the time-h-map T_h defined by

$$T_h(x) := e^{Ah}x + \int_0^h e^{A(h-\tau)} f(\varphi_\tau x)\, d\tau =: e^{Ah}x + g(x)\,.$$

Since g is continuous and $\sup_{x\in\mathbb{R}^d} \|g(x)\| \leq C\int_0^h \|e^{A(h-\tau)}\|\, d\tau < \infty$, the map T_h is of the form (4.14). By Lemma 3.6, for almost all $h > 0$ the matrix e^{Ah} does not have b-resonant spectrum, and clearly $|\lambda| > 1$ for all $\lambda \in \sigma(e^{Ah})$. The claim thus follows from Theorem 4.1 together with [14, Thm. 9.6]. □

Remark 4.4. (i) As in the discrete-time case, under the conditions of Theorem 4.3, bounded components of the solution $(\varphi_t x_0)_{t\geq 0}$ of (4.15) may or may not be Benford functions.

(ii) Theorems 4.1 and 4.3 are extensions of one-dimensional results in [6].

4.2. **Maps with polynomial growth.** In this section, each component of T is assumed to have precisely one dominating polynomial term. Under mild assumptions, this particular structure of T ensures the generation of Benford sequences for almost all sufficiently large initial points. Specifically, assume that for $j = 1, \ldots, d$ the j-th component of $T : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$T(x)^{(j)} = \gamma_j \big(x^{(1)}\big)^{a_{j,1}} \cdot \ldots \cdot \big(x^{(d)}\big)^{a_{j,d}} \big(1 + f_j(x)\big)\,, \tag{4.16}$$

where $\gamma_j \in \mathbb{R}\setminus\{0\}$, $a_{j,k} \in \mathbb{N}_0$, and the f_j denote C^1 functions with $|f_j(x)| \to 0$ as $\|x\| \to \infty$, for all j. To avoid trivialities, it is natural to require $\sum_k a_{j,k} \geq 1$ for all j in (4.16). Thus each component of T vanishes on some coordinate hyperplane $x^{(k)} \equiv 0$, and Benford sequences may be generated from initial points which are sufficiently far away from any of these hyperplanes. Therefore, for $\alpha > 0$ define the cone $C_\alpha := \big\{x \in \mathbb{R}^d : \min_{j=1}^d |x^{(j)}| \geq \alpha\big\}$, and let $D_x f$ denote the Jacobian of f at the point x, that is $(D_x f)_{jk} = \frac{\partial f_j}{\partial x^{(k)}}(x)$. The following theorem provides a natural generalization of Theorem 4.1 in [6].

Theorem 4.5. *Let T be given by (4.16), and assume that the matrix $A = (a_{j,k})_{j,k=1}^{d}$ is hyperbolic and invertible, i.e. $\sigma(A) \cap (S^1 \cup \{0\}) = \emptyset$. Also assume that the C^1 functions f_j satisfy $\sup_{x \in C_\alpha} |f_j(x)| \to 0$ as $\alpha \to \infty$ for all j, as well as $\sup_{x \in C_{\alpha_o}} \|x\|^{1+\varepsilon} \|D_x f\| < \infty$ for some $\varepsilon > 0$, $\alpha_o > 0$. Then for $\alpha \geq \alpha_o$ sufficiently large, each component of $O_T(x)$ is a strict Benford sequence for almost all $x \in C_\alpha$.*

Proof. First observe that without loss of generality $|\gamma_j| = 1$ for all j: otherwise by rescaling $x^{(j)}$ as $\alpha_j \widetilde{x}^{(j)}$ and by requiring that

$$|\gamma_j| \alpha_1^{a_{j,1}} \cdot \ldots \cdot \alpha_{j-1}^{a_{j,j-1}} \alpha_j^{a_{j,j}-1} \alpha_{j+1}^{a_{j,j+1}} \cdot \ldots \cdot \alpha_d^{a_{j,d}} = 1 \qquad \text{for all } j \,, \tag{4.17}$$

(4.16) may be transformed to that special form. Since A does not have 1 as an eigenvalue, (4.17) has precisely one solution $(\alpha_1, \ldots, \alpha_d)$ with $\alpha_j > 0$ for all j.

