Most linear flows on $\mathbb{R}^d$ are Benford

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Abstract

A necessary and sufficient condition ("exponential nonresonance") is established for every signal obtained from a linear flow on $\mathbb{R}^d$ by means of a linear observable to either vanish identically or else exhibit a strong form of Benford's Law (logarithmic distribution of significant digits). The result extends and unifies all previously known (sufficient) conditions. Exponential nonresonance is shown to be typical for linear flows, both from a topological and a measure-theoretical point of view.

Keywords. Benford function, (continuous) uniform distribution mod 1, linear flow, observable, $\mathbb{Q}$-independence, (exponentially) nonresonant set.

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1 Introduction

Let ϕ be a flow on $X = \mathbb{R}^d$ endowed with the usual topology, i.e., $\phi : \mathbb{R} \times X \to X$ is continuous, and $\phi(0, x) = x$ as well as $\phi\big(s, \phi(t, x)\big) = \phi(s + t, x)$ for all $x \in X$ and $s, t \in \mathbb{R}$. Denoting the homeomorphism $x \mapsto \phi(t, x)$ of X simply by ϕ_t and the space of all linear maps $A : X \to X$ by $\mathcal{L}(X)$, as usual, call the flow ϕ *linear* if each ϕ_t is linear, that is, $\phi_t \in \mathcal{L}(X)$ for every $t \in \mathbb{R}$. Given a linear flow ϕ on X, fix any linear functional $H : \mathcal{L}(X) \to \mathbb{R}$ and consider the function $H(\phi_\bullet)$. The main goal of this article is to completely describe the distribution of numerical values for the real-valued functions thus generated.

To see why this distribution may be of interest, recall that throughout science and engineering, flows on the phase space $X = \mathbb{R}^d$ are often used to provide models for real-worlds processes; e.g., see [1]. From a scientist's or engineer's perspective, it may not be desirable or even possible to observe a flow ϕ in its entirety, especially if d is large. Rather, what matters is the behaviour of certain functions ("signals") distilled from ϕ. Adopting terminology used similarly in e.g. quantum mechanics and ergodic theory [9, 18], call any function $h : X \to \mathbb{R}$ an *observable* (on X).

With this, what really matters from a scientist's or engineer's point of view are properties of signals $h\big(\phi(\bullet, x)\big)$ for specific observables h and points $x \in X$ that are relevant to the process being modelled by ϕ. In the case of linear flows, a special role is naturally played by *linear* observables. Note that if ϕ and h both are linear then $h\big(\phi(t,x)\big) \equiv H(\phi_t)$, where $H : \mathcal{L}(X) \to \mathbb{R}$ is the linear functional with $H(A) = h(Ax)$ for all $A \in \mathcal{L}(X)$. Given any linear flow ϕ on X, it makes sense, therefore, to more generally consider signals $H(\phi_\bullet)$ where $H : \mathcal{L}(X) \to \mathbb{R}$ is any linear functional; by a slight abuse of terminology, such functionals will henceforth be referred to as *linear observables* $\big($on $\mathcal{L}(X)\big)$ as well.

What, if anything, can be said about the distribution of values for signals $H(\phi_\bullet)$, where ϕ and H are a linear flow on X and a linear observable on $\mathcal{L}(X)$, respectively? As indicated below and demonstrated rigorously through the results of this article, for the overwhelming majority of linear flows this question has a surprisingly simple, though perhaps somewhat counter-intuitive answer: Except for the trivial case of $H(\phi_\bullet) = 0$, that is, $H(\phi_t) = 0$ for all $t \in \mathbb{R}$, the values of $H(\phi_\bullet)$ always exhibit *one and the same* distribution, regardless of d, ϕ and H. As it turns out, this distinguished distribution is nothing other than *Benford's Law* (BL), the logarithmic law for significant digits.

Within the study of (digits of) numerical data generated by dynamical processes — a classical subject that continues to attract interest from disciplines as diverse as ergodic and number theory [2, 10, 12, 21, 24], analysis [8, 25], and statistics [14, 17, 26] — the astounding ubiquity of BL is a recurring, popular theme. The most well-known special case of BL is the so-called (*decimal*) *first-digit law* which asserts that

$$\mathbb{P}(\textit{leading digit}_{10} = \ell) = \log_{10}\left(1 + \ell^{-1}\right) \quad \forall \ell = 1, \ldots, 9\,, \tag{1.1}$$

where *leading digit*$_{10}$ refers to the leading (or first significant) decimal digit, and $\log_{10}$ is the base-10 logarithm (see Section 2 for rigorous definitions); for example, the leading decimal digit of $e = 2.718$ is 2, whereas the leading digit of $-e^e = -15.15$ is 1. Note that (1.1) is heavily skewed towards the smaller digits: For instance, the leading decimal digit is almost six times more likely to equal 1 (probability $\log_{10} 2 = 30.10\%$) than to equal 9 (probability $1 - \log_{10} 9 = 4.57\%$). Ever since first recorded by Newcomb [28] in 1881 and re-discovered by Benford [3] in 1938, examples of data and systems conforming to (1.1) in one form or another have been discussed extensively, notably for real-life data (e.g. [15, 29]) as well as in stochastic (e.g. [31]) and deterministic processes (e.g. the Lorenz flow [33] and certain unimodal maps [7, 32]). As of this writing, an online database [4] devoted exclusively to BL lists more than 800 references.

Given any (Borel) measurable function $f : \mathbb{R}^+ \to \mathbb{R}$, arguably the simplest and most natural notion of f conforming to (1.1) is to require that

$$\lim\nolimits_{T\to+\infty} \frac{\lambda(\{t \le T : \textit{leading digit}_{10} f(t) = \ell\})}{T} = \log_{10}\left(1 + \ell^{-1}\right) \quad \forall \ell = 1, \ldots, 9\,; \tag{1.2}$$

here and throughout, λ denotes Lebesgue measure on $\mathbb{R}^+$, or on parts thereof. With this, the central question studied herein is this: Does (1.2) hold for $f = H(\phi_\bullet)$ where ϕ is a linear flow on X and H is any linear observable on $\mathcal{L}(X)$? Several attempts to answer this question are recorded in the literature; e.g., see [5, 20, 27, 33]. All these attempts, however, seem to have led only to *sufficient* conditions for (1.2) that are either restrictive or complicated to state. In contrast, Theorem 3.2 below, one of the main results of this article, provides a simple *necessary and sufficient* condition for every non-trivial signal $f = H(\phi_\bullet)$ to satisfy (1.2), and in fact to conform to BL in an even stronger sense. All results in the literature alluded to earlier are but simple special cases of this theorem.

To see why it is plausible for signals $f = H(\phi_\bullet)$ to satisfy (1.2), pick any real number $\alpha \neq 0$ and consider as an extremely simple but also quite compelling example the function $f(t) = e^{\alpha t}$. Obviously, $x = f$ is a solution of $\dot{x} = \alpha x$, and $f(t) \equiv \phi_t \in \mathcal{L}(\mathbb{R}^1)$ for the linear flow generated by this differential equation. A short elementary calculation shows that, for all $T > 0$ and $1 \leq \ell \leq 9$,

$$\left| \frac{\lambda(\{t \leq T : \textit{leading digit}_{10} e^{\alpha t} = \ell\})}{T} - \log_{10}\left(1 + \ell^{-1}\right) \right| < \frac{1}{|\alpha| T}\,, \tag{1.3}$$

and hence (1.2) holds for $f(t) = e^{\alpha t}$ whenever $\alpha \neq 0$. (Trivially, it does not hold if $\alpha = 0$.) However, already for the linear flow ϕ on $\mathbb{R}^2$ generated by

$$\dot{x} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} x\,,$$

with $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$, a brute-force calculation is of little use in deciding whether all non-trivial signals $f = H(\phi_\bullet)$ satisfy (1.2). Theorem 3.2 shows that indeed they do, provided that $\alpha\pi/(\beta \ln 10)$ is irrational; see Example 3.4.

This article is organized as follows. Section 2 introduces the formal definitions and analytic tools required for the analysis. In Section 3, the main results characterizing conformance to BL for linear flows are stated and proved, based upon a tailor-made notion of exponential nonresonance (Definition 2.9). Several examples are presented in order to illustrate this notion as well as the main results. Section 4 establishes the fact that, as suggested by the simple examples in the preceding paragraph, exponential nonresonance, and hence conformance to BL as well, is generic for linear flows on $\mathbb{R}^d$. Given the widespread use of linear differential equations as models throughout the sciences, the results of this article may contribute to a better understanding of, and deeper appreciation for, BL and its applications across a wide range of disciplines.

2 Definitions and tools

The following, mostly standard notation and terminology is used throughout. The symbols $\mathbb{N}$, $\mathbb{Z}^+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of, respectively, positive integer, non-negative integer, integer, rational, non-negative real, real, and complex numbers, and $\varnothing$ is the empty set. Recall that Lebesgue measure on $\mathbb{R}^+$ or subsets thereof is written simply as λ. For each integer $b \geq 2$, the logarithm base b of $x > 0$ is denoted $\log_b x$, and $\ln x$ is the natural logarithm (base e) of x; for convenience, let $\log_b 0 := 0$ for every b, and $\ln 0 := 0$. Given any $x \in \mathbb{R}$, the largest integer not larger than x is symbolized by $\lfloor x \rfloor$. The real part, imaginary part, complex conjugate, and absolute value (modulus) of any $z \in \mathbb{C}$ is $\Re z$, $\Im z$, $\overline{z}$, and $|z|$, respectively. For each $z \in \mathbb{C} \setminus \{0\}$, there exists a unique number $-\pi < \arg z \leq \pi$ with $z = |z|e^{\imath \arg z}$. Given any $w \in \mathbb{C}$ and $\mathcal{Z} \subset \mathbb{C}$, define $w + \mathcal{Z} := \{w + z : z \in \mathcal{Z}\}$ and $w\mathcal{Z} := \{wz : z \in \mathcal{Z}\}$. Thus with the unit circle $\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$, for example, $w + \mathbb{S} = \{z \in \mathbb{C} : |z - w| = 1\}$ and $w\mathbb{S} = \{z \in \mathbb{C} : |z| = |w|\}$ for each $w \in \mathbb{C}$. The cardinality (number of elements) of any finite set $\mathcal{Z} \subset \mathbb{C}$ is $\#\mathcal{Z}$.

Recall throughout that b is an integer with $b \geq 2$, informally referred to as a *base*. Given a base b and any $x \neq 0$, there exists a unique real number $S_b(x)$ with $1 \leq S_b(x) < b$ and a unique integer k such that $|x| = S_b(x)b^k$. The number $S_b(x)$ is the *significand* or *mantissa* (base b) of x; for convenience, define $S_b(0) := 0$ for every base b. The integer $\lfloor S_b(x) \rfloor$ is the *first significant digit* (base b) of x; note that $\lfloor S_b(x) \rfloor \in \{1, \ldots, b-1\}$ whenever $x \neq 0$.

In this article, conformance to BL for real-valued functions, specifically for signals generated by linear flows, is studied via the following definition.

Definition 2.1. Let $b \in \mathbb{N} \setminus \{1\}$. A (Borel) measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ is a b-*Benford function*, or b-*Benford* for short, if

$$\lim_{T \to +\infty} \frac{\lambda\bigl(\{t \leq T : S_b\bigl(f(t)\bigr) \leq s\}\bigr)}{T} = \log_b s \quad \forall s \in [1, b)\,.$$

The function f is a *Benford function*, or simply *Benford*, if it is b-Benford for every $b \in \mathbb{N} \setminus \{1\}$.

Note that (1.2) holds whenever f is 10-Benford. The converse is not true in general since, for instance, the (piecewise constant) function $\lfloor S_{10}(2^\bullet) \rfloor$ only attains the values $1, \ldots, 9$ and hence clearly is not 10-Benford, yet (1.3) with $\alpha = \ln 2$ shows that it does satisfy (1.2).

The subsequent analysis of the Benford property for signals generated by linear flows is greatly facilitated by a few basic facts from the theory of uniform distribution, reviewed here for the reader's convenience; e.g., see [16, 23] for authoritative accounts of the subject. Throughout, the symbol d denotes a positive integer, usually unspecified or clear from the context. The d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ is symbolized by $\mathbb{T}^d$, its elements being represented as $\langle x \rangle = x + \mathbb{Z}^d$ with $x \in \mathbb{R}^d$; for simplicity write $\mathbb{T}$ instead of $\mathbb{T}^1$. Endow $\mathbb{T}^d$ with its usual (quotient) topology and σ-algebra $\mathcal{B}(\mathbb{T}^d)$ of Borel sets, and let $\mathcal{P}(\mathbb{T}^d)$ be the set of all probability measures on $\bigl(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d)\bigr)$. Denote the Haar (probability) measure of the compact Abelian group $\mathbb{T}^d$ by $\lambda_{\mathbb{T}^d}$. Call a set $\mathcal{J} \subset \mathbb{T}$ an *arc* if $\mathcal{J} = \langle \mathcal{I} \rangle := \{\langle x \rangle : x \in \mathcal{I}\}$ for some interval $\mathcal{I} \subset \mathbb{R}$. With this, a (Borel) measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ is *continuously uniformly distributed modulo one*, henceforth abbreviated *c.u.d.* mod 1, if

$$\lim_{T \to +\infty} \frac{\lambda\bigl(\{t \leq T : \langle f(t) \rangle \in \mathcal{J}\}\bigr)}{T} = \lambda_{\mathbb{T}}(\mathcal{J}) \quad \text{for every arc } \mathcal{J} \subset \mathbb{T}\,.$$

Equivalently, $\lim_{T\to+\infty} \frac{1}{T}\int_0^T F\big(\langle f(t)\rangle\big)\,\mathrm{d}t = \int_{\mathbb{T}} F\,\mathrm{d}\lambda_{\mathbb{T}}$ for every continuous (or just Riemann integrable) function $F:\mathbb{T}\to\mathbb{C}$. In particular, therefore, if the function f is c.u.d. mod 1 then $\lim_{T\to+\infty}\frac{1}{T}\int_0^T e^{2\pi \imath k f(t)}\mathrm{d}t = 0$ for every $k\in\mathbb{Z}\setminus\{0\}$, and the converse is also true (Weyl's criterion [23, Thm.I.9.2]).

