A characterization of Benford's Law in discrete-time linear systems

Arno Berger and Gideon Eshun

Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta, CANADA

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Abstract

A necessary and sufficient condition ("nonresonance") is established for every solution of an autonomous linear difference equation, or more generally for every sequence $(x^\top A^n y)$ with $x, y \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$, to be either trivial or else conform to a strong form of Benford's Law (logarithmic distribution of significands). This condition contains all pertinent results in the literature as special cases. Its number-theoretical implications are discussed in the context of specific examples, and so are its possible extensions and modifications.

Keywords. Benford sequence, uniform distribution mod 1, $\mathbb{Q}$-independence, nonresonant set.

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1 Introduction

The study of digits generated by dynamical processes is a classical subject that continues to attract interest from many disciplines, including ergodic and number theory [1, 13, 14, 23, 27], analysis [11, 29] and statistics [16, 20, 30]. A recurring theme across the disciplines is the surprising ubiquity of a logarithmic distribution of digits often referred to as *Benford's Law* (BL). The most well-known special case of BL is the so-called (*decimal*) *first-digit law* which asserts that

$$\mathbb{P}(\textit{leading digit}_{10} = d_1) = \log_{10}\left(1 + d_1^{-1}\right) , \quad \forall d_1 = 1, \ldots, 9 , \tag{1.1}$$

where *leading digit*$_{10}$ refers to the leading (or first significant) decimal digit, and $\log_{10}$ is the base-10 logarithm (see Section 2 for rigorous definitions); for example, the leading decimal digit of $e = 2.718$ is 2, whereas the leading digit of $-e^e = -15.15$ is 1. Note that (1.1) is heavily skewed towards the smaller digits: For instance, the leading decimal digit is almost seven times as likely to equal 1 (probability $\log_{10} 2 = 30.10\%$) as it is to equal 9 (probability $1 - \log_{10} 9 = 4.57\%$).

Ever since first recorded by Newcomb [33] in 1881 and re-discovered by Benford [2] in 1938, examples of data and systems conforming to (1.1) in one form or another have been discussed extensively, for instance in real-life data (e.g. [17, 35]), stochastic processes (e.g. [37]) and deterministic sequences (e.g. $(n!)$ and the prime numbers [15]). There now exists a large body of literature devoted to the mechanisms whereby mathematical objects, such as e.g. sequences or random variables, do or do not satisfy (1.1) or variants thereof. As of this writing, an online database [3] devoted exclusively to BL lists more than 800 references.

Due to their important role as elementary models throughout science, *linear difference equations* have, from very early on, been studied for their conformance to (1.1). A simple but prominent case in point is the sequence $(x_n) = (1, 1, 2, 3, 5, \ldots)$ of Fibonacci numbers, which has long been known [10, 19, 25, 39] to conform to (1.1) in the sense that

$$\lim_{N\to\infty} \frac{\#\{n \le N : \textit{leading digit}_{10}(x_n) = d_1\}}{N} = \log_{10}(1 + d_1^{-1}) , \quad \forall d_1 = 1, \ldots 9 . \tag{1.2}$$

Recall that (x_n) is a solution of a (very simple) *autonomous linear difference equation*, namely $x_n = x_{n-1} + x_{n-2}$ for all $n \ge 3$. This article provides a comprehensive theory of BL for such equations. Specifically, the central question addressed (and answered) herein is this: Given $d \in \mathbb{N}$ and real numbers $a_1, a_2, \ldots, a_{d-1}, a_d$ with $a_d \ne 0$, consider the (autonomous, d-th order) linear difference equation

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_{d-1} x_{n-d+1} + a_d x_{n-d} , \quad \forall n \ge d + 1 . \tag{1.3}$$

Under which conditions on $a_1, a_2, \ldots, a_{d-1}, a_d$, and presumably also on the initial values $x_1, \ldots, x_d$, does the solution (x_n) of (1.3) satisfy (1.2)? There already exists a sizeable literature addressing this question; see e.g. [4, 22, 32, 36]. All previous work, however, seems to have led merely to *sufficient* conditions that are either restrictive or difficult to state. By contrast, the main result in this paper (Theorem 3.16) provides an easy-to-state, *necessary and sufficient* condition for every nontrivial solution of (1.3) to satisfy (1.2), and in fact to conform to (1.1) in an even stronger sense. The classical results in the literature are then but simple corollaries.

To illustrate the main result, consider specifically the second-order difference equation

$$x_n = 2\gamma x_{n-1} - 5x_{n-2} , \quad \forall n \ge 3 , \tag{1.4}$$

where γ is a real parameter with $|\gamma| < \sqrt{5}$. Given any initial values $x_1, x_2 \in \mathbb{R}$, does the solution (x_n) of (1.4) satisfy (1.2)? Theorem 3.16 asserts that the answer to this question is positive provided that the set $\mathcal{Z}_\gamma = \{z^2 = 2\gamma z - 5\} = \{\gamma \pm \imath\sqrt{5 - \gamma^2}\}$ has a certain number-theoretical property ("nonresonance"). For example, if $\gamma = \sqrt{5}\cos(\pi/\sqrt{8}) = 0.9928$ then $\mathcal{Z}_\gamma$ turns out to be nonresonant, and (1.2) holds for every solution (x_n) of (1.4), unless $x_1 = x_2 = 0$, in which case $x_n \equiv 0$. On the other hand, if $\gamma = \sqrt{5}\cos(\frac{1}{2}\pi \log_{10} 5) = 1.018$ then $\mathcal{Z}_\gamma$ fails to be nonresonant, and correspondingly (1.2) does not hold for *any* solution of (1.4). Finally, if $\gamma = 1$ then (x_n) either satisfies (1.2) for *all* initial values x_1, x_2 (unless $x_1 = x_2 = 0$) or for *none* at all, and experimental evidence seems to support the former alternative; see Figure 1 and also Example 3.18 below.

	1	2	3	4	5	6	7	8	9
$\gamma = 0.9928$	29.99	17.43	12.82	9.58	7.93	6.70	6.01	5.02	4.52
$\gamma = 1.018$	43.68	7.77	7.20	6.87	6.68	6.62	6.71	6.99	7.48
$\gamma = 1$	29.99	17.23	12.78	9.51	7.92	6.61	6.01	5.19	4.76
exact BL	30.10	17.60	12.49	9.69	7.91	6.69	5.79	5.11	4.57

Figure 1: Relative frequencies (in percent) of the leading decimal digits for the first 10000 terms of the solution (x_n) of (1.4) with $x_1 = x_2 = 1$, for different values of the parameter γ; the bottom row shows the exact BL probabilities $100 \cdot \log_{10}(1 + d_1^{-1})$.

This article is organized as follows. Section 2 introduces the formal definitions and analytic tools required for the analysis. In Section 3, the main results are stated and proved, based upon a tailor-made notion of nonresonance (Definition 3.1). Several examples are presented in order to illustrate this notion as well as the main results. Finally, Section 4 briefly discusses possible extensions and modifications of the latter. Given the widespread usage of discrete-time linear systems and linear difference equations as models throughout the sciences, the results of this article may contribute to a better understanding of, and appreciation for BL and its applications in many disciplines. For the reader's convenience, several analytical facts of an auxiliary nature are deferred to an appendix, including the plausible but lengthy-to-prove Theorem A.4 which in turn implies the crucial Lemma 2.7.

2 Basic definitions and tools

Throughout this article, the following, mostly standard notation and terminology is used. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of, respectively, positive integer, nonnegative integer, integer, rational, positive real, real and complex numbers, and $\varnothing$ is the empty set. For every integer $b \ge 2$, the logarithm base b of $x \in \mathbb{R}^+$ is denoted $\log_b x$, and $\ln x$ is the natural logarithm (base e) of x; for convenience, let $\log_b 0 := 0$ for every b, and $\ln 0 := 0$. Given any $x \in \mathbb{R}$, the largest integer not larger than x is symbolized by $\lfloor x \rfloor$. The real part, imaginary part, complex conjugate and absolute value (modulus) of any $z \in \mathbb{C}$ is $\Re z$, $\Im z$, $\overline{z}$ and $|z|$, respectively. For every $z \in \mathbb{C} \setminus \{0\}$ there exists a unique number $-\pi < \arg z \le \pi$

with $z = |z|e^{\imath \arg z}$. Given any $w \in \mathbb{C}$ and $\mathcal{Z} \subset \mathbb{C}$, define $w + \mathcal{Z} := \{w + z : z \in \mathcal{Z}\}$ and $w\mathcal{Z} := \{wz : z \in \mathcal{Z}\}$. Thus with the unit circle $\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}$, for example, $w + \mathbb{S} = \{z \in \mathbb{C} : |z - w| = 1\}$ and $w\mathbb{S} = \{z \in \mathbb{C} : |z| = |w|\}$ for every $w \in \mathbb{C}$. The cardinality (number of elements) of any finite set $\mathcal{Z} \subset \mathbb{C}$ is $\#\mathcal{Z}$.

The symbol d throughout denotes a positive integer, usually unspecified or clear from the context. The d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ is symbolized by $\mathbb{T}^d$, its elements being represented as $\langle x \rangle = x + \mathbb{Z}^d$ with $x \in \mathbb{R}^d$; for simplicity write $\mathbb{T}$ instead of $\mathbb{T}^1$. The compact Abelian group $\mathbb{T}^d$ can be identified with the d-fold product $\mathbb{S} \times \ldots \times \mathbb{S}$, via the identification $\langle x \rangle = \langle (x_1, \ldots, x_d) \rangle \leftrightarrow (e^{2\pi \imath x_1}, \ldots, e^{2\pi \imath x_d})$ which is both a homeomorphism (of compact spaces) and an isomorphism (of groups). Denote the Haar (probability) measure on $\mathbb{T}^d$ by $\lambda_{\mathbb{T}^d}$. Call a set $\mathcal{J} \subset \mathbb{T}$ an *arc* if $\mathcal{J} = \langle \mathcal{I} \rangle := \{\langle x \rangle : x \in \mathcal{I}\}$ for some interval $\mathcal{I} \subset \mathbb{R}$. With this, a sequence (x_n) of real numbers is *uniformly distributed modulo one*, henceforth abbreviated as *u.d.* mod 1, if

$$\lim_{N\to\infty} \frac{\#\{n \le N : \langle x_n \rangle \in \mathcal{J}\}}{N} = \lambda_{\mathbb{T}}(\mathcal{J}) \quad \text{for every arc } \mathcal{J} \subset \mathbb{T}\,.$$

Equivalently, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\langle x_n \rangle) = \int_{\mathbb{T}} f \, \mathrm{d}\lambda_{\mathbb{T}}$ holds for every continuous (or merely Riemann integrable) function $f : \mathbb{T} \to \mathbb{C}$.

Recall that throughout b is an integer with $b \ge 2$, informally referred to as a *base*. Given a base b and any $x \ne 0$, there exists a unique real number $1 \le S_b(x) < b$ and a unique integer k such that $|x| = S_b(x) b^k$. The number $S_b(x)$, referred to as the (base-b) *significand* (or *mantissa*) of x, can be written explicitly as

$$S_b(x) = b^{\log_b |x| - \lfloor \log_b |x| \rfloor}\,;$$

in addition, let $S_b(0) := 0$ for every base b. The integer $\lfloor S_b(x) \rfloor$ is the *first significant digit* (base b) of x; note that $\lfloor S_b(x) \rfloor \in \{1, \ldots, b-1\}$ whenever $x \ne 0$.

In this article, conformance to BL for sequences of real numbers is studied via the following basic definition.

Definition 2.1. A sequence (x_n) in $\mathbb{R}$ is a b*-Benford sequence*, or b*-Benford* for short, with $b \in \mathbb{N} \setminus \{1\}$, if

$$\lim_{N\to\infty} \frac{\#\{n \le N : S_b(x_n) \le s\}}{N} = \log_b s\,, \quad \forall s \in [1, b)\,.$$

The sequence (x_n) is a *Benford sequence*, or simply *Benford*, if it is b-Benford for every $b \in \mathbb{N} \setminus \{1\}$.

Specifically, note that (1.2) holds whenever (x_n) is 10-Benford, whereas the converse is not true in general since, for instance, the sequence of first significant digits of (2^n), i.e. $\big(\lfloor S_{10}(2^n) \rfloor\big) = (2, 4, 8, 1, 3, \ldots)$, is clearly not 10-Benford yet can easily be shown to satisfy (1.2).

Though very simple, the following observation is fundamental for the purpose of this work because it enables the application of a host of tools from the theory of uniform distribution.

Proposition 2.2. [15, Thm.1] *Let $b \in \mathbb{N} \setminus \{1\}$. A sequence (x_n) in $\mathbb{R}$ is b-Benford if and only if the sequence $(\log_b |x_n|)$ is u.d.* mod 1*.*

To prepare for the application of Proposition 2.2, several basic facts from the theory of uniform distribution are reviewed here for the convenience of the reader who, for an authoritative account on the theory in general, may also wish to consult [18, 26].

Lemma 2.3. *The following are equivalent for every sequence (x_n) in $\mathbb{R}$:*

(i) *(x_n) is u.d.* mod 1*;*

(ii) *For every $\varepsilon > 0$ there exists a uniformly distributed sequence $(\widetilde{x}_n)$ with*

$$\overline{\lim}_{N\to\infty} \frac{\#\{n \le N : |x_n - \widetilde{x}_n| > \varepsilon\}}{N} < \varepsilon\,;$$

(iii) *Whenever (y_n) converges in $\mathbb{R}$ then $(x_n + y_n)$ is u.d.* mod 1*;*

(iv) *(kx_n) is u.d.* mod 1 *for every $k \in \mathbb{Z} \setminus \{0\}$;*

(v) *$(x_n + \alpha \ln n)$ is u.d.* mod 1 *for every $\alpha \in \mathbb{R}$.*

Proof. Clearly (i)⇒(ii), and the converse is analogous to [4, Lem.2.3]. Also, each of the statements (iii), (iv), and (v) trivially implies (i), while the reverse implication is [26, Thm.I.1.2], [26, Exc.I.2.4], and [4, Lem.2.8], respectively. □

Lemma 2.4. *Let (x_n) be a sequence in $\mathbb{R}$, and $L \in \mathbb{N}$. If $(x_{nL+\ell})$ is u.d.* mod 1 *for every $\ell \in \{1, \ldots, L\}$ then (x_n) is u.d.* mod 1 *as well.*

Proof. This follows directly from Weyl's criterion [26, Thm.I.2.1]: For every $k \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned}
\left| \frac{1}{N} \sum\nolimits_{n=1}^{N} e^{2\pi\imath k x_n} \right| &\le \left| \frac{1}{N} \sum\nolimits_{n=1}^{L\lfloor N/L \rfloor} e^{2\pi\imath k x_n} \right| + \left| \frac{1}{N} \sum\nolimits_{n=L\lfloor N/L\rfloor+1}^{N} e^{2\pi\imath k x_n} \right| \\
&\le \left| \frac{1}{N} \sum\nolimits_{\ell=1}^{L} \sum\nolimits_{n=0}^{\lfloor N/L \rfloor - 1} e^{2\pi\imath k x_{nL+\ell}} \right| + \frac{L}{N} \\
&\le \frac{1}{L} \sum\nolimits_{\ell=1}^{L} \left| \frac{1}{\lfloor N/L \rfloor} \sum\nolimits_{n=0}^{\lfloor N/L \rfloor - 1} e^{2\pi\imath k x_{nL+\ell}} \right| + \frac{L}{N} \stackrel{N\to\infty}{\longrightarrow} 0\,,
\end{aligned}$$

because $\lim_{M\to\infty} \dfrac{1}{M} \sum\nolimits_{n=0}^{M-1} e^{2\pi\imath k x_{nL+\ell}} = 0$ for every ℓ, by assumption. □

When combined with the well-known fact that $(n\vartheta)$ is u.d. mod 1 precisely if $\vartheta \in \mathbb{R}$ is irrational [26, Exp.I.2.1], Lemma 2.3 and 2.4 immediately yield

Lemma 2.5. *Let $\alpha, \vartheta \in \mathbb{R}$, $L \in \mathbb{N}$, and assume the sequence (y_n) in $\mathbb{R}$ has the property that $(y_{nL+\ell})$ converges for every $\ell \in \{1, \ldots, L\}$. Then $(n\vartheta + \alpha \ln n + y_n)$ is u.d.* mod 1 *if and only if $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$.*

The remaining two results in this section deal with sequences of a particular form that are going to appear naturally in later sections. For a concise formulation, given any $\mathcal{Z} \subset \mathbb{C}$, denote by $\mathrm{span}_{\mathbb{Q}}\mathcal{Z}$ the smallest subspace of $\mathbb{C}$ (over $\mathbb{Q}$) containing $\mathcal{Z}$; equivalently, if $\mathcal{Z} \neq \varnothing$ then $\mathrm{span}_{\mathbb{Q}}\mathcal{Z}$ is the set of all finite *rational* linear combinations of elements of $\mathcal{Z}$, i.e.

$$\mathrm{span}_{\mathbb{Q}}\mathcal{Z} = \big\{\rho_1 z_1 + \ldots + \rho_n z_n : n \in \mathbb{N}, \rho_1, \ldots, \rho_n \in \mathbb{Q}, z_1, \ldots, z_n \in \mathcal{Z}\big\};$$

note that $\mathrm{span}_{\mathbb{Q}}\varnothing = \{0\}$. With this terminology, recall that $z_1, \ldots, z_n \in \mathbb{C}$ are $\mathbb{Q}$-*independent* (or *rationally independent*) if $\mathrm{span}_{\mathbb{Q}}\{z_1, \ldots, z_n\}$ is n-dimensional, or equivalently if $\sum_{j=1}^n k_j z_j = 0$ with integers $k_1, \ldots, k_n$ implies $k_1 = \ldots = k_n = 0$. The following result is a generalization of [4, Lem.2.9].

Lemma 2.6. *Let $d \in \mathbb{N}$, $\vartheta_0, \vartheta_1, \ldots, \vartheta_d \in \mathbb{R}$, and assume $f : \mathbb{T}^d \to \mathbb{C}$ is continuous, and non-zero $\lambda_{\mathbb{T}^d}$-almost everywhere. If the $d+2$ numbers $1, \vartheta_0, \vartheta_1, \ldots, \vartheta_d$ are $\mathbb{Q}$-independent then the sequence*

$$\Big(n\vartheta_0 + \alpha \ln n + \beta \ln \big|f\big(\langle(n\vartheta_1, \ldots, n\vartheta_d)\rangle\big) + z_n\big|\Big)$$

is u.d. mod 1 *for every $\alpha, \beta \in \mathbb{R}$ and every sequence (z_n) in $\mathbb{C}$ with* $\lim_{n\to\infty} z_n = 0$.

Proof. For convenience, let

$$x_n := n\vartheta_0 + \alpha \ln n + \beta \ln \big|f\big(\langle(n\vartheta_1, \ldots, n\vartheta_d)\rangle\big) + z_n\big|, \quad \forall n \in \mathbb{N}.$$

The function $g := \beta \ln |f|$ is continuous on a set of full $\lambda_{\mathbb{T}^d}$-measure, and so [4, Cor.2.6] together with Lemma 2.3(v) shows that the sequence $(\widetilde{x}_n)$ with

$$\widetilde{x}_n := n\vartheta_0 + \alpha \ln n + \beta \ln \big|f\big(\langle(n\vartheta_1, \ldots, n\vartheta_d)\rangle\big)\big|, \quad \forall n \in \mathbb{N},$$

is u.d. mod 1 for every $\alpha, \beta \in \mathbb{R}$. Given $0 < \varepsilon \le 1$, choose $0 < \delta < \frac{1}{2}\varepsilon/(1+|\beta|)$ so small that $\lambda_{\mathbb{T}^d}\big(\{t \in \mathbb{T}^d : |f(t)| \le \delta\}\big) < \varepsilon$. There exists $\mathcal{T} \subset \mathbb{T}^d$ such that $\mathcal{T}$ is a finite union of open balls, $\mathcal{T} \supset \{t \in \mathbb{T}^d : |f(t)| \le \delta\}$, and $\lambda_{\mathbb{T}^d}(\mathcal{T}) < \varepsilon$. Observe now that if $\langle(n\vartheta_1, \ldots, n\vartheta_d)\rangle \notin \mathcal{T}$ and $|z_n| < \delta^2$ then

$$|x_n - \widetilde{x}_n| = |\beta| \left| \ln \left| 1 + \frac{z_n}{f\big(\langle(n\vartheta_1, \ldots, n\vartheta_d)\rangle\big)} \right| \right| \le 2|\beta|\delta < \varepsilon.$$

By the $\mathbb{Q}$-independence of $1, \vartheta_1, \ldots, \vartheta_d$, the sequence $\big((n\vartheta_1, \ldots, n\vartheta_d)\big)$ is u.d. mod 1 in $\mathbb{R}^d$, see e.g. [26, Exp.I.6.1], and so

$$\lim_{N\to\infty} \frac{\#\{n \le N : \langle(n\vartheta_1, \ldots, n\vartheta_d)\rangle \in \mathcal{T}\}}{N} = \lambda_{\mathbb{T}^d}(\mathcal{T}) < \varepsilon.$$

With this and $\lim_{n\to\infty} z_n = 0$, it follows that

$$\begin{aligned}
&\overline{\lim}_{N\to\infty} \frac{\#\{n \le N : |x_n - \widetilde{x}_n| > \varepsilon\}}{N} \\
&\le \overline{\lim}_{N\to\infty} \frac{\#\{n \le N : \langle(n\vartheta_1, \ldots, n\vartheta_d)\rangle \in \mathcal{T} \text{ or } |z_n| \ge \delta^2\}}{N} \\
&\le \overline{\lim}_{N\to\infty} \frac{\#\{n \le N : \langle(n\vartheta_1, \ldots, n\vartheta_d)\rangle \in \mathcal{T}\}}{N} + \overline{\lim}_{N\to\infty} \frac{\#\{n \le N : |z_n| \ge \delta^2\}}{N} \\
&= \lambda_{\mathbb{T}^d}(\mathcal{T}) + 0 < \varepsilon,
\end{aligned}$$

and an application of Lemma 2.3(ii) completes the proof. $\square$

The assertion of the next, final lemma is very plausible indeed. Its proof, however, is somewhat technical and hence deferred to an appendix for the reader's convenience.