Although iteration of T may make points jump between different components of C_α, since there are only finitely many components, the sequence of components containing $T^n(x)$ is eventually periodic. The considerations below will make it evident that the analysis may thus be restricted to $C_\alpha^+ := C_\alpha \cap (\mathbb{R}^+)^d$ and to the case $\gamma_j = 1$ for all j.

It will now be shown that $T^N(C_\alpha^+) \subseteq C_\alpha^+$ for some $N \in \mathbb{N}$ and α sufficiently large. (Notice that the latter statement does not hold in general if A lacks hyperbolicity.) To this end, fix a base b and introduce new coordinates $\bigl(y^{(1)}, \ldots, y^{(d)}\bigr)$ on C_α^+ by setting $y^{(j)} := \log_b x^{(j)}$, $j = 1, \ldots, d$. In these new coordinates, T induces the map S_b with

$$S_b(y)^{(j)} = \log_b T^{(j)}\bigl(b^{y^{(1)}}, \ldots, b^{y^{(d)}}\bigr) = a_{j,1} y^{(1)} + \ldots + a_{j,d} y^{(d)} + g_b^{(j)}(y) \,,$$

where $g_b^{(j)}(y) := \log_b\bigl(1 + f_j(b^{y^{(1)}}, \ldots, b^{y^{(d)}})\bigr)$. Clearly $\sup_{y \in C_\beta^+} \|g_b(y)\| \to 0$ as $\beta \to \infty$, and g_b is C^1 on C_β^+ for β sufficiently large. The matrix A is non-negative, so $AC_\beta^+ \subseteq C_\beta^+$ for all $\beta > 0$, and A has a dominant real eigenvalue which by hyperbolicity is larger than one. Consequently, there exists $N \in \mathbb{N}$ such that $A^N(1, \ldots, 1) \in C_2^+$. Now take $\beta > 1$ so large that $\sup_{y \in C_\beta^+} \|g_b(y)\| \leq N^{-1} \|A\|_\infty^{-N}$, where $\|A\|_\infty := \max_j \sum_k |a_{j,k}| \geq 1$. For each $y \in C_{2\beta}^+$ therefore $S_b^l(y) \in C_\beta^+$ for all $l = 0, \ldots, N$. But

$$S_b^N(y) = A^N y + A^{N-1} g_b(y) + A^{N-2} g_b \circ S_b(y) + \ldots + A g_b \circ S_b^{N-2}(y) + g_b \circ S_b^{N-1}(y) \,,$$

and so, for all j and $y \in C_{2\beta}^+$,

$$S_b^N(y)^{(j)} \geq (A^N y)^{(j)} - N^{-1}(1 + \|A\|_\infty^{-1} + \ldots + \|A\|_\infty^{-N+1}) \geq 4\beta - 1 > 3\beta \,,$$

that is, $S_b^N(C_{2\beta}^+) \subseteq C_{3\beta}^+$, which shows that $T^N(C_\alpha^+) \subseteq C_\alpha^+$ with $\alpha = b^{2\beta}$.

By the hyperbolicity of A, there exists an invariant splitting of $\mathbb{R}^d$ into a stable and an unstable part, i.e. $\mathbb{R}^d = E_s \oplus E_u$ with $AE_s \subseteq E_s$, $AE_u \subseteq E_u$ such that the restriction of A to E_s and E_u has only eigenvalues of absolute value smaller and larger than one, respectively. (Notice that E_s may be trivial, but E_u is not.) Denote by π_s, π_u the projections induced by the direct sum $\mathbb{R}^d = E_s \oplus E_u$. Now fix a sufficiently large β and consider the map

$$h_b : y \mapsto y + \sum_{l=0}^{\infty} A^{-(l+1)} \pi_u \circ g_b \circ S_b^l(y) \,, \tag{4.18}$$

which is well-defined and continuous on C_β^+ since $\sum_{l=0}^\infty \|A^{-(l+1)}\pi_u\| < \infty$. An explicit calculation yields