The importance of uniform distribution concepts for the present article stems from the following fact which, though very simple, is nevertheless fundamental for all that follows; see [13, Thm.1] for a discrete-time analogue, and [7, Sec.4.1] for a full discussion.

Proposition 2.2. *Let $b\in\mathbb{N}\setminus\{1\}$. A measurable function $f:\mathbb{R}^+\to\mathbb{R}$ is b-Benford if and only if the function $\log_b|f|$ is c.u.d.* mod 1.

In order to enable the effective application of Proposition 2.2, a few basic facts from the theory of uniform distribution are re-stated here. In this context, the following discrete-time analogue of continuous uniform distribution is also useful: A sequence (x_n) of real numbers, by definition, is *uniformly distributed modulo one* (*u.d.* mod 1) if the (piecewise constant) function $f = x_{1+\lfloor\bullet\rfloor}$ is c.u.d. mod 1, or equivalently, if

$$\lim_{N\to\infty}\frac{\#\{n\le N:\langle x_n\rangle\in\mathcal{J}\}}{N} = \lambda_{\mathbb{T}}(\mathcal{J}) \quad \text{for every arc } \mathcal{J}\subset\mathbb{T}\,.$$

Lemma 2.3. *For each measurable function $f:\mathbb{R}^+\to\mathbb{R}$ the following are equivalent:*

(i) *f is c.u.d.* mod 1*;*

(ii) *If $g:\mathbb{R}^+\to\mathbb{R}$ is measurable and $\lim_{t\to+\infty}\big(g(t)-f(t)\big)$ exists (in $\mathbb{R}$) then g is c.u.d.* mod 1*;*

(iii) *kf is c.u.d.* mod 1 *for every $k\in\mathbb{Z}\setminus\{0\}$;*

(iv) *$f+\alpha\ln t$ is c.u.d.* mod 1 *for every $\alpha\in\mathbb{R}$.*

Proof. Clearly, (ii), (iii), and (iv) each implies (i), and the converse is [23, Exc.I.9.4], [23, Exc.I.9.6], and [5, Lem.2.8], respectively. □

The next result is a slight generalization of [23, Thm.I.9.6(b)].

Lemma 2.4. *Let the function $f:\mathbb{R}^+\to\mathbb{R}$ be measurable, and $\delta_0>0$. If, for some measurable, bounded $F:\mathbb{T}\to\mathbb{C}$ and $z\in\mathbb{C}$,*

$$\lim_{N\to\infty}\frac{1}{N}\sum\nolimits_{n=1}^{N} F\big(\langle f(n\delta)\rangle\big) = z \quad \text{for almost all } 0<\delta<\delta_0\,,$$

then also

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T F\big(\langle f(t)\rangle\big)\,\mathrm{d}t = z\,. \tag{2.1}$$

In particular, if the sequence $\big(f(n\delta)\big)$ is u.d. mod 1 *for almost all $0<\delta<\delta_0$ then f is c.u.d.* mod 1.

Proof. For each $n \in \mathbb{N}$, let $z_n = \int_{\delta_0(n-1)}^{\delta_0 n} F\big(\langle f(t)\rangle\big)\,\mathrm{d}t$. By the Dominated Convergence Theorem,

$$\lim_{N\to\infty} \frac{1}{\delta_0}\int_0^{\delta_0} \frac{1}{N}\sum\nolimits_{n=1}^{N} F\big(\langle f(n\delta)\rangle\big)\,\mathrm{d}\delta = z\,.$$

On the other hand,

$$\begin{aligned}\frac{1}{\delta_0}\int_0^{\delta_0} \frac{1}{N}\sum\nolimits_{n=1}^{N} F\big(\langle f(n\delta)\rangle\big)\,\mathrm{d}\delta &= \frac{1}{\delta_0 N}\sum\nolimits_{n=1}^{N} \frac{1}{n}\int_0^{\delta_0 n} F\big(\langle f(t)\rangle\big)\,\mathrm{d}t \\ &= \frac{1}{\delta_0 N}\sum\nolimits_{n=1}^{N} \frac{1}{n}\sum\nolimits_{\ell=1}^{n} z_\ell\,,\end{aligned}$$

and since the sequence (z_n) is bounded, a well-known Tauberian theorem [19, Thm.92] implies that

$$\begin{aligned} z &= \lim_{N\to\infty}\frac{1}{\delta_0 N}\sum\nolimits_{n=1}^{N} z_n = \lim_{N\to\infty}\frac{1}{\delta_0 N}\int_0^{\delta_0 N} F\big(\langle f(t)\rangle\big)\,\mathrm{d}t \\ &= \lim_{T\to+\infty}\frac{1}{T}\int_0^{T} F\big(\langle f(t)\rangle\big)\,\mathrm{d}t\,.\end{aligned}$$

The second assertion now follows immediately by considering specifically the functions $F(\langle x\rangle) = e^{2\pi\imath kx}$ for $k \in \mathbb{Z}$, together with Weyl's criterion. □

The following result pertains to very particular functions that map $\mathbb{T}^d$ into $\mathbb{T}$; such functions will appear naturally in the next section. Concretely, let $p_1, \ldots, p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R}\setminus\{0\}$, and consider the function

$$P_u : \begin{cases} \mathbb{T}^d & \to \quad \mathbb{T}\,, \\ \langle x\rangle & \mapsto \quad \big\langle p_1x_1 + \ldots + p_dx_d + \alpha\ln|u_1\cos(2\pi x_1) + \ldots + u_d\cos(2\pi x_d)|\big\rangle\,; \end{cases}$$

here $u \in \mathbb{R}^d\setminus\{0\}$ may be thought of as a parameter. (Recall the convention that $\ln 0 = 0$.) Note that P_u is measurable (in fact, differentiable $\lambda_{\mathbb{T}^d}$-a.e.), and so each $\mu \in \mathcal{P}(\mathbb{T}^d)$ induces a well-defined element $\mu\circ P_u^{-1}$ of $\mathcal{P}(\mathbb{T})$, via $\mu\circ P_u^{-1}(B) := \mu\big(P_u^{-1}(B)\big)$ for all $B \in \mathcal{B}(\mathbb{T})$. It is easy to see that $\mu\circ P_u^{-1}$ is absolutely continuous (w.r.t. $\lambda_{\mathbb{T}}$) whenever μ is absolutely continuous (w.r.t. $\lambda_{\mathbb{T}^d}$). For the purpose of this work, only the case $\mu = \lambda_{\mathbb{T}^d}$ is of further interest. Observe that $\lambda_{\mathbb{T}^d}\circ P_u^{-1}$ is equivalent to (i.e., has the same nullsets as) $\lambda_{\mathbb{T}}$. Moreover, for $\mathcal{P}(\mathbb{T})$ endowed with the topology of weak convergence, the Dominated Convergence Theorem implies that the $\mathcal{P}(\mathbb{T})$-valued function $u \mapsto \lambda_{\mathbb{T}^d}\circ P_u^{-1}$ is continuous on $\mathbb{R}^d\setminus\{0\}$, for any fixed $p_1,\ldots,p_d \in \mathbb{Z}$ and $\alpha \in \mathbb{R}\setminus\{0\}$. The arguments in [6, Sec.5] show that this function is non-constant, as might be expected.

Proposition 2.5. [6, Thm.5.4] *Given $p_1,\ldots,p_d \in \mathbb{Z}$, $\alpha \in \mathbb{R}\setminus\{0\}$, and any $\nu \in \mathcal{P}(\mathbb{T})$, there exists $u \in \mathbb{R}^d\setminus\{0\}$ such that $\lambda_{\mathbb{T}^d}\circ P_u^{-1} \neq \nu$.*

Remark 2.6. Specifically for the case $\nu = \lambda_{\mathbb{T}}$, it has been conjectured in [6] that $\lambda_{\mathbb{T}^d}\circ P_u^{-1} = \lambda_{\mathbb{T}}$ (if and) only if $\prod_{j:p_j\neq 0} u_j = 0$, and hence $\lambda_{\mathbb{T}^d}\circ P_u^{-1} \neq \lambda_{\mathbb{T}}$ for *most* $u \in \mathbb{R}^d\setminus\{0\}$.

The remainder of this section reviews tools and terminology concerning certain elementary number-theoretical properties of sets $\mathcal{Z} \subset \mathbb{C}$. Specifically, denote by $\text{span}_{\mathbb{Q}}\mathcal{Z}$ the smallest subspace of $\mathbb{C}$ (over $\mathbb{Q}$) containing $\mathcal{Z}$; equivalently, if $\mathcal{Z} \neq \varnothing$ then $\text{span}_{\mathbb{Q}}\mathcal{Z}$ is the set of all finite *rational* linear combinations of elements of $\mathcal{Z}$, i.e.,

$$\text{span}_{\mathbb{Q}}\mathcal{Z} = \left\{ \rho_1 z_1 + \ldots + \rho_n z_n : n \in \mathbb{N}, \rho_1, \ldots, \rho_n \in \mathbb{Q}, z_1, \ldots, z_n \in \mathcal{Z} \right\};$$

note that $\text{span}_{\mathbb{Q}}\varnothing = \{0\}$. With this terminology, recall that $z_1, \ldots, z_L \in \mathbb{C}$ are $\mathbb{Q}$-*independent* if $\text{span}_{\mathbb{Q}}\{z_1, \ldots, z_L\}$ is L-dimensional, or equivalently if $\sum_{\ell=1}^{L} p_\ell z_\ell = 0$ with integers $p_1, \ldots, p_L$ implies $p_1 = \ldots = p_L = 0$. The notion of $\mathbb{Q}$-independence is crucial for the distribution mod 1 of certain sequences and functions, and hence, via Proposition 2.2, also for the study of BL. A simple but useful fact in this regard is as follows.

Proposition 2.7. [6, Lem.2.6] *Let $\vartheta_0, \vartheta_1, \ldots, \vartheta_d \in \mathbb{R}$, and assume that the function $F : \mathbb{T}^d \to \mathbb{C}$ is continuous, and non-zero $\lambda_{\mathbb{T}^d}$-almost everywhere. If the $d+2$ numbers $1, \vartheta_0, \vartheta_1, \ldots, \vartheta_d$ are $\mathbb{Q}$-independent then the sequence*

$$\Big(n\vartheta_0 + \alpha \ln n + \beta \ln \big| F\big(\langle (n\vartheta_1, \ldots, n\vartheta_d) \rangle \big) + z_n \big| \Big)$$

is u.d. mod 1 *for every $\alpha, \beta \in \mathbb{R}$ and every sequence (z_n) in $\mathbb{C}$ with* $\lim_{n\to\infty} z_n = 0$.

The following definitions of nonresonance and exponential nonresonance have been introduced in [6] and [7], respectively. As will become clear in the next section, they owe their specific form to Propositions 2.2, 2.5, and 2.7.

Definition 2.8. Let $b \in \mathbb{N} \setminus \{1\}$. A non-empty set $\mathcal{Z} \subset \mathbb{C}$ with $|z| = r$ for some $r > 0$ and all $z \in \mathcal{Z}$, i.e. $\mathcal{Z} \subset r\mathbb{S}$, is *b-nonresonant* if the associated set

$$\Delta_{\mathcal{Z}} := \left\{ 1 + \frac{\arg z - \arg w}{2\pi} : z, w \in \mathcal{Z} \right\} \subset \mathbb{R}$$

has the following two properties:

(i) $\Delta_{\mathcal{Z}} \cap \mathbb{Q} = \{1\}$;

(ii) $\log_b r \notin \text{span}_{\mathbb{Q}}\Delta_{\mathcal{Z}}$.

An arbitrary set $\mathcal{Z} \subset \mathbb{C}$ is b-nonresonant if, for every $r > 0$, the set $\mathcal{Z} \cap r\mathbb{S}$ is either b-nonresonant or empty; otherwise, $\mathcal{Z}$ is *b-resonant.*

Definition 2.9. Let $b \in \mathbb{N} \setminus \{1\}$. A set $\mathcal{Z} \subset \mathbb{C}$ is *exponentially b-nonresonant* if the set $e^{t\mathcal{Z}} := \{e^{tz} : z \in \mathcal{Z}\}$ is b-nonresonant for some $t \in \mathbb{R}^+$; otherwise, $\mathcal{Z}$ is *exponentially b-resonant.*

Example 2.10. The empty set $\varnothing$ is b-nonresonant and exponentially b-resonant for every b. The singleton $\{z\}$ with $z \in \mathbb{C}$ is b-nonresonant if and only if either $z = 0$ or $\log_b |z| \notin \mathbb{Q}$, and it is exponentially b-nonresonant precisely if $\Re z \neq 0$. Similarly, any set $\{z, \overline{z}\}$ with $z \in \mathbb{C} \setminus \mathbb{R}$ is b-nonresonant if and only if 1, $\log_b |z|$ and $\frac{1}{2\pi} \arg z$ are $\mathbb{Q}$-independent, and it is exponentially b-nonresonant precisely if $\Re z\pi/(\Im z \ln b) \notin \mathbb{Q}$.