Lemma 2.7. *Let $d \in \mathbb{N}$, $p_1, \ldots, p_d \in \mathbb{Z}$, and $\beta \in \mathbb{R} \setminus \{0\}$. Then there exists $u \in \mathbb{R}^d$ such that the sequence*

$$\Big(p_1 n\vartheta_1 + \ldots + p_d n\vartheta_d + \beta \ln \big| u_1 \cos(2\pi n\vartheta_1) + \ldots + u_d \cos(2\pi n \vartheta_d) \big| \Big)$$

is not u.d. mod 1 *whenever $\vartheta_1, \ldots, \vartheta_d \in \mathbb{R}$ and the $d+1$ numbers $1, \vartheta_1, \ldots, \vartheta_d$ are $\mathbb{Q}$-independent.*

Proof. See Appendix A. $\square$

3 A Characterization of Benford's Law

Given a positive integer d and real numbers $a_1, a_2, \ldots, a_{d-1}, a_d$ with $a_d \neq 0$, consider the autonomous, d-th order linear difference equation (or recursion)

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_{d-1} x_{n-d+1} + a_d x_{n-d}\,, \quad \forall n \geq d+1\,. \tag{1.3}$$

The goal of this section is to provide a necessary and sufficient condition on the coefficients $a_1, a_2, \ldots, a_{d-1}, a_d$ guaranteeing that every solution (x_n) of (1.3) is either Benford or trivial (identically zero); see Theorem 3.16 below. To make the analysis as transparent as possible, a standard matrix-vector approach is utilized. Thus associate with (1.3) the matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_{d-1} & a_d \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{d \times d}\,, \tag{3.1}$$

which is invertible since $a_d \neq 0$, and recall that, given initial values $x_1, \ldots, x_d \in \mathbb{R}$, the solution (x_n) of (1.3) can be expressed in the form

$$x_n = (e^{(d)})^\top A^n y\,, \quad \text{where } \; y = A^{-1} \begin{bmatrix} x_d \\ \vdots \\ x_1 \end{bmatrix} \in \mathbb{R}^d\,; \tag{3.2}$$

here $e^{(1)}, \ldots, e^{(d)}$ represent the standard basis of $\mathbb{R}^d$; A^n is the n-th power of A, i.e. $A^n = AA^{n-1}$ for $n \geq 1$ and $A^0 = I_d$, the $d \times d$-identity matrix; and $x^\top$ denotes the transpose of $x \in \mathbb{R}^d$, with $x^\top y$ being understood as the real number $\sum_{j=1}^d x_j y_j$. As suggested by (3.2), in what follows, conditions are studied under which $(x^\top A^n y)$

is b-Benford, where $x, y \in \mathbb{R}^d$ and A is any given real $d \times d$-matrix. Towards the end of this section, these conditions are, via (3.1) and (3.2), specialized to solutions (x_n) of the linear difference equation (1.3). Note that with A given by (3.1), the sequence $(x^\top A^n y)$ is a solution of (1.3) for *every* $x, y \in \mathbb{R}^d$; see also the proof of Lemma 3.6 below.

As throughout the entire article, in the subsequent analysis of powers of matrices, d always denotes a fixed but otherwise unspecified positive integer. For every $x \in \mathbb{R}^d$, the number $|x| \ge 0$ is the *Euclidean norm* of x, i.e. $|x| = \sqrt{x^\top x} = \sqrt{\sum_{j=1}^d x_j^2}$. A vector $x \in \mathbb{R}^d$ is a *unit* vector if $|x| = 1$. For every matrix $A \in \mathbb{R}^{d\times d}$, its spectrum, i.e. the set of its eigenvalues, is denoted by $\sigma(A)$. Thus $\sigma(A) \subset \mathbb{C}$ is non-empty, contains at most d numbers and is symmetric w.r.t. the real axis, i.e., all non-real elements of $\sigma(A)$ come in complex-conjugate pairs. The number $r_\sigma(A) := \max\{|\lambda| : \lambda \in \sigma(A)\} \ge 0$ is the *spectral radius* of A. Note that $r_\sigma(A) > 0$ unless A is *nilpotent*, i.e. unless $A^N = 0$ for some $N \in \mathbb{N}$; in the latter case $A^d = 0$ as well. For every $A \in \mathbb{R}^{d\times d}$, the number $|A|$ is the (*spectral*) *norm* of A as induced by $|\cdot|$, i.e. $|A| = \max\{|Ax| : |x| = 1\}$. It is well-known that $|A| = \sqrt{r_\sigma(A^\top A)} \ge r_\sigma(A) = \lim_{n\to\infty} |A^n|^{1/n}$.

As will become clear shortly, some Benford properties related to linear difference equations can be characterized in terms of the spectrum of an associated matrix. The following terminology turns out to be useful in this context.

Definition 3.1. Let $b \in \mathbb{N} \setminus \{1\}$. A non-empty set $\mathcal{Z} \subset \mathbb{C}$ with $|z| = r$ for some $r > 0$ and all $z \in \mathcal{Z}$, i.e. $\mathcal{Z} \subset r\mathbb{S}$, is *b-nonresonant* if the associated set

$$\Delta_{\mathcal{Z}} := \left\{1 + \frac{\arg z - \arg w}{2\pi} : z, w \in \mathcal{Z}\right\} \subset \mathbb{R} \tag{3.3}$$

satisfies both of the following conditions:

(i) $\Delta_{\mathcal{Z}} \cap \mathbb{Q} = \{1\}$;

(ii) $\log_b r \notin \mathrm{span}_{\mathbb{Q}} \Delta_{\mathcal{Z}}$.

An arbitrary set $\mathcal{Z} \subset \mathbb{C}$ is b-nonresonant if, for every $r > 0$, the set $\mathcal{Z} \cap r\mathbb{S}$ is either b-nonresonant or empty; otherwise, $\mathcal{Z}$ is *b-resonant.*

Note that the set $\Delta_{\mathcal{Z}}$ in (3.3) automatically satisfies $1 \in \Delta_{\mathcal{Z}} \subset (0, 2)$ and is symmetric w.r.t. the point 1, i.e. $\Delta_{\mathcal{Z}} = 2 - \Delta_{\mathcal{Z}}$. The empty set $\varnothing$ and the singleton $\{0\}$ are b-nonresonant for every $b \in \mathbb{N} \setminus \{1\}$. Also, if $\mathcal{Z}$ is b-nonresonant then so is every $\mathcal{W} \subset \mathcal{Z}$. On the other hand, $\mathcal{Z} \subset \mathbb{C}$ is certainly b-resonant for every b if either $\#(\mathcal{Z} \cap r\mathbb{S} \cap \mathbb{R}) = 2$ for some $r > 0$, in which case (i) is violated, or $\mathcal{Z} \cap \mathbb{S} \ne \varnothing$, which causes (ii) to fail.

Example 3.2. The singleton $\{z\}$ with $z \in \mathbb{C}$ is b-nonresonant if and only if either $z = 0$ or $\log_b |z| \notin \mathbb{Q}$. Similarly, any set $\{z, \overline{z}\}$ with $z \in \mathbb{C} \setminus \mathbb{R}$ is b-nonresonant if and only if 1, $\log_b |z|$ and $\frac{1}{2\pi} \arg z$ are $\mathbb{Q}$-independent.

Remark 3.3. (i) If $\mathcal{Z} \subset r\mathbb{S}$ then, for every $z \in \mathcal{Z}$,

$$\mathrm{span}_{\mathbb{Q}}\Delta_{\mathcal{Z}} = \mathrm{span}_{\mathbb{Q}}\left(\{1\} \cup \left\{\frac{\arg z - \arg w}{2\pi} : w \in \mathcal{Z}\right\}\right),$$

which shows that the dimension of $\mathrm{span}_{\mathbb{Q}}\Delta_{\mathcal{Z}}$ as a linear space over $\mathbb{Q}$ is at most $\#\mathcal{Z}$. Also, if $\mathcal{Z} \subset r\mathbb{S}$ is (non-empty and) symmetric w.r.t. the real axis, i.e. if $\overline{\mathcal{Z}} = \mathcal{Z} \neq \varnothing$, then condition (ii) in Definition 3.1 is equivalent to $\log_b r \notin \mathrm{span}_{\mathbb{Q}}(\{1\} \cup \{\frac{1}{2\pi}\arg z : z \in \mathcal{Z}\})$; cf. [4, Def.3.1].

(ii) The number 1 in (3.3) and part (i) of Definition 3.1 has been chosen for convenience only; for the purpose of this work, it could be replaced by any non-zero rational number.

Recall that for the sequence $(xa^n y)$ with any $x, y \in \mathbb{R}$ and $a \in \mathbb{R}\backslash\{0\}$ to be either b-Benford (if $xy \neq 0$) or trivial (if $xy = 0$) it is necessary and sufficient that $\log_b |a|$ be irrational. (This follows immediately e.g. from Proposition 2.2 and Lemma 2.5.) The following theorem, the first main result of this article, extends this simple fact to arbitrary (finite) dimension by characterizing the b-Benford property of $(x^\top A^n y)$ for any $x, y \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$. To concisely formulate this and subsequent results, call $(x^\top A^n y)$ and $(|A^n x|)$ with $x, y \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$ *terminating* if, respectively, $x^\top A^n y = 0$ or $A^n x = 0$ for all $n \geq d$; similarly, $(|A^n|)$ is terminating if $A^n = 0$ for all $n \geq d$. Also, recall that the asymptotic behaviour of (A^n) is completely determined by the eigenvalues of A, together with the corresponding (generalized) eigenvectors. As far as Benford's Law base b is concerned, the key question turns out to be whether or not the set $\sigma(A)$ is b-nonresonant. Notice that for $A = [a] \in \mathbb{R}^{1\times 1}$ with $a \neq 0$ the set $\sigma(A) = \{a\}$ is b-nonresonant if and only if $\log_b |a|$ is irrational.

Theorem 3.4. *Let $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{N} \setminus \{1\}$. Then the following are equivalent:*

(i) *For every $x, y \in \mathbb{R}^d$ the sequence $(x^\top A^n y)$ is either b-Benford or terminating;*

(ii) *The set $\sigma(A)$ is b-nonresonant.*

The proof of Theorem 3.4 is facilitated by two simple observations, the first of which is an elementary fact from linear algebra.

Lemma 3.5. *Let $L \in \{1, \ldots, d\}$ and assume $y^{(1)}, \ldots, y^{(L)} \in \mathbb{R}^d$ are linearly independent. Then, given any $u \in \mathbb{R}^L$, there exists $x \in \mathbb{R}^d$ such that $x^\top y^{(\ell)} = u_\ell$ for every $1 \leq \ell \leq L$.*

Proof. The function

$$\Phi : \left\{ \begin{array}{ccc} \mathbb{R}^d & \to & \mathbb{R}^L, \\ x & \mapsto & \sum_{\ell=1}^L (x^\top y^{(\ell)})e^{(\ell)}, \end{array} \right.$$

is linear, and since the (Gram) determinant

$$\det[\Phi(y^{(1)}), \ldots, \Phi(y^{(L)})] = \det[(y^{(\ell)})^\top y^{(k)}]_{\ell,k=1}^L$$

is non-zero, Φ is also onto. $\square$

A second observation clarifies the role of condition (i) in Definition 3.1 and may also be of independent interest. Recall that a set $\mathcal{N} \subset \mathbb{N}$ *has density* if

$$\rho(\mathcal{N}) := \lim_{N\to\infty} \frac{\#\{n \le N : n \in \mathcal{N}\}}{N}$$

exists. In this case, $\rho(\mathcal{N})$ is called the *density* of $\mathcal{N}$. Clearly, $0 \le \rho(\mathcal{N}) \le 1$ whenever $\mathcal{N}$ has density. Not all subsets of $\mathbb{N}$ have density, but those most relevant for Theorem 3.4 do.

Lemma 3.6. *For every $A \in \mathbb{R}^{d\times d}$ and $x, y \in \mathbb{R}^d$, let*

$$\mathcal{N}_{A,x,y} := \{n \in \mathbb{N} : x^\top A^n y = 0\}. \tag{3.4}$$

Then $\mathcal{N}_{A,x,y}$ has density, and $\rho(\mathcal{N}_{A,x,y}) \in \mathbb{Q} \cap [0, 1]$.

Proof. By the Cayley–Hamilton Theorem, there exist $a_1, a_2, \ldots, a_{d-1}, a_d \in \mathbb{R}$ such that

$$A^d = a_1 A^{d-1} + a_2 A_{d-2} + \ldots + a_{d-1} A + a_d I_d .$$

Thus, for every $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} x^\top A^{n+d} y &= x^\top (a_1 A^{n+d-1} + a_2 A^{n+d-2} + \ldots + a_{d-1} A^{n+1} + a_d A^n) y \\ &= a_1 x^\top A^{n+d-1} y + a_2 x^\top A^{n+d-2} y + \ldots + a_{d-1} x^\top A^{n+1} y + a_d x^\top A^n y , \end{aligned}$$

showing that $(x^\top A^n y)$ satisfies a linear d-step recursion relation with constant coefficients. By the Skolem–Mahler–Lech Theorem [31, Thm.A], the set $\mathcal{N}_{A,x,y}$ is the union of a finite (possibly empty) set $\mathcal{N}_0$ and a finite (possibly zero) number of lattices, i.e.

$$\mathcal{N}_{A,x,y} = \mathcal{N}_0 \cup \bigcup\nolimits_{\ell=1}^{L} \{nN_\ell + M_\ell : n \in \mathbb{N}\} , \tag{3.5}$$

where L is a nonnegative integer, and $M_\ell, N_\ell \in \mathbb{N}$ for $1 \le \ell \le L$. From (3.5) it is clear that $\mathcal{N}_{A,x,y}$ has density, and $\rho(\mathcal{N}_{A,x,y})$ is a rational number, in fact $\rho(\mathcal{N}_{A,x,y}) \cdot \mathrm{lcm}\{N_1, \ldots, N_L\}$ is a (nonnegative) integer. □

By using information about $\sigma(A)$, more can be said about the possible values of $\rho(\mathcal{N}_{A,x,y})$ in Lemma 3.6. In order to concisely state the following observation, call a set $\mathcal{N} \subset \mathbb{N}$ *co-finite* if $\mathbb{N} \setminus \mathcal{N}$ is finite. With this, $(x^\top A^n y)$ is terminating precisely if $\mathcal{N}_{A,x,y}$ is co-finite.

Lemma 3.7. *For every $A \in \mathbb{R}^{d\times d}$ the following three statements are equivalent:*

(i) *For every $x, y \in \mathbb{R}^d$ the set $\mathcal{N}_{A,x,y}$ in* (3.4) *is either finite or co-finite;*

(ii) $\rho(\mathcal{N}_{A,x,y}) \in \{0, 1\}$ *for every $x, y \in \mathbb{R}^d$;*

(iii) *For every $r > 0$ either $\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q} = \{1\}$ or $\sigma(A) \cap r\mathbb{S} = \varnothing$.*

Proof. Clearly (i)⇒(ii), because $\rho(\mathcal{N}) = 0$ or $\rho(\mathcal{N}) = 1$ whenever $\mathcal{N}$ is finite or co-finite, respectively.

Next, to establish the implication (ii)⇒(iii), assume (ii) but suppose (iii) did not hold. (Note that this is possible only if $d \ge 2$.) Thus $\#(\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q}) \ge 2$ for some $r > 0$, which in turn entails one of the following three possibilities: Either

$$\text{both } -r \text{ and } r \text{ are eigenvalues of } A, \tag{3.6}$$

or

$$A \text{ has an eigenvalue } \lambda \in \mathbb{C}\setminus\mathbb{R} \text{ with } |\lambda| = r \text{ and } \tfrac{1}{2\pi}\arg\lambda > 0 \text{ rational}, \tag{3.7}$$

or

$$\begin{array}{l} A \text{ has two eigenvalues } \lambda_1,\lambda_2 \in \mathbb{C}\setminus\mathbb{R} \text{ with } |\lambda_1| = |\lambda_2| = r \text{ and } \arg\lambda_1 > \arg\lambda_2 > 0 \\ \text{such that at least one of the two numbers } \frac{1}{2\pi}(\arg\lambda_1 \pm \arg\lambda_2) \text{ is rational.} \end{array} \tag{3.8}$$

Note that these cases are not mutually exclusive, and (3.8) can occur only for $d \ge 4$.

In case (3.6), let $u, v \in \mathbb{R}^d$ be eigenvectors of A corresponding to the eigenvalues $-r, r$, respectively. Pick $x \in \mathbb{R}^d$ such that $x^\top u = x^\top v = 1$. This is possible because u, v are linearly independent; see Lemma 3.5. Then, with $y := u + v$,

$$x^\top A^n y = x^\top\big((-r)^n u + r^n v\big) = r^n\big((-1)^n + 1\big), \quad \forall n \in \mathbb{N},$$

showing that $\mathcal{N}_{A,x,y} = \{2n-1 : n \in \mathbb{N}\}$. Thus $\rho(\mathcal{N}_{A,x,y}) = \frac{1}{2} \notin \{0,1\}$, contradicting (ii).

In case (3.7), let $w \in \mathbb{C}^d$ be an eigenvector of A corresponding to the eigenvalue λ, and observe that, for every $n \in \mathbb{N}$,

$$\begin{aligned} A^n \Re w &= r^n\big(\cos(n\arg\lambda)\Re w - \sin(n\arg\lambda)\Im w\big), \\ A^n \Im w &= r^n\big(\sin(n\arg\lambda)\Re w + \cos(n\arg\lambda)\Im w\big). \end{aligned} \tag{3.9}$$

Again, since $\Re w, \Im w \in \mathbb{R}^d$ are linearly independent, it is possible to choose $x \in \mathbb{R}^d$ such that $x^\top \Re w = 1$ and $x^\top \Im w = 0$. With $y := \Im w$, therefore,

$$x^\top A^n y = r^n \sin(n\arg\lambda), \quad \forall n \in \mathbb{N}.$$

Since $\frac{1}{\pi}\arg\lambda$ is rational and strictly between 0 and 1, the set $\mathcal{N}_{A,x,y}$ equals $N\mathbb{N}$ for some integer $N \ge 2$. Thus $0 < \rho(\mathcal{N}_{A,x,y}) = \frac{1}{N} < 1$, again contradicting (ii).

Lastly, in case (3.8) let $w^{(1)}, w^{(2)} \in \mathbb{C}^d$ be eigenvectors of A corresponding to the eigenvalues λ_1, λ_2, respectively. As seen in (3.9) above, for every $n \in \mathbb{N}$,

$$\begin{aligned} A^n \Re(w^{(1)} + w^{(2)}) = r^n\big(&\cos(n\arg\lambda_1)\Re w^{(1)} - \sin(n\arg\lambda_1)\Im w^{(1)} + \\ &+ \cos(n\arg\lambda_2)\Re w^{(2)} - \sin(n\arg\lambda_2)\Im w^{(2)}\big). \end{aligned}$$

Again, $\Re w^{(1)}, \Im w^{(1)}, \Re w^{(2)}, \Im w^{(2)} \in \mathbb{R}^d$ are linearly independent, and so by Lemma 3.5 it is possible to choose $x \in \mathbb{R}^d$ such that $x^\top \Re w^{(1)} = -1$, $x^\top \Im w^{(1)} = x^\top \Im w^{(2)} = 0$, and $x^\top \Re w^{(2)} = 1$. Then, with $y := \Re(w^{(1)} + w^{(2)})$,

$$\begin{aligned} x^\top A^n y &= r^n \big(\cos(n \arg \lambda_2) - \cos(n \arg \lambda_1)\big) \\ &= 2r^n \sin\left(\pi n \frac{\arg \lambda_1 - \arg \lambda_2}{2\pi}\right) \sin\left(\pi n \frac{\arg \lambda_1 + \arg \lambda_2}{2\pi}\right) . \end{aligned}$$

Since both numbers $\frac{1}{2\pi}(\arg \lambda_1 \pm \arg \lambda_2)$ are strictly between 0 and 1 and at least one of them is rational, the set $\mathcal{N}_{A,x,y}$ once more has a rational density that equals neither 0 nor 1: From $\mathcal{N}_{A,x,y} = N_1\mathbb{N} \cup N_2\mathbb{N}$ with two (not necessarily different) integers $N_1, N_2 \geq 2$, it follows that

$$0 < \frac{1}{\min\{N_1, N_2\}} \leq \rho(\mathcal{N}_{A,x,y}) \leq 1 - \frac{1}{\operatorname{lcm}\{N_1, N_2\}} < 1 .$$

Once again this contradicts (ii) and hence completes the proof that indeed (ii)⇒(iii).