$$h_b \circ S_b^n(y) - A^n h_b(y) = \sum_{l=0}^{n-1} A^{n-1-l}\pi_s \circ g_b \circ S_b^l(y) \in E_s \tag{4.19}$$

for all $n \in \mathbb{N}$. It is easily deduced from (4.19) and the preceding calculations that $\|h_b \circ S^n(y) - A^n h_b(y)\| \to 0$ as $n \to \infty$. Furthermore, definition (4.18) shows that $\sup_{y\in C_\beta^+} \|h_b(y) - y\| \to 0$ as $\beta \to \infty$. On the other hand, for every $y \in C_\beta^+$ the properties of A imply that $A^n y^{(j)} \to \infty$ for all $j = 1, \ldots, d$, so $S_b^n(y)^{(j)} \to \infty$ for all j and $y \in C_\beta^+$.

The map $\zeta \mapsto A\zeta \pmod 1$ on the torus $\mathbb{T}^d$ is ergodic with respect to $\lambda_{\mathbb{T}^d}$, see [13]. For almost all $y \in (\mathbb{R}^+)^d$, therefore, each component of $(A^n y)_{y\in\mathbb{N}_0}$ is u.d. mod 1. From the readily checked fact that

$$\sum_{l=0}^\infty \|A^{-1}\|^l \|A\|^l \|D_{S_b^l(y)} g_b\| < \infty$$

and by means of termwise differentiation of (4.18), it is straightforward to show that for sufficiently large β the map h_b is in fact a local diffeomorphism on C_β^+; in particular, h_b maps sets of measure zero to sets of measure zero. For almost all $y \in C_\beta^+$ with β sufficiently large $O_{S_b}^{(j)}(y)$ therefore is u.d. mod 1 for all $j = 1, \ldots, d$, and $O_T^{(j)}(x)$ is a b-Benford sequence for almost all $x \in C_\alpha^+$ with $\alpha = b^\beta$. Since $T^n(x)^{(j)} \to \infty$ for all j and $x \in C_\alpha^+$, there exists, for each base b, a set $B_b \subset C_\alpha^+$ such that $\lambda^d(C_\alpha^+ \setminus B_b) = 0$, and $O_T^{(j)}(x)$ is a b-Benford sequence for all j and $x \in B_b$. Setting $B := \bigcap_{b\geq 2} B_b$ yields $\lambda^d(C_\alpha^+ \setminus B) = 0$, and so $O_T^{(j)}(x)$ is a strict Benford sequence for all j and all $x \in B_b$. □

Unlike for the linear and the linearly dominated case, with T given by (4.16) there may exist points x for which all components of $O_T(x)$ are unbounded and nevertheless lack the Benford property.

Corollary 4.6. *Let T be as in* (4.16)*, and assume that T satisfies the assumptions of Theorem* 4.5*. Then for $\alpha \geq \alpha_o$ sufficiently large, there exists a dense set $E \subseteq C_\alpha$ such that no component of $O_T(x)$ is a Benford sequence for any $x \in E$.*

Proof. Again assume without loss of generality that $\gamma_j = 1$ for all j, and restrict to C_α^+. Observing that the map $\zeta \mapsto A\zeta \pmod 1$ on the torus $\mathbb{T}^d$ has a dense set P of periodic points [13], the claim follows immediately as in the proof of Theorem 4.5. Indeed, with the notation of that proof it suffices to fix a base b and define

$$E := \left\{ \left(b^{y^{(1)}}, \ldots, b^{y^{(d)}}\right) \,:\, h_b(y) \in C^+_{\log_b \alpha - 1} \cap (P + \mathbb{Z}^d) \right\} \cap C_\alpha^+ \,,$$

where α is sufficiently large so that the above arguments (ensuring in particular the existence and properties of the map h_b as stated there) remain valid, and also $\sup_{y\in C^+_{\log_b \alpha}} \|h_b(y) - y\| < 1$. □

Example 4.7. For the sake of lucidity, the components of $x \in \mathbb{R}^2$ will be denoted by u, v rather than $x^{(1)}, x^{(2)}$ throughout this example.