Note that if $\mathcal{Z}$ is (exponentially) b-nonresonant then so are the sets $-\mathcal{Z} := (-1)\mathcal{Z}$ and $\overline{\mathcal{Z}} := \{\overline{z} : z \in \mathcal{Z}\}$, as well as every $\mathcal{W} \subset \mathcal{Z}$. Also, for each $n \in \mathbb{N}$ the set $\mathcal{Z}^n := \{z^n : z \in \mathcal{Z}\}$ is b-nonresonant whenever $\mathcal{Z}$ is. The converse fails since, for instance, $\mathcal{Z} = \{-e, e\}$ is b-resonant whereas $\mathcal{Z}^2 = \{e^2\}$ is b-nonresonant. Similarly, if $\mathcal{Z}$ is exponentially b-nonresonant then so is $t\mathcal{Z}$ for all $t \in \mathbb{R} \setminus \{0\}$. On the other hand, a set $\mathcal{Z}$ is certainly b-resonant if $\mathcal{Z} \cap \mathbb{S} \neq \varnothing$, and it is exponentially b-resonant whenever $\mathcal{Z} \cap \imath\mathbb{R} \neq \varnothing$.

The following simple observation establishes an alternative description of exponential b-nonresonance. Recall that a set is *countable* if it is either finite (possibly empty) or countably infinite.

Lemma 2.11. *Let $b \in \mathbb{N} \setminus \{1\}$. Assume that the set $\mathcal{Z} \subset \mathbb{C}$ is countable and symmetric w.r.t. the real axis, i.e., $\overline{\mathcal{Z}} = \mathcal{Z}$. Then the following are equivalent:*

(i) *$\mathcal{Z}$ is exponentially b-nonresonant;*

(ii) *For every $z \in \mathcal{Z}$,*

$$\Re z \notin \mathrm{span}_{\mathbb{Q}} \left\{ \frac{\ln b}{\pi} \Im w : w \in \mathcal{Z}, \Re w = \Re z \right\} . \tag{2.2}$$

Moreover, if (i) *and* (ii) *hold then the set* $\{t \in \mathbb{R}^+ : e^{t\mathcal{Z}}$ is b-resonant$\}$ *is countable.*

Proof. To show (i)⇒(ii), suppose there exist different elements $w_1, \ldots, w_L$ of $\mathcal{Z}$ with $\Re w_1 = \ldots = \Re w_L$, as well as $p_1, \ldots, p_L \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$\Re w_1 = \sum\nolimits_{\ell=1}^{L} \frac{p_\ell}{q} \frac{\ln b}{\pi} \Im w_\ell .$$

Pick any $t > 0$, let $r := e^{t\Re w_1}$, and note that

$$\log_b r = \frac{t\Re w_1}{\ln b} = \sum\nolimits_{\ell=1}^{L} \frac{p_\ell}{q} \frac{t\Im w_\ell}{\pi} .$$

On the other hand, since $\mathcal{Z}$ is symmetric w.r.t. the real axis, and since $\arg e^{tw_\ell}$ differs from $t\Im w_\ell$ by an integer multiple of 2π,

$$\mathrm{span}_{\mathbb{Q}} \Delta_{e^{t\mathcal{Z}} \cap r\mathbb{S}} \supset \mathrm{span}_{\mathbb{Q}} \left(\{1\} \cup \left\{ \frac{t\Im w_\ell}{\pi} : \ell = 1, \ldots, L \right\} \right) .$$

Thus $\log_b r \in \mathrm{span}_{\mathbb{Q}} \Delta_{e^{t\mathcal{Z}} \cap r\mathbb{S}}$, showing that $e^{t\mathcal{Z}}$ is b-resonant for all $t > 0$. Since clearly $e^{0\mathcal{Z}} = \{1\}$ is b-resonant as well, $\mathcal{Z}$ is exponentially b-resonant, contradicting (i). Hence (i)⇒(ii); note that the countability of $\mathcal{Z}$ has not been used here.

To establish the reverse implication (ii)⇒(i), suppose the set $\mathcal{Z}$ is exponentially b-resonant. In this case, for every $t > 0$ there exists $r = r(t) > 0$ such that $e^{t\mathcal{Z}} \cap r\mathbb{S}$ is b-resonant, and so either $\Delta_{e^{t\mathcal{Z}} \cap r\mathbb{S}} \cap \mathbb{Q} \neq \{1\}$ or $\log_b r \in \mathrm{span}_{\mathbb{Q}} \Delta_{e^{t\mathcal{Z}} \cap r\mathbb{S}}$, or both. In the first case, there exist elements $w_1 = w_1(t)$ and $w_2 = w_2(t)$ of $\mathcal{Z}$ with $\Re w_1 = \Re w_2$ but $w_1 \neq w_2$ such that $t(\Im w_1 - \Im w_2) \in \pi\mathbb{Q} \setminus \{0\}$. In particular, therefore,

$$t \in \bigcup\nolimits_{z \in \mathcal{Z}} \bigcup\nolimits_{w \in \mathcal{Z} \setminus \{z\} : \Re w = \Re z} \frac{\pi}{\Im w - \Im z} \mathbb{Q} \; =: \; \Omega_1 . \tag{2.3}$$

In the second case, for some positive integer $L = L(t)$ and some $w_1(t), \ldots, w_L(t) \in \mathcal{Z}$ with $\Re w_1 = \ldots = \Re w_L = t^{-1} \ln r$,

$$\log_b r = \frac{t \Re w_1}{\ln b} \in \mathrm{span}_{\mathbb{Q}} \Delta_{e^{t\mathcal{Z}} \cap r\mathbb{S}} \subset \mathrm{span}_{\mathbb{Q}} \left(\{1\} \cup \left\{ \frac{t \Im w_\ell}{\pi} : \ell = 1, \ldots, L \right\} \right) .$$

With the appropriate $p_0(t), p_1(t), \ldots, p_L(t) \in \mathbb{Z}$ and $q(t) \in \mathbb{N}$, therefore,

$$t \left(q \Re w_1 - \sum\nolimits_{\ell=1}^{L} p_\ell \frac{\ln b}{\pi} \Im w_\ell \right) = p_0 \ln b \, . \tag{2.4}$$

Since $\mathcal{Z}$ is countable, the set Ω_1 in (2.3) is countable as well. Consequently, if $\mathcal{Z}$ is exponentially b-resonant then (2.4) must hold for all but countably many $t > 0$. Hence there exist $t_2 > t_1 > 0$ such that $L(t_2) = L(t_1)$, $w_\ell(t_2) = w_\ell(t_1)$ and similarly $p_\ell(t_2) = p_\ell(t_1)$ for all $\ell = 1, \ldots, L$, as well as $q(t_2) = q(t_1)$. This in turn implies

$$\Re w_1 = \sum\nolimits_{\ell=1}^{L} \frac{p_\ell}{q} \frac{\ln b}{\pi} \Im w_\ell \, ,$$

which clearly contradicts (2.2). For countable $\mathcal{Z}$, therefore, (ii) fails whenever (i) fails, that is, (ii)$\Rightarrow$(i); note that the symmetry of $\mathcal{Z}$ has not been used here.

Finally, if (i) and (ii) hold, and if $e^{t\mathcal{Z}}$ is b-resonant for some $t > 0$ then, as seen in the previous paragraph, either $t \in \Omega_1$ or else, by (2.4),

$$\frac{\ln b}{t} \in \bigcup\nolimits_{z \in \mathcal{Z}} \mathrm{span}_{\mathbb{Q}} \left(\{\Re z\} \cup \left\{ \frac{\ln b}{\pi} \Im w : w \in \mathcal{Z}, \Re w = \Re z \right\} \right) =: \Omega_2 \, .$$

Since $\mathcal{Z}$ is countable, so are Ω_1 and Ω_2, and hence $\{t \in \mathbb{R}^+ : e^{t\mathcal{Z}} \text{ is } b\text{-resonant}\}$ is countable as well. □

Remark 2.12. The symmetry and countability assumptions are essential in Lemma 2.11. If $\mathcal{Z}$ is not symmetric w.r.t. the real axis then the implication (i)$\Rightarrow$(ii) may fail, as is seen e.g. for $\mathcal{Z} = \{1 + \imath\pi/\ln 10\}$ which is exponentially 10-nonresonant by Example 2.10, yet does not satisfy (ii) for $b = 10$. Conversely, if $\mathcal{Z}$ is uncountable then (ii)$\Rightarrow$(i) may fail. To see this, simply take $\mathcal{Z} = \mathbb{R} \setminus \{0\}$ which satisfies (ii) for all b, and yet $e^{t\mathcal{Z}}$ is b-resonant for every $t \in \mathbb{R}^+$.

Deciding whether a set $\mathcal{Z} \subset \mathbb{C}$ is b-resonant may be difficult in practice, even if $\#\mathcal{Z} = 2$. For example, it is unknown whether $\{z \in \mathbb{C} : z^2 + 2z + 3 = 0\}$ is 10-resonant; see [7, Ex.7.27]. In many situations of practical interest, the situation regarding *exponential* b-resonance is much simpler. Recall that a number $z \in \mathbb{C}$ is *algebraic* (over $\mathbb{Q}$) if it is the root of some non-constant polynomial with integer coefficients.

Lemma 2.13. *Let $b \in \mathbb{N} \setminus \{1\}$. Assume every element of $\mathcal{Z} \subset \mathbb{C}$ is algebraic. Then $\mathcal{Z}$ is exponentially b-nonresonant if and only if $\mathcal{Z} \cap \imath\mathbb{R} = \varnothing$.*

Proof. The "only if" part is obvious since, as seen earlier, $\mathcal{Z} \cap \imath\mathbb{R} \neq \varnothing$ always renders the set $\mathcal{Z}$ exponentially b-resonant. To prove the "if" part, suppose that $\mathcal{Z} \cap \imath\mathbb{R} = \varnothing$ yet $\mathcal{Z}$ is exponentially b-resonant. Since all of its elements are algebraic, the set $\mathcal{Z}$

is countable. By Lemma 2.11 there exist $z_1, \ldots, z_L \in \mathcal{Z}$ with $\Re z_1 = \ldots = \Re z_L$, as well as $p_1, \ldots, p_L \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$\Re z_1 = \sum\nolimits_{\ell=1}^{L} \frac{p_\ell}{q} \frac{\ln b}{\pi} \Im z_\ell \,.$$

(Recall that the proof of the implication (ii)⇒(i) in that lemma does not require $\mathcal{Z}$ to be symmetric w.r.t. the real axis.) Since $\Re z_1 \neq 0$, it follows that

$$\frac{\pi}{\ln b} = \sum\nolimits_{\ell=1}^{L} \frac{p_\ell}{q} \frac{\Im z_\ell}{\Re z_1} \,,$$

which in turn implies that $\pi/\ln b$ is algebraic. However, by the Gel'fond–Schneider Theorem [34, Thm.1.4], the number $\pi/\ln b$ is *not* algebraic for any $b \in \mathbb{N}\setminus\{1\}$. This contradiction shows that $\mathcal{Z}$ cannot be exponentially b-resonant if $\mathcal{Z} \cap \imath\mathbb{R} = \varnothing$. $\square$

3 Characterizing BL for linear flows on $\mathbb{R}^d$

Let ϕ be a linear flow on $X = \mathbb{R}^d$. Recall that there exists a unique $A_\phi \in \mathcal{L}(X)$ such that $\phi_t = e^{tA_\phi}$ for all $t \in \mathbb{R}$. (The linear map A_ϕ is sometimes referred to as the *generator* of ϕ.) In fact, with $I_X \in \mathcal{L}(X)$ denoting the identity map, $A_\phi = \lim_{t\to 0} t^{-1}(\phi_t - I_X) = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t|_{t=0}$, and ϕ is simply the flow generated by the (autonomous) linear differential equation $\dot{x} = A_\phi x$. Since conversely $(t,x) \mapsto e^{tA}x$ defines, for each $A \in \mathcal{L}(X)$, a linear flow on X with generator A, there is a one-to-one correspondence between the family of all linear flows on X and the space $\mathcal{L}(X)$. Thus it makes sense to define the *spectrum* of ϕ as

$$\sigma(\phi) := \sigma(A_\phi) = \{z \in \mathbb{C} : z \text{ is an eigenvalue of } A_\phi\} \,.$$

Note that $\sigma(\phi) \subset \mathbb{C}$ is non-empty, countable (in fact, finite with $\#\sigma(\phi) \le d$) and symmetric w.r.t. the real axis.

Recall that the symbol H is used throughout to denote a linear observable (i.e., a linear functional) on $\mathcal{L}(X)$. For convenience, let $\mathcal{O}(X)$ be the space of all such observables, i.e., $\mathcal{O}(X)$ is simply the dual of $\mathcal{L}(X)$, endowed with the usual topology. The following is a basic linear algebra observation.