Finally, to show that (iii)⇒(i), denote the "upper half" of $\sigma(A)$ by

$$\sigma^+(A) := \{\lambda \in \sigma(A) : \Im\lambda \geq 0\} \setminus \{0\} .$$

Note that $\sigma^+(A) = \varnothing$ if and only if A is nilpotent, in which case clearly $\mathcal{N}_{A,x,y}$ is co-finite for all $x, y, \in \mathbb{R}^d$. From now on, therefore, assume that $\sigma^+(A) \neq \varnothing$. Recall that A^n can be written in the form

$$A^n = \Re\left(\sum\nolimits_{\lambda \in \sigma^+(A)} P_\lambda(n)\lambda^n\right) , \quad \forall n \geq d , \tag{3.10}$$

where P_λ is, for every $\lambda \in \sigma^+(A)$, a (possibly non-real) matrix-valued polynomial of degree at most $d-1$, i.e. $P_\lambda \in \mathbb{C}^{d\times d}$, and for all $j, k \in \{1, \ldots, d\}$ the entry $[P_\lambda]_{jk} = (e^{(j)})^\top P_\lambda e^{(k)}$ is a complex polynomial in n of degree at most $d-1$. Moreover, P_λ is real, i.e. $P_\lambda \in \mathbb{R}^{d\times d}$, whenever $\lambda \in \mathbb{R}$. The representation (3.10) follows for instance from the Jordan Normal Form Theorem. Deduce from (3.10) that

$$x^\top A^n y = \Re\left(\sum\nolimits_{\lambda \in \sigma^+(A)} x^\top P_\lambda(n) y \lambda^n\right) =: \Re\left(\sum\nolimits_{\lambda \in \sigma^+(A)} p_\lambda(n)\lambda^n\right) , \quad \forall n \geq d , \tag{3.11}$$

with $p_\lambda = x^\top P_\lambda y$ being, for every $\lambda \in \sigma^+(A)$, a (possibly non-real) polynomial in n of degree at most $d-1$. Clearly, if $p_\lambda = 0$ for every $\lambda \in \sigma^+(A)$ then $\mathcal{N}_{A,x,y}$ is co-finite. From now on, therefore, assume that $p_\lambda \neq 0$ for at least one $\lambda \in \sigma^+(A)$, i.e.

$$r := \max\{|\lambda| : \lambda \in \sigma^+(A), p_\lambda \neq 0\} > 0 .$$

Denote by $k \in \mathbb{N}_0$ the maximal degree of the polynomials p_λ for which $|\lambda| = r$, i.e., let $k = \max\{\deg p_\lambda : \lambda \in \sigma^+(A), |\lambda| = r\}$, and consider the (non-empty) subset σ^{++} of $\sigma^+(A)$ given by

$$\sigma^{++} = \{\lambda \in \sigma^+(A) : |\lambda| = r, \deg p_\lambda = k\} .$$

Note that $c_\lambda := \lim_{n\to\infty} p_\lambda(n)/n^k$ exists for every $\lambda \in \sigma^{++}$ and is non-zero. With this, it follows from (3.11) that

$$x^\top A^n y = r^n n^k \Re\left(\sum\nolimits_{\lambda\in\sigma^+(A)} \frac{p_\lambda(n)}{n^k}\left(\frac{\lambda}{r}\right)^n\right) = r^n n^k \Re\left(\sum\nolimits_{\lambda\in\sigma^{++}} c_\lambda e^{\imath n \arg\lambda} + z_n\right), \tag{3.12}$$

where (nz_n) is a bounded sequence in $\mathbb{C}$. Assume now that (iii) holds but suppose $\mathcal{N}_{A,x,y}$ was infinite. Then, by (3.5),

$$\mathcal{N}_{A,x,y} \supset \{nN + M : n \in \mathbb{N}\}, \tag{3.13}$$

with the appropriate $M, N \in \mathbb{N}$. Since $\lim_{n\to\infty} z_n = 0$, it follows from (3.12) and (3.13) that

$$\lim\nolimits_{n\to\infty} \Re\left(\sum\nolimits_{\lambda\in\sigma^{++}} c_\lambda e^{\imath M\arg\lambda}\left(e^{\imath N\arg\lambda}\right)^n\right) = 0. \tag{3.14}$$

Since $c_\lambda e^{\imath M\arg\lambda} \neq 0$ for every $\lambda \in \sigma^{++}$, Lemma A.3 implies that either $e^{\imath N\arg\lambda_1} = e^{\pm\imath N\arg\lambda_2}$ for some $\lambda_1, \lambda_2 \in \sigma^{++}$ with $\lambda_1 \neq \lambda_2$, or else $e^{\imath N\arg\lambda_1} = \pm 1$ for some $\lambda_1 \in \sigma^{++}$. In the former case, at least one of the two numbers $\frac{N}{2\pi}(\arg\lambda_1 \pm \arg\lambda_2)$ is a non-zero integer, which in turn shows that $\#(\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q}) \geq 2$ and hence contradicts the assumed validity of (iii). In the latter case, note first that $\#\sigma^{++} \geq 2$ because otherwise either $c_{\lambda_1} = 0$ (if $\sigma^{++} = \{\lambda_1\} \subset \mathbb{R}$), which is impossible by the very definition of σ^{++}, or else $\frac{N}{\pi}\arg\lambda_1$ is a non-zero integer (if $\sigma^{++} = \{\lambda_1\} \subset \mathbb{C}\backslash\mathbb{R}$), which again contradicts (iii). But $e^{\imath N\arg\lambda_1} = \pm 1$, together with $\#\sigma^{++} \geq 2$ and (3.14), leads to

$$\lim\nolimits_{n\to\infty} \Re\left(\sum\nolimits_{\lambda\in\sigma^{++}\backslash\{\lambda_1\}} c_\lambda e^{\imath M\arg\lambda}\left(e^{2\imath N\arg\lambda}\right)^n\right) = -\Re\left(c_{\lambda_1} e^{\imath M\arg\lambda_1}\right),$$

and hence by Lemma A.3 either $e^{2\imath N\arg\lambda_2} = e^{\pm 2\imath N\arg\lambda_3}$ for some $\lambda_2, \lambda_3 \in \sigma^{++} \setminus \{\lambda_1\}$ with $\lambda_2 \neq \lambda_3$, or else $e^{2\imath N\arg\lambda_2} = \pm 1$ for some $\lambda_2 \in \sigma^{++} \setminus \{\lambda_1\}$. As before, in the former case at least one of the two numbers $\frac{N}{\pi}(\arg\lambda_2 \pm \arg\lambda_3)$ is a non-zero integer, contradicting (iii) again. Similarly, in the latter case, $\frac{N}{\pi}\arg\lambda_1$ and $\frac{2N}{\pi}\arg\lambda_2$ are both integers, hence $\frac{1}{2\pi}(\arg\lambda_1 - \arg\lambda_2)$ is rational and non-zero, and this once more violates (iii). In summary, if (iii) holds then the set $\mathcal{N}_{A,x,y}$ is necessarily finite whenever $p_\lambda \neq 0$ for at least one $\lambda \in \sigma^+(A)$, and, as seen earlier, it is co-finite otherwise. Thus (iii)$\Rightarrow$(i), and the proof is complete. $\square$

Proof of Theorem 3.4: To prove (i)$\Rightarrow$(ii), assume $\sigma(A)$ is b-resonant. Then, for some $r > 0$, either $\#(\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q}) \geq 2$ or $\log_b r \in \operatorname{span}_{\mathbb{Q}} \Delta_{\sigma(A)\cap r\mathbb{S}}$, or both. In the former case, Lemma 3.7 guarantees the existence of $x, y \in \mathbb{R}^d$ for which $0 < \rho(\mathcal{N}_{A,x,y}) < 1$ and hence $(x^\top A^n y)$ is neither b-Benford nor terminating. As this clearly contradicts (i), it only remains to consider the case where $\#(\Delta_{\sigma(A)\cap r\mathbb{S}} \cap \mathbb{Q}) \leq 1$ for every $r > 0$ yet $\log_b r_0 \in \operatorname{span}_{\mathbb{Q}} \Delta_{\sigma(A)\cap r_0\mathbb{S}}$ for some $r_0 > 0$. Label the elements of $\sigma(A) \cap r_0\mathbb{S}$ as $\lambda_1, \ldots, \lambda_L$. Since $\overline{\sigma(A)} = \sigma(A)$,

$$\log_b r_0 \in \operatorname{span}_{\mathbb{Q}} \Delta_{\sigma(A)\cap r_0\mathbb{S}} = \operatorname{span}_{\mathbb{Q}}\left(\{1\} \cup \left\{\frac{\arg\lambda_\ell}{2\pi} : 1 \leq \ell \leq L\right\}\right);$$

see Remark 3.3(i). Let L_0+1 be the dimension (over $\mathbb{Q}$) of $\text{span}_{\mathbb{Q}}\Delta_{\sigma(A)\cap r_0\mathbb{S}}$. Hence $L_0 \le L$, and $L_0 \in \mathbb{N}$ unless $\frac{1}{2\pi}\arg\lambda_\ell$ is rational for every $1 \le \ell \le L$, in which case $L_0 = 0$. (For instance, the latter inevitably occurs if $d=1$.)

First consider the case of $L_0 = 0$. Here, $\log_b r_0$ and $\frac{1}{2\pi}\arg\lambda_1$ are both rational, and in fact $\lambda_1 \in \mathbb{R}$ because otherwise $\#(\Delta_{\sigma(A)\cap r_0\mathbb{S}} \cap \mathbb{Q}) \ge 2$. But then taking x to be any eigenvector of A corresponding to the eigenvalue λ_1 yields

$$\log_b |x^\top A^n x| = \log_b (r_0^n |x|^2) = n \log_b r_0 + 2\log_b |x| ,$$

which is periodic modulo one. Hence $(x^\top A^n x)$ is neither b-Benford nor terminating, a fact obviously contradicting (i).

Assume from now on that $L_0 \ge 1$. In this case, by re-labelling the eigenvalues $\lambda_1, \ldots, \lambda_L$, it can be assumed that $1, \frac{1}{2\pi}\arg\lambda_1, \ldots, \frac{1}{2\pi}\arg\lambda_{L_0}$ are $\mathbb{Q}$-independent, and so

$$\log_b r_0 = \frac{p_0}{q} + \frac{p_1}{q}\frac{\arg\lambda_1}{2\pi} + \ldots + \frac{p_{L_0}}{q}\frac{\arg\lambda_{L_0}}{2\pi} , \tag{3.15}$$

with the appropriate $p_0, p_1, \ldots, p_{L_0} \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $w^{(1)}, \ldots, w^{(L_0)} \in \mathbb{C}^d$ be eigenvectors of A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{L_0}$, respectively. Note that $\lambda_1, \ldots, \lambda_{L_0}$ are all non-real, and consequently the $2L_0$ vectors $\Re w^{(1)}, \Im w^{(1)}, \ldots, \Re w^{(L_0)}, \Im w^{(L_0)}$ are linearly independent. Lemma 3.5 guarantees that, given any $u \in \mathbb{R}^{L_0}$, it is possible to pick $x \in \mathbb{R}^d$ such that $x^\top \Re w^{(\ell)} = u_\ell$ and $x^\top \Im w^{(\ell)} = 0$ for all $1 \le \ell \le L_0$. With $y := \Re(w^{(1)} + \ldots + w^{(L_0)})$, therefore,

$$x^\top A^n y = r_0^n \big(u_1 \cos(n\arg\lambda_1) + \ldots + u_{L_0}\cos(n\arg\lambda_{L_0})\big) , \quad \forall n \in \mathbb{N} .$$

As $(x^\top A^n y)$ is not terminating whenever $u \ne 0 \in \mathbb{R}^{L_0}$, Lemma 3.7 shows that $x^\top A^n y \ne 0$ for all sufficiently large n, and (3.15) leads to

$$\begin{aligned} q\log_b |x^\top A^n y| = p_0 n + p_1 n \frac{\arg\lambda_1}{2\pi} + \ldots + p_{L_0} n \frac{\arg\lambda_{L_0}}{2\pi} + \\ + \frac{q}{\ln b}\ln\left| u_1 \cos\left(2\pi n \frac{\arg\lambda_1}{2\pi}\right) + \ldots + u_{L_0}\cos\left(2\pi n\frac{\arg\lambda_{L_0}}{2\pi}\right)\right| . \end{aligned}$$

Since $1, \frac{1}{2\pi}\arg\lambda_1, \ldots, \frac{1}{2\pi}\arg\lambda_{L_0}$ are $\mathbb{Q}$-independent, by Lemma 2.7 one can specifically choose $u \in \mathbb{R}^{L_0}$ such that $(q\log_b|x^\top A^n y|)$ is not u.d. mod 1, and hence $(\log_b|x^\top A^n y|)$ is not u.d. mod 1 either, by Lemma 2.3(iv). Thus $(x^\top A^n y)$ is neither Benford nor terminating, a fact once again contradicting (i). Overall, therefore, (i)$\Rightarrow$(ii), as claimed.

To prove the reverse implication (ii)$\Rightarrow$(i), let $\sigma(A)$ be b-nonresonant. Given $x, y \in \mathbb{R}^d$, deduce from (3.11) that $(x^\top A^n y)$ is either terminating, or else

$$x^\top A^n y = |\lambda_1|^n n^k \Re\left(c_1 e^{\imath n \arg\lambda_1} + \ldots + c_L e^{\imath n\arg\lambda_L} + z_n\right) , \quad \forall n \in \mathbb{N} , \tag{3.16}$$

where $k \in \mathbb{N}_0$ and $L \in \mathbb{N}$; the numbers $\lambda_1, \ldots, \lambda_L$ are appropriate (different) eigenvalues of A with $|\lambda_1| = \ldots = |\lambda_L| > 0$ and $\Im\lambda_\ell \ge 0$ for all $1 \le \ell \le L$; the numbers

$c_1, \ldots, c_L \in \mathbb{C}$ are all non-zero; and (nz_n) is a bounded sequence in $\mathbb{C}$. By the assumption of $\sigma(A)$ being b-nonresonant,

$$\log_b |\lambda_1| \notin \mathrm{span}_{\mathbb{Q}} \Delta_{\sigma(A)\cap|\lambda_1|\mathbb{S}} \supset \mathrm{span}_{\mathbb{Q}} \left(\{1\} \cup \left\{ \frac{\arg \lambda_\ell}{2\pi} : 1 \le \ell \le L \right\} \right) .$$

As before, let $L_0 + 1$ be the dimension of $\mathrm{span}_{\mathbb{Q}} \left(\{1\} \cup \{\frac{1}{2\pi} \arg \lambda_\ell : 1 \le \ell \le L\}\right)$, and consider first the case of $L_0 = 0$, that is, $\frac{1}{2\pi} \arg \lambda_\ell$ is rational for every $1 \le \ell \le L$. As $\sigma(A)$ would be b-resonant otherwise, this implies that $L = 1$ and $\lambda_1 \in \mathbb{R}$. Since λ_1 is real, so is c_1, and for all $n \in \mathbb{N}$,

$$|x^\top A^n y| = |\lambda_1|^n n^k \left| \Re \left(c_1 e^{\imath n \arg \lambda_1} + z_n \right) \right| = |\lambda_1|^n n^k |c_1| \left| 1 + c_1^{-1} e^{-\imath n \arg \lambda_1} \Re z_n \right| .$$

For all sufficiently large n, therefore,

$$\log_b |x^\top A^n y| = n \log_b |\lambda_1| + \frac{k}{\ln b} \ln n + \log_b |c_1| + \log_b \left| 1 + c_1^{-1} e^{-\imath n \arg \lambda_1} \Re z_n \right|$$

and since $\log_b |\lambda_1|$ is irrational, Lemmas 2.3 and 2.5 imply that $(x^\top A^n y)$ is b-Benford.

It remains to consider the case of $L_0 \ge 1$. In this case, assume w.l.o.g. that $1, \frac{1}{2\pi} \arg \lambda_1, \ldots, \frac{1}{2\pi} \arg \lambda_{L_0}$ are $\mathbb{Q}$-independent. Hence there exists $q \in \mathbb{N}$ and, for every $\ell \in \{L_0 + 1, \ldots, L\}$, an integer $p_{0\ell}$ as well as a vector $p^{(\ell)} \in \mathbb{Z}^{L_0}$ such that

$$\frac{\arg \lambda_\ell}{2\pi} = \frac{p_{0\ell}}{q} + \frac{p_1^{(\ell)}}{q} \frac{\arg \lambda_1}{2\pi} + \ldots + \frac{p_{L_0}^{(\ell)}}{q} \frac{\arg \lambda_{L_0}}{2\pi} , \quad \forall \ell \in \{L_0 + 1, \ldots, L\} . \tag{3.17}$$

Note that $p^{(\ell)} = 0 \in \mathbb{Z}^{L_0}$ for at most one ℓ, and the $2L - L_0$ vectors

$$qe^{(1)}, \ldots, qe^{(L_0)}, \pm p^{(L_0+1)}, \ldots, \pm p^{(L)} \in \mathbb{Z}^{L_0}$$

are all different because otherwise $\sigma(A)$ would be b-resonant. As a consequence, for every $w \in \mathbb{C}^L$ the multi-variate trigonometric polynomial $f_w : \mathbb{T}^{L_0} \to \mathbb{R}$ given by

$$f_w(t) = \Re \left(\sum\nolimits_{\ell=1}^{L_0} w_\ell e^{2\pi \imath q t_\ell} + \sum\nolimits_{\ell=L_0+1}^{L} w_\ell e^{2\pi \imath t^\top p^{(\ell)}} \right)$$

is non-constant, and so $f_w(t) \ne 0$ for $\lambda_{\mathbb{T}^{L_0}}$-almost all $t \in \mathbb{T}^{L_0}$, provided that at least one of the L_0 numbers $w_1, \ldots, w_{L_0}$ is non-zero.

Fix now any $m \in \{1, \ldots, q\}$ and deduce from (3.16) and (3.17) that

$$\begin{aligned} x^\top A^{nq+m} y &= |\lambda_1|^{nq+m} (nq+m)^k \Re \left(\sum\nolimits_{\ell=1}^{L} c_\ell e^{\imath (nq+m) \arg \lambda_\ell} + z_{nq+m} \right) \\ &= |\lambda_1|^{nq} n^k |\lambda_1|^m \left(q + \frac{m}{n} \right)^k \Re \left(\sum\nolimits_{\ell=1}^{L_0} c_\ell e^{\imath m \arg \lambda_\ell} e^{\imath n q \arg \lambda_\ell} + \right. \\ &\qquad \left. + \sum\nolimits_{\ell=L_0+1}^{L} c_\ell e^{\imath m \arg \lambda_\ell} \prod\nolimits_{k=1}^{L_0} e^{\imath n p_k^{(\ell)} \arg \lambda_k} + z_{nq+m} \right) \\ &= |\lambda_1|^{nq} n^k |\lambda_1|^m \left(q + \frac{m}{n} \right)^k \left(f_w \left(n \frac{\arg \lambda_1}{2\pi}, \ldots, n \frac{\arg \lambda_{L_0}}{2\pi} \right) + \Re z_{nq+m} \right) , \end{aligned}$$

where $w \in \mathbb{C}^L$ is given by $w_\ell = c_\ell e^{\imath m \arg \lambda_\ell} \neq 0$ for all $\ell \in \{1, \ldots, L\}$. Recall that by assumption the $L_0 + 2$ numbers $1, q \log_b |\lambda_1|, \frac{1}{2\pi} \arg \lambda_1, \ldots, \frac{1}{2\pi} \arg \lambda_{L_0}$ are $\mathbb{Q}$-independent. Since $\lim_{n\to\infty} z_{nq+m} = 0$ as well, Lemma 2.3 and 2.6 applied to

$$\log_b |x^\top A^{nq+m} y| = nq \log_b |\lambda_1| + \frac{k}{\ln b} \ln n + m \log_b |\lambda_1| + k \log_b \left(q + \frac{m}{n}\right) + \\ + \frac{1}{\ln b} \ln \left| f_w \left(n \frac{\arg \lambda_1}{2\pi}, \ldots, n \frac{\arg \lambda_{L_0}}{2\pi} \right) + \Re z_{nq+m} \right|$$

show that $(\log_b |x^\top A^{nq+m} y|)$ is u.d. mod 1. As $m \in \{1, \ldots, q\}$ was arbitrary, $(\log_b |x^\top A^n y|)$ is u.d. mod 1, by Lemma 2.4, i.e., $(x^\top A^n y)$ is b-Benford. In summary, therefore, (ii)⇒(i), and the proof is complete. □

Remark 3.8. For *invertible* A the important formula (3.10) holds for all $n \in \mathbb{N}$. In this case, "terminating" in Theorem 3.4(i) can be replaced by "identically zero"; see also Corollary 3.12 below.

Example 3.9. (i) The spectrum of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is $\sigma(A) = \{\varphi, -\varphi^{-1}\}$ with $\varphi = \frac{1}{2}(1+\sqrt{5})$. Since A is invertible and $\log_b \varphi$ is irrational (in fact, transcendental) for every $b \in \mathbb{N} \setminus \{1\}$, the sequence $(x^\top A^n y)$ is, for every $x, y \in \mathbb{R}^2$, either Benford or identically zero. The latter alternative occurs if and only if x and y are multiples of the (orthogonal) eigenvectors corresponding, respectively, to the eigenvalues φ and $-\varphi^{-1}$, or vice versa.

(ii) Consider the (integer) 3×3-matrix

$$B = \begin{bmatrix} -3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 6 \end{bmatrix},$$

the characteristic polynomial of which is

$$\chi_B(\lambda) = \det(B - \lambda I_3) = -\lambda^3 + 3\lambda^2 + 20\lambda - 3 .$$

Since B is symmetric, all three eigenvalues of B are real, and from $\chi_B(0) < 0 < \chi_B(1)$ it is clear that they are all different. They also have different absolute values. To show that $\sigma(B)$ is b-nonresonant for every $b \in \mathbb{N} \setminus \{1\}$, assume that $|\lambda| = b^{p/q}$ for some $\lambda \in \sigma(B)$ and relatively prime $p \in \mathbb{Z} \setminus \{0\}$, $q \in \mathbb{N}$. If $p > 0$ then b^p is an eigenvalue of B^q or $-B^q$ and hence divides $|\det B^q| = 3^q$. This is only possible if $b = 3^N$ for some $N \in \mathbb{N}$. Similarly, if $p < 0$ then $3^q b^{|p|}$ is an eigenvalue of one of the two integer matrices $\pm(3B^{-1})^q$ and hence divides $|\det(3B^{-1})^q| = 3^{2q}$. Again, this leaves only the possibility of $b = 3^N$ for some $N \in \mathbb{N}$. To analyse the latter, assume now that $|\lambda| = 3^{p/q}$ with relatively prime $p \in \mathbb{Z} \setminus \{0\}$, $q \in \mathbb{N}$, possibly different from before. Consider first the case of $p > 0$. In this case, λ is a root of one of the two irreducible polynomials $\lambda^q \pm 3^p$ which in turn is a factor of χ_B. Thus $q \leq 3$, and since 3^p is an eigenvalue of one of the two matrices $\pm B^q$, it follows that $p \leq q$.

It can now be checked easily, e.g. by computing χ_{B^2} and χ_{B^3}, or by means of row reductions, that none of the four numbers $\pm 3, \pm 3^2$ is an eigenvalue of any of the three matrices B, B^2, B^3. The possibility of $|\lambda| = 3^{p/q}$ with $p < 0$ is ruled out in a completely similar manner. In summary, $\log_b |\lambda|$ is irrational for every $\lambda \in \sigma(B)$ and every $b \in \mathbb{N} \setminus \{1\}$, and $\sigma(B)$ is b-nonresonant. (Note that in order to draw this conclusion, it is not necessary to explicitly know any eigenvalue of B.) By Theorem 3.4 and Remark 3.8, the sequence $(x^\top B^n y)$ is, for every $x, y \in \mathbb{R}^3$, either Benford or identically zero. As in (i), the latter case occurs precisely if x and y (or vice versa) are, respectively, proportional and orthogonal to the *same* eigenvector of B.

(iii) For the (invertible) matrix $C = \frac{1}{2}\begin{bmatrix} 1+\pi & 1-\pi \\ 1-\pi & 1+\pi \end{bmatrix}$ the spectrum $\sigma(C) = \{1, \pi\}$ is b-resonant for every $b \in \mathbb{N} \setminus \{1\}$. By Theorem 3.4, there exist $x, y \in \mathbb{R}^2$ for which $(x^\top C^n y)$ is neither b-Benford nor identically zero. Indeed, with $x = y = e^{(1)} + e^{(2)}$, for instance, $x^\top C^n y \equiv 2$. Similarly, $|C^n x| \equiv \sqrt{2}$, so $(|C^n x|)$ as well is neither b-Benford nor trivial. On the other hand, $(|C^n|) = (\pi^n)$ is Benford. Theorems 3.10 and 3.11 below relate these two simple observations to the fact that $\sigma(C^n) = \{1, \pi^n\}$ is b-resonant for every $n \in \mathbb{N}$, whereas $\sigma(C) \cap r_\sigma(C)\mathbb{S} = \{\pi\}$ is not.