(i) For the map

$$T : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} 2v - ve^{-u^2v^2} \\ 3uv + \dfrac{uv}{u^2+v^2+1} \end{pmatrix}$$

all the hypotheses of Theorem 4.5 are satisfied with $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Notice that this theorem does not require all eigenvalues of A to lie outside the unit circle.

(ii) Unlike for the one-dimensional case, a multi-variate polynomial map T generally does *not* have a dominant term of polynomial growth such that it may be written in the form (4.16). Even if T can be written in that form, the quantity $\|x\|^{1+\varepsilon}\|D_xf\|$ need not be bounded on C_{α_o} for any $\varepsilon > 0$, $\alpha_o > 0$, as can be seen from the simple example

$$T : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} uv+1 \\ uv^3 - v^2 \end{pmatrix} .$$

However, if the leading term of T dominates more significantly in the sense that

$$\lim_{x \in C_\alpha, \|x\| \to \infty} x^{(1)} \cdot \ldots \cdot x^{(d)} f_j(x) = 0 \qquad \text{for all } j\,,$$

then the condition $\sup_{x\in C_{\alpha_o}} \|x\|^{1+\varepsilon}\|D_xf\| < \infty$ holds with $\varepsilon = 1$. In this case an application of Theorem 4.5 only requires the hyperbolicity and invertibility of A to be checked. An illustrating example is provided by the map

$$T : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u^2v^2 + 2 \\ -3u^4v^3 + 4u^2v - 5v^2 \end{pmatrix} ,$$

for which $A = \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}$ satisfies all the assumptions of Theorem 4.5.

4.3. **Complex analytic maps.** As explained earlier, with regards to Benford's law, the analysis of the dynamics of complex analytic maps naturally focuses on systems which have 0 or ∞ (or both) as an attractor. (For a comprehensive view on complex analytic dynamics see e.g. [3].) Since an analytic map $T : \mathbb{C} \to \mathbb{C}$ having ∞ as an attractor necessarily is a rational function, without loss of generality the attractor may be assumed to be at the origin. In this section, therefore, the following situation will be analyzed: the analytic map T has 0 as a stable attracting fixed point, i.e. $T(0) = 0$ and $|\gamma| < 1$, where $\gamma := T'(0)$. As Benford's law is a statement about real sequences, this map will be dealt with as a two-dimensional real map. Similarly, all metric statements below refer to the two-dimensional Lebesgue measure λ^2. It proves helpful to distinguish the two cases $|\gamma| > 0$ and $\gamma = 0$, respectively. The linearly dominated case $0 < |\gamma| < 1$ can be analyzed by means of the results for linear maps in Section 3, together with another shadowing argument.

Theorem 4.8. *Let T be analytic at* 0*, and assume $T(0) = 0$ and $0 < |\gamma| < 1$ with $\gamma := T'(0)$. If $1, \log_b|\gamma|, \frac{1}{2\pi}\arg\gamma$ are $\mathbb{Q}$-independent, then for all z sufficiently close to* 0*, the sequences $\Re O_T(z)$, $\Im O_T(z)$ are both b-Benford sequences. Furthermore, $|O_T(z)|$ is a b-Benford sequence if and only if $\log_b|\gamma|$ is irrational.*

Proof. The argument is similar to the one in [6, Sect. 3]. Rewrite T as $T(z) = \gamma z\big(1 - f(z)\big)$, where f is analytic, and $f(0) = 0$. It is readily checked that the map

H with

$$H(z) := \frac{z}{1 + z \sum_{l=0}^{\infty} \frac{\gamma^l}{1 - f \circ T^l(z)} \cdot \frac{f \circ T^l(z)}{T^l(z)}}$$

is analytic at 0, and $H \circ T = \gamma H$ holds sufficiently close to the origin; furthermore, $\lim_{z \to 0} z^{-1} H(z) = 1$, so

$$\lim_{n \to \infty} \frac{\gamma^n H(z)}{T^n(z)} = 1 ,$$

and the assertion about the b-Benford property of $|O_T(z)|$ follows immediately since

$$n \log_b |\gamma| + \log_b |H(z)| - \log_b |T^n(z)| \to 0 \qquad \text{as } n \to \infty .$$