Lemma 3.1. *Let ϕ be a linear flow on X. Given any non-empty set $\mathcal{Z} \subset \sigma(\phi)$ and any vector $u \in \mathbb{R}^{\mathcal{Z}}$, there exists $H \in \mathcal{O}(X)$ such that*

$$H(\phi_t) = \sum\nolimits_{z\in\mathcal{Z}} e^{t\Re z} u_z \cos(t\Im z) \quad \forall t \in \mathbb{R} \,. \tag{3.1}$$

Proof. For every real $z \in \mathcal{Z}$, pick $v_z \in X \setminus \{0\}$ such that $A_\phi v_z = z v_z$ and let $\widetilde{v}_z := v_z$. For every non-real $z \in \mathcal{Z}$, pick $v_z, \widetilde{v}_z \in X \setminus \{0\}$ such that

$$A_\phi v_z = v_z \Re z - \widetilde{v}_z \Im z \,, \quad A_\phi \widetilde{v}_z = v_z \Im z + \widetilde{v}_z \Re z \,;$$

note that $v_z, \widetilde{v}_z$ are linearly independent. With this, for each $z \in \mathcal{Z}$,

$$\begin{aligned} \phi_t v_z &= e^{t\Re z}\big(v_z \cos(t\Im z) - \widetilde{v}_z \sin(t\Im z)\big) \\ \phi_t \widetilde{v}_z &= e^{t\Re z}\big(v_z \sin(t\Im z) + \widetilde{v}_z \cos(t\Im z)\big) \end{aligned} \quad \forall t \in \mathbb{R} \,. \tag{3.2}$$

For each $z \in \mathcal{Z}$, pick a linear functional h_z on X such that $h_z(v_z) = h_z(\widetilde{v}_z) = \frac{1}{2}u_z$. Using (3.2) it is easily verified that $H \in \mathcal{O}(X)$ given by

$$H(A) = \sum\nolimits_{z \in \mathcal{Z}} \bigl(h_z(Av_z) + h_z(A\widetilde{v}_z)\bigr) \quad \forall A \in \mathcal{L}(X)\,,$$

does indeed satisfy (3.1). □

As it turns out, the set $\sigma(\phi)$ controls the Benford property for all signals $H(\phi_\bullet)$. This is the first main result of the present article.

Theorem 3.2. *Let $b \in \mathbb{N} \setminus \{1\}$. For each linear flow ϕ on X the following are equivalent:*

(i) *The set $\sigma(\phi)$ is exponentially b-nonresonant;*

(ii) *For every $H \in \mathcal{O}(X)$ either the function $H(\phi_\bullet)$ is b-Benford, or $H(\phi_\bullet) = 0$.*

The proof of Theorem 3.2 makes use of a discrete-time analogue established in [6].

Proposition 3.3. [6, Thm.3.4] *Let $b \in \mathbb{N} \setminus \{1\}$. For each invertible $A \in \mathcal{L}(X)$ the following are equivalent:*

(i) *The set $\sigma(A)$ is b-nonresonant;*

(ii) *For every $H \in \mathcal{O}(X)$ either the sequence $\bigl(\log_b |H(A^n)|\bigr)$ is u.d.* mod 1*, or $H(A^n) \equiv 0$.*

Proof of Theorem 3.2. To prove (i)⇒(ii), let $\sigma(\phi)$ be exponentially b-nonresonant and fix any $H \in \mathcal{O}(X)$. Note that $\sigma(\phi_\delta) = e^{\delta\sigma(\phi)}$ is b-nonresonant for all but countably many $\delta > 0$, by Lemma 2.11. Since ϕ_δ is invertible for all $\delta > 0$, Proposition 3.3 implies that either $H(\phi_\delta^n) = H(\phi_{n\delta}) \equiv 0$, or else the sequence $\bigl(\log_b |H(\phi_{n\delta})|\bigr)$ is u.d. mod 1. Let

$$\delta_0 := \inf\{\delta > 0 : H(\phi_{n\delta}) \equiv 0\} \geq 0\,,$$

with $\inf \varnothing := +\infty$. If $\delta_0 = 0$ then there exists a sequence (δ_n) with $\delta_n \searrow 0$ and $H(\phi_{\delta_n}) \equiv 0$. Since $t \mapsto H(\phi_t)$ is analytic, it follows that $H(\phi_\bullet) = 0$. If, on the other hand, $\delta_0 > 0$ then, for almost all $0 < \delta < \delta_0$, the sequence $\bigl(f(n\delta)\bigr)$ is u.d. mod 1, with the measurable function $f = \log_b |H(\phi_\bullet)|$. Hence by Lemma 2.4, f is c.u.d. mod 1, i.e., $H(\phi_\bullet)$ is b-Benford.

To prove (ii)⇒(i), let $\sigma(\phi)$ be exponentially b-resonant. By Lemma 2.11, there exists $z_1 \in \sigma(\phi)$ such that

$$\Re z_1 \in \operatorname{span}_{\mathbb{Q}} \left\{ \frac{\ln b}{\pi} \Im w : w \in \sigma(\phi), \Re w = \Re z_1 \right\}. \tag{3.3}$$

Let L be the dimension of the $\mathbb{Q}$-linear space in (3.3).

If $L = 0$ then $z_1 = 0$, and picking any $v \in X \setminus \{0\}$ with $A_\phi v = 0$ yields $\phi_t v \equiv v$. With any linear functional h on X that satisfies $h(v) = 1$, and with the linear observable H defined as $H(A) = h(Av)$ for all $A \in \mathcal{L}(X)$, therefore, $H(\phi_t) \equiv 1$ is neither b-Benford nor zero, i.e., (ii) fails.

For $L \geq 1$, it is possible to choose $z_1, \ldots, z_L \in \sigma(\phi)$ with $\Re z_1 = \ldots = \Re z_L$ and $0 \leq \Im z_1 < \ldots < \Im z_L$ such that the L numbers $\frac{1}{\pi}\Im z_1, \ldots, \frac{1}{\pi}\Im z_L$ are $\mathbb{Q}$-independent. By (3.3) there exist $p_1, \ldots, p_L \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$\Re z_1 = \sum\nolimits_{\ell=1}^{L} \frac{p_\ell}{2q}\frac{\ln b}{\pi}\Im z_\ell\,.$$

Use Proposition 2.5, with $d = L$ and $\alpha = q/\ln b$, to chose $u \in \mathbb{R}^L \setminus \{0\}$ such that $\nu := \lambda_{\mathbb{T}^L} \circ P_u^{-1} \neq \lambda_{\mathbb{T}}$, and use Lemma 3.1 to pick $H \in \mathcal{O}(X)$ with

$$H(\phi_t) = e^{t\Re z_1}\sum\nolimits_{\ell=1}^{L} u_\ell \cos(t\Im z_\ell) \quad \forall t \in \mathbb{R}\,.$$

Since $t \mapsto H(\phi_t)$ is analytic and non-constant, the set $\{t \in \mathbb{R}^+ : H(\phi_t) = 0\}$ is countable. For all but countably many $\delta > 0$, therefore, $H(\phi_{n\delta}) \neq 0$ for all $n \in \mathbb{N}$. Consequently, for almost all $\delta > 0$ and all $n \in \mathbb{N}$,

$$\begin{aligned}
\langle \log_b |H(\phi_{n\delta})^q| \rangle &= \left\langle \frac{nq\delta\Re z_1}{\ln b} + \frac{q}{\ln b}\ln\left|\sum\nolimits_{\ell=1}^{L} u_\ell \cos(n\delta\Im z_\ell)\right|\right\rangle \\
&= \left\langle \sum\nolimits_{\ell=1}^{L} p_\ell \frac{n\delta\Im z_\ell}{2\pi} + \frac{q}{\ln b}\ln\left|\sum\nolimits_{\ell=1}^{L} u_\ell \cos\left(2\pi\frac{n\delta\Im z_\ell}{2\pi}\right)\right|\right\rangle \\
&= P_u\left(\left\langle\left(\frac{n\delta\Im z_1}{2\pi}, \ldots, \frac{n\delta\Im z_L}{2\pi}\right)\right\rangle\right)\,.
\end{aligned}$$

The $L+1$ numbers $1, \frac{1}{2\pi}\delta\Im z_1, \ldots, \frac{1}{2\pi}\delta\Im z_L$ are $\mathbb{Q}$-independent for all but countably many $\delta > 0$, and whenever they are, the sequence $\left(\left\langle\left(\frac{1}{2\pi}n\delta\Im z_1, \ldots, \frac{1}{2\pi}n\delta\Im z_L\right)\right\rangle\right)$ is uniformly distributed on $\mathbb{T}^L$; e.g., see [23, Exp.I.6.1]. Since the function $e^{2\pi\imath k P_u}$ is Riemann integrable for each $k \in \mathbb{Z}$, it follows that for almost all $\delta > 0$,

$$\begin{aligned}
\frac{1}{N}\sum\nolimits_{n=1}^{N} e^{2\pi\imath k \log_b |H(\phi_{n\delta})^q|} &= \frac{1}{N}\sum\nolimits_{n=1}^{N} e^{2\pi\imath k P_u\left(\left\langle(n\delta\Im z_1/(2\pi), \ldots, n\delta\Im z_L/(2\pi))\right\rangle\right)} \\
&\overset{N\to\infty}{\longrightarrow} \int_{\mathbb{T}^L} e^{2\pi\imath k P_u}\,\mathrm{d}\lambda_{\mathbb{T}^L} = \int_{\mathbb{T}} e^{2\pi\imath k y}\,\mathrm{d}\nu(y) \quad \forall k \in \mathbb{Z}\,.
\end{aligned}$$

Recall that $\nu \neq \lambda_{\mathbb{T}}$, so $\int_{\mathbb{T}} e^{2\pi\imath k^* y}\,\mathrm{d}\nu(y) \neq 0$ for some integer $k^* \neq 0$, and Lemma 2.4 shows that

$$\lim\nolimits_{T\to+\infty}\frac{1}{T}\int_0^T e^{2\pi\imath k^* \log_b |H(\phi_t)^q|}\,\mathrm{d}t = \int_{\mathbb{T}} e^{2\pi\imath k^* y}\,\mathrm{d}\nu(y) \neq 0\,.$$

Thus $\log_b |H(\phi_\bullet)^q| = q\log_b |H(\phi_\bullet)|$ is not c.u.d. mod 1, and neither is $\log_b |H(\phi_\bullet)|$, by Lemma 2.3. In other words, $H(\phi_\bullet)$ is not b-Benford (and clearly $H(\phi_\bullet) \neq 0$). Overall, (ii) fails whenever (i) fails, that is, (ii)⇒(i). □

Example 3.4. By utilizing Example 2.10, the examples mentioned in the Introduction are easily reviewed in the light of Theorem 3.2.

(i) For the linear flow ϕ on $\mathbb{R}^1$ generated by the scalar equation $\dot{x} = \alpha x$, the set $\sigma(\phi) = \{\alpha\}$ is exponentially b-nonresonant if and only if $\alpha \neq 0$.

(ii) For the linear flow ϕ on $\mathbb{R}^2$ generated by

$$\dot{x} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} x\,,$$

with $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$, the set $\sigma(\phi) = \{\alpha \pm \imath\beta\}$ is exponentially b-nonresonant if and only if $\alpha\pi/(\beta \ln b) \notin \mathbb{Q}$, and whenever it is, $H(\phi_\bullet)$ is b-Benford for every $H \in \mathcal{O}(\mathbb{R}^2)$ unless $H(I_{\mathbb{R}^2}) = H\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = 0$; in the latter case, $H(\phi_\bullet) = 0$.

Example 3.5. Let $p \neq 0$ be any real polynomial, and $\alpha \in \mathbb{R}$. The function $f(t) = p(t)e^{\alpha t}$ is b-Benford if and only if $\alpha \neq 0$. To see this, simply note that f solves a linear differential equation with constant coefficients and order equal to the degree of p plus one. Thus, $f = H(\phi)$ for the appropriate linear observable H and linear flow ϕ with $\sigma(\phi) = \{\alpha\}$, and the claim follows from Theorem 3.2.

Example 3.6. Theorem 3.2 remains valid if (ii) is required to hold more generally for all observables on $\mathcal{L}(X)$ of the form $p \circ H$, where p is any real polynomial with $p(0) = 0$ and $H \in \mathcal{O}(X)$. To illustrate this, consider the linear flow ϕ generated on $X = \mathbb{R}^3$ by

$$\dot{x} = \begin{bmatrix} 1 & -\pi & 0 \\ \pi & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} x \,, \tag{3.4}$$

with $\alpha = \ln 10 - \frac{1}{2} = 1.802$. Clearly, $\sigma(\phi) = \{1 \pm \imath\pi, \alpha\}$ is exponentially b-nonresonant for all $b \in \mathbb{N} \setminus \{1\}$. Taking for instance $p(x) = x^2$, the generalized version of Theorem 3.2 just mentioned implies that $[\phi_\bullet]_{j,k}^2$ is Benford or trivial for all $1 \leq j, k \leq 3$. Note that by Lemma 2.3,

$$t \mapsto \sqrt{\sum\nolimits_{j,k=1}^{3} [\phi_t]_{j,k}^2} = \sqrt{2e^{2t} + e^{2\alpha t}} = e^{\alpha t}\sqrt{1 + 2e^{-2(\alpha - 1)t}}$$

is Benford as well. Though this does not follow from even the generalized theorem, it nevertheless suggests that $\|\phi_\bullet\|$ may also be b-Benford for some or even all norms $\|\cdot\|$ on $\mathcal{L}(X)$. In fact, to guarantee the latter, exponential b-nonresonance of an appropriate subset of $\sigma(\phi)$ suffices; see Theorem 3.15 below.