In addition to sequences of the form $(x^\top A^n y)$ in Theorem 3.4(i), which may be thought of as *linear* observables of the process (A^n), some *non-linear* observables may also be of interest. The next theorem establishes the Benford property specifically for $(|A^n x|)$ with $x \in \mathbb{R}^d$. For the formulation of the result, note that if $\mathcal{Z} \subset \mathbb{C}$ is b-nonresonant then so is $\mathcal{Z}^n := \{z^n : z \in \mathcal{Z}\}$ for every $n \in \mathbb{N}$. The converse does not hold in general (unless $\#\mathcal{Z} \le 1$), as the example of the b-resonant set $\mathcal{Z} = \{-\pi, \pi\}$ shows, for which $\mathcal{Z}^2 = \{\pi^2\}$ is b-nonresonant. Furthermore, this example illustrates the easily established fact that $\mathcal{Z} \subset r\mathbb{S}$ satisfies (ii) of Definition 3.1 if and only if $\mathcal{Z}^N$ is b-nonresonant for some $N \in \mathbb{N}$. Also, recall that $\sigma(A^n) = \sigma(A)^n$ for every $A \in \mathbb{R}^{d\times d}$ and $n \in \mathbb{N}$.

Theorem 3.10. *Let $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{N} \setminus \{1\}$. If $\sigma(A^N)$ is b-nonresonant for some $N \in \mathbb{N}$ then, for every $x \in \mathbb{R}^d$, the sequence $(|A^n x|)$ is either b-Benford or terminating.*

Proof. Assume that $\sigma(A^N)$ is b-nonresonant and, as in the proof of Lemma 3.7, consider the set $\sigma^+(A) = \{\lambda \in \sigma(A) : \Im\lambda \ge 0\} \setminus \{0\}$. If $\sigma^+(A) = \varnothing$ then A is nilpotent, $A^n x = 0$ for all $n \ge d$, and $(|A^n x|)$ is terminating. From now on, therefore, assume that $\sigma^+(A) \ne \varnothing$, and hence also $\sigma^+(A^N) \ne \varnothing$. Fix any $m \in \{1, \dots, N\}$. Given $x \in \mathbb{R}^d$, deduce from (3.10) with A replaced by A^N that

$$A^{nN+m}x = \Re\left(\sum\nolimits_{\lambda\in\sigma^+(A^N)} P_\lambda(n) A^m x \lambda^n\right) =: \Re\left(\sum\nolimits_{\lambda\in\sigma^+(A^N)} q_\lambda(n)\lambda^n\right), \tag{3.18}$$

for all $n \ge d$, where each q_λ now is a (possibly non-real) vector-valued polynomial of degree at most $d-1$, i.e., $q_\lambda(n) \in \mathbb{C}^d$, and every component of q_λ is a polynomial in n of degree no larger than $d-1$. By the identical reasoning as in the proof of the (iii)⇒(i) part in Lemma 3.7, deduce from (3.18) that either $A^{nN+m}x = 0$ for all $n \ge d$, in which case $(|A^n x|)$ is terminating, or else, with the appropriate non-empty set $\sigma^{++} \subset \sigma^+(A^N)$ and $c_\lambda \in \mathbb{C}^d \setminus \{0\}$ for every $\lambda \in \sigma^{++}$,

$$A^{nN+m}x = r^n n^k \left(\Re \left(\sum\nolimits_{\lambda \in \sigma^{++}} c_\lambda e^{\imath n \arg \lambda} \right) + u_n \right), \tag{3.19}$$

where $r > 0$, $k \in \mathbb{N}_0$, and (u_n) is a sequence in $\mathbb{R}^d$ for which $(n|u_n|)$ is bounded. (Note that σ^{++}, and hence r, k, c_λ and (u_n) as well, may depend on x and m.) Since $\sigma(A^N)$ is b-nonresonant,

$$\log_b r \notin \operatorname{span}_{\mathbb{Q}} \Delta_{\sigma(A^N) \cap r\mathbb{S}} \supset \operatorname{span}_{\mathbb{Q}} \left(\{1\} \cup \left\{ \frac{\arg \lambda}{2\pi} : \lambda \in \sigma^{++} \right\} \right).$$

The argument now proceeds as in the proof of Theorem 3.4: Let $L_0 + 1$ be the dimension of $\operatorname{span}_{\mathbb{Q}} \left(\{1\} \cup \left\{ \frac{1}{2\pi} \arg \lambda : \lambda \in \sigma^{++} \right\} \right)$. If $L_0 = 0$ then $\sigma^{++} = \{\lambda_1\}$ for some $\lambda_1 \in \mathbb{R} \setminus \{0\}$. (Otherwise $\sigma(A^N)$ would be b-resonant.) In this case, c_{λ_1} is real as well, i.e. $c_{\lambda_1} \in \mathbb{R}^d$, and (3.19) implies

$$\log_b |A^{nN+m}x| = n \log_b r + \frac{k}{\ln b} \ln n + \log_b |c_{\lambda_1} + e^{-\imath n \arg \lambda_1} u_n|, \quad \forall n \ge d.$$

Since $\log_b r$ is irrational, $(\log_b |A^{nN+m}x|)$ is u.d. mod 1 by Lemma 2.5. The same argument can be applied for every $m \in \{1, \ldots, N\}$, and so $(\log_b |A^n x|)$ is u.d. mod 1 as well, by Lemma 2.4. In other words, $(|A^n x|)$ is b-Benford.

Consider in turn the case of $L_0 \ge 1$. Label the elements of σ^{++} as $\lambda_1, \ldots, \lambda_L$ and assume w.l.o.g. that $1, \frac{1}{2\pi} \arg \lambda_1, \ldots, \frac{1}{2\pi} \arg \lambda_{L_0}$ are $\mathbb{Q}$-independent. With the same notation as in (3.17), and given any vectors $w^{(1)}, \ldots, w^{(L)} \in \mathbb{C}^d$, the vector-valued trigonometric polynomial $f : \mathbb{T}^{L_0} \to \mathbb{R}^d$ given by

$$f_{w^{(1)},\ldots,w^{(L)}}(t) = \Re \left(\sum\nolimits_{\ell=1}^{L_0} w^{(\ell)} e^{2\pi \imath q t_\ell} + \sum\nolimits_{\ell=L_0+1}^{L} w^{(\ell)} e^{2\pi \imath t^\top p^{(\ell)}} \right)$$

is non-constant, provided that $w^{(\ell)} \neq 0$ for at least one $\ell \in \{1, \ldots, L_0\}$. In this case, $f_{w^{(1)},\ldots,w^{(L)}}(t) \neq 0$, and hence also $|f_{w^{(1)},\ldots,w^{(L)}}(t)| \neq 0$ for $\lambda_{\mathbb{T}^{L_0}}$-almost all $t \in \mathbb{T}^{L_0}$. Note that $|f_{w^{(1)},\ldots,w^{(L)}}| : \mathbb{T}^{L_0} \to \mathbb{R}$ is continuous. Fix now any $l \in \{1, \ldots, q\}$, and deduce from (3.19) that

$$\begin{aligned} A^{(nq+l)N+m}x &= r^{nq} n^k r^l \left(q + \frac{l}{n} \right)^k \Bigg(\Re \Bigg(\sum\nolimits_{\ell=1}^{L_0} c_{\lambda_\ell} e^{\imath l \arg \lambda_\ell} e^{\imath n q \arg \lambda_\ell} + \\ &\qquad + \sum\nolimits_{\ell=L_0+1}^{L} c_{\lambda_\ell} e^{\imath l \arg \lambda_\ell} \prod\nolimits_{\nu=1}^{L_0} e^{\imath n p_\nu^{(\ell)} \arg \lambda_\nu} \Bigg) + u_{nq+l} \Bigg) \\ &= r^{nq} n^k r^l \left(q + \frac{l}{n} \right)^k \left(f_{w^{(1)},\ldots,w^{(L)}} \left(n \frac{\arg \lambda_1}{2\pi}, \ldots, n \frac{\arg \lambda_{L_0}}{2\pi} \right) + u_{nq+l} \right), \end{aligned}$$

with $w^{(\ell)} = c_{\lambda_\ell} e^{\imath l \arg \lambda_\ell} \in \mathbb{C}^d \setminus \{0\}$ for every $\ell \in \{1, \ldots, L\}$. It follows that

$$|A^{(nq+l)N+m}x| = r^{nq} n^k r^l \left(q + \frac{l}{n} \right)^k \left| \left| f_{w^{(1)},\ldots,w^{(L)}} \left(n \frac{\arg \lambda_1}{2\pi}, \ldots, n \frac{\arg \lambda_{L_0}}{2\pi} \right) \right| + z_n \right|,$$

where the (real) sequence (z_n) is given by

$$z_n = \left| f_{w^{(1)},\ldots,w^{(L)}}\left(n\frac{\arg\lambda_1}{2\pi}, \ldots, n\frac{\arg\lambda_{L_0}}{2\pi} \right) + u_{nq+l} \right| \\ - \left| f_{w^{(1)},\ldots,w^{(L)}}\left(n\frac{\arg\lambda_1}{2\pi}, \ldots, n\frac{\arg\lambda_{L_0}}{2\pi} \right) \right| .$$

Clearly, $|z_n| \le |u_{nq+l}|$, and so $\lim_{n\to\infty} z_n = 0$. Lemmas 2.3 and 2.6 now show that $(\log_b |A^{(nq+l)N+m}x|)$ is u.d. mod 1. Since the number $l \in \{1, \ldots q\}$ was arbitrary, $(\log_b |A^{nN+m}x|)$ is u.d. mod 1 as well, by Lemma 2.4. Moreover, the same argument can be applied for every $m \in \{1, \ldots, N\}$, hence $(\log_b |A^n x|)$, too, is u.d. mod 1, i.e., $(|A^n x|)$ is b-Benford. □

In analogy to Theorem 3.10, the next result adresses the b-Benford property of the sequence $(|A^n|)$. For a concise statement, the following terminology is useful. Given any eigenvalue λ of $A \in \mathbb{R}^{d\times d}$, let $k(\lambda) \in \{0, \ldots, d-1\}$ be the largest integer for which

$$\operatorname{rank}(A - \lambda I_d)^{k+1} < \operatorname{rank}(A - \lambda I_d)^k \quad \text{if } \lambda \in \mathbb{R} ,$$

and

$$\operatorname{rank}(A^2 - 2\Re\lambda A + |\lambda|^2 I_d)^{k+1} < \operatorname{rank}(A^2 - 2\Re\lambda A + |\lambda|^2 I_d)^k \quad \text{if } \lambda \in \mathbb{C} \setminus \mathbb{R} .$$

Equivalently, $k(\lambda)+1$ is the size of the largest block associated with the eigenvalue λ in the Jordan Normal Form (over $\mathbb{C}$) of A. With this, define the *extremal peripheral spectrum* of A, henceforth denoted $\sigma_{EP}(A)$, to be the set

$$\sigma_{EP}(A) = \left\{ \lambda \in \sigma(A) \cap r_\sigma(A)\mathbb{S} : k(\lambda) = k_{\max} \right\} , \tag{3.20}$$

where $k_{\max} = k_{\max}(A) = \max\{k(\lambda) : \lambda \in \sigma(A) \cap r_\sigma(A)\mathbb{S}\}$. Clearly $\sigma_{EP}(A) \subset \sigma(A)$, and just as $\sigma(A)$, the set $\sigma_{EP}(A)$ is non-empty and symmetric w.r.t. the real axis. Also, $\sigma_{EP}(A^n) = \sigma_{EP}(A)^n$ for every $n \in \mathbb{N}$.

Theorem 3.11. *Let $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{N} \setminus \{1\}$. If $\sigma_{EP}(A^N)$ is b-nonresonant for some $N \in \mathbb{N}$ then either $(|A^n|)$ is b-Benford or A is nilpotent.*

Proof. Clearly, $(|A^n|)$ is terminating if and only if A is nilpotent. Assume henceforth that A is not nilpotent, thus $r_\sigma(A) > 0$, and let $\sigma_{EP}(A^N)$ be b-nonresonant. Fix any $m \in \{1, \ldots, N\}$ and recall from (3.10) that, in analogy to (3.18) and (3.19) above,

$$\begin{aligned} A^{nN+m} &= \Re\left(\sum\nolimits_{\lambda\in\sigma^+(A^N)} P_\lambda(n) A^m \lambda^n \right) \\ &= r^n n^k \left(\Re\left(\sum\nolimits_{\lambda\in\sigma^+(A^N)\cap r\mathbb{S}} C_\lambda e^{in\arg\lambda} \right) + D_n \right) , \quad \forall n \ge d , \end{aligned} \tag{3.21}$$

where $0 < r \le r_\sigma(A^N) = r_\sigma(A)^N$ and $k \in \{0, \ldots, d-1\}$ with $k \le k_{\max}(A) = k_{\max}(A^N) =: k_{\max}$, $C_\lambda \in \mathbb{C}^{d\times d}$, and (D_n) is a sequence in $\mathbb{R}^{d\times d}$ for which $(n|D_n|)$

is bounded. (As in (3.19) the quantities r, k, C_λ and (D_n) may all depend on m.) From (3.21), it follows that

$$|A^{nN+m}| \le r^n n^k a\,, \quad \forall n \in \mathbb{N}\,, \tag{3.22}$$

with the appropriate $a > 0$. On the other hand, there exist $x, y \in \mathbb{R}^d$ for which

$$\limsup_{n\to\infty} \frac{|x^\top A^{nN+m} y|}{r_\sigma(A)^{nN+m}(nN+m)^{k_{\max}}} \ge 1\,. \tag{3.23}$$

Combining (3.22) and (3.23) yields

$$\begin{aligned} 1 &\le \limsup_{n\to\infty} \frac{|x^\top A^{nN+m} y|}{r_\sigma(A)^{nN+m}(nN+m)^{k_{\max}}} \\ &\le \frac{|x||y|a}{r_\sigma(A)^m N^{k_{\max}}} \limsup_{n\to\infty} \left(\frac{r}{r_\sigma(A)^N}\right)^n n^{k-k_{\max}}\,, \end{aligned}$$

which in turn shows that $r = r_\sigma(A)^N$ and $k = k_{\max}$. With $\sigma_{EP}^+(A^N) := \{\lambda \in \sigma_{EP}(A^N) : \Im\lambda \ge 0\}$, therefore, (3.21) can be re-written as

$$A^{nN+m} = r_\sigma(A)^{nN} n^{k_{\max}} \left(\Re\left(\sum\nolimits_{\lambda\in\sigma_{EP}^+(A^N)} C_\lambda e^{in\arg\lambda}\right) + E_n\right)\,, \quad \forall n \in \mathbb{N}\,, \tag{3.24}$$

where $C_\lambda \ne 0$ for some $\lambda \in \sigma_{EP}^+(A^N)$, and $(n|E_n|)$ is bounded. Using (3.24) and the b-nonresonance of $\sigma_{EP}(A^N)$, completely analogous arguments as in the proof of Theorem 3.10 show that $(\log_b |A^{nN+m}|)$ is u.d. mod 1. Since $m \in \{1, \ldots, N\}$ was arbitrary, $(\log_b |A^n|)$ is u.d. mod 1 as well, i.e., $(|A^n|)$ is b-Benford. □

Corollary 3.12. *Let $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{N}\setminus\{1\}$. Assume that A is invertible and $\sigma(A)$ is b-nonresonant. Then:*

(i) *For every $x, y \in \mathbb{R}^d$ the sequence $(x^\top A^n y)$ is b-Benford or identically zero;*

(ii) *For every $x \in \mathbb{R}^d \setminus \{0\}$ the sequence $(|A^n x|)$ is b-Benford;*

(iii) *The sequence $(|A^n|)$ is b-Benford.*

Remark 3.13. (i) Theorems 3.10 and 3.11, and hence also Corollary 3.12(ii,iii), hold similarly with $|\cdot|$ replaced by any norm on $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$, respectively.

(ii) When comparing Theorems 3.10 and 3.11 to Theorem 3.4, the reader may wonder what would happen to the latter if in its statement (ii) b-nonresonance was assumed merely for $\sigma(A^N)$ with some $N \ge 2$, rather than for $\sigma(A)$. The answer is simple: With (ii) thus modified, (i)⇒(ii) of Theorem 3.4 would remain unchanged whereas the converse (ii)⇒(i) would fail because unlike its analogues (3.19) and (3.24), the representation (3.16) with A^n replaced by A^{nN+m} may no longer be valid. Note that this is in perfect agreement with the fact, following from Lemma 3.7, that if $\sigma(A^N)$ is nonresonant for some $N \ge 2$ yet $\sigma(A)$ is resonant then there exist $x, y \in \mathbb{R}^d$ with $0 < \rho(\mathcal{N}_{A,x,y}) < 1$.

The converses of Theorems 3.10 and 3.11 do not hold in general: Even if $\sigma(A^n)$ and $\sigma_{EP}(A^n)$, respectively, are b-resonant for all $n \in \mathbb{N}$, the sequence $(|A^n x|)$ nevertheless may, for every $x \in \mathbb{R}^d$, be b-Benford or terminating, and $(|A^n|)$ may be b-Benford. In fact, as the next example shows, it is impossible to characterize the b-Benford property of $(|A^n x|)$ and $(|A^n|)$ solely in terms of $\sigma(A)$ and $\sigma_{EP}(A)$, respectively — except, of course, for the trivial case of $d = 1$.

Example 3.14. For convenience, fix $b = 10$ and consider the (invertible) 2×2-matrix

$$A = 10^{\pi} \left[\begin{array}{cc} \cos(\pi^2) & -\sin(\pi^2) \\ \sin(\pi^2) & \cos(\pi^2) \end{array} \right] .$$

The set $\sigma(A^n) = \sigma_{EP}(A^n) = \{10^{\pi n} e^{\pm \pi^2 \imath n}\}$ is b-resonant for every $n \in \mathbb{N}$ because

$$\pi n = \log_{10} 10^{\pi n} \in \operatorname{span}_{\mathbb{Q}} \Delta_{\sigma(A^n)} = \operatorname{span}_{\mathbb{Q}} \{1, \pi\} .$$

Nevertheless, $10^{-\pi n} A^n$ is simply a rotation, hence $|A^n x| = 10^{\pi n} |x|$ for every $x \in \mathbb{R}^2$, and since $\log_{10} 10^{\pi} = \pi$ is irrational, $(|A^n x|)$ is 10-Benford whenever $x \neq 0$. Similarly, $(|A^n|) = (10^{\pi n})$ is 10-Benford. Thus the nonresonance assumptions in Theorems 3.10 and 3.11, respectively, are not necessary for the conclusion.

Consider now also the (invertible) matrix

$$B = \frac{10^{\pi}}{\sqrt{3}} \left[\begin{array}{cc} \sqrt{3}\cos(\pi^2) & -3\sin(\pi^2) \\ \sin(\pi^2) & \sqrt{3}\cos(\pi^2) \end{array} \right] ,$$

for which $\sigma(B) = \sigma_{EP}(B) = \{10^{\pi} e^{\pm \pi^2 \imath}\} = \sigma(A)$, and so $\sigma(B^n) = \sigma_{EP}(B^n) = \sigma_{EP}(A^n)$ is b-resonant for every $n \in \mathbb{N}$. As far as spectral data are concerned, therefore, the matrices A and B are indistinguishable. (In fact, they are similar.) However, from

$$B^n = \frac{10^{\pi n}}{\sqrt{3}} \left[\begin{array}{cc} \sqrt{3}\cos(\pi^2 n) & -3\sin(\pi^2 n) \\ \sin(\pi^2 n) & \sqrt{3}\cos(\pi^2 n) \end{array} \right] , \quad \forall n \in \mathbb{N}_0 ,$$

it follows for instance that

$$|B^n e^{(2)}| = 10^{\pi n} \sqrt{2 - \cos(2\pi^2 n)} , \quad \forall n \in \mathbb{N}_0 ,$$

and consequently

$$\left\langle \log_{10} |B^n e^{(2)}| \right\rangle = \left\langle \pi n + \tfrac{1}{2} \log_{10}\bigl(2 - \cos(2\pi^2 n)\bigr) \right\rangle = f(\langle n\pi \rangle) ,$$

with the smooth function $f : \mathbb{T} \to \mathbb{T}$ given by

$$f(t) = t + \tfrac{1}{2} \log_{10}\bigl(2 - \cos(2\pi t)\bigr) .$$

Recall that $(n\pi)$ is u.d. mod 1. Since f is a diffeomorphism of $\mathbb{T}$ with non-constant derivative, it follows that $\bigl(f(\langle n\pi \rangle)\bigr)$ is not u.d. mod 1, basically because $\lambda_{\mathbb{T}} \circ f^{-1} \neq$

$\lambda_{\mathbb{T}}$ (cf. Appendix A). Thus $(|B^n e^{(2)}|)$, and in fact $(|B^n x|)$ for *every* $x \in \mathbb{R}^2 \setminus \{0\}$, is neither 10-Benford nor identically zero. Similarly,

$$|B^n| = \frac{10^{\pi n}}{\sqrt{3}} \sqrt{4 - \cos(2\pi^2 n) + |\sin(\pi^2 n)|\sqrt{14 - 2\cos(2\pi^2 n)}}\,, \quad \forall n \in \mathbb{N}_0\,,$$

and a completely analogous argument shows that $(|B^n|)$ is not 10-Benford either.

Example 3.15. Let again $b = 10$ for convenience and consider the 6×6-matrix

$$A = \operatorname{diag}\left[\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}, \frac{2}{\sqrt{3}} \begin{bmatrix} \sqrt{3}\cos(\pi \log_{10} 2) & -3\sin(\pi \log_{10} 2) \\ \sin(\pi \log_{10} 2) & \sqrt{3}\cos(\pi \log_{10} 2) \end{bmatrix} \right],$$

for which $\sigma(A) = \{\pm 2, 2e^{\pm \pi \imath \log_{10} 2}\} \subset 2\mathbb{S}$. Since

$$\log_{10} 2^n = n \log_{10} 2 \in \operatorname{span}_{\mathbb{Q}}\{1, \log_{10} 2\} \subset \operatorname{span}_{\mathbb{Q}} \Delta_{\sigma(A^n)}\,,$$

the set $\sigma(A^n)$ is b-resonant for every $n \in \mathbb{N}$. Correspondingly, there exist $x, y \in \mathbb{R}^6$ for which the sequence $(x^\top A^n y)$, and in fact $(|A^n x|)$ as well, is neither 10-Benford nor terminating. Essentially the same calculation as in Example 3.14 shows that one can take for instance $x = y = e^{(6)}$. Note, however, that $(x^\top A^n y)$ is 10-Benford whenever $|x_1 y_2| \neq |x_3 y_4|$, hence for *most* $x, y \in \mathbb{R}^6$; see also Theorem 4.1 below.