Clearly, $H(z) \neq 0$ for all $z \neq 0$ sufficiently close to the origin. Since $\frac{1}{2\pi} \arg \gamma$ is irrational, the sequences $\Re\big(\gamma^n H(z)\big)$ and $\Im\big(\gamma^n H(z)\big)$ do not vanish eventually. Splitting $z \mapsto \gamma z$ into real and imaginary part yields the realified system

$$x \mapsto (\gamma)_{\mathbb{R}} x = \begin{pmatrix} \Re\gamma & -\Im\gamma \\ \Im\gamma & \Re\gamma \end{pmatrix} x ,$$

to which Theorem 3.3 applies. Thus the two sequences $\big(\Re(\gamma^n H(z))\big)_{n \in \mathbb{N}_0}$ and $\big(\Im(\gamma^n H(z))\big)_{n \in \mathbb{N}_0}$ are b-Benford for all $z \neq 0$ sufficiently close to the origin. To prove that the same is true for $\Re O_T(z)$, write

$$\gamma^n H(z) = |\gamma^n H(z)| e^{i\left(n \arg \gamma + \arg H(z)\right)} \quad \text{and} \quad T^n(z) = |T^n(z)| e^{i(n \arg \gamma + \arg H(z) + \psi_n)}$$

with $\lim_{n \to \infty} \psi_n = 0$. But then

$$\begin{aligned} \log_b \big|\Re\big(\gamma^n H(z)\big)\big| - \log_b \big|\Re T^n(z)\big| = & \log_b \left|\frac{\gamma^n H(z)}{T^n(z)}\right| + \log_b \big|\cos\big(n \arg \gamma + \arg H(z)\big)\big| \\ & - \log_b \big|\cos\big(n \arg \gamma + \arg H(z) + \psi_n\big)\big| , \end{aligned}$$

and the claim follows from Lemma 2.3, because for any given $\varepsilon > 0$ the right-hand side is (in absolute value) less than ε for all n in a set $J_\varepsilon \subseteq \mathbb{N}_0$ with lower density at least $1 - \varepsilon$ (by an argument similar to the one in Lemma 2.9).

The assertion concerning the imaginary part $\Im O_T(z)$ is verified in a completely analogous manner. □

Remark 4.9. A rational value of $\frac{1}{2\pi} \arg \gamma$ need not rule out the emergence of Benford sequences, as the (linear) map $z \mapsto ie^{-1} z$ shows. This is a direct analogue of Example 3.5(ii) because $(ie^{-1})_{\mathbb{R}} = A^{-1}$ with the matrix A used there. Therefore $\Re O_T(z)$ and $\Im O_T(z)$ are strict Benford sequences for every z not in $\mathbb{R} \cup i\mathbb{R}$. For $z \in \mathbb{R} \cup i\mathbb{R}$ the mantissa distribution of $\Re O_T(z)$ and $\Im O_T(z)$ is a convex combination of Benford's distribution and an atom at 0.

The situation encountered above is fairly general: if, within the setting of Theorem 4.8, $\log_b |\gamma|$ is irrational but $\frac{1}{2\pi} \arg \gamma = \frac{p}{2q}$ for relatively prime integers p, q and $0 < |p| < q$, then for $|z|$ sufficiently small, $\Re O_T(z)$ and $\Im O_T(z)$ are b-Benford sequences unless

$$H(z) \in \bigcup_{l=1}^{q} (\mathbb{R} \cup i\mathbb{R}) e^{\pi i q^{-1} l} ,$$

that is, unless $H(z)$ belongs to a union of at most $2q$ lines intersecting at the origin. In the latter case, in addition to the logarithmic distribution, an atom at 0 occurs in the base b mantissa distribution of $\Re O_T(z)$ or $\Im O_T(z)$.