While $h(\phi_\bullet)$ thus is Benford for *some* non-linear observables h on $\mathcal{L}(X)$ also, it should be noted that, on the other hand, $h(\phi_\bullet)$ may fail to be b-Benford even for very simple polynomial observables h, despite $\sigma(\phi)$ being exponentially b-nonresonant. Concretely, the implication (i)⇒(ii) in Theorem 3.2 fails if the linear observable H in (ii) is replaced by $h = p \circ H_1 + p \circ H_2$ where p is a real polynomial with $p(0) = 0$, and $H_1, H_2 \in \mathcal{O}(X)$. To see this, let ϕ be again the linear flow on $\mathbb{R}^3$ generated by (3.4), and take $p(x) = x^3$ as well as $H_1 = [\,\cdot\,]_{1,1} - [\,\cdot\,]_{3,3}$ and $H_2 = [\,\cdot\,]_{3,3}$. Then

$$\begin{aligned} h(\phi_t) &= ([\phi_t]_{1,1} - [\phi_t]_{3,3})^3 + [\phi_t]_{3,3}^3 \\ &= e^{(1+2\alpha)t}\cos(\pi t)\left(3 - 3e^{-(\alpha-1)t}\cos(\pi t) + e^{-2(\alpha-1)t}\cos(\pi t)^2\right) \quad \forall t \in \mathbb{R}\,, \end{aligned}$$

and it is straightforward to see that $h(\phi_\bullet) \neq 0$ is not 10-Benford.

Finally, while the implication (i)⇒(ii) in Theorem 3.2 remains valid if the linear observable H in (ii) is replaced by $p \circ H$ for any polynomial p with $p(0) = 0$, this implication may fail if, only slightly more generally, H is instead replaced by $\varphi \circ H$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is real-analytic with $\varphi(0) = 0$. For a simple example illustrating this with ϕ as above, let $\varphi(x) = x^2/(1+x^2)$ and $H = [\,\cdot\,]_{3,3}$. Then

$$\varphi \circ H(\phi_t) = \frac{[\phi_t]_{3,3}^2}{1 + [\phi_t]_{3,3}^2} = \frac{1}{1 + e^{-2\alpha t}} \quad \forall t \in \mathbb{R}\,,$$

and since $\lim_{t\to+\infty} \varphi \circ H(\phi_t) = 1$, clearly $\varphi \circ H(\phi_\bullet) \neq 0$ is not b-Benford for any b.

By combining it with Lemma 2.13, Theorem 3.2 can be given a simpler form that applies in many situations of practical interest. To this end, call a linear flow ϕ on X *algebraically generated* if there exists a basis $v_1, \ldots, v_d$ of X such that the (uniquely determined) numbers $a_{jk} \in \mathbb{R}$ with $A_\phi v_j = \sum_{k=1}^d a_{jk} v_k$ for all $j = 1, \ldots, d$ are all algebraic. In other words, all entries of the coordinate matrix of A_ϕ relative to the basis $v_1, \ldots, v_d$ are algebraic numbers. Note that ϕ is algebraically generated if and only if $\sigma(\phi)$ consists of algebraic numbers only. The following, then, is an immediate consequence of Lemma 2.13 and Theorem 3.2.

Corollary 3.7. *For each algebraically generated linear flow ϕ on X the following are equivalent:*

(i) $\sigma(\phi) \cap \imath\mathbb{R} = \varnothing$;

(ii) *For every $H \in \mathcal{O}(X)$ either the function $H(\phi_\bullet)$ is Benford, or $H(\phi_\bullet) = 0$.*

Remark 3.8. A linear flow ϕ with $\sigma(\phi) \cap \imath\mathbb{R} = \varnothing$ is commonly referred to as *hyperbolic*; e.g., see [1]. Thus, an algebraically generated linear flow exhibits the Benford–or–trivial dichotomy of Corollary 3.7(ii) if and only if it is hyperbolic.

Example 3.9. In order to decide whether $\sigma(\phi) \cap \imath\mathbb{R} = \varnothing$, it is not necessary to explicitly determine $\sigma(\phi)$. For instance, if ϕ is a linear flow on $\mathbb{R}^2$ then $\sigma(\phi) \cap \imath\mathbb{R} = \varnothing$ if and only if

$$\operatorname{trace} A_\phi \det A_\phi \neq 0 \quad \text{or} \quad \det A_\phi < 0\,. \tag{3.5}$$

For a concrete example, let $\alpha, \beta \in \mathbb{R}$ be algebraic and consider the linear second-order equation

$$\ddot{y} + \alpha \dot{y} + \beta y = 0\,. \tag{3.6}$$

Since $y = H(\phi_\bullet)$ with the appropriate $H \in \mathcal{O}(\mathbb{R}^2)$ and the linear flow ϕ on $\mathbb{R}^2$ generated by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} x\,,$$

Corollary 3.7, together with (3.5), shows that every solution $y \neq 0$ of (3.6) is Benford if and only if $\alpha\beta \neq 0$ or $\beta < 0$, or equivalently, if and only if $(1+\alpha^2)|\beta| > \beta$.

To motivate the second main result of this section, Theorem 3.11 below, note that even if $\sigma(\phi)$ is exponentially b-resonant, the signals $H(\phi_\bullet)$ may nevertheless be b-Benford for some or in fact for *most* linear observables H.

Example 3.10. Let ϕ be the linear flow on $X = \mathbb{R}^2$ generated by $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since $\sigma(\phi) = \{0, 2\}$ is exponentially b-resonant for every $b \in \mathbb{N} \setminus \{1\}$, there exists a linear observable H for which $H(\phi_\bullet) \neq 0$ is not b-Benford. A simple example is $H = [\,\cdot\,]_{1,1} - [\,\cdot\,]_{1,2}$ yielding $H(\phi_t) \equiv 1$. However, from the explicit formula

$$\phi_t = \tfrac{1}{2} e^{2t} A - \tfrac{1}{2}(A - 2I_{\mathbb{R}^2}) \quad \forall t \in \mathbb{R},$$

it is clear that $H(\phi_\bullet)$ is Benford *unless* $H(A) = 0$. For *most* $H \in \mathcal{O}(X)$, therefore, $H(\phi_\bullet)$ is Benford. On the other hand, for the time-reversed flow ψ, i.e., for $\psi_t \equiv \phi_{-t}$, a signal $H(\psi_\bullet)$ can be b-Benford *only* if $H(A - 2I_{\mathbb{R}^2}) = 0$. Thus $H(\psi_\bullet)$ is, for most $H \in \mathcal{O}(X)$, neither b-Benford nor trivial.

To formalize the observation made in Example 3.10, recall first that the space $\mathcal{O}(X)$ can, upon choosing a basis, be identified with $\mathbb{R}^{d^2}$. In particular, therefore, the notion of a property holding for (Lebesgue) *almost every* $H \in \mathcal{O}(X)$ is well-defined and independent of the choice of basis. Given any $A \in \mathcal{L}(X)$, for each $z \in \sigma(A)$ define $k_z \geq 0$ to be the maximal integer for which

$$\operatorname{rank}(A - zI_X)^{k+1} < \operatorname{rank}(A - zI_X)^k \quad \text{if } z \in \mathbb{R},$$

and

$$\operatorname{rank}(A^2 - 2\Re z A + |z|^2 I_X)^{k+1} < \operatorname{rank}(A^2 - 2\Re z A + |z|^2 I_X)^k \quad \text{if } z \in \mathbb{C} \setminus \mathbb{R}.$$

Equivalently, $1 \leq k_z + 1 \leq d$ is the size of the largest block associated with the eigenvalue z in the Jordan Normal Form (over $\mathbb{C}$) of A. Denote by (r_A, k_A) the (unique) element of $\{(\Re z, k_z) : z \in \sigma(A)\}$ that is maximal in the lexicographic order on $\mathbb{R} \times \mathbb{Z}$, and define the *dominant spectrum* of A as

$$\sigma_{\text{dom}}(A) := \{z \in \sigma(A) : \Re z = r_A, k_z = k_A\}.$$

Thus $\sigma_{\text{dom}}(A) \subset \sigma(A)$ consists of all right-most eigenvalues of A that have a Jordan block of maximal size associated with them. As it turns out, for every linear flow ϕ on X, the set $\sigma_{\text{dom}}(\phi) := \sigma_{\text{dom}}(A_\phi)$, though usually constituting but a small part of $\sigma(\phi)$, controls the Benford property of $H(\phi_\bullet)$ for most linear observables H.

Theorem 3.11. *Let $b \in \mathbb{N} \setminus \{1\}$. For each linear flow ϕ on X the following are equivalent:*

(i) *The set $\sigma_{\text{dom}}(\phi)$ is exponentially b-nonresonant;*

(ii) *For almost every $H \in \mathcal{O}(X)$ the function $H(\phi_\bullet)$ is b-Benford.*

The proof of Theorem 3.11 makes use of the following two observations which are a direct analogue of Lemma 3.1 and an immediate consequence of [6, Lem.5.3], respectively. The routine verification of both assertions is left to the reader.

Lemma 3.12. *Let ϕ be a linear flow on X. Given any non-empty set $\mathcal{Z} \subset \sigma_{\rm dom}(\phi)$ and any vector $u \in \mathbb{R}^{\mathcal{Z}}$, there exists $H \in \mathcal{O}(X)$ such that, with $r = r_{A_\phi} \in \mathbb{R}$ and $k = k_{A_\phi} \in \{0, \ldots, d-1\}$,*

$$H(\phi_t) = e^{rt}t^k \sum\nolimits_{z\in\mathcal{Z}} u_z \cos(t\Im z) \quad \forall t \in \mathbb{R}\,.$$

Lemma 3.13. *Let $\Omega \subset \mathbb{R}^+$ be finite. For each function $f : \Omega \to \mathbb{C}$ the following are equivalent:*

(i) $\lim_{t\to+\infty} \Re\left(\sum_{\omega\in\Omega} f(\omega)e^{\imath\omega t}\right)$ *exists;*

(ii) $f(\omega) = 0$ *for every* $\omega \in \Omega \setminus \{0\}$.

Proof of Theorem 3.11. For convenience, define $\sigma^+_{\rm dom} := \{z \in \sigma_{\rm dom}(\phi) : \Im z \geq 0\}$, and let $r = r_{A_\phi}$ and $k = k_{A_\phi}$; clearly, the set $\sigma^+_{\rm dom} \subset r + \imath\mathbb{R}$ is non-empty, and is exponentially b-nonresonant if and only if $\sigma_{\rm dom}(\phi)$ is. Let L be the dimension of $\mathrm{span}_{\mathbb{Q}}\left\{\frac{\ln b}{\pi}\Im z : z \in \sigma^+_{\rm dom}\right\}$, and observe that $L = 0$ if and only if $\sigma^+_{\rm dom} = \{r\}$. Recall that, as a consequence of, for instance, the Jordan Normal Form Theorem, there exists a family U_z, V_z $(z \in \sigma^+_{\rm dom})$ in $\mathcal{L}(X)$ such that

$$\phi_t = e^{rt}t^k \left(\sum\nolimits_{z\in\sigma^+_{\rm dom}} \big(U_z\cos(t\Im z) + V_z \sin(t\Im z)\big) + G(t)\right) \quad \forall t > 0\,, \tag{3.7}$$

where G is continuous with $\lim_{t\to+\infty} G(t) = 0$, and $V_r = 0$ in case $r \in \sigma^+_{\rm dom}$. Moreover, $U_z \neq 0$ or $V_z \neq 0$ in (3.7) for at least one $z \in \sigma^+_{\rm dom}$, since otherwise $\lim_{t\to+\infty} e^{-rt}t^{-k}H(\phi_t) = 0$ for every $H \in \mathcal{O}(X)$, whereas Lemma 3.12 guarantees, for each $z \in \sigma^+_{\rm dom}$, the existence of an H with $e^{-rt}t^{-k}H(\phi_t) \equiv \cos(t\Im z)$, an obvious contradiction.

Consider first the case $L = 0$. Here $\sigma^+_{\rm dom} = \{r\}$, and (3.7) yields

$$H(\phi_t) = e^{rt}t^k\big(H(U_r) + H \circ G(t)\big) \quad \forall t > 0\,,$$

where $U_r \neq 0$. If $H(U_r) \neq 0$ then, for all sufficiently large $t > 0$,

$$\log_b |H(\phi_t)| = \frac{rt}{\ln b} + \frac{k}{\ln b}\ln t + \frac{1}{\ln b}\ln|H(U_r) + H\circ G(t)|\,.$$

Note that $\sigma_{\rm dom} = \{r\}$ is exponentially b-resonant if and only if $r \neq 0$. Lemma 2.3 shows that $\log_b|H(\phi_\bullet)|$ is c.u.d. mod 1 if and only if $t \mapsto rt/\ln b$ is. Lemma 2.4 and Proposition 2.7 imply that the latter is the case if $r \neq 0$, while it is obviously not the case if $r = 0$. Provided that $H(U_r) \neq 0$, therefore, $H(\phi_\bullet)$ is b-Benford precisely if $r \neq 0$. Since $\{H : H(U_r) = 0\}$ is a nullset (in fact, a proper subspace) in $\mathcal{O}(X)$, it follows that (i)$\Leftrightarrow$(ii) whenever $L = 0$.