On the other hand, since $k(\pm 2) = 2$ and $k(2e^{\pm \pi \imath \log_{10} 2}) = 1$, the set $\sigma_{EP}(A)$ equals $\{\pm 2\}$ which is also b-resonant, yet $\sigma_{EP}(A^2) = \{4\}$ is b-nonresonant. By Theorem 3.11, therefore, the sequence $(|A^n|)$ is 10-Benford. This could also have been demonstrated by means of Lemma 2.5 and an explicit calculation yielding

$$|A^n| = 2^{n-1} n (1 + \alpha_n)\,, \quad \forall n \in \mathbb{N}\,,$$

where (α_n) is a sequence in $\mathbb{R}$ with $\lim_{n\to\infty} n^2 \alpha_n = 4$.

The final theorem in this section characterizes the b-Benford property of solutions (x_n) to linear difference equations (1.3). The result, which has informally been mentioned already in the Introduction, follows directly from Theorem 3.4.

Theorem 3.16. *Let $a_1, a_2, \ldots, a_{d-1}, a_d$ be real numbers with $a_d \neq 0$, and $b \in \mathbb{N} \setminus \{1\}$. Then the following are equivalent:*

(i) *Every solution (x_n) of (1.3) is b-Benford unless $x_1 = x_2 = \ldots = x_d = 0$;*

(ii) *With the polynomial $p(z) = z^d - a_1 z^{d-1} - a_2 z^{d-2} - \ldots - a_{d-1} z - a_d$, the set $\{z \in \mathbb{C} : p(z) = 0\}$ is b-nonresonant.*

Proof. For convenience, let $\mathcal{Z} := \{z \in \mathbb{C} : p(z) = 0\}$. Note that $\mathcal{Z} = \sigma(A)$ for the matrix A associated with (1.3) via (3.1) because

$$\begin{aligned} \chi_A(z) = \det(A - zI_d) &= (-1)^d (z^d - a_1 z^{d-1} - a_2 z^{d-2} - \ldots - a_{d-1} z - a_d) \\ &= (-1)^d p(z)\,. \end{aligned}$$

To prove (i)$\Rightarrow$(ii), assume $\mathcal{Z}$ is b-resonant. By Theorem 3.4 there exist $x, y \in \mathbb{R}^d$ for which $(x^\top A^n y)$ is neither b-Benford nor terminating. Recall (e.g. from the proof of Lemma 3.6) that (x_n) with $x_n := x^\top A^n y$ for all $n \in \mathbb{N}$ is a solution of (1.3). By the choice of x, y, the sequence (x_n) is neither b-Benford nor terminating, let alone identically zero. Hence (i) fails whenever (ii) fails, that is, (i)$\Rightarrow$(ii).

To establish the reverse implication (ii)$\Rightarrow$(i), recall from (3.2) that

$$x_n = (e^{(d)})^\top A^{n-1} y\,, \quad \forall n \in \mathbb{N}\,,$$

where $y = \sum_{j=1}^d x_{d+1-j} e^{(j)}$. As A is invertible, if $\mathcal{Z} = \sigma(A)$ is b-nonresonant then, by Corollary 3.12, (x_n) is either b-Benford or identically zero. $\square$

Example 3.17. The set associated, via Theorem 3.16, with the familiar difference equation

$$x_n = x_{n-1} + x_{n-2}\,, \quad \forall n \geq 3\,, \tag{3.25}$$

i.e. $\{z \in \mathbb{C} : z^2 - z - 1 = 0\} = \{\varphi, -\varphi^{-1}\}$, is b-nonresonant for every $b \in \mathbb{N} \setminus \{1\}$, see Example 3.9(i). Except for the trivial solution $x_n \equiv 0$, therefore, every solution (x_n) of (3.25) is Benford. This contains as special cases the well-known sequences of Fibonacci and Lucas numbers corresponding to the initial values $x_1 = x_2 = 1$ and $x_1 = 2$, $x_2 = 1$, respectively.

Example 3.18. This example reviews, in the light of Theorem 3.16, the second-order difference equation (1.4) for the three specific values of the parameter $\gamma \in \mathbb{R}$ already considered in the Introduction (recall Figure 1). For convenience, let $b = 10$ throughout. Note that the set associated with (1.4) is $\mathcal{Z} = \mathcal{Z}_\gamma = \{z \in \mathbb{C} : z^2 = 2\gamma z - 5\} = \{\gamma \pm \imath\sqrt{5 - \gamma^2}\}$, and so for $|\gamma| < \sqrt{5}$ equals $\{\sqrt{5}e^{\pm \imath \arg z}\} \subset \sqrt{5}\mathbb{S}$ with $\arg z = \arccos(\gamma/\sqrt{5}) \in (0, \pi)$.

(i) Let $\gamma = \sqrt{5}\cos(\pi/\sqrt{8}) = 0.9928$. Then $\arg z = \pi/\sqrt{8}$, and since

$$\log_{10} 5 \notin \operatorname{span}_{\mathbb{Q}} \Delta_{\mathcal{Z}_\gamma} = \operatorname{span}_{\mathbb{Q}}\{1, \sqrt{2}\}\,,$$

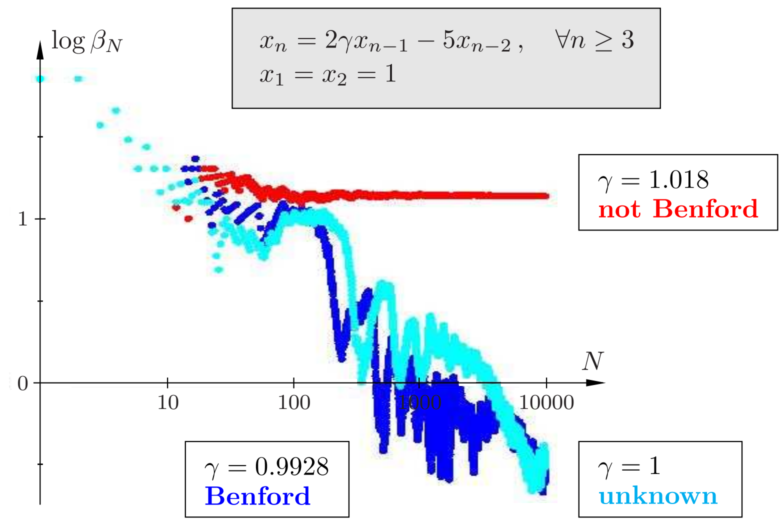
the set $\mathcal{Z}_\gamma$ is b-nonresonant. By Theorem 3.16, except for $x_n \equiv 0$, *every* solution (x_n) of (1.4) is 10-Benford.

(ii) Next, consider the case of $\gamma = \sqrt{5}\cos(\frac{1}{2}\pi \log_{10} 5) = 1.018$. Now $\arg z = \frac{1}{2}\pi \log_{10} 5$, and since obviously

$$\log_{10} 5 \in \operatorname{span}_{\mathbb{Q}} \Delta_{\mathcal{Z}_\gamma} = \operatorname{span}_{\mathbb{Q}}\{1, \log_{10} 5\}\,,$$

the set $\mathcal{Z}_\gamma$ is b-resonant. It is clear that *no* solution of (1.4) is 10-Benford in this case.

(iii) Finally, let $\gamma = 1$. Here $\arg z = \arccos(1/\sqrt{5}) = \arctan 2$. It is not hard to see that $\frac{1}{\pi}\arctan 2$ is irrational (as is, of course, $\log_{10} 5$). Thus, the b-nonresonance of $\mathcal{Z}_\gamma$ is equivalent to $\log_{10} 5 \notin \operatorname{span}_{\mathbb{Q}}\{1, \frac{1}{\pi}\arctan 2\}$. It appears to be unknown, however, whether the three numbers 1, $\log_{10} 5$, $\frac{1}{\pi}\arctan 2$ are $\mathbb{Q}$-independent. If they are, then every non-trivial solution of (1.4) is 10-Benford; otherwise none is. As seen in Figure 2, numerical evidence seems to be in support of the former alternative. (Rational independence of 1, $\log_{10} 5$, and $\frac{1}{\pi}\arctan 2$, and thus 10-nonresonance of $\mathcal{Z}_\gamma$ for $\gamma = 1$ would follow immediately from Schanuel's conjecture, a prominent but as yet unproven assertion in number theory [38, Sec.1.4].)



$$\beta_N = \max_{d_1=1}^{9}\left|\frac{\#\{n \leq N : \textit{leading digit}_{10}(x_n) = d_1\}}{N} - \log_{10}(1 + d_1^{-1})\right|$$

Figure 2: For different values of the parameter γ, the solutions (x_n) of (1.4) may or may not be 10-Benford; see Example 3.18 and also Figure 1.

Remark 3.19. Earlier, weaker forms and variants of the implication (ii)⇒(i) in Theorems 3.4 and 3.16, or special cases thereof, can be traced back at least to [32] and may also be found in [4, 6, 9, 22, 36]. The reverse implication (i)⇒(ii) seems to have been addressed previously only for $d < 4$; see [6, Thm.5.37]. For the special case of $b = 10$, partial proofs of Theorems 3.4 and 3.16 have been presented in [5, 7].

4 Further examples and concluding remarks

This final section illustrates how key results of this article (Theorems 3.4 and 3.16) may take a significantly different (and arguably simpler) form if either their conclusion is weakened slightly or one additional assumption is imposed. Concretely, Theorem 3.4 for instance may be weakened in that its b-Benford-or-terminating dichotomy (i) is assumed to hold only for (Lebesgue) *almost all* $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Alternatively, it may be assumed that the matrix A^N is *positive* for some $N \in \mathbb{N}$. As detailed below, either of these modifications gives rise to new forms of the results that may be of independent interest.

As throughout, $b \geq 2$ is a positive integer, and given any $A \in \mathbb{R}^{d\times d}$, let

$$\mathbb{B}_b(A) := \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : (x^\top A^n y) \text{ is } b\text{-Benford}\right\}.$$

Denote Lebesgue measure on $\mathbb{R}^d \times \mathbb{R}^d$ by $\mathrm{Leb}_{d,d}$. Also recall from (3.20) the definition of the extremal peripheral spectrum $\sigma_{EP}(A)$. Although $\sigma_{EP}(A)$ may constitute only a small part of $\sigma(A)$, it nevertheless controls the Benford property of *most* sequence $(x^\top A^n y)$. More precisely, the following variant of Theorem 3.4 holds.

Theorem 4.1. *Let $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{N}\setminus\{1\}$. Assume A is not nilpotent. Then the following are equivalent:*

(i) *For almost every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ the sequence $(x^\top A^n y)$ is b-Benford, i.e., $\mathbb{R}^d \times \mathbb{R}^d \setminus \mathbb{B}_b(A)$ is a $\mathrm{Leb}_{d,d}$-nullset;*

(ii) *The set $\sigma_{EP}(A^N)$ is b-nonresonant for some $N \in \mathbb{N}$.*

Proof. To demonstrate (i)$\Rightarrow$(ii), assume that $\sigma_{EP}(A^n) = \sigma_{EP}(A)^n$ is b-resonant for every $n \in \mathbb{N}$, and hence $\log_b r_\sigma(A) \in \mathrm{span}_{\mathbb{Q}}\Delta_{\sigma_{EP}(A)}$. In analogy to (3.24), write

$$A^n = r_\sigma(A)^n n^{k_{\max}} \left(\Re\left(\sum\nolimits_{\lambda\in\sigma^+_{EP}(A)} C_\lambda e^{\imath n \arg\lambda}\right) + E_n\right), \quad \forall n \in \mathbb{N}, \tag{4.1}$$

where $C_\lambda \in \mathbb{C}^{d\times d}$ for every $\lambda \in \sigma^+_{EP}(A)$, and (E_n) is a sequence in $\mathbb{R}^{d\times d}$ for which $(n|E_n|)$ is bounded. If $C_\lambda = 0$ for all $\lambda \in \sigma^+_{EP}(A)$, then (4.1) would imply that

$$\lim\nolimits_{n\to\infty} \frac{|A^n|}{r_\sigma(A)^n n^{k_{\max}}} = 0,$$

whereas on the other hand there always exist $x, y \in \mathbb{R}^d$ with

$$1 \leq \limsup\nolimits_{n\to\infty} \frac{|x^\top A^n y|}{r_\sigma(A)^n n^{k_{\max}}} \leq |x|\,|y| \limsup\nolimits_{n\to\infty} \frac{|A^n|}{r_\sigma(A)^n n^{k_{\max}}}.$$

This contradiction shows that $C_\lambda \neq 0$ for some $\lambda \in \sigma^+_{EP}(A)$.

Similarly to the proofs in the previous section, let $L_0 + 1$ be the dimension of $\mathrm{span}_{\mathbb{Q}}\Delta_{\sigma_{EP}(A)}$ and consider first the case of $L_0 = 0$. Here, with the appropriate $q \in \mathbb{N}$, the numbers $q \log_b r_\sigma(A)$ and $q\frac{1}{2\pi}\arg\lambda$ for all $\lambda \in \sigma^+_{EP}(A)$ are integers, and so (4.1) takes the form

$$A^n = r_\sigma(A)^n n^{k_{\max}}(B_n + E_n), \quad \forall n \in \mathbb{N}, \tag{4.2}$$

where the sequence (B_n) in $\mathbb{R}^{d\times d}$ is q-periodic, i.e. $B_{n+q} = B_n$ for all $n \in \mathbb{N}$. Suppose that $B_\ell = 0$ for some $\ell \in \{1, \ldots, q\}$. Then

$$\lim\nolimits_{n\to\infty} \frac{|A^{nq+\ell}|}{r_\sigma(A)^{nq+\ell}(nq+\ell)^{k_{\max}}} = 0,$$

whereas similarly as before,

$$\limsup\nolimits_{n\to\infty} \frac{|x^\top A^{nq+\ell} y|}{r_\sigma(A)^{nq+\ell}(nq+\ell)^{k_{\max}}} \geq 1$$

with the appropriate $x, y \in \mathbb{R}^d$. This contradiction shows that $B_\ell \neq 0$ for every $\ell \in \{1, \ldots, q\}$. Consequently, for each ℓ the set

$$\mathcal{R}_\ell := \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x^\top B_\ell \, y = 0\right\}$$

is a $\mathrm{Leb}_{d,d}$-nullset, and so is $\mathcal{R} := \bigcup_{\ell=1}^q \mathcal{R}_\ell$. Whenever $(x, y) \notin \mathcal{R}$, it follows from (4.2) that

$$\log_b |x^\top A^n y| = n \log_b r_\sigma(A) + k_{\max} \log_b n + \log_b |x^\top B_n y + x^\top E_n y|$$

for all sufficiently large n, and since $\log_b r_\sigma(A)$ is rational and $(x^\top B_n y)$ is periodic, Lemma 2.5 shows that $(x^\top A^n y)$ is not b-Benford. In other words, $\mathbb{B}_b(A) \subset \mathcal{R}$, so in particular $\mathbb{R}^d \times \mathbb{R}^d \setminus \mathbb{B}_b(A)$ is not a nullset, i.e., (i) fails.

It remains to consider the case of $L_0 \geq 1$. In this case, label the elements of $\sigma^+_{EP}(A)$ as $\lambda_1, \ldots, \lambda_L$ with $L \geq L_0$ and assume w.l.o.g. that the $L_0 + 1$ numbers $1, \frac{1}{2\pi} \arg \lambda_1, \ldots, \frac{1}{2\pi} \arg \lambda_{L_0}$ are $\mathbb{Q}$-independent. Given any $u \in \mathbb{R}^{L_0}$, there exist $x_u, y_u \in \mathbb{R}^d$ such that

$$\begin{aligned} x_u^\top A^n y_u &= r_\sigma(A)^n n^{k_{\max}} \left(\sum\nolimits_{\ell=1}^{L_0} u_\ell \cos(n \arg \lambda_\ell) + z_n\right) \\ &= r_\sigma(A)^n n^{k_{\max}} \left(\Re\left(\sum\nolimits_{\ell=1}^{L_0} u_\ell e^{\imath n \arg \lambda_\ell}\right) + z_n\right), \quad \forall n \in \mathbb{N}, \end{aligned} \tag{4.3}$$

where $(n z_n)$ is a bounded sequence in $\mathbb{R}$. On the other hand, (4.1) implies

$$x_u^\top A^n y_u = r_\sigma(A)^n n^{k_{\max}} \left(\Re\left(\sum\nolimits_{\lambda \in \sigma^+_{EP}(A)} x_u^\top C_\lambda y_u e^{\imath n \arg \lambda}\right) + x_u^\top E_n y_u\right), \ \forall n \in \mathbb{N}. \tag{4.4}$$

Comparing (4.3) and (4.4) yields

$$\Re\left(\sum\nolimits_{\ell=1}^{L_0} u_\ell e^{\imath n \arg \lambda_\ell} - \sum\nolimits_{\lambda \in \sigma^+_{EP}(A)} x_u^\top C_\lambda y_u e^{\imath n \arg \lambda}\right) \stackrel{n \to \infty}{\longrightarrow} 0.$$

Lemma A.2 shows that $x_u^\top C_{\lambda_\ell} y_u = u_\ell$ for every $\ell \in \{1, \ldots, L_0\}$, and $x_u^\top C_{\lambda_\ell} y_u = 0$ for every $\ell \in \{L_0 + 1, \ldots, L\}$. Recall now that $\log_b r_{\sigma(A)} \in \mathrm{span}_{\mathbb{Q}} \Delta_{\sigma^+_{EP}(A)}$. Lemma 2.7 guarantees that it is possible to choose $u \in \mathbb{R}^{L_0}$ in such a way that the sequence $(x_u^\top A^n y_u)$ in (4.3) is neither b-Benford nor terminating. The continuity of the map

$$\left\{ \begin{array}{ccc} \mathbb{R}^d \times \mathbb{R}^d & \to & \mathbb{C}^L \\ (x, y) & \mapsto & (x^\top C_{\lambda_1} y, \ldots, x^\top C_{\lambda_L} y) \end{array} \right.$$

implies that $(x^\top A^n y)$ is not b-Benford whenever x and y are sufficiently close to x_u and y_u, respectively. Thus $\mathbb{R}^d \times \mathbb{R}^d \setminus \mathbb{B}_b(A)$ contains a non-empty open set, and so again (i) fails. This completes the proof of (i)⇒(ii).

To establish the reverse implication (ii)⇒(i), let $\sigma_{EP}(A^N)$ be b-nonresonant and fix any $m \in \{1, \ldots, N\}$. It follows from (4.1) that

$$A^{nN+m} = r_\sigma(A)^{nN} n^{k_{\max}} \left(\Re\left(\sum\nolimits_{\lambda \in \sigma^{++}} C_\lambda e^{\imath n \arg \lambda}\right) + E_n\right), \quad \forall n \in \mathbb{N},$$

where $\sigma^{++} \subset \sigma_{EP}^{+}(A^N)$ is non-empty, $C_\lambda \in \mathbb{C}^{d\times d} \setminus \{0\}$ for every $\lambda \in \sigma^{++}$, and $(n|E_n|)$ is bounded. (Once again it should be noted that the set σ^{++}, the matrices C_λ and the sequence (E_n) may all vary with m.) The set

$$\mathcal{R}_{m,\lambda} := \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x^\top C_\lambda y = 0\right\}$$

is a $\text{Leb}_{d,d}$-nullset, and so is $\mathcal{R} := \bigcup_{m=1}^N \bigcup_{\lambda\in\sigma^{++}} \mathcal{R}_{m,\lambda}$. Whenever $(x,y) \notin \mathcal{R}$, an argument completely analogous to the one establishing (ii)⇒(i) in Theorem 3.4 shows that $(x^\top A^n y)$ is b-Benford. Thus $\mathbb{R}^d \times \mathbb{R}^d \setminus \mathbb{B}_b(A) \subset \mathcal{R}$, and the proof is complete. □

Example 4.2. (i) As seen in Example 3.9(iii) the matrix $A = \frac{1}{2}\begin{bmatrix} 1+\pi & 1-\pi \\ 1-\pi & 1+\pi \end{bmatrix}$ has $\sigma(A) = \{1, \pi\}$ b-resonant for every b. However, $\sigma_{EP}(A) = \{\pi\}$ is b-nonresonant, and since

$$A^n = \frac{\pi^n}{\pi - 1}(A - I_2) + \frac{1}{\pi - 1}(\pi I_2 - A) \tag{4.5}$$

for all $n \in \mathbb{N}$, it is clear that

$$\begin{aligned}\mathbb{R}^2 \times \mathbb{R}^2 \setminus \mathbb{B}_b(A) &= \left\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x^\top (A - I_2) y = 0\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : (x_1 - x_2)(y_1 - y_2) = 0\right\}\end{aligned}$$

is a nullset. Also, $(|A^n x|)$ is Benford unless $Ax = x$.

(ii) Let $B := A^{-1}$. Then $\sigma(B) = \{\pi^{-1}, 1\}$, so $\sigma_{EP}(B^n) = \{1\}$ is b-resonant for every b and $n \in \mathbb{N}$. Since (4.5) actually holds for all $n \in \mathbb{Z}$, the sequence $(x^\top B^n y)$ can only be b-Benford if $x^\top(\pi I_2 - A)y = 0$, i.e.

$$\begin{aligned}\mathbb{B}_b(B) &\subset \left\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x^\top (\pi I_2 - A) y = 0\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : (x_1 + x_2)(y_1 + y_2) = 0\right\},\end{aligned}$$

showing that $\mathbb{B}_b(B)$ is a nullset in this case. Similarly, $(|B^n x|)$ can only be Benford if $Bx = \pi^{-1}x$.

Remark 4.3. Recall from Theorem 3.10 that $(|A^n x|)$ is b-Benford provided that $\sigma(A^N)$ is b-nonresonant for some $N \in \mathbb{N}$. If A is not nilpotent then $(|A^n x|)$ is terminating only if x is an element of the proper subspace (and hence nullset) $\ker A^d$. For *almost* all $x \in \mathbb{R}^d$, therefore, $(|A^n x|)$ is b-Benford. As it turns out, a much weaker assumption suffices to guarantee the latter conclusion: Similarly to Theorem 4.1, it can be shown that b-nonresonance of $\sigma_{EP}(A^N)$ for some N implies that $(|A^n x|)$ is b-Benford for almost all $x \in \mathbb{R}^d$. Unlike in Theorem 4.1 (yet much like in Theorem 3.10), the converse does not hold in general. In fact, as demonstrated already by Example 3.14, it is impossible to characterize the b-Benford property of $(|A^n x|)$ for almost all $x \in \mathbb{R}^d$ using only $\sigma(A)$, let alone $\sigma_{EP}(A)$.