Corollary 4.10. *Under the hypotheses of Theorem* 4.8*, for almost all γ with $0 < |\gamma| < 1$, the sequences $\Re O_T(z)$, $\Im O_T(z)$ and $|O_T(z)|$ are strict Benford sequences for all sufficiently small $|z|$.*

If $T'(0) = 0$, then the dynamics of T near 0 is essentially non-linear. As a simple example consider the map $T : z \mapsto z^2$. By [6], $\Re O_T(z)$ is a strict Benford sequence for λ^1-almost all $z \in \mathbb{R}$, whereas $\Im T^n(z)$ vanishes identically for such z. Since $T^n(iz) = T^n(z)$ for $n \geq 2$ and all z, it is natural to expect that the real as well as the imaginary part of $O_T(z)$ is typically a Benford sequence. Notice that this assertion cannot be proved by falling back on Theorem 4.5 because T, when considered as a two-dimensional real map, does not have a dominant polynomial term. Nevertheless, a rather complete analysis is again provided by means of a shadowing argument.

Theorem 4.11. *Let $T \not\equiv 0$ be analytic at 0, with $T(0) = 0$ and $T'(0) = 0$. For almost all z sufficiently close to the origin, $\Re O_T(z)$, $\Im O_T(z)$ and $|O_T(z)|$ are strict Benford sequences.*

Proof. Rewrite T as $T(z) = \gamma z^p\bigl(1 - f(z)\bigr)$, where $\gamma \in \mathbb{C} \setminus \{0\}$, $p \in \mathbb{N} \setminus \{1\}$ and f is an analytic function with $f(0) = 0$. Let ρ denote a complex number with $\rho^{p-1} = \gamma$, and define a map H by setting

$$H(z) := \rho z \prod_{l=0}^{\infty} \bigl(1 - f \circ T^l(z)\bigr)^{p^{-(l+1)}}, \tag{4.20}$$

where for every $l \in \mathbb{N}$ the p^l-th root is understood as that branch of the root which maps the positive real axis into itself. It is easily checked that (4.20) defines an analytic function near the origin, with $\lim_{z\to 0} z^{-1}H(z) = \rho \neq 0$. Therefore H is locally a conformal map with $H \circ T(z) = H^p(z)$ for all z sufficiently close to 0. Consequently,

$$\lim_{n\to\infty} \frac{H(z)^{p^n}}{T^n(z)} = \rho\,,$$

from which the assertion about the strict Benford property of $|O_T(z)|$ follows immediately, because for all bases b

$$p^n \log_b |H(z)| - \log_b |T^n(z)| \to \log_b |\rho| \qquad \text{as } n \to \infty\,,$$

and $(p^n x)_{n\in\mathbb{N}_0}$ is u.d. mod 1 for λ^1-almost all $x \in \mathbb{R}$, see [9, 12].

To prove that $\Re O_T(z)$ is a strict Benford sequence for almost all z, it is sufficient to show that the same is true for $\bigl(\Re\rho^{-1}H(z)^{p^n}\bigr)_{n\in\mathbb{N}_0}$, as is seen by writing

$$\bigl|\Re\bigl(\rho^{-1}H(z)^{p^n}\bigr)\bigr| = \bigl|\rho^{-1}H(z)^{p^n}\bigr| \cdot \bigl|\cos(p^n \arg H(z) - \arg\rho)\bigr| \tag{4.21}$$

and

$$|\Re T^n(z)| = |T^n(z)| \cdot \bigl|\cos(p^n \arg H(z) - \arg\rho + \psi_n)\bigr| \tag{4.22}$$

with $\lim_{n\to\infty} \psi_n = 0$. From (4.21) and (4.22) it follows that

$$\begin{aligned}\log_b\bigl|\Re\bigl(\rho^{-1}H(z)^{p^n}\bigr)\bigr| - \log_b |\Re T^n(z)| = \log_b \left|\frac{H(z)^{p^n}}{T^n(z)\rho}\right| + \\ + \log_b\bigl|\cos(p^n \arg H(z) - \arg\rho)\bigr| - \log_b\bigl|\cos(p^n \arg H(z) - \arg\rho + \psi_n)\bigr|\,.\end{aligned}$$

Provided that $\frac{1}{2\pi}\arg H(z)$ is a p-normal number [12, 14], the right-hand side is less than a given $\varepsilon > 0$ for all n in a set $J_\varepsilon \subseteq \mathbb{N}$ which has lower density at least

$1-\varepsilon$. Therefore, if $\frac{1}{2\pi}\arg H(z)$ is p-normal and $\big(\Re(\rho^{-1}H(z)^{p^n})\big)_{n\in\mathbb{N}_0}$ is a b-Benford sequence, then so is $\Re O_T(z)$.