It remains to consider the case $L \geq 1$. In this case, pick $z_1, \ldots, z_L \in \sigma^+_{\rm dom}$ such that $\frac{\ln b}{\pi}\Im z_1, \ldots, \frac{\ln b}{\pi}\Im z_L$ are $\mathbb{Q}$-independent; for convenience, let $\mathcal{Z} := \{z_1, \ldots, z_L\}$. For every $z \in \sigma^+_{\rm dom}\setminus\mathcal{Z}$ there exists an integer L-tuple $p^{(z)}$, i.e., $p^{(z)} \in \mathbb{Z}^L$, such that

$$\Im z = \sum\nolimits_{\ell=1}^{L} \frac{p^{(z)}_\ell}{q}\Im z_\ell = \frac{p^{(z)}\cdot(\Im z_1, \ldots, \Im z_L)}{q}\,, \tag{3.8}$$

with the appropriate $q \in \mathbb{N}$ independent of z; here $u \cdot v$ denotes the standard inner product on $\mathbb{R}^L$, that is, $u \cdot v = \sum_{\ell=1}^{L} u_\ell v_\ell$. In addition, for every $1 \le j, \ell \le L$ let

$$p_j^{(z_\ell)} := \begin{cases} q & \text{if } j = \ell\,, \\ 0 & \text{otherwise}\,; \end{cases}$$

with this, (3.8) is valid for all $z \in \sigma_{\rm dom}^+$. Note that the set $\{\pm p^{(z)} : z \in \sigma_{\rm dom}^+\} \subset \mathbb{Z}^L$ contains at least $2\#\sigma_{\rm dom}^+ - 1$ different elements, and hence $z \mapsto p^{(z)}$ is one-to-one.

With these ingredients, given any $H \in \mathcal{O}(X)$, deduce from (3.7) that

$$\begin{aligned} H(\phi_{qt}) &= e^{rqt}(qt)^k \left(\sum\nolimits_{z \in \sigma_{\rm dom}^+} \big(H(U_z)\cos(qt\Im z) + H(V_z)\sin(qt\Im z)\big) + H \circ G(qt) \right) \\ &= e^{rqt}(qt)^k \left(F_H \left(\left\langle \left(\frac{t\Im z_1}{2\pi}, \ldots, \frac{t\Im z_L}{2\pi} \right) \right\rangle \right) + H \circ G(qt) \right) \quad \forall t > 0\,, \end{aligned} \tag{3.9}$$

where the smooth function $F_H : \mathbb{T}^L \to \mathbb{R}$ is given by

$$F_H(\langle x \rangle) = \Re \left(\sum\nolimits_{z \in \sigma_{\rm dom}^+} \big(H(U_z) - \imath H(V_z)\big) e^{2\pi \imath p^{(z)} \cdot x} \right) .$$

Note that $F_H = 0$ only if $H(U_z) = H(V_z) = 0$ for all $z \in \sigma_{\rm dom}^+$, whereas otherwise the set $\{\langle x \rangle : F_H(\langle x \rangle) = 0\}$ is a $\lambda_{\mathbb{T}^L}$-nullset. Thus, $F_H(\langle x \rangle) \ne 0$ for $\lambda_{\mathbb{T}^L}$-almost all $\langle x \rangle \in \mathbb{T}^L$ if and only if

$$H(U_z) \ne 0 \ \text{ or } \ H(V_z) \ne 0 \ \text{ for some } z \in \sigma_{\rm dom}^+ . \tag{3.10}$$

To establish (i)⇒(ii), assume that $\sigma_{\rm dom}(\phi)$ is exponentially b-nonresonant, fix any $H \in \mathcal{O}(X)$, and let $f_H := H(\phi_\bullet)$. Deduce from (3.9) that, for all $n \in \mathbb{N}$ and $\delta > 0$,

$$f_H(qn\delta) = e^{rqn\delta} q^k n^k \delta^k \left(F_H \left(\left\langle \left(n\frac{\delta \Im z_1}{2\pi}, \ldots, n\frac{\delta \Im z_L}{2\pi} \right) \right\rangle \right) + H \circ G(qn\delta) \right) .$$

Observe that the $L+2$ numbers $1, rq\delta/\ln b, \frac{1}{2\pi}\delta\Im z_1, \ldots, \frac{1}{2\pi}\delta\Im z_L$ are $\mathbb{Q}$-independent for all but countably many $\delta > 0$, and whenever they are, $f_H(qn\delta) \ne 0$ for all sufficiently large n. Hence by Proposition 2.7, with $d = L$, $\vartheta_0 = rq\delta/\ln b$ and $\vartheta_\ell = \frac{1}{2\pi}\delta\Im z_\ell$ for $\ell = 1, \ldots, L$, and with

$$\alpha = \frac{k}{\ln b}\,, \quad \beta = \frac{1}{\ln b}\,, \quad F = q^k \delta^k F_H\,, \quad (z_n) = \big(q^k \delta^k H \circ G(qn\delta) \big)\,,$$

the sequence $\big(\log_b |f_H(qn\delta)|\big)$ is u.d. mod 1 for almost all $\delta > 0$. Lemma 2.4 shows that $\log_b |f_H|$ is c.u.d. mod 1, i.e., $f_H = H(\phi_\bullet)$ is b-Benford. In summary, given any $H \in \mathcal{O}(X)$, the signal $H(\phi_\bullet)$ is b-Benford whenever (3.10) holds. Since $U_z \ne 0$ or $V_z \ne 0$ for at least one $z \in \sigma_{\rm dom}^+$, the set

$$\big\{ H : H(U_z) = H(V_z) = 0 \ \forall z \in \sigma_{\rm dom}^+ \big\} \ \subset \ \mathcal{O}(X)$$

is a nullset (in fact, a proper subspace) in $\mathcal{O}(X)$. For (Lebesgue) almost every $H \in \mathcal{O}(X)$, therefore, $H(\phi_\bullet)$ is b-Benford.

To prove (ii)⇒(i), assume that $\sigma_{\rm dom}(\phi)$ is exponentially b-resonant. By Lemma 2.11, there exist integers $\widetilde{p}_1, \ldots, \widetilde{p}_L$ and $\widetilde{q} \in \mathbb{N}$ such that

$$r = \sum\nolimits_{\ell=1}^{L} \frac{\widetilde{p}_\ell}{\widetilde{q}} \frac{\ln b}{\pi} \Im z_\ell \, .$$

Use Proposition 2.5 with $d = L$, $p_\ell = 2\widetilde{p}_\ell$ for $\ell = 1, \ldots, L$, and $\alpha = \widetilde{q}/\ln b$ to choose $u^* \in \mathbb{R}^L \setminus \{0\}$ such that $\lambda_{\mathbb{T}^L} \circ P_{u^*}^{-1} \neq \lambda_{\mathbb{T}}$, and use Lemma 3.12 to pick $H^* \in \mathcal{O}(X)$ with

$$H^*(\phi_t) = e^{rt} t^k \sum\nolimits_{\ell=1}^{L} u_\ell^* \cos(t \Im z_\ell) \quad \forall t \in \mathbb{R} \, .$$

It follows that

$$e^{-rqt}(qt)^{-k} H^*(\phi_{qt}) = \sum\nolimits_{\ell=1}^{L} u_\ell^* \cos(qt\Im z_\ell) = \Re \left(\sum\nolimits_{\ell=1}^{L} u_\ell^* e^{\imath qt \Im z_\ell} \right) \quad \forall t > 0 \, ,$$

whereas (3.9) yields

$$e^{-rqt}(qt)^{-k} H^*(\phi_{qt}) = \Re \left(\sum\nolimits_{z \in \sigma_{\rm dom}^+} \big(H^*(U_z) - \imath H^*(V_z) \big) e^{\imath qt \Im z} \right) + H^* \circ G(qt) \, .$$

Since $\lim_{t \to +\infty} H^* \circ G(qt) = 0$ and $\Im z_\ell > 0$ for all $\ell = 1, \ldots, L$, Lemma 3.13 shows that for each $z \in \sigma_{\rm dom}^+$,

$$H^*(U_z) = \begin{cases} u_\ell^* & \text{if } z = z_\ell \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad H^*(V_z) = 0 \, .$$

Next, pick any $H \in \mathcal{O}(X)$ that satisfies (3.10), and consider the function $g_H(t) := \widetilde{q} \log_b |t^{-k} H(\phi_t)|$ for $t > 0$. It follows from (3.9) that, for almost all $\delta > 0$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} \langle g_H(qn\delta) \rangle &= \langle \widetilde{q} \log_b |(qn\delta)^{-k} H(\phi_{qn\delta})| \rangle \\ &= \left\langle qn\delta \frac{\widetilde{q} r}{\ln b} + \frac{\widetilde{q}}{\ln b} \ln \left| F_H \left(\left\langle \left(n \frac{\delta \Im z_1}{2\pi}, \ldots, n \frac{\delta \Im z_L}{2\pi} \right) \right\rangle \right) + H \circ G(qn\delta) \right| \right\rangle \\ &= \left\langle Q_H \left(\left\langle \left(n \frac{\delta \Im z_1}{2\pi}, \ldots, n \frac{\delta \Im z_L}{2\pi} \right) \right\rangle \right) + y_n \right\rangle \, , \end{aligned}$$

with the (measurable) function $Q_H : \mathbb{T}^L \to \mathbb{T}$ given by

$$Q_H(\langle x \rangle) = \left\langle \sum\nolimits_{\ell=1}^{L} 2\widetilde{p}_\ell q x_\ell + \frac{\widetilde{q}}{\ln b} \ln |F_H(\langle x \rangle)| \right\rangle \, ,$$

and with an appropriate sequence (y_n) in $\mathbb{R}$ that satisfies $\lim_{n \to \infty} y_n = 0$. The $L+1$ numbers $1, \frac{1}{2\pi}\delta \Im z_1, \ldots, \frac{1}{2\pi}\delta \Im z_L$ are $\mathbb{Q}$-independent for all but countably many $\delta > 0$, and whenever they are,

$$\frac{1}{N} \sum\nolimits_{n=1}^{N} e^{2\pi \imath k g_H(nq\delta)} \stackrel{N \to \infty}{\longrightarrow} \int_{\mathbb{T}^L} e^{2\pi \imath k Q_H} \, {\rm d}\lambda_{\mathbb{T}^L} \quad \forall k \in \mathbb{Z} \, .$$

By Lemma 2.4, this means that

$$\lim\nolimits_{T \to +\infty} \frac{1}{T} \int_0^T e^{2\pi \imath k g_H(t)} \, {\rm d}t = \int_{\mathbb{T}^L} e^{2\pi \imath k Q_H} \, {\rm d}\lambda_{\mathbb{T}^L} \quad \forall k \in \mathbb{Z} \, .$$

Note that $Q_{H^*} = P_{u^*} \circ M_q$, with the map $M_q : \mathbb{T}^L \to \mathbb{T}^L$ given by $M_q(\langle x \rangle) = \langle qx \rangle$. Observe that $\lambda_{\mathbb{T}^L} \circ M_q^{-1} = \lambda_{\mathbb{T}^L}$, and recall that $\lambda_{\mathbb{T}^L} \circ P_{u^*}^{-1} \neq \lambda_{\mathbb{T}}$, hence

$$\lambda_{\mathbb{T}^L} \circ Q_{H^*}^{-1} = (\lambda_{\mathbb{T}^L} \circ M_q^{-1}) \circ P_{u^*}^{-1} = \lambda_{\mathbb{T}^L} \circ P_{u^*}^{-1} \neq \lambda_{\mathbb{T}} \,,$$

and so $\int_{\mathbb{T}^L} e^{2\pi\imath k^* Q_{H^*}} \, \mathrm{d}\lambda_{\mathbb{T}^L} \neq 0$ for some $k^* \in \mathbb{Z} \setminus \{0\}$. By the Dominated Convergence Theorem, the $\mathcal{P}(\mathbb{T})$-valued function $H \mapsto \lambda_{\mathbb{T}^L} \circ Q_H^{-1}$ is continuous on the (non-empty open) set $\{H : (3.10) \text{ holds}\} \subset \mathcal{O}(X)$. Since (3.10) holds in particular with $H = H^*$,

$$\lim\nolimits_{H \to H^*} \int_{\mathbb{T}^L} e^{2\pi\imath k^* Q_H} \, \mathrm{d}\lambda_{\mathbb{T}^L} = \int_{\mathbb{T}^L} e^{2\pi\imath k^* Q_{H^*}} \, \mathrm{d}\lambda_{\mathbb{T}^L} \neq 0 \,,$$

and consequently, for every H sufficiently close to H^*,

$$\lim\nolimits_{T \to +\infty} \frac{1}{T} \int_0^T e^{2\pi\imath k^* g_H(t)} \, \mathrm{d}t \neq 0 \,,$$

which in turn shows that g_H is not c.u.d. mod 1, and neither is $\log_b |H(\phi_\bullet)|$, by Lemma 2.3. In summary, $H(\phi_\bullet)$ is not b-Benford whenever H is sufficiently close to H^*. Consequently, the set $\{H : H(\phi_\bullet) \text{ is not } b\text{-Benford}\}$ contains a non-empty open set, and hence is not a nullset in $\mathcal{O}(X)$. Thus (ii)⇒(i), and the proof is complete. ∎

Example 3.14. The observations made for the flows ϕ and ψ on $\mathbb{R}^2$ in Example 3.10 are fully consistent with Theorem 3.11: The set $\sigma_{\mathrm{dom}}(\phi) = \{2\}$ is exponentially b-nonresonant for all b, while $\sigma_{\mathrm{dom}}(\psi) = \{0\}$ is exponentially b-resonant. Hence $H(\phi_\bullet)$ is Benford for almost all $H \in \mathcal{O}(\mathbb{R}^2)$ whereas $H(\psi_\bullet)$ is not.