The following variant of Theorem 3.16 is motivated by Theorem 4.1. Recall that $\mathcal{Z}^n = \{z^n : z \in \mathcal{Z}\}$ for any $\mathcal{Z} \subset \mathbb{C}$. If $p = p(z)$ is a non-constant polynomial and

$\mathcal{Z} = \{z \in \mathbb{C} : p(z) = 0\}$, let $\zeta := \max_{z\in\mathcal{Z}} |z|$ and, for each $z \in \mathcal{Z}$, let $k(z)$ be the multiplicity of z as a root of p, that is, $k(z) = \min\{n \in \mathbb{N} : p^{(n)}(z) \neq 0\}$. In analogy to the extremal peripheral spectrum, define

$$\mathcal{Z}_{EP} := \{z \in \mathbb{C} : p(z) = 0\}_{EP} := \{z \in \mathcal{Z} \cap \zeta\mathbb{S} : k(z) = k_{\max}\} ,$$

where $k_{\max} := \max\{k(z) : z \in \mathcal{Z} \cap \zeta\mathbb{S}\}$.

Theorem 4.4. *Let $a_1, a_2, \ldots, a_{d-1}, a_d$ be real numbers with $a_d \neq 0$, and $b \in \mathbb{N}\backslash\{1\}$. Then the following are equivalent:*

(i) *The solution (x_n) of (1.3) is b-Benford for almost all $(x_1, \ldots, x_d) \in \mathbb{R}^d$;*

(ii) *With the polynomial $p(z) = z^d - a_1z^{d-1} - a_2z^{d-2} - \ldots - a_{d-1}z - a_d$, the set $\{z \in \mathbb{C} : p(z) = 0\}^N_{EP}$ is b-nonresonant for some $N \in \mathbb{N}$.*

Proof. As seen in the proof of Theorem 3.16, for the matrix A associated with (1.3) via (3.1), $\sigma(A) = \{z \in \mathbb{C} : p(z) = 0\}$, and in fact $\sigma_{EP}(A^n) = \{z \in \mathbb{C} : p(z) = 0\}^n_{EP}$ for every $n \in \mathbb{N}$. With this as well as (3.2) and (4.1), the argument is completely analogous to the proof of Theorem 4.1; details are left to the reader. $\square$

Example 4.5. (i) For convenience let $b = 10$ and consider the third-order equation

$$x_n = 5x_{n-1} - 11x_{n-2} + 15x_{n-3} , \quad \forall n \geq 4 . \tag{4.6}$$

With the associated set

$$\mathcal{Z} = \{z \in \mathbb{C} : z^3 - 5z^2 + 11z - 15 = 0\} = \{z \in \mathbb{C} : (z - 3)(z^2 - 2z + 5) = 0\} ,$$

clearly $\zeta = 3$, and $\mathcal{Z}_{EP} = \{3\}$ is b-nonresonant. For almost all $(x_1, x_2, x_3) \in \mathbb{R}^3$, therefore, the solution (x_n) of (4.6) is 10-Benford. In fact, since $\lim_{n\to\infty} 3^{-n}x_n = \frac{1}{24}(x_3 - 2x_2 + 5x_1)$, the sequence (x_n) is 10-Benford unless $x_3 = 2x_2 - 5x_1$. Note that in the latter case, (x_n) solves the *second*-order equation $x_n = 2x_{n-1} - 5x_{n-2}$, i.e. (1.4) with $\gamma = 1$, and as seen in Example 3.18, except for the trivial case of $x_n \equiv 0$ it is not known whether (x_n) is 10-Benford.

(ii) The set $\mathcal{Z}$ associated with the second-order equation

$$x_n = \pi^{-2}x_{n-2} , \quad \forall n \geq 3 , \tag{4.7}$$

i.e. $\mathcal{Z} = \{\pm\pi^{-1}\}$ is b-resonant for all $b \in \mathbb{N} \setminus \{1\}$. However, with $\zeta = \pi^{-1}$, the set $\mathcal{Z}^2 = \mathcal{Z}^2_{EP} = \{\pi^{-2}\}$ is b-nonresonant. Hence the solution (x_n) of (4.7) is Benford for almost all $(x_1, x_2) \in \mathbb{R}^2$. Again, it is easy to check that in fact (x_n) is Benford if and only if $x_1x_2 \neq 0$.

(iii) As a variant of (4.7), consider the recursion

$$x_n = (1 - \pi^{-2})x_{n-1} + \pi^{-2}x_{n-2} , \quad \forall n \geq 3 . \tag{4.8}$$

Now $\mathcal{Z} = \{-\pi^{-2}, 1\}$, hence $\zeta = 1$, and $\mathcal{Z}^n_{EP} = \{1\}$ is b-resonant for every $n \in \mathbb{N}$. By Theorem 4.4, the solution (x_n) of (4.8) is not Benford for almost all $(x_1, x_2) \in \mathbb{R}^2$. In fact, (x_n) can only be Benford if $x_1 + \pi^2x_2 = 0$, hence $\{(x_1, x_2) \in \mathbb{R}^2 : (x_n) \text{ is Benford}\}$ is a nullset.

Remark 4.6. In light of the above examples, it may be conjectured that in the context of Theorem 4.1, $\mathbb{B}_b(A)$ is actually a nullset if $\sigma_{EP}(A^n)$ is b-resonant for all $n \in \mathbb{N}$. Similarly, the solution (x_n) of (1.3) may for almost all $(x_1, \ldots, x_d) \in \mathbb{R}^d$ not be b-Benford whenever $\{z \in \mathbb{C} : p(z) = 0\}_{EP}^n$ is b-resonant for every n. Using Lemmas A.8 and A.10, it is not hard to verify this conjecture for $d \in \{1, 2, 3\}$. However, the authors do not know of any proof of, or counter-example to the conjecture for $d \geq 4$; cf. Remark A.12(i).

Clearly, if $\sigma(A^N)$ is b-nonresonant for some $N \in \mathbb{N}$ then so is $\sigma_{EP}(A^N)$, and unless A is nilpotent, this in turn implies that $\log_b r_\sigma(A)$ is irrational. As the next result shows, even the latter, seemingly much weaker condition alone suffices to recover a strong form of Theorem 3.4 — provided that some power of A is positive. Recall that $A \in \mathbb{R}^{d\times d}$ is *positive* (*nonnegative*), in symbols $A > 0$ ($A \geq 0$), if $[A]_{jk} > 0$ ($[A]_{jk} \geq 0$) for all $j, k \in \{1, \ldots, d\}$; here $[A]_{jk}$ denotes the entry of A at position (j, k), i.e. in the j-th row and k-th column, thus $[A]_{jk} = (e^{(j)})^\top A e^{(k)}$. For convenience, write $x > 0$ ($x \geq 0$) for $x \in \mathbb{R}^d$ if $x_j > 0$ ($x_j \geq 0$) for all $j \in \{1, \ldots, d\}$. A proof of the following result can be found in [5, Sec.3] for $b = 10$, but the argument given there immediately carries over to arbitrary base b.

Proposition 4.7. *Let $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{N} \setminus \{1\}$. Assume that $A^N > 0$ for some $N \in \mathbb{N}$. Then the following four statements are equivalent:*

(i) *For every $x, y \in \mathbb{R}^d \setminus \{0\}$ with $x \geq 0$, $y \geq 0$ the sequence $(x^\top A^n y)$ is b-Benford;*

(ii) *For every $x \in \mathbb{R}^d \setminus \{0\}$ with $x \geq 0$ the sequence $(|A^n x|)$ is b-Benford;*

(iii) *The sequence $(|A^n|)$ is b-Benford;*

(iv) $\log_b r_\sigma(A)$ *is irrational.*

Example 4.8. (i) For the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 1 & 0 \end{bmatrix},$$

one finds $\sigma(A) = \{-1 \pm \imath\sqrt{2}, 2\}$, and so $\sigma(A)$ is b-resonant whenever $b \in \{2^n, 3^n : n \in \mathbb{N}\}$. For any other base b, and similarly to Example 3.18(iii), it is apparently unknown whether $\sigma(A)$ is b-resonant. Note, however, that $A \geq 0$ and $A^5 > 0$, hence Proposition 4.7 applies with $r_\sigma(A) = 2$. For every b not an integer power of 2, therefore, and for all $x, y \in \mathbb{R}^3 \setminus \{0\}$ with $x, y \geq 0$, the sequences $(x^\top A^n y)$ and $(|A^n x|)$ are b-Benford. This nicely complements the fact that $(x^\top A^n y)$ and $(|A^n x|)$ are b-Benford in this case for almost all $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ (by Theorem 4.1) and almost all $x \in \mathbb{R}^3$ (by Remark 4.3), respectively. Also, $(|A^n|)$ is b-Benford by Theorem 3.11.

(ii) For the matrix

$$B = \begin{bmatrix} -3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 6 \end{bmatrix},$$

it is easily checked that $B^8 > 0$. An argument similar to, but simpler than the one in Example 3.9(ii) shows that $\log_b r_\sigma(B)$ is irrational for every b. Hence again Proposition 4.7 applies. Note that in order to reach this conclusion it is not necessary to explicitly determine the value of $r_\sigma(B)$.

Remark 4.9. For nonnegative $A \in \mathbb{R}^{d\times d}$, it is well-known that $A^N > 0$ for some $N \in \mathbb{N}$ (if and) only if $A^{d^2-2d+2} > 0$; see e.g. [21, Prop.8.5]. On the other hand, for $d \geq 3$ and *arbitrary* $A \in \mathbb{R}^{d\times d}$, the minmal number N for which $A^N > 0$, if at all existant, may be arbitrarily large; see [5, Sec.3].

Proposition 4.7 has a counterpart for difference equations which is a variant of Theorem 3.16 under the assumption of *positivity*, both for the coefficients and the initial data; for a proof the reader is again referred to [5].

Proposition 4.10. *Let $a_1, a_2, \ldots, a_{d-1}, a_d$ be positive real numbers, and $b \in \mathbb{N}\backslash\{1\}$. Then the following are equivalent:*

(i) *Every solution (x_n) of (1.3) with $x_1, \ldots, x_d \geq 0$ and $\max_{j=1}^d x_j > 0$ is b-Benford;*

(ii) *$\log_b \zeta$ is irrational where $z = \zeta$ is the right-most root of $p(z) = 0$ with the polynomial $p(z) = z^d - a_1 z^{d-1} - a_2 z^{d-2} - \ldots - a_{d-1} z - a_d$.*

To finally put Theorems 3.4 and 4.1 as well as Corollary 3.12 in perspective, recall that, informally put, b-Benford sequences are prevalent among the sequences $(x^\top A^n y)$, $(|A^n x|)$, and $(|A^n|)$ derived from (A^n) whenever $\sigma(A^N)$ is b-nonresonant for some $N \in \mathbb{N}$. For *most* matrices $A \in \mathbb{R}^{d\times d}$ the set $\sigma(A)$ is b-nonresonant for every b, as are $\sigma(A^n)$ and $\sigma_{EP}(A^n)$ for all $n \in \mathbb{N}$, and $\log_b r_\sigma(A)$ is irrational. More formally, let

$$\mathcal{G}_{d,b} := \left\{ A \in \mathbb{R}^{d\times d} : A \text{ is invertible and } \sigma(A) \text{ is } b\text{-nonresonant} \right\}.$$

With this, it can be shown that while the set $\mathbb{R}^{d\times d} \setminus \mathcal{G}_{d,b}$ is dense in $\mathbb{R}^{d\times d}$, it nevertheless is a first-category set (i.e. a countable union of nowhere dense sets) and has (Lebesgue) measure zero. The same, therefore, is true for $\bigcup_{b\in\mathbb{N}\backslash\{1\}}(\mathbb{R}^{d\times d} \setminus \mathcal{G}_{d,b})$. In other words, most real $d \times d$-matrices, both in a topological and measure-theoretical sense, belong to $\bigcap_{b\in\mathbb{N}\backslash\{1\}} \mathcal{G}_{d,b}$, and thus are invertible with their spectrum b-nonresonant for every b; see e.g. [4, 8, 6] for details. This observation may help explain the conformance to BL often observed empirically across a wide range of scientific disciplines.

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A Some auxiliary results

The purpose of this appendix is to provide proofs for several analytical facts that have been used in establishing the main results of this article. Throughout, let d be a fixed positive integer.

Lemma A.1. *Given any* $z_1, \ldots, z_d \in \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$*, the following are equivalent:*

(i) *If* $c_1, \ldots, c_d \in \mathbb{C}$ *and* $\lim_{n\to\infty}(c_1 z_1^n + \ldots + c_d z_d^n)$ *exists then* $c_1 = \ldots = c_d = 0$*;*

(ii) $z_j \notin \{1\} \cup \{z_k : k \neq j\}$ *for every* $1 \le j \le d$*.*

Proof. Clearly (i)⇒(ii) because if $z_j = 1$ for some j simply let $c_j = 1$ and $c_\ell = 0$ for all $\ell \neq j$, whereas if $z_j = z_k$ for some $j \neq k$ take $c_j = 1$, $c_k = -1$, and $c_\ell = 0$ for all $\ell \in \{1, \ldots, d\} \setminus \{j, k\}$. To show that (ii)⇒(i) as well, proceed by induction. Trivially, if $d = 1$ then $(c_1 z_1^n)$ with $z_1 \in \mathbb{S}$ converges only if $c_1 = 0$ or $z_1 = 1$. Assume now that (ii)⇒(i) has been established already for some $d \in \mathbb{N}$, let $z_1, \ldots, z_{d+1} \in \mathbb{S}$, and assume that $z_j \notin \{1\} \cup \{z_k : k \neq j\}$ for every $1 \le j \le d+1$. If $\lim_{n\to\infty}(c_1 z_1^n + \ldots + c_{d+1} z_{d+1}^n)$ exists then, as $z_{d+1} \neq 1$,

$$\begin{aligned}
&\left\{ c_1 \left(\frac{z_1}{z_{d+1}} \right)^n \frac{z_1 - 1}{z_{d+1} - 1} + \ldots + c_d \left(\frac{z_d}{z_{d+1}} \right)^n \frac{z_d - 1}{z_{d+1} - 1} + c_{d+1} \right\} z_{d+1}^n (z_{d+1} - 1) \\
&\qquad = c_1 z_1^n (z_1 - 1) + \ldots + c_{d+1} z_{d+1}^n (z_{d+1} - 1) \\
&\qquad = c_1 z_1^{n+1} + \ldots + c_{d+1} z_{d+1}^{n+1} - \left(c_1 z_1^n + \ldots + c_{d+1} z_{d+1}^n \right) \ \stackrel{n\to\infty}{\longrightarrow} \ 0\,,
\end{aligned}$$

which in turn yields

$$\lim_{n\to\infty} \left\{ c_1 \frac{z_1 - 1}{z_{d+1} - 1} \left(\frac{z_1}{z_{d+1}} \right)^n + \ldots + c_d \frac{z_d - 1}{z_{d+1} - 1} \left(\frac{z_d}{z_{d+1}} \right)^n \right\} = -c_{d+1}\,.$$

Note that $\dfrac{z_j}{z_{d+1}} \notin \{1\} \cup \left\{ \dfrac{z_k}{z_{d+1}} : k \neq j \right\}$ for every $1 \le j \le d$. By the induction assumption, $c_j \dfrac{z_j - 1}{z_{d+1} - 1} = 0$ for all $1 \le j \le d$. Hence $c_1 = \ldots = c_d = 0$, and clearly $c_{d+1} = 0$ as well. $\square$

Two simple consequences of Lemma A.1 have been used repeatedly.

Lemma A.2. *Let* $0 = t_0 < t_1 < \ldots < t_d < t_{d+1} = \pi$ *and* $c_0, c_1 \ldots, c_d, c_{d+1} \in \mathbb{C}$*. If*

$$\lim_{n\to\infty} \Re(c_0 e^{\imath n t_0} + c_1 e^{\imath n t_1} + \ldots + c_d e^{\imath n t_d} + c_{d+1} e^{\imath n t_{d+1}}) = 0\,,$$

then $\Re c_0 = \Re c_{d+1} = 0$ *and* $c_1 = \ldots = c_d = 0$*.*

Proof. For every $j \in \{1, \dots, 2d+1\}$ let

$$z_j = \begin{cases} e^{\imath t_j} & \text{if } 1 \le j \le d+1\,, \\ e^{-\imath t_{2d+2-j}} & \text{if } d+2 \le j \le 2d+1\,, \end{cases}$$

and note that $z_j \notin \{1\} \cup \{z_k : k \neq j\}$. Since

$$\begin{aligned} 2\lim_{n\to\infty} \Re\left(c_0 e^{\imath n t_0} + c_1 e^{\imath n t_1} + \ldots + c_d e^{\imath n t_d} + c_{d+1} e^{\imath n t_{d+1}}\right) - 2\Re c_0 \\ = 2\lim_{n\to\infty} \Re\left(c_1 e^{\imath n t_1} + \ldots + c_d e^{\imath n t_d} + c_{d+1} e^{\imath n t_{d+1}}\right) \\ = \lim_{n\to\infty}\left(\sum\nolimits_{j=1}^{d} c_j z_j^n + 2(\Re c_{d+1}) z_{d+1}^n + \sum\nolimits_{j=d+2}^{2d+1} \overline{c_{2d+2-j}} z_j^n\right) \end{aligned}$$

exists by assumption, Lemma A.1 shows that $c_1 = \ldots = c_d = 0$ and $\Re c_{d+1} = 0$, and so clearly $\Re c_0 = 0$ as well. $\square$

Lemma A.3. *Given any $z_1, \dots, z_d \in \mathbb{S}$, the following are equivalent:*

(i) *If $c_1, \dots, c_d \in \mathbb{C}$ and $\lim_{n\to\infty} \Re(c_1 z_1^n + \ldots + c_d z_d^n)$ exists then $c_1 = \ldots = c_d = 0$;*

(ii) *$z_j \notin \{-1, 1\} \cup \{z_k, \overline{z_k} : k \neq j\}$ for every $1 \le j \le d$.*

Proof. Clearly (i)⇒(ii) because if $z_j \in \{-1, 1\}$ for some $1 \le j \le d$ simply let $c_j = \imath$ and $c_\ell = 0$ for all $\ell \neq j$, whereas if $z_j \in \{z_k, \overline{z_k}\}$ for some $j \neq k$, take $c_j = 1$, $c_k = -1$, and $c_\ell = 0$ for all $\ell \in \{1, \dots, d\} \setminus \{j, k\}$. Conversely, if

$$\lim_{n\to\infty} \Re(c_1 z_1^n + \ldots + c_d z_d^n) = \tfrac{1}{2}\lim_{n\to\infty}(c_1 z_1^n + \overline{c_1}\,\overline{z_1}^n + \ldots + c_d z_d^n + \overline{c_d}\,\overline{z_d}^n)$$

exists then, by Lemma A.1, $c_1 = \ldots = c_d = 0$ unless either $z_j = 1$ or $z_j = \overline{z_j}$ (and hence $z_j \in \{-1, 1\}$) for some j, or else $z_j \in \{z_k, \overline{z_k}\}$ for some $j \neq k$. Overall, $c_1 = \ldots = c_d = 0$ unless $z_j \in \{-1, 1, z_k, \overline{z_k}\}$ for some $j \neq k$. Thus (ii)⇒(i), as claimed. $\square$

Let $\vartheta_1, \dots, \vartheta_d$ and $\beta \neq 0$ be real numbers, and $p_1, \dots, p_d$ integers. With these ingredients, consider the sequence (x_n) of real numbers given by

$$x_n = p_1 n\vartheta_1 + \ldots + p_d n\vartheta_d + \beta \ln\left|u_1 \cos(2\pi n\vartheta_1) + \ldots + u_d \cos(2\pi n\vartheta_d)\right|, \quad \forall n \in \mathbb{N}\,, \tag{A.1}$$

where $u \in \mathbb{R}^d$. Recall that Lemma 2.7, which has been instrumental in the proof of Theorem 3.4, asserts that it is possible to choose $u \in \mathbb{R}^d$ in such a way that (x_n) is *not* u.d. mod 1 whenever the $d+1$ numbers $1, \vartheta_1, \dots, \vartheta_d$ are $\mathbb{Q}$-independent. The remainder of this appendix is devoted to providing a rigorous proof of Lemma 2.7.

To prepare for the argument, recall that $\mathbb{T}^d$ denotes the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$, together with the σ-algebra $\mathcal{B}(\mathbb{T}^d)$ of its Borel sets. Let $\mathcal{P}(\mathbb{T}^d)$ be the set of all probability measures on $\left(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d)\right)$, and given any $\mu \in \mathcal{P}(\mathbb{T}^d)$, associate with it the family $\left(\widehat{\mu}(k)\right)_{k\in\mathbb{Z}^d}$ of its *Fourier coefficients*, defined as

$$\widehat{\mu}(k) = \int_{\mathbb{T}^d} e^{2\pi\imath k^\top t}\,\mathrm{d}\mu(t) = \int_{\mathbb{T}^d} e^{2\pi\imath(k_1 t_1 + \ldots + k_d t_d)}\,\mathrm{d}\mu(t_1, \dots, t_d)\,, \quad \forall k \in \mathbb{Z}^d\,.$$

Recall that $\mu \mapsto \big(\widehat{\mu}(k)\big)_{k\in\mathbb{Z}^d}$ is one-to-one, i.e., the Fourier coefficients determine μ uniquely. Arguably the most prominent element in $\mathcal{P}(\mathbb{T}^d)$ is the Haar measure $\lambda_{\mathbb{T}^d}$ for which, with $\mathrm{d}\lambda_{\mathbb{T}^d}(t)$ abbreviated $\mathrm{d}t$ as usual,

$$\widehat{\lambda_{\mathbb{T}^d}}(k) = \int_{\mathbb{T}^d} e^{2\pi\imath(k_1t_1+\ldots+k_dt_d)}\,\mathrm{d}t = \prod\nolimits_{j=1}^d \int_{\mathbb{T}} e^{2\pi\imath k_j t}\mathrm{d}t = \begin{cases} 1 & \text{if } k = 0\in\mathbb{Z}^d\,, \\ 0 & \text{if } k\neq 0\,. \end{cases}$$

Given $\mu\in\mathcal{P}(\mathbb{T}^d)$, therefore, to show that $\mu\neq\lambda_{\mathbb{T}^d}$ it is (necessary and) sufficient to find at least one $k\in\mathbb{Z}^d\setminus\{0\}$ for which $\widehat{\mu}(k)\neq 0$. Recall also that, given any (Borel) measurable map $T:\mathbb{T}^d\to\mathbb{T}$, each $\mu\in\mathcal{P}(\mathbb{T}^d)$ induces a unique $\mu\circ T^{-1}\in\mathcal{P}(\mathbb{T})$, via

$$\mu\circ T^{-1}(B) = \mu\big(T^{-1}(B)\big)\,, \quad \forall B\in\mathcal{B}(\mathbb{T})\,.$$

Note that the Fourier coefficients of $\mu\circ T^{-1}$ are simply

$$\widehat{\mu\circ T^{-1}}(k) = \int_{\mathbb{T}} e^{2\pi\imath kt}\mathrm{d}(\mu\circ T^{-1})(t) = \int_{\mathbb{T}^d} e^{2\pi\imath kT(t)}\mathrm{d}\mu(t)\,, \quad k\in\mathbb{Z}\,.$$

If in particular $d=1$ and $\mu\circ T^{-1}=\mu$ then μ is said to be T-*invariant* (and T is μ-*preserving*).