Since H is a conformal map near the origin, and the countable union of sets of measure zero has measure zero itself, it is enough to establish the Benford property for $\big(\Re(\rho^{-1}z^{p^n})\big)_{n\in\mathbb{N}_0}$ for almost every z. However, since

$$\log_b\big|\Re(\rho^{-1}z^{p^n})\big| = p^n\log_b|z| - \log_b|\rho| + \log_b\big|\cos(p^n\arg z - \arg\rho)\big|\,,$$

this follows immediately from Lemma 2.7.

The proof of the strict Benford property for $\Im O_T(z)$ is completely analogous. $\square$

In the following corollary the term *exceptional point* refers to any point z for which neither of the sequences $\Re O_T(z)$, $\Im O_T(z)$ nor $|O_T(z)|$ is a strict Benford sequence.

Corollary 4.12. *Let $T\not\equiv 0$ be analytic at 0, with $T(0)=0$ and $T'(0)=0$. Then every sufficiently small disc centered at the origin contains an uncountable dense set of exceptional points.*

Proof. Fix a base $b\in\mathbb{N}\setminus\{1\}$, and consider the (uncountable, dense) set

$$E_b := \{z\in\mathbb{C}\setminus\{0\}\,:\,\log_b|z| = kp^l \text{ with } k,l\in\mathbb{Z}\,,\ \frac{1}{2\pi}\arg z \text{ is } p\text{-normal}\,\}\,.$$

Since the map $x\mapsto\log_b|\cos 2\pi x|$ (mod 1) does not preserve $\lambda_{\mathbb{T}^1}$ for any base b, taking an appropriate neighborhood U of the origin, it follows as in the proof of Theorem 4.11, in particular (4.21) and (4.22), that neither of the sequences $\Re O_T(z)$, $\Im O_T(z)$ nor $|O_T(z)|$ is a b-Benford sequence for any $z\in H^{-1}(E_b\cap U)$. $\square$

Remark 4.13. (i) For any domain $U\subseteq\mathbb{C}$ containing the origin, the family $\mathcal{H}_0(U)$ of analytic maps T on U with $T(0)=0$ is a Polish space when endowed with the topology of uniform convergence on compact sets, and so are the open subset $\mathcal{A}(U):=\{T\in\mathcal{H}_0(U):|T'(0)|<1\}$ and the closed subset $\mathcal{A}_0(U):=\{T\in\mathcal{A}(U):T'(0)=0\}$, which represent the family of analytic maps having 0 as a stable attracting and a super-attracting fixed point, respectively, see e.g. [5]. Let $\mathcal{B}(U)$ denote the set of maps $T\in\mathcal{A}(U)$ which under iteration typically produce strict Benford sequences near the origin, more precisely

$$\begin{aligned}\mathcal{B}(U):=\{T\in\mathcal{A}(U):\exists\,\varepsilon>0 \text{ such that } &\Re O_T(z),\Im O_T(z),|O_T(z)| \text{ are strict}\\ &\text{Benford sequences for (at least) almost all } z \text{ with } |z|<\varepsilon\}\,.\end{aligned}$$

Theorem 4.11 implies that $\mathcal{A}_0(U)\subseteq\mathcal{B}(U)$, and the set $\mathcal{A}(U)\setminus\mathcal{B}(U)$ is of first category by Theorem 4.8. The generation of Benford sequences is thus a generic phenomenon in $\mathcal{A}(U)$.

(ii) The set of p-normal numbers on the real line is of first category [12], so the sets of full measure referred to in Theorem 4.11 may be of first category, too.

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