As indicated already in Example 3.6, the Benford property may be of interest for some *non-linear* observables also. A simple natural example are norms on $\mathcal{L}(X)$.

Theorem 3.15. *Let $b \in \mathbb{N} \setminus \{1\}$ and $\|\cdot\|$ any norm on $\mathcal{L}(X)$. If $\sigma_{\mathrm{dom}}(\phi)$ is exponentially b-nonresonant for the linear flow ϕ on X then $\|\phi_\bullet\|$ is b-Benford.*

Proof. Using the same notation as in the proof of Theorem 3.11 above, let $f(t) := \log_b t^{-k} \|\phi_t\|$ for all $t > 0$, and deduce from (3.7) and (3.8) that

$$f(qt) = \frac{rqt}{\ln b} + \frac{1}{\ln b} \ln \left\| E\left(\left\langle \left(\frac{t\Im z_1}{2\pi}, \ldots, \frac{t\Im z_L}{2\pi} \right) \right\rangle \right) + G(qt) \right\| \quad \forall t > 0 \,,$$

where the smooth function $E : \mathbb{T}^L \to \mathcal{L}(X)$ is given by

$$E(\langle x \rangle) = \sum\nolimits_{z \in \sigma_{\mathrm{dom}}^+} \left(U_z \cos(p^{(z)} \cdot x) + V_z \sin(p^{(z)} \cdot x) \right) .$$

Recall that $U_z \neq 0$ or $V_z \neq 0$ for at least one $z \in \sigma_{\mathrm{dom}}^+$, which in turn implies that $E(\langle x \rangle) \neq 0$, and hence also $\|E(\langle x \rangle)\| \neq 0$, for $\lambda_{\mathbb{T}^L}$-almost all $\langle x \rangle \in \mathbb{T}^L$. The argument is now analogous to the one establishing (i)⇒(ii) in Theorem 3.11: For all but countably many $\delta > 0$, the $L+2$ numbers $1, rq\delta/\ln b, \frac{1}{2\pi}\delta\Im z_1, \ldots, \frac{1}{2\pi}\delta\Im z_L$ are $\mathbb{Q}$-independent, and whenever they are, the sequence $\big(f(qn\delta)\big)$ is u.d. mod 1 by

Proposition 2.7, with $d = L$, $\vartheta_0 = rq\delta/\ln b$ and $\vartheta_\ell = \frac{1}{2\pi}\delta\Im z_\ell$ for $\ell = 1, \ldots, L$, as well as

$$\alpha = 0\,, \quad \beta = \frac{1}{\ln b}\,, \quad F = \|E\|\,, \quad (z_n) = \big(\|E_n + G(qn\delta)\| - \|E_n\|\big)\,,$$

where $E_n = E\left(\langle\left(\frac{1}{2\pi}n\delta\Im z_1, \ldots, \frac{1}{2\pi}n\delta\Im z_L\right)\rangle\right)$. As before, it follows that f is c.u.d. mod 1, and so is $\log_b \|\phi_\bullet\|$, i.e., $\|\phi_\bullet\|$ is b-Benford. □

Unlike in Theorems 3.2 and 3.11, the converse in Theorem 3.15 is not true in general: The signal $\|\phi_\bullet\|$ may be b-Benford even if $\sigma_{\text{dom}}(\phi)$ is exponentially b-resonant. In fact, as the next example demonstrates, except for the trivial case of $d = 1$, it is impossible to characterize the Benford property of $\|\phi_\bullet\|$ solely in terms of $\sigma(\phi)$, let alone $\sigma_{\text{dom}}(\phi)$.

Example 3.16. Let $b = 10$ for convenience and denote by $|\cdot|$ the Euclidean (or spectral) norm on $\mathcal{L}(X)$, induced by the standard Euclidean norm $|\cdot|$ on X, i.e., $|A| = \max\{|Ax| : x \in X, |x| = 1\}$ with $|x| = \sqrt{x \cdot x}$. Consider the linear flow ϕ on $X = \mathbb{R}^2$ generated by

$$\dot{x} = \begin{bmatrix} 1 & -2\pi/\ln 10 \\ 2\pi/\ln 10 & 1 \end{bmatrix} x\,.$$

Since $\sigma(\phi) = \sigma_{\text{dom}}(\phi) = \{1 \pm 2\imath\pi/\ln 10\}$ is exponentially 10-resonant, by Theorem 3.11 the signal $H(\phi_\bullet)$ fails to be 10-Benford for many (in fact, most) $H \in \mathcal{O}(\mathbb{R}^2)$. To see this explicitly, note that $H(\phi_\bullet) = 0$ if and only if

$$H(I_{\mathbb{R}^2}) = 0 \quad \text{and} \quad H\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right) = 0\,, \tag{3.11}$$

and otherwise, with the appropriate $\rho > 0$ and $0 \le \eta < 1$,

$$H(\phi_t) = e^t \rho \cos\big(2\pi(t/\ln 10 - \eta)\big) \quad \forall t \in \mathbb{R}\,.$$

For all but countably many $\delta > 0$ and all sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} \langle \log_{10}|H(\phi_{n\delta})|\rangle &= \big\langle n\delta/\ln 10 + \log_{10}\rho + \log_{10}\big|\cos\big(2\pi(n\delta/\ln 10 - \eta)\big)\big|\big\rangle \\ &= \big\langle P(\langle n\delta/\ln 10 - \eta\rangle) + \eta + \log_{10}\rho\big\rangle\,, \end{aligned}$$

with the map $P : \mathbb{T} \to \mathbb{T}$ given by

$$P(\langle x\rangle) = \big\langle x + \log_{10}|\cos(2\pi x)|\big\rangle\,.$$

Since the sequence $(n\delta/\ln 10 - \eta)$ is u.d. mod 1 for all but countably many $\delta > 0$, and since, as is easily checked, $\lambda_{\mathbb{T}} \circ P^{-1} \ne \lambda_{\mathbb{T}}$, Lemma 2.4 shows that $H(\phi_\bullet)$ is not Benford. Whenever (3.11) fails, therefore, $H(\phi_\bullet)$ is neither 10-Benford nor trivial. On the other hand, $|\phi_t| \equiv e^t$ is Benford. Thus the implication in Theorem 3.15 can not in general be reversed.

Consider now also the linear flow ψ on X generated by

$$\dot{x} = \begin{bmatrix} 1 & -4\pi/\ln 10 \\ \pi/\ln 10 & 1 \end{bmatrix} x \, .$$

Note that A_ψ and A_ϕ are similar, so $\sigma(\psi) = \sigma(\phi)$ and also $\sigma_{\rm dom}(\phi) = \sigma_{\rm dom}(\psi)$. A short calculation confirms that

$$|\psi_t| = \frac{e^t}{4}\sqrt{25 - 9\cos(4\pi t/\ln 10) + 3|\sin(2\pi t/\ln 10)|\sqrt{82 - 18\cos(4\pi t/\ln 10)}} \, ,$$

and hence, for any $\delta > 0$ and $n \in \mathbb{N}$,

$$\langle \log_{10} |\psi_{n\delta}| \rangle = \bigl\langle Q(\langle n\delta/\ln 10\rangle) - \log_{10} 4 \bigr\rangle \, ,$$

with the (piecewise smooth) map $Q : \mathbb{T} \to \mathbb{T}$ given by

$$Q(\langle x \rangle) = \Bigl\langle x + \tfrac{1}{2}\log_{10}\bigl(25 - 9\cos(4\pi x) + 3|\sin(2\pi x)|\sqrt{82 - 18\cos(4\pi x)}\bigr) \Bigr\rangle \, .$$

As before, it is straightforward to see that $\lambda_{\mathbb{T}} \circ Q^{-1} \neq \lambda_{\mathbb{T}}$, and Lemma 2.4 implies that $|\psi_\bullet|$ is not 10-Benford. In summary, even though the linear flows ϕ and ψ have identical spectra and dominant spectra, the signal $|\phi_\bullet|$ is 10-Benford whereas the signal $|\psi_\bullet|$ is not.

Remark 3.17. From Examples 3.14 and 3.16, it may be conjectured that if $\sigma_{\rm dom}(\phi)$ is exponentially b-resonant then $\{H : H(\phi_\bullet) \text{ is } b\text{-Benford}\}$ actually is a nullset in $\mathcal{O}(X)$. By means of a stronger variant of Proposition 2.5 established in [6], it is not difficult to see that this is indeed the case for $1 \le d \le 4$. However, the author does not know of a proof of, or counter-example to, this conjecture for $d \ge 5$.

4 Most linear flows are Benford

As seen in the previous section, if $\sigma(\phi)$ is exponentially nonresonant for the linear flow ϕ on $X = \mathbb{R}^d$, then the Benford-or-trivial dichotomy of Theorem 3.2(ii) holds for every signal $H(\phi_\bullet)$. In fact, $H(\phi_\bullet)$ is Benford unless

$$H(A_\phi^j) = 0 \quad \forall j = 0, \ldots, d-1 \, ; \tag{4.1}$$

here, as usual, $A^0 := I_X$ for all $A \in \mathcal{L}(X)$. Note that the linear observables satisfying (4.1), and hence $H(\phi_\bullet) = 0$, constitute a proper subspace of $\mathcal{O}(X)$. In fact, as seen in the proof of Theorem 3.11, even if exponential nonresonance holds only for $\sigma_{\rm dom}(\phi)$, the signal $H(\phi_\bullet)$ is still Benford, provided that H does not belong to one distinguished proper subspace of $\mathcal{O}(X)$ that is independent of H. Put differently, if $\sigma_{\rm dom}(\phi)$ or even $\sigma(\phi)$ is exponentially nonresonant then BL is the only relevant digit distribution that can be distilled from ϕ by means of linear observables. The purpose of this short section is to demonstrate in turn that $\sigma(\phi)$, and hence also $\sigma_{\rm dom}(\phi)$, is exponentially b-nonresonant for all $b \in \mathbb{N}\setminus\{1\}$ and most linear flows ϕ, both from a topological and a measure-theoretical point of view.

Recall that every linear flow ϕ on X can be identified, via $\phi \leftrightarrow A_\phi$, with a unique element of $\mathcal{L}(X)$. The latter space has a natural linear and topological structure making it isomorphic and homeomorphic to $\mathbb{R}^{d^2}$, and hence it will be convenient to phrase the results of this section as statements regarding $\mathcal{L}(X)$. Specifically, for every $b \in \mathbb{N} \setminus \{1\}$ consider the set of linear maps

$$\mathcal{R}_b := \left\{ A \in \mathcal{L}(X) : \sigma(A) \text{ is exponentially } b\text{-resonant} \right\};$$

also let $\mathcal{R} := \bigcup_{b \in \mathbb{N}\setminus\{1\}} \mathcal{R}_b$. Recall that a subset of a topological space is *meagre* (or *of first category*) if it is the countable union of nowhere dense sets. According to the Baire Category Theorem, in a complete metric space (such as, e.g., $\mathcal{L}(X)$ endowed with any norm), meagre sets are, in a sense, topologically negligible. The goal of this section, then, is to establish the following fact which, informally put, shows that $\mathcal{R}$ is a negligible set, both topologically and measure-theoretically.

Theorem 4.1. *The set $\mathcal{R}$ is a meagre nullset in $\mathcal{L}(X)$.*

A crucial ingredient in the proof of Theorem 4.1 presented below is the real-analyticity of certain functions. Recall that a function $f : \mathcal{U} \to \mathbb{C}$, with $\mathcal{U} \neq \varnothing$ denoting a connected open subset of $\mathbb{R}^L$ for some $L \in \mathbb{N}$, is *real-analytic* (on $\mathcal{U}$) if it can be, in a neighbourhood of each point of $\mathcal{U}$, represented as a convergent power series. An important property of real-analytic functions not shared by arbitrary $\mathbb{C}$-valued C^∞-functions on $\mathcal{U}$ is the following fact regarding their zero-locus, which apparently is part of analysis folklore; e.g., see [22, p.83].

Proposition 4.2. *Let $f : \mathcal{U} \to \mathbb{C}$ be real-analytic, and $N_f := \{x \in \mathcal{U} : f(x) = 0\}$. Then either $N_f = \mathcal{U}$, or else N_f is a (Lebesgue) nullset.*

Next consider any monic polynomial $p_a : \mathbb{C} \to \mathbb{C}$ of degree $L \geq 2$, i.e.,

$$p_a(z) = z^L + a_1 z^{L-1} + \ldots + a_{L-1} z + a_L \,,$$

where $a = (a_1, \ldots, a_L) \in \mathbb{R}^L$, and recall that p_a has, for most $a \in \mathbb{R}^L$, only simple roots. More formally, there exists a non-constant real-analytic function $g_L : \mathbb{R}^L \to \mathbb{R}$ with the property that if p_a has a multiple root, i.e., $p_a(z_0) = p_a'(z_0) = 0$ for some $z_0 \in \mathbb{C}$, then $g_L(a) = 0$. In fact, the function g_L can be chosen as a polynomial with integer coefficients and degree $2L-2$; e.g., see [11, Lem.3.3.4]. Whenever $g_L(a) \neq 0$, therefore, the equation $p_a(z) = 0$ has exactly L different solutions which, by (the real-analytic version of) the Implicit Function Theorem [22, Thm.2.3.5] depend real-analytically on a. To put these facts together in a form facilitating a proof of Theorem 4.1, for every $A_0 \in \mathcal{L}(X)$ and $\varepsilon > 0$, denote by $B_\varepsilon(A_0)$ the open ball with radius ε centered at A_0, that is, $B_\varepsilon(A_0) = \{A \in \mathcal{L}(X) : \|A - A_0\| < \varepsilon\}$, where $\|\cdot\|$ is any fixed norm on $\mathcal{L}(X)$.