With a view towards Lemma 2.7, for any $p_1,\ldots,p_d\in\mathbb{Z}$ and $\beta\in\mathbb{R}$ consider the map

$$\Lambda_u : \begin{cases} \mathbb{T}^d & \to \quad \mathbb{T}\,, \\ t & \mapsto \quad \big\langle p_1t_1+\ldots+p_dt_d+\beta\ln\big|u_1\cos(2\pi t_1)+\ldots+u_d\cos(2\pi t_d)\big|\big\rangle\,; \end{cases} \tag{A.2}$$

here $u\in\mathbb{R}^d$ may be thought of as a parameter. (Recall the convention, adhered to throughout, that $\ln 0 = 0$.) Note that each map Λ_u is (Borel) measurable, in fact differentiable outside a set of $\lambda_{\mathbb{T}^d}$-measure zero. For every $\mu\in\mathcal{P}(\mathbb{T}^d)$, therefore, the measure $\mu\circ\Lambda_u^{-1}$ is a well-defined element of $\mathcal{P}(\mathbb{T})$. Lemma 2.7 is a consequence of the following fact which may also be of independent interest.

Theorem A.4. *For every $p_1,\ldots,p_d\in\mathbb{Z}$ and $\beta\in\mathbb{R}\setminus\{0\}$, there exists $u\in\mathbb{R}^d$ such that $\lambda_{\mathbb{T}^d}\circ\Lambda_u^{-1}\neq\lambda_{\mathbb{T}}$, with Λ_u given by* (A.2).

To see that Theorem A.4 does indeed imply Lemma 2.7, let $p_1,\ldots,p_d\in\mathbb{Z}$ and $\beta\in\mathbb{R}\setminus\{0\}$ be given, and pick $u\in\mathbb{R}^d$ such that $\lambda_{\mathbb{T}^d}\circ\Lambda_u^{-1}\neq\lambda_{\mathbb{T}}$. Consequently, there exists a continuous function $f:\mathbb{T}\to\mathbb{C}$ for which $\int_{\mathbb{T}} f\,\mathrm{d}(\lambda_{\mathbb{T}^d}\circ\Lambda_u^{-1})\neq\int_{\mathbb{T}} f\,\mathrm{d}\lambda_{\mathbb{T}}$. Note that $f\circ\Lambda_u:\mathbb{T}^d\to\mathbb{C}$ is continuous $\lambda_{\mathbb{T}^d}$-almost everywhere as well as bounded, hence Riemann integrable. Also recall that the sequence $\big((n\vartheta_1,\ldots,n\vartheta_d)\big)$ is u.d. mod 1 in $\mathbb{R}^d$ whenever $1,\vartheta_1,\ldots,\vartheta_d$ are $\mathbb{Q}$-independent [26, Exp.I.6.1]. In the latter case, therefore,

$$\begin{aligned}\lim\nolimits_{N\to\infty}\frac{1}{N}\sum\nolimits_{n=1}^N f(\langle x_n\rangle) &= \lim\nolimits_{N\to\infty}\frac{1}{N}\sum\nolimits_{n=1}^N f\circ\Lambda_u\big(\langle(n\vartheta_1,\ldots,n\vartheta_d)\rangle\big) \\ &= \int_{\mathbb{T}^d} f\circ\Lambda_u\,\mathrm{d}\lambda_{\mathbb{T}^d} = \int_{\mathbb{T}} f\,\mathrm{d}(\lambda_{\mathbb{T}^d}\circ\Lambda_u^{-1}) \neq \int_{\mathbb{T}} f\,\mathrm{d}\lambda_{\mathbb{T}}\,,\end{aligned}$$

showing that (x_n) is *not* u.d. mod 1.

Thus it remains to prove Theorem A.4. Though the assertion of the latter is quite plausible intuitively, the authors do not know of any simple but rigorous justification. The proof presented here is computational and proceeds in essentially two steps: First the case of $d = 1$ is analyzed in detail. Specifically, it is shown that $\lambda_{\mathbb{T}} \circ \Lambda_u^{-1} \neq \lambda_{\mathbb{T}}$ unless $p_1 \neq 0$ and $\beta u_1 = 0$. For itself, this could be seen directly by noticing that the map $\Lambda_u : \mathbb{T} \to \mathbb{T}$ has a non-degenerate critical point whenever $\beta u_1 \neq 0$, and hence cannot possibly preserve $\lambda_{\mathbb{T}}$, see e.g. [5, Lem.2.6] or [6, Ex.5.27(iii)]. The more elaborate calculation given here, however, is useful also in the second step of the proof, i.e. the analysis for $d \geq 2$. As it turns out, the case of $d \geq 2$ can, in essence, be reduced to calculations already done for $d = 1$.

To concisely formulate the subsequent results, recall that the Euler Gamma function, denoted $\Gamma = \Gamma(z)$ as usual, is a meromorphic function with poles precisely at $z \in -\mathbb{N}_0 = \{0, -1, -2, \ldots\}$, and $\Gamma(z+1) = z\Gamma(z) \neq 0$ for every $z \in \mathbb{C} \setminus (-\mathbb{N}_0)$. Also, for convenience every "empty sum" is understood to equal zero, e.g. $\sum_{2 \leq j \leq 1} j^2 = 0$, whereas every "empty product" is understood to equal 1, e.g. $\prod_{2 \leq j \leq 1} j^2 = 1$. Finally, the standard (ascending) Pochhammer symbol $(z)_n$ will be used where, given any $z \in \mathbb{C}$,

$$(z)_n := z(z+1)\ldots(z+n-1) = \prod\nolimits_{\ell=0}^{n-1} (z+\ell)\,, \quad \forall n \in \mathbb{N}\,,$$

and $(z)_0 := 1$, in accordance with the convention on empty products. Note that $(z)_n = \Gamma(z+n)/\Gamma(z)$ whenever $z \notin \mathbb{C} \setminus (-\mathbb{N}_0)$.

For every $p \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, consider now the integral

$$I_{p,\beta} := \int_{\mathbb{T}} e^{4\pi\imath pt + 2\imath\beta \ln|\cos(2\pi t)|}\,\mathrm{d}t\,. \tag{A.3}$$

The specific form of $I_{p,\beta}$ is suggested by the Fourier coefficients of $\lambda_{\mathbb{T}} \circ \Lambda_u^{-1}$ in the case of $d = 1$; see the proof of Lemma A.6 below. Not surprisingly, the value of $I_{p,\beta}$ can be expressed explicitly by means of special functions.

Lemma A.5. *For every $p \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$,*

$$I_{p,\beta} = (-1)^p e^{-\imath\beta \ln 4} \frac{2\imath\beta\Gamma(2\imath\beta)}{\big(\imath\beta\Gamma(\imath\beta)\big)^2} \cdot \frac{(-\imath\beta)_{|p|}}{(1+\imath\beta)_{|p|}}\,, \tag{A.4}$$

and hence in particular

$$|I_{p,\beta}|^2 = \frac{\beta \tanh(\pi\beta)}{\pi(p^2+\beta^2)} > 0\,. \tag{A.5}$$

Proof. Substituting $-t$ for t in (A.3) shows that $I_{p,\beta} = I_{|p|,\beta}$, and a straightforward calculation, with T_ℓ denoting the ℓ-th Chebyshev polynomial ($\ell \in \mathbb{N}_0$), yields

$$\begin{aligned}
I_{p,\beta} &= \int_{\mathbb{T}} e^{4\pi\imath|p|t+2\imath\beta\ln|\cos(2\pi t)|}\,\mathrm{d}t = \int_0^1 e^{2\pi\imath|p|x+2\imath\beta\ln|\cos(\pi x)|}\,\mathrm{d}x \\
&= \int_0^{\frac12} 2\cos(2\pi|p|x)e^{2\imath\beta\ln|\cos(\pi x)|}\,\mathrm{d}x = 2\int_0^{\frac12} T_{2|p|}\big(\cos(\pi x)\big)e^{2\imath\beta\ln|\cos(\pi x)|}\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^1 \frac{T_{2|p|}(x)}{\sqrt{1-x^2}}e^{2\imath\beta\ln x}\,\mathrm{d}x = \frac{2}{\pi}\int_0^{+\infty} T_{2|p|}\left(\frac{1}{\sqrt{1+x^2}}\right)\frac{e^{-\imath\beta\ln(1+x^2)}}{1+x^2}\,\mathrm{d}x\,.
\end{aligned}$$

As the polynomial $T_{2|p|}$ can, for every $p \in \mathbb{Z}$ and $y \neq 0$, be written as

$$T_{2|p|}(y) = y^{2|p|}\sum\nolimits_{\ell=0}^{|p|}\binom{2|p|}{2\ell}(1-y^{-2})^{\ell}\,,$$

it follows that

$$\begin{aligned}
I_{p,\beta} &= \frac{2}{\pi}\sum\nolimits_{\ell=0}^{|p|}(-1)^{\ell}\binom{2|p|}{2\ell}\int_0^{+\infty}\frac{x^{2\ell}}{(1+x^2)^{1+|p|+\imath\beta}}\,\mathrm{d}x \\
&= \frac{1}{\pi}\sum\nolimits_{\ell=0}^{|p|}(-1)^{\ell}\binom{2|p|}{2\ell}\int_0^{+\infty}\frac{x^{\ell-\frac12}}{(1+x)^{1+|p|+\imath\beta}}\,\mathrm{d}x \\
&= \frac{1}{\pi\Gamma(1+|p|+\imath\beta)}\sum\nolimits_{\ell=0}^{|p|}(-1)^{\ell}\binom{2|p|}{2\ell}\Gamma(\tfrac12+\ell)\Gamma(\tfrac12+|p|-\ell+\imath\beta)\,.
\end{aligned}$$

Note that Γ is finite and non-zero for each argument appearing in this sum. Recall that

$$\Gamma(\tfrac12+\ell) = \frac{(2\ell)!\sqrt{\pi}}{\ell!\,2^{2\ell}}\,,\quad \forall\ell\in\mathbb{N}_0\,,$$

and so

$$\begin{aligned}
I_{p,\beta} &= \frac{(-1)^p(2|p|)!}{\sqrt{\pi}2^{2|p|}\Gamma(1+|p|+\imath\beta)}\sum\nolimits_{\ell=0}^{|p|}\left\{(-1)^{\ell}\frac{2^{2\ell}\Gamma(\frac12+\ell+\imath\beta)}{(2\ell)!(|p|-\ell)!}\right\} \\
&= \frac{(-1)^p\Gamma(\frac12+|p|)\Gamma(\frac12+\imath\beta)}{\pi\Gamma(1+|p|+\imath\beta)}\sum\nolimits_{\ell=0}^{|p|}\left\{(-1)^{\ell}\binom{|p|}{\ell}\prod\nolimits_{k=1}^{\ell}\frac{2k-1+2\imath\beta}{2k-1}\right\} \\
&= \frac{(-1)^p\Gamma(\frac12+\imath\beta)}{\sqrt{\pi}2^{|p|}\Gamma(1+|p|+\imath\beta)}\cdot \\
&\qquad\cdot\sum\nolimits_{\ell=0}^{|p|}\left\{(-1)^{\ell}\binom{|p|}{\ell}\prod\nolimits_{k=1}^{\ell}(2k-1+2\imath\beta)\prod\nolimits_{k=\ell+1}^{|p|}(2k-1)\right\} \\
&= \frac{(-1)^p\Gamma(\frac12+\imath\beta)}{\sqrt{\pi}2^{|p|}\Gamma(1+|p|+\imath\beta)}R_{|p|}(2\imath\beta)\,,
\end{aligned}$$

where, for every $m\in\mathbb{N}_0$, the polynomial R_m is given by

$$R_m(z) = \sum\nolimits_{\ell=0}^{m}\left\{(-1)^{\ell}\binom{m}{\ell}\prod\nolimits_{k=1}^{\ell}(2k-1+z)\prod\nolimits_{k=\ell+1}^{m}(2k-1)\right\}\,. \tag{A.6}$$

Thus for example $R_0(z) \equiv 1$, $R_1(z) = -z$, $R_2(z) = -2z + z^2$. Note that the degree of R_m equals m, and for every $m \in \mathbb{N}$ and $j \in \{0, 1, \dots, m-1\}$,

$$\begin{aligned} R_m(2j) &= \sum\nolimits_{\ell=0}^{m} \left\{ (-1)^\ell \binom{m}{\ell} \prod\nolimits_{k=1}^{\ell} (2k-1+2j) \prod\nolimits_{k=\ell+1}^{m} (2k-1) \right\} \\ &= \sum\nolimits_{\ell=0}^{m} \left\{ (-1)^\ell \binom{m}{\ell} \prod\nolimits_{k=j+1}^{m} (2k-1) \prod\nolimits_{k=\ell+1}^{\ell+j} (2k-1) \right\} \\ &= \left\{ \prod\nolimits_{k=j+1}^{m} (2k-1) \right\} \sum\nolimits_{\ell=0}^{m} \left\{ (-1)^\ell \binom{m}{\ell} \prod\nolimits_{k=1}^{j} (2\ell+2k-1) \right\} = 0 \, . \end{aligned}$$

Here the elementary fact has been used that $\sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} Q(\ell) = 0$ holds for every polynomial Q of degree less than m. As the polynomial R_m has degree m, it cannot have any further roots besides $0, 2, 4, \dots, 2m-2$, and so

$$R_m(z) = c_m \prod\nolimits_{\ell=0}^{m-1} (z - 2\ell) \, , \tag{A.7}$$

with a constant c_m yet to be determined. The correct value of c_m is readily found by observing that (A.7) yields

$$R_m(-1) = c_m \prod\nolimits_{\ell=0}^{m-1} (-1-2\ell) = c_m (-1)^m \cdot 1 \cdot 3 \cdot \ldots \cdot (2m-1) \, ,$$

whereas, by the very definition (A.6) of R_m,

$$R_m(-1) = \sum\nolimits_{\ell=0}^{m} \left\{ (-1)^\ell \binom{m}{\ell} \prod\nolimits_{k=1}^{\ell} (2k-2) \prod\nolimits_{k=\ell+1}^{m} (2k-1) \right\} = \prod\nolimits_{k=1}^{m} (2k-1) \, .$$

Thus $c_m = (-1)^m$, and overall

$$R_m(z) = (-1)^m \prod\nolimits_{\ell=0}^{m-1} (z - 2\ell) = \prod\nolimits_{\ell=0}^{m-1} (2\ell - z) = 2^m \left(-\tfrac{1}{2} z\right)_m \, .$$

With this, one obtains

$$\begin{aligned} I_{p,\beta} &= \frac{(-1)^p \Gamma(\frac{1}{2} + \imath\beta)}{\sqrt{\pi} 2^{|p|} \Gamma(1 + |p| + \imath\beta)} \prod\nolimits_{\ell=0}^{|p|-1} (2\ell - 2\imath\beta) \\ &= \frac{2(-1)^{p+1} e^{-\imath\beta \ln 4}}{|p| - \imath\beta} \cdot \frac{\Gamma(2\imath\beta)}{\Gamma(\imath\beta)^2} \prod\nolimits_{\ell=1}^{|p|} \frac{\ell - \imath\beta}{\ell + \imath\beta} \\ &= (-1)^p e^{-\imath\beta \ln 4} \frac{2\imath\beta \Gamma(2\imath\beta)}{\big(\imath\beta\Gamma(\imath\beta)\big)^2} \cdot \frac{(-\imath\beta)_{|p|}}{(1 + \imath\beta)_{|p|}} \, , \end{aligned}$$

where the so-called Legendre duplication formula for the Γ-function has been used in the form

$$\Gamma(\imath\beta)\Gamma(\tfrac{1}{2} + \imath\beta) = 2^{1-2\imath\beta} \sqrt{\pi} \, \Gamma(2\imath\beta) \, , \quad \forall \beta \in \mathbb{R} \setminus \{0\} \, .$$

Thus (A.4) has been established, and together with the standard fact

$$|\Gamma(\imath\beta)|^2 = \frac{\pi}{\beta \sinh(\pi\beta)}\,, \quad \forall \beta \in \mathbb{R} \setminus \{0\}\,,$$

this immediately yields

$$|I_{p,\beta}|^2 = \frac{4}{p^2+\beta^2} \cdot \frac{|\Gamma(2\imath\beta)|^2}{|\Gamma(\imath\beta)|^4} = \frac{4\beta^2\pi}{2\beta\sinh(2\pi\beta)} \cdot \frac{\sinh^2(\pi\beta)}{\pi^2(p^2+\beta^2)} = \frac{\beta\tanh(\pi\beta)}{\pi(p^2+\beta^2)}\,,$$

i.e., (A.5) holds as claimed. □

An immediate consequence of Lemma A.5 is that for $d=1$ the map Λ_u does typically not preserve $\lambda_{\mathbb{T}}$. Notice that the following result is much stronger than (and hence obviously proves) Theorem A.4 for $d=1$.

Lemma A.6. *Let $p_1 \in \mathbb{Z}$, $\beta \in \mathbb{R}$ and $u_1 \in \mathbb{R}$. Then $\lambda_{\mathbb{T}} \circ \Lambda_u^{-1} = \lambda_{\mathbb{T}}$, where Λ_u is given by* (A.2) *with $d=1$, if and only if $p_1 \neq 0$ and $\beta u_1 = 0$.*

Proof. Simply note that for $\beta u_1 = 0$ and every $k \in \mathbb{Z}$,

$$\widehat{\lambda_{\mathbb{T}} \circ \Lambda_u^{-1}}(k) = \begin{cases} 1 & \text{if } kp_1 = 0\,, \\ 0 & \text{if } kp_1 \neq 0\,, \end{cases}$$

and hence $\lambda_{\mathbb{T}} \circ \Lambda_u^{-1} = \lambda_{\mathbb{T}}$ precisely if $p_1 \neq 0$. On the other hand, for $\beta u_1 \neq 0$,

$$\begin{aligned} \widehat{\lambda_{\mathbb{T}} \circ \Lambda_u^{-1}}(2) &= \int_{\mathbb{T}} e^{4\pi\imath t}\, \mathrm{d}(\lambda_{\mathbb{T}} \circ \Lambda_u^{-1})(t) = \int_{\mathbb{T}} e^{4\pi\imath(p_1 t + \beta \ln|u_1 \cos(2\pi t)|)}\, \mathrm{d}t \\ &= e^{4\pi\imath\beta \ln|u_1|} I_{p_1, 2\pi\beta} \neq 0\,, \end{aligned}$$

showing that $\lambda_{\mathbb{T}} \circ \Lambda_u^{-1} \neq \lambda_{\mathbb{T}}$ in this case. □

As indicated earlier, the case of $d \geq 2$ of Theorem A.4 is now going to be studied and, in a way, reduced to the case of $d=1$. To this end, let again $p \in \mathbb{Z}$ and $\beta \in \mathbb{R}$ be given, and consider the function $i_{p,\beta} : \mathbb{R} \to \mathbb{C}$ with

$$i_{p,\beta}(x) = \int_{\mathbb{T}} e^{4\pi\imath pt + 2\imath\beta \ln|x+\cos(2\pi t)|}\, \mathrm{d}t\,, \quad \forall x \in \mathbb{R}\,. \tag{A.8}$$

A few elementary properties of $i_{p,\beta}$ are contained in

Lemma A.7. *For every $p \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, the function $i_{p,\beta}$ is continuous and even, with $|i_{p,\beta}(x)| \leq 1$ for all $x \in \mathbb{R}$. Moreover, $i_{p,\beta}(0) = I_{p,\beta}$ and $i_{p,\beta}(1) = e^{\imath\beta \ln 4} I_{2p,2\beta}$; in particular, $i_{p,\beta}(0) \neq i_{p,\beta}(1)$ whenever $\beta \neq 0$.*

Proof. Since for every $x \in \mathbb{R}$,

$$\lim\nolimits_{y \to x} \ln|y + \cos(2\pi t)| = \ln|x + \cos(2\pi t)|$$

holds for all but (at most) two $t \in \mathbb{T}$, the continuity of $i_{p,\beta}$ follows from the Dominated Convergence Theorem. Clearly, $i_{p,\beta}$ is even, with $|i_{p,\beta}(x)| \leq \int_{\mathbb{T}} 1\, \mathrm{d}\lambda_{\mathbb{T}} = 1$ for every $x \in \mathbb{R}$, and $i_{p,\beta}(0) = I_{p,\beta}$. Finally, it follows from

$$i_{p,\beta}(1) = e^{\imath\beta \ln 4} \int_{\mathbb{T}} e^{4\pi\imath pt + 4\imath\beta \ln|\cos(\pi t)|}\, \mathrm{d}t = e^{\imath\beta \ln 4} I_{2p,2\beta}\,,$$

and (A.5) that, for every $p \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$,

$$\left| \frac{i_{p,\beta}(1)}{i_{p,\beta}(0)} \right|^2 = \frac{|I_{2p,2\beta}|^2}{|I_{p,\beta}|^2} = \frac{2\beta \tanh(2\pi\beta)}{4p^2 + 4\beta^2} \cdot \frac{p^2 + \beta^2}{\beta \tanh(\pi\beta)} = \frac{1}{2} \left(1 + \frac{1}{\cosh(2\pi\beta)} \right) < 1 \, ,$$

and hence $i_{p,\beta}(1) \neq i_{p,\beta}(0)$. □

The subsequent analysis crucially depends on the fact that $i_{p,\beta}$ is actually much smoother than Lemma A.7 seems to suggest. Recall that a function $f : \mathbb{R}^m \to \mathbb{C}$ is *real-analytic* on an open set $\mathcal{U} \subset \mathbb{R}^m$ if f can, in a neighbourhood of each point in $\mathcal{U}$, be represented as a convergent power series. As will become clear soon, the ultimate proof of Theorem A.4 relies heavily on the following refinement of Lemma A.7.

Lemma A.8. *For every $p \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, the function $i_{p,\beta}$ is real-analytic on $(-1,1)$.*

Proof. As $i_{p,0}$ is constant, and thus trivially real-analytic, henceforth assume $\beta \neq 0$. By Lemma A.7, the function $f : \mathbb{T} \to \mathbb{C}$ with $f(t) = i_{p,\beta}\big(\cos(\pi t)\big)$ is well-defined and continuous. Hence it can be represented, at least in the $L^2(\lambda_{\mathbb{T}})$-sense, as a Fourier series $f(t) \sim \sum_{k \in \mathbb{Z}} c_k e^{2\pi\imath kt}$ where, for every $k \in \mathbb{Z}$,

$$\begin{aligned} c_k &= \int_{\mathbb{T}} f(t) e^{-2\pi\imath kt} \, \mathrm{d}t = \int_{\mathbb{T}^2} e^{-2\pi\imath kt_1 + 4\pi\imath|p|t_2 + 2\imath\beta \ln|\cos(\pi t_1) + \cos(2\pi t_2)|} \, \mathrm{d}t \\ &= \int_{\mathbb{T}^2} e^{4\pi\imath|p|(t_1 - t_2) - 4\pi\imath k(t_1 + t_2) + 2\imath\beta \ln|2\cos(2\pi t_1)\cos(2\pi t_2)|} \, \mathrm{d}t \\ &= e^{\imath\beta \ln 4} \int_{\mathbb{T}} e^{4\pi\imath(|p| - k)t + 2\imath\beta \ln|\cos(2\pi t)|} \, \mathrm{d}t \int_{\mathbb{T}} e^{4\pi\imath(|p| + k)t + 2\imath\beta \ln|\cos(2\pi t)|} \, \mathrm{d}t \\ &= e^{\imath\beta \ln 4} I_{|p|-k,\beta} I_{|p|+k,\beta} \, . \end{aligned}$$

Since $c_{-k} = c_k$, the Fourier series of f is

$$c_0 + 2\sum\nolimits_{n \in \mathbb{N}} c_n \cos(2\pi nt) = c_0 + 2\sum\nolimits_{n=1}^{\infty} c_n T_{2n}\big(\cos(\pi t)\big) \, ,$$

and since furthermore

$$|c_n| = |I_{n-|p|,\beta} I_{n+|p|,\beta}| = \frac{\beta \tanh(\pi\beta)}{\pi\sqrt{(n^2 + p^2 + \beta^2)^2 - 4n^2p^2}} = \mathcal{O}(n^{-2}) \, , \quad \text{as } n \to \infty \, ,$$

and hence $\sum_{n=1}^{\infty} |c_n| < +\infty$, this series converges uniformly on $\mathbb{T}$, by the Weierstrass M-test. It follows that $i_{p,\beta}(x) = c_0 + 2\sum_{n=1}^{\infty} c_n T_{2n}(x)$ uniformly in $x \in [-1,1]$.

For every $y \in (-1,1)$, consider now the auxiliary function

$$h(x,y) := 2\sum\nolimits_{n=1+|p|}^{\infty} c_n T_{2n}(x) y^n \, .$$

Note that $i_{p,\beta}(x) = c_0 + 2\sum_{n=1}^{|p|} c_n T_{2n}(x) + \lim_{y \uparrow 1} h(x,y)$ uniformly in $x \in [-1,1]$. In addition, introduce an analytic function on the open unit disc as

$$H(z) := \sum\nolimits_{n=1+|p|}^{\infty} c_n z^n \, , \quad \forall z \in \mathbb{C} : |z| < 1 \, , \tag{A.9}$$

and observe that

$$
\begin{aligned}
H(z) &= z^{1+|p|}\sum\nolimits_{n=0}^{\infty} c_{n+1+|p|}z^n = e^{\imath\beta\ln 4}z^{1+|p|}\sum\nolimits_{n=0}^{\infty} I_{n+1,\beta}I_{n+1+2|p|,\beta}z^n \\
&= e^{-\imath\beta\ln 4}z^{1+|p|}\frac{\big(2\imath\beta\Gamma(2\imath\beta)\big)^2}{\big(\imath\beta\Gamma(\imath\beta)\big)^4}\sum\nolimits_{n=0}^{\infty}\frac{(-\imath\beta)_{n+1}(-\imath\beta)_{n+1+2|p|}}{(1+\imath\beta)_{n+1}(1+\imath\beta)_{n+1+2|p|}}z^n \\
&= e^{-\imath\beta\ln 4}z^{1+|p|}\frac{\big(2\imath\beta\Gamma(2\imath\beta)\big)^2}{\big(\imath\beta\Gamma(\imath\beta)\big)^4}\cdot\frac{(\imath\beta)^2}{(1+\imath\beta)^2}\cdot\frac{(1-\imath\beta)_{2|p|}}{(2+\imath\beta)_{2|p|}}\cdot \\
&\qquad\cdot\sum\nolimits_{n=0}^{\infty}\frac{(1-\imath\beta)_n(1+2|p|-\imath\beta)_n}{(2+\imath\beta)_n(2+2|p|+\imath\beta)_n}z^n \\
&= \frac{4e^{-\imath\beta\ln 4}\Gamma(2\imath\beta)^2(1-\imath\beta)_{2|p|}}{(1+\imath\beta)^2\Gamma(\imath\beta)^4(2+\imath\beta)_{2|p|}}\cdot \\
&\qquad\cdot {}_3F_2(1-\imath\beta, 1+2|p|-\imath\beta, 1; 2+\imath\beta, 2+2|p|+\imath\beta; z)z^{1+|p|};
\end{aligned}
$$

here the standard notation for (generalized) hypergeometric functions has been used, see e.g. [28, Ch.II] or [34, Ch.16]. Recall that ${}_3F_2$ is an analytic function on $\mathbb{C}\setminus[1,+\infty)$, that is, on the entire complex plane minus a cut from 1 to ∞ along the positive real axis. Hence H as given by (A.9) can be extended analytically to $\mathbb{C}\setminus[1,+\infty)$ as well. Observe now that

$$
\begin{aligned}
H(e^{2\pi\imath t}y)+H(e^{-2\pi\imath t}y) &= 2\sum\nolimits_{n=1+|p|}^{\infty}c_nT_{2n}\big(\cos(\pi t)\big)y^n \\
&= h\big(\cos(\pi t), y\big), \quad \forall t\in\mathbb{T}, y\in(-1,1).
\end{aligned}
$$

It follows that, for all $x\in[-1,1]$,

$$
\begin{aligned}
i_{p,\beta}(x) &= c_0 + 2\sum\nolimits_{n=1}^{|p|}c_nT_{2n}(x) + \\
&\quad + \lim_{y\uparrow 1}\Big\{H\big((2x^2-1+2\imath x\sqrt{1-x^2})y\big)+H\big((2x^2-1-2\imath x\sqrt{1-x^2})y\big)\Big\} \\
&= c_0 + 2\sum\nolimits_{n=1}^{|p|}c_nT_{2n}(x) + \\
&\quad + H\big(2x^2-1+2\imath x\sqrt{1-x^2}\,\big)+H\big(2x^2-1-2\imath x\sqrt{1-x^2}\,\big).
\end{aligned}
$$

Note now that $2z^2-1\pm 2\imath z\sqrt{1-z^2}\notin[1,+\infty)$ whenever $|z|<1$. The function

$$
z\mapsto c_0+2\sum\nolimits_{n=1}^{|p|}c_nT_{2n}(z)+H\big(2z^2-1+2\imath z\sqrt{1-z^2}\,\big)+H\big(2z^2-1-2\imath z\sqrt{1-z^2}\,\big),
$$

therefore, is analytic on the open unit disc and coincides with $i_{p,\beta}$ on $\{z : |z|<1\}\cap\mathbb{R}=(-1,1)$. Thus $i_{p,\beta}$ is real-analytic on $(-1,1)$, and in fact $i_{p,\beta}(x)=\sum_{n=0}^{\infty}i_{p,\beta}^{(n)}(0)x^n/n!$ for all $x\in(-1,1)$. □

Remark A.9. Since $t\mapsto x+\cos(2\pi t)$ does not change sign on $\mathbb{T}$ whenever $|x|>1$, it is clear from (A.8) that the function $i_{p,\beta}$ is real-analytic on $\mathbb{R}\setminus[-1,1]$ as well.

For every $d\in\mathbb{N}$, define a non-empty open subset of $\mathbb{R}^d$ as

$$
\mathcal{E}_d := \Big\{u\in\mathbb{R}^d : \exists j\in\{1,\ldots,d\} \text{ with } |u_j|>\sum\nolimits_{k\neq j}|u_k|\Big\}.
$$

Geometrically, $\mathcal{E}_d$ is the disjoint union of $2d$ open cones. For example, $\mathcal{E}_1 = \mathbb{R} \setminus \{0\}$ and $\mathcal{E}_2 = \{u \in \mathbb{R}^2 : |u_1| \neq |u_2|\}$, hence $\mathcal{E}_d$ is also dense in $\mathbb{R}^d$ for $d = 1, 2$. For $d \geq 3$ this is no longer the case. In fact, a simple calculation shows that

$$\frac{\mathrm{Leb}(\mathcal{E}_d \cap [-1,1]^d)}{\mathrm{Leb}([-1,1]^d)} = \frac{2^d/\Gamma(d)}{2^d} = \frac{1}{\Gamma(d)}, \quad \forall d \in \mathbb{N},$$

and so the (relative) portion of $\mathbb{R}^d$ taken up by $\mathcal{E}_d$ decays rapidly with growing d.

In order to utilize Lemma A.8 for a proof of Theorem A.4, given any $p_1, \ldots, p_d \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, recall the map Λ_u from (A.2) and consider the integral

$$\begin{aligned} J = J(u) = \widehat{\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1}}(2) &= \int_{\mathbb{T}} e^{4\pi\imath t} \, \mathrm{d}(\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1})(t) \\ &= \int_{\mathbb{T}^d} e^{4\pi\imath(p_1 t_1 + \ldots p_d t_d + \beta \ln |u_1 \cos(2\pi t_1) + \ldots + u_d \cos(2\pi t_d)|)} \, \mathrm{d}t \,. \end{aligned} \tag{A.10}$$

An important consequence of Lemma A.8 is

Lemma A.10. *For every $p_1, \ldots, p_d \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$, the function $u \mapsto J(u)$ given by* (A.10) *is real-analytic and non-constant on each connected component of $\mathcal{E}_d$.*

Proof. If $d = 1$ then, as seen in essence already in the proof of Lemma A.6,

$$u_1 \mapsto J(u_1) = \int_{\mathbb{T}} e^{4\pi\imath p_1 t + 4\pi\imath\beta \ln |u_1 \cos(2\pi t)|} \, \mathrm{d}t = e^{4\pi i \beta \ln |u_1|} I_{p_1, 2\pi\beta}$$

is real-analytic and non-constant on each of the two connected parts of $\mathbb{R} \setminus \{0\} = \mathcal{E}_1$.

Assume in turn that $d \geq 2$. As the roles of $t_1, \ldots, t_d$ can be interchanged in (A.10), assume w.l.o.g. that $u_d \neq 0$. Since $J(\pm u_1, \ldots, \pm u_d) = J(u_1, \ldots, u_d)$ for all $u \in \mathbb{R}^d$ and every possible combination of $+$ and $-$ signs, and since also

$$J(u) = e^{4\pi i \beta \ln |u_d|} J\left(\frac{u_1}{u_d}, \ldots, \frac{u_{d-1}}{u_d}, 1\right),$$

it suffices to show that $\widetilde{J} = \widetilde{J}(u) := J(u_1, \ldots, u_{d-1}, 1)$ is real-analytic and non-constant on $\widetilde{\mathcal{E}}_{d-1} := \{u \in \mathbb{R}^{d-1} : \sum_{j=1}^{d-1} |u_j| < 1\}$. To this end note first that

$$\widetilde{J}(u) = \int_{\mathbb{T}^{d-1}} e^{4\pi\imath(p_1 t_1 + \ldots + p_{d-1} t_{d-1})} i_{p_d, 2\pi\beta}\big(u_1 \cos(2\pi t_1) + \ldots + u_{d-1} \cos(2\pi t_{d-1})\big) \, \mathrm{d}t \,.$$

With Lemma A.7 and the Dominated Convergence Theorem, it is clear that $\widetilde{J}$ is continuous on $\mathbb{R}^{d-1}$. Recall from the proof of Lemma A.8 that $i_{p,\beta}$ can be represented by a power series, namely $i_{p,\beta}(x) = \sum_{n=0}^{\infty} i_{p,\beta}^{(n)}(0) x^n / n!$ for all $p \in \mathbb{Z}$, $\beta \in \mathbb{R}$ and $|x| < 1$. For every $u \in \widetilde{\mathcal{E}}_{d-1}$, therefore,

$$\begin{aligned} \widetilde{J}(u) &= \int_{\mathbb{T}^{d-1}} e^{4\pi\imath(p_1 t_1 + \ldots + p_{d-1} t_{d-1})} \sum\nolimits_{n=0}^{\infty} \frac{i_{p_d, 2\pi\beta}^{(n)}(0)}{n!} \left(\sum\nolimits_{j=1}^{d-1} u_j \cos(2\pi t_j) \right)^n \\ &= \sum\nolimits_{n=0}^{\infty} \frac{i_{p_d, 2\pi\beta}^{(2n)}(0)}{2^{2n}} \sum\nolimits_{|\nu| = n} \left\{ \prod\nolimits_{j=1}^{d-1} \frac{u_j^{2\nu_j}}{(2\nu_j)!} \binom{2\nu_j}{\nu_j + |p_j|} \right\}, \end{aligned} \tag{A.11}$$

where the standard notation for multi-indices $\nu = (\nu_1, \ldots, \nu_{d-1}) \in (\mathbb{N}_0)^{d-1}$ has been used, see e.g. [24, pp.25–29]. Thus $\widetilde{J}$ is real-analytic on $\widetilde{\mathcal{E}}_{d-1}$, by [24, Prop.2.2.7].

It remains to show that $\widetilde{J}$ is non-constant on $\widetilde{\mathcal{E}}_{d-1}$. Consider first the case of $d = 2$, for which (A.11) takes the form

$$\widetilde{J}(u_1) = \sum\nolimits_{n=|p_1|}^{\infty} \frac{i^{(2n)}_{p_2,2\pi\beta}(0)}{2^{2n}} \begin{pmatrix} 2n \\ n+|p_1| \end{pmatrix} \frac{u_1^{2n}}{(2n)!}\,, \quad \forall u_1 \in \widetilde{\mathcal{E}}_1 = (-1, 1)\,. \tag{A.12}$$

Recall that $u_1 \mapsto \widetilde{J}(u_1)$ is continuous. If $p_1 \neq 0$ then $\widetilde{J}(0) = 0$ whereas

$$\begin{aligned}\widetilde{J}(1) &= \int_{\mathbb{T}^2} e^{4\pi\imath(p_1t_1+p_2t_2+\beta\ln|\cos(2\pi t_1)+\cos(2\pi t_2)|)}\,\mathrm{d}t \\ &= \int_{\mathbb{T}^2} e^{4\pi\imath(p_1(t_1-t_2)+p_2(t_1+t_2)+\beta\ln|2\cos(2\pi t_1)\cos(2\pi t_2)|)}\,\mathrm{d}t \\ &= e^{4\pi\imath\beta\ln 2} I_{p_1+p_2,2\pi\beta} I_{p_1-p_2,2\pi\beta} \neq 0\,,\end{aligned}$$

since $\beta \neq 0$. If, on the other hand, $p_1 = 0$ then $\widetilde{J}(0) = I_{p_2,2\pi\beta}$, while $\widetilde{J}(1) = e^{4\pi\imath\beta\ln 2} I^2_{p_2,2\pi\beta} \neq \widetilde{J}(0)$. In either case, therefore, $u_1 \mapsto \widetilde{J}(u_1)$ is non-constant on $\widetilde{\mathcal{E}}_1 = (-1, 1)$. This concludes the proof for $d = 2$.

Finally, to deal with the case of $d \geq 3$, note first that the above argument for $d = 2$ really shows that, given any $p \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$, the number $i^{(2n)}_{p,2\pi\beta}(0)$ is non-zero for infinitely many $n \in \mathbb{N}_0$. (Otherwise, by (A.12), the function $u_1 \mapsto \widetilde{J}(u_1)$ would be constant for $|p_1|$ sufficiently large, which has just been shown not to be the case.) But then

$$\widetilde{J}(u) = \sum\nolimits_{n=|p_1|+\ldots+|p_{d-1}|}^{\infty} \frac{i^{(2n)}_{p_d,2\pi\beta}(0)}{2^{2n}} \sum\nolimits_{|\nu|=n} \left\{ \prod\nolimits_{j=1}^{d-1} \frac{u_j^{2\nu_j}}{(2\nu_j)!} \begin{pmatrix} 2\nu_j \\ \nu_j+|p_j| \end{pmatrix} \right\}$$

is obviously non-constant on $\widetilde{\mathcal{E}}_{d-1}$. $\square$

Given $p_1, \ldots, p_d \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, denote by $\mathcal{D}_d$ the set of all $u \in \mathbb{R}^d$ for which $\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1}$ coincides with $\lambda_{\mathbb{T}}$, i.e., let $\mathcal{D}_d = \{u \in \mathbb{R}^d : \lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1} = \lambda_{\mathbb{T}}\}$. An immediate consequence of Lemma A.10 is

Lemma A.11. *For every $p_1, \ldots, p_d \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$ the set $\mathcal{D}_d \cap \mathcal{E}_d \subset \mathbb{R}^d$ is nowhere dense and has Lebesgue measure zero.*

Proof. This is clear from the fact that $\mathcal{D}_d \cap \mathcal{E}_d \subset \{u \in \mathcal{E}_d : J(u) = 0\}$. As $u \mapsto J(u)$ is real-analytic and non-constant on each component of $\mathcal{E}_d$, the zero-locus of J on $\mathcal{E}_d$ is nowhere dense and has Lebesgue measure zero; see e.g. [8, Lem.19] or [24, Sec.4.1]. $\square$

At long last, the *Proof of Theorem A.4* has become very simple: Since $\mathcal{D}_d \cap \mathcal{E}_d$ is nowhere dense, $\mathcal{E}_d \setminus \mathcal{D}_d \neq \varnothing$, and $\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1} \neq \lambda_{\mathbb{T}}$ for every $u \in \mathcal{E}_d \setminus \mathcal{D}_d$, by the definition of $\mathcal{D}_d$. $\square$

Remark A.12. (i) Since $\mathcal{E}_1$ and $\mathcal{E}_2$ are dense in $\mathbb{R}$ and $\mathbb{R}^2$, respectively, the set $\mathcal{D}_d$ is nowhere dense in $\mathbb{R}^d$ for $d = 1, 2$ whenever $\beta \neq 0$. It may be conjectured that $\mathcal{D}_d$ is nowhere dense (and has Lebesgue measure zero) for $d \geq 3$ also; no proof of, or counter-example to this conjecture is known to the authors.

(ii) Note that $\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1} = \lambda_{\mathbb{T}}$ if, for some $j \in \{1, \ldots, d\}$, both $p_j \neq 0$ and $\beta u_j = 0$. Thus

$$\bigcup\nolimits_{j:p_j\neq 0} \left\{ u \in \mathbb{R}^d : \beta u_j = 0 \right\} \subset \mathcal{D}_d \, , \tag{A.13}$$

and hence for $\beta \neq 0$ the set $\mathcal{D}_d$ contains the union of at most d coordinate hyperplanes. Beyond the conjecture formulated in (i), it is tempting to speculate whether in fact equality holds in (A.13) always, i.e. for any $p_1, \ldots, p_d \in \mathbb{Z}$ and $\beta \in \mathbb{R}$ — as it does for $\beta = 0$ (trivial) and $d = 1$ (Lemma A.6). Obviously, equality in (A.13) would establish a much stronger version of Theorem A.4.

(iii) Even if the set $\mathcal{D}_d \subset \mathbb{R}^d$ is indeed nowhere dense and has Lebesgue measure zero for every $d \in \mathbb{N}$, as conjectured in (i), for large values of d the equality $\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1} = \lambda_{\mathbb{T}}$, though generically false, is nevertheless often true approximately — in some sense, and quite independently of the specific values of $p_1, \ldots, p_d \in \mathbb{Z}$ and $\beta \in \mathbb{R} \setminus \{0\}$. Under mild conditions on these parameters, this observation can easily be made rigorous as follows: Assume, for instance, that the integer sequence (p_n) is not identically zero, say $p_1 \neq 0$ for convenience, and $\beta \neq 0$. Also assume that

$$(u_n) \text{ is a bounded sequence in } \mathbb{R} \text{ with } \sum\nolimits_{n=1}^{\infty} u_n^2 = +\infty \, . \tag{A.14}$$

If $u_1 = 0$ then $\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1} = \lambda_{\mathbb{T}}$ for all $d \in \mathbb{N}$. On the other hand, if $u_1 \neq 0$, let $\sigma_d := \sqrt{1 + \sum_{j=2}^d u_j^2}$ and observe that, for every $k \in \mathbb{Z} \setminus \{0\}$,

$$\widehat{\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1}}(k) = e^{2\pi \imath k \beta \ln \sigma_d} \cdot \\ \cdot \int_{\mathbb{T}^d} e^{2\pi \imath k \left(p_1 t_1 + \sum_{j=2}^d p_j t_j + \beta \ln \left| u_1/\sigma_d \cos(2\pi t_1) + \sum_{j=2}^d u_j/\sigma_d \cos(2\pi t_j) \right| \right)} \, dt \, .$$

Since $\sigma_d \to +\infty$ as $d \to \infty$ yet (u_n) is bounded, it follows from the Central Limit Theorem (see e.g. [12, Sec.9.1]) that $\lim_{d\to\infty} \widehat{\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1}}(k) = 0$. Under the mild assumption (A.14), therefore, $\lim_{d\to\infty} \lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1} = \lambda_{\mathbb{T}}$ in $\mathcal{P}(\mathbb{T})$ in the sense of weak convergence of probability measures. Informally put, the probability measure $\lambda_{\mathbb{T}^d} \circ \Lambda_u^{-1}$ typically differs but little from $\lambda_{\mathbb{T}}$ whenever d is large.

(iv) The above *proof* of Theorem A.4 relies heavily on specific properties of the logarithm, notably on the fact that $\ln |xy| = \ln |x| + \ln |y|$ whenever $xy \neq 0$. It seems plausible, however, that the *conclusion* of that theorem may remain valid if the function $\ln |\cdot|$ in (A.2) is replaced by virtually any non-constant function that is real-analytic on $\mathbb{R} \setminus \{0\}$ and has 0 as a mild singularity. Establishing such a much more general version of Theorem A.4 will likely require a conceptual approach quite different from the rather computational strategy pursued herein.

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