Lemma 4.3. *There exists a closed nullset $\mathcal{N} \subset \mathcal{L}(X)$ with the following property: For each $A_0 \in \mathcal{L}(X) \setminus \mathcal{N}$ there exist $\varepsilon > 0$ and d real-analytic functions $\lambda_1, \ldots, \lambda_d : B_\varepsilon(A_0) \to \mathbb{C}$ such that, for all $A \in B_\varepsilon(A_0)$,*

(i) $\sigma(A) = \{\lambda_1(A), \ldots, \lambda_d(A)\}$;

(ii) $\lambda_j(A) \neq \lambda_k(A)$ *whenever* $j \neq k$;

(iii) $\lambda_j(A) \neq \overline{\lambda_k(A)}$ *whenever* $j \neq k$, *unless* $\lambda_j = \overline{\lambda_k}$ *on* $B_\varepsilon(A_0)$.

Proof. For $d = 1$ simply take $\mathcal{N} = \varnothing$ and $\lambda_1([a]) = a$. For $d \geq 2$, note that

$$p_A(z) := \det(zI_X - A) = z^d + a_1(A)z^{d-1} + \ldots + a_{d-1}(A)z + a_d(A)\,,$$

with real-analytic (in fact, polynomial) functions $a_1, \ldots, a_d : \mathcal{L}(X) \to \mathbb{R}$; for example, $a_1(A) = -\mathrm{trace}\, A$ and $a_d(A) = (-1)^d \det A$. Thus the function $g := g_d(a_1, \ldots, a_d) : \mathcal{L}(X) \to \mathbb{R}$ is real-analytic and non-constant, and so

$$\mathcal{N} := \bigl\{A \in \mathcal{L}(X) : A \text{ has a multiple eigenvalue}\bigr\} = \{A \in \mathcal{L}(X) : g(A) = 0\}$$

is a closed nullset, by Proposition 4.2. For each $A_0 \in \mathcal{L}(X) \setminus \mathcal{N}$ there exists $\varepsilon > 0$ such that $B_\varepsilon(A_0) \cap \mathcal{N} = \varnothing$, and for ε sufficiently small, by the Implicit Function Theorem, there also exist d real-analytic functions $\lambda_1, \ldots, \lambda_d : B_\varepsilon(A_0) \to \mathbb{C}$ with $\sigma(A) = \{\lambda_1(A), \ldots, \lambda_d(A)\}$ for all $A \in B_\varepsilon(A_0)$. Clearly, $\lambda_j(A) \neq \lambda_k(A)$ whenever $j \neq k$, since otherwise $g(A) = 0$. Finally, if $\lambda_j(A_1) = \overline{\lambda_k(A_1)}$ for some $A_1 \in B_\varepsilon(A_0)$ then $\overline{\lambda_k(A)}$ is, for every A sufficiently close to A_1, an eigenvalue of A that, by continuity, must coincide with $\lambda_j(A)$. Hence $\lambda_j(A) = \overline{\lambda_k(A)}$ for all A close to A_1, and therefore, by Proposition 4.2, for all $A \in B_\varepsilon(A_0)$ as well. □

Proof of Theorem 4.1. Since for $d = 1$ clearly $\mathcal{R}_b = \{0\}$ for all b, the set $\mathcal{R} = \{0\}$ is a meagre nullset in $\mathcal{L}(X) = \mathbb{R}$, and only the case $d \geq 2$ has to be considered henceforth. Fix $b \in \mathbb{N} \setminus \{1\}$, and given any $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d$, $q \in \mathbb{N}$, and non-empty set $J \subset \{1, \ldots, d\}$, define a real-analytic (in fact, polynomial) function $f_{p,q,J} : \mathbb{R}^{2d} \to \mathbb{R}$ as

$$f_{p,q,J}(x) := \sum\nolimits_{j,k \in J} (x_j - x_k)^2 + \sum\nolimits_{j \in J} \Bigl(\pi q x_j - \ln b \sum\nolimits_{k \in J} p_k x_{d+k}\Bigr)^2 .$$

For each $A_0 \in \mathcal{L}(X) \setminus \mathcal{N}$, pick $\varepsilon > 0$ and $\lambda_1, \ldots, \lambda_d : B_\varepsilon(A_0) \to \mathbb{C}$ as in Lemma 4.3. Observe that if $\sigma(A)$ is exponentially b-resonant for some $A \in B_\varepsilon(A_0)$ then

$$F_{p,q,J}(A) := f_{p,q,J}\bigl(\Re\lambda_1(A), \ldots, \Re\lambda_d(A), \Im\lambda_1(A), \ldots, \Im\lambda_d(A)\bigr) = 0 \qquad (4.2)$$

for the appropriate p, q, and J. Clearly, every function $F_{p,q,J} : B_\varepsilon(A_0) \to \mathbb{R}$ is real-analytic. Moreover, if $F_{p,q,J}(A_1) = 0$ for some $A_1 \in B_\varepsilon(A_0)$ then also $A_1 + \delta I_X \in B_\varepsilon(A_0)$ for all sufficiently small $\delta > 0$, and $F_{p,q,J}(A_1 + \delta I_X) = \pi^2 q^2 \delta^2 \# J > 0$. Thus $F_{p,q,J} \neq 0$, and hence the set

$$\mathcal{N}_{p,q,J,A_0} := \bigl\{A \in B_\varepsilon(A_0) : F_{p,q,J}(A) = 0\bigr\}$$

is a closed nullset, by Proposition 4.2; in particular, $\mathcal{N}_{p,q,J,A_0}$ is nowhere dense, and (4.2) implies that

$$\mathcal{R}_b \cap B_\varepsilon(A_0) = \bigcup\nolimits_{p,q,J} \mathcal{N}_{p,q,J,A_0} =: \mathcal{N}_{A_0}\,.$$

Being the countable union of nowhere dense nullsets, the set $\mathcal{N}_{A_0}$ is itself a meagre nullset. Since $\mathcal{L}(X)$ is separable, there exists a sequence $(A_{0,n})$ in $\mathcal{L}(X) \setminus \mathcal{N}$ and a sequence (ε_n) in $\mathbb{R}$ with $\varepsilon_n > 0$ for all n, such that

$$\mathcal{L}(X) \setminus \mathcal{N} = \bigcup\nolimits_{n \in \mathbb{N}} B_{\varepsilon_n}(A_{0,n}) .$$

It follows that

$$\mathcal{R}_b \subset \mathcal{N} \cup \bigcup\nolimits_{n \in \mathbb{N}} \bigl(\mathcal{R}_b \cap B_{\varepsilon_n}(A_{0,n})\bigr) = \mathcal{N} \cup \bigcup\nolimits_{n \in \mathbb{N}} \mathcal{N}_{A_{0,n}} ,$$

which shows that $\mathcal{R}_b$ is a meagre nullset as well, and so is $\mathcal{R} = \bigcup_{b \in \mathbb{N}\setminus\{1\}} \mathcal{R}_b$. □

Remark 4.4. Despite being a meagre nullset, the set $\mathcal{R}$ could nevertheless be dense in $\mathcal{L}(X)$. This, however, is not the case: Lemmas 2.11 and 4.3 imply that $\mathcal{L}(X) \setminus \mathcal{R}$ contains the non-empty open set $\bigl\{A \in \mathcal{L}(X) \setminus \mathcal{N} : \sigma(A) \subset \mathbb{R} \setminus \{0\}\bigr\}$.

Informally put, Theorems 3.2 and 4.1 together show that for a generic linear flow ϕ on $X = \mathbb{R}^d$, the set $\sigma(\phi)$ is exponentially b-nonresonant for all bases b, and so for each linear observable H on $\mathcal{L}(X)$ the signal $H(\phi_\bullet)$ is Benford unless (4.1) holds, in which case $H(\phi_\bullet) = 0$. This may provide yet another explanation as to why BL is so often observed for even the simplest dynamical models in science and engineering.

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References

[1] H. Amann, *Ordinary differential equations: an introduction to non-linear analysis*, deGruyter, 1990.

[2] T.C. Anderson, L. Rolen, and R. Stoehr, Benford's law for coefficients of modular forms and partition functions, *Proc. Amer. Math. Soc.* **139**(2011), 1533–1541.

[3] F. Benford, The law of anomalous numbers, *Proc. Amer. Philos. Soc.* **78**(1938), 551–572.

[4] Benford Online Bibliography, `http://www.benfordonline.net`.

[5] A. Berger, Multi-dimensional dynamical systems and Benford's Law, *Discrete Contin. Dyn. Syst.* **13**(2005), 219–237.

[6] A. Berger and G. Eshun, A characterization of Benford's Law in discrete-time linear systems, to appear in *J. Dynam. Differential Equations*, 2015.

[7] A. Berger and T.P. Hill, *An Introduction to Benford's Law*, Princeton University Press, 2015.

[8] R. Bumby and E. Ellentuck, Finitely additive measures and the first digit problem, *Fund. Math.* **65**(1969), 33–42.

[9] N. Chernov, Decay of correlations, *Scholarpedia*, 3(4):4862, 2008.

[10] D.I.A. Cohen and T.M. Katz, Prime numbers and the first digit phenomenon, *J. Number Theory* **18**(1984), 261–268.

[11] H. Cohen, *A Course in Computational Algebraic Number Theory* (third, corr. print.), Springer, 1996.

[12] K. Dajani and C. Kraikamp, Ergodic theory of numbers, Carus Mathematical Monographs **29**, Mathematical Association of America, Washington DC (2002).

[13] P. Diaconis, The Distribution of Leading Digits and Uniform Distribution Mod 1, *Annals of Probability* **5**(1977), 72–81.

[14] A. Diekmann, Not the first digit! Using Benford's law to detect fraudulent scientific data, *J. Appl. Stat.* **34**(2007), 321–329.

[15] S. Docampo, M. del Mar Trigo, M.J. Aira, B. Cabezudo, and A. Flores-Moya, Benford's law applied to aerobiological data and its potential as a quality control tool, *Aerobiologia* **25**(2009), 275–283.

[16] M. Drmota and R. Tichy, *Sequences, Discrepancies, and Applications*, Springer Lecture Notes in Mathematics **1651**(1997).

[17] C.L. Geyer and P.P. Williamson, Detecting Fraud in Data Sets Using Benford's Law, *Communications in Statistics: Simulation and Computation* **33**(2004), 229–246.

[18] S.J. Gustafson and I.M. Sigal, *Mathematical Concepts of Quantum Mechanics*, Springer, 2003.

[19] G.H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949.

[20] S. Kanemitsu, K. Nagasaka, G. Rauzy, and J.-S. Shiue, On Benford's law: the first digit problem, Springer Lecture Notes in Mathematics **1299**(1988), 158–169.

[21] A.V. Kontorovich and S.J. Miller, Benford's law, values of L-functions and the $3x + 1$ problem, *Acta Arith.* **120**(2005), 269–297.

[22] S.G. Krantz and H.R. Parks, *A Primer of Real Analytic Functions* (second ed.), Birkhäuser, 2002.

[23] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley, 1974.

[24] J.C. Lagarias and K. Soundararajan, Benford's law for the $3x + 1$ function, *J. London Math. Soc.* **74**(2006), 289–303.

[25] B. Massé and D. Schneider, A survey on weighted densities and their connection with the first digit phenomenon, *Rocky Mountain J. Math.* **41**(2011), 1395–1415.

[26] S.J. Miller and M.J. Nigrini, Order statistics and Benford's law, *Int. J. Math. Math. Sci.* Art. ID 382948 (2008).

[27] K. Nagasaka and J.-S. Shiue, Benford's law for linear recurrence sequences, *Tsukuba J. Math.* **11**(1987), 341–351.

[28] S. Newcomb, Note on the frequency of use of the different digits in natural numbers, *Amer. J. Math.* **4**(1881), 39–40.

[29] M. Sambridge, H. Tkalčić, and A. Jackson, Benford's law in the natural sciences, *Geophysical Research Letters* **37**(2010), L22301.

[30] P. Schatte, On the uniform distribution of certain sequences and Benford's law, *Math. Nachr.* **136**(1988), 271–273.

[31] K. Schürger, Extensions of Black–Scholes processes and Benford's law, *Stochastic Process. Appl.* **118**(2008), 1219–1243.

[32] M.A. Snyder, J.H. Curry, and A.M. Dougherty, Stochastic aspects of one-dimensional discrete dynamical systems: Benford's law, *Phys. Rev. E*, 64:026222, 2001.

[33] C.R. Tolle, J.L. Budzien, and R.A. LaViolette, Do dynamical systems follow Benford's law? *Chaos* **10**(2000), 331–336.

[34] M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups. Transcendence Properties of the Exponential Function in Several Variables*, Springer, Berlin, 2000.