Math 411: Honours Complex Variables

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Contents

Chapter 1

The Complex Numbers

Definition. The *complex numbers*—denoted by $\mathbb{C}$—are $\mathbb{R}^2$ equipped with the operations

$$\begin{aligned}(x,y)+(u,v) &:= (x+u, y+v),\\ (x,y)(u,v) &:= (xu-yv, xv+yu)\end{aligned}$$

for $x, y, u, v \in \mathbb{R}$.

Theorem 1.1 ($\mathbb{C}$ is a Field)**.** *The complex numbers are a field. Specifically, we have:*

- $(0,0)$ *is the identity element of addition;*
- $-(x,y) = (-x,-y)$ *for* $x, y \in \mathbb{R}$*;*
- $(1,0)$ *is the identity element of multiplication;*
- $(x,y)^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ *for* $x, y \in \mathbb{R}$ *with* $(x,y) \neq (0,0)$*.*

Proof (of the last claim only). Let $x, y \in \mathbb{R}$ be such that $(x,y) \neq (0,0)$, and note that

$$\begin{aligned}(x,y)\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) &= \left(\frac{x^2}{x^2+y^2} - \frac{-y^2}{x^2+y^2}, \frac{-xy}{x^2+y^2} + \frac{xy}{x^2+y^2}\right)\\ &= \left(\frac{x^2+y^2}{x^2+y^2}, \frac{-xy+xy}{x^2+y^2}\right)\\ &= (1,0).\end{aligned}$$

□

Proposition 1.1. *The set* $\{(x,0) : x \in \mathbb{R}\}$ *is a subfield of* $\mathbb{C}$*, and the map*

$$\theta\colon \mathbb{R} \to \mathbb{C}, \quad x \mapsto (x,0)$$

is an isomorphism onto its image.

Proposition 1.1 is often worded as:

$$\mathbb{R} \text{ “is” a subfield of } \mathbb{C}.$$

Set $1 := (1, 0)$ and $i := (0, 1)$. Then, for any $z = (x, y) \in \mathbb{C}$, we have

$$z = (x, 0) + (0, y) = (1, 0)(x, 0) + (0, 1)(y, 0) = x + iy.$$

We write

$$\operatorname{Re} z := x = \text{“the } \textit{real part} \text{ of } z\text{”}$$

and

$$\operatorname{Im} z := y = \text{“the } \textit{imaginary part} \text{ of } z\text{”}.$$

The complex number i is called the *imaginary unit* and satisfies

$$i^2 = (0, 1)^2 = (-1, 0) = -1.$$

Unlike $\mathbb{R}$, the set $\mathbb{C} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$ is not *ordered*; there is no notion of positive and negative (greater than or less than) on the complex plane. For example, if i were positive or zero, then $i^2 = -1$ would have to be positive or zero. If i were negative, then $-i$ would be positive, which would imply that $(-i)^2 = i^2 = -1$ is positive. It is thus not possible to divide the complex numbers into of negative, zero, and positive numbers.

The frequently appearing notation $\sqrt{-1}$ for i is misleading and should be avoided, because the rule $\sqrt{xy} = \sqrt{x}\sqrt{y}$ (which one might anticipate) does not hold for negative x and y, as the following contradiction illustrates:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i^2 = -1.$$

Furthermore, by definition $\sqrt{x} \geq 0$, but one cannot write $i \geq 0$, since $\mathbb{C}$ is not ordered.

Definition. For $z = x + iy \in \mathbb{C}$, its *complex conjugate* is defined as $\bar{z} = x - iy$.

Proposition 1.2. *For $z, w \in \mathbb{C}$, the following hold true:*

(i) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$ *and* $\operatorname{Im} z = \frac{1}{2i}(z - \bar{z})$*;*

(ii) $\overline{z + w} = \bar{z} + \bar{w}$*;*

(iii) $\overline{zw} = \bar{z}\bar{w}$*;*

(iv) $\overline{z^{-1}} = \bar{z}^{-1}$ *if* $z \neq 0$*.*

Proof. (i): If $z = x + iy$, then $\bar{z} = x - iy$, so that $2x = z + \bar{z}$; this yields the claim for $\operatorname{Re} z$. The assertion for $\operatorname{Im} z$ is proven similarly.

(ii) is obvious.

(iii): Let $z = x + iy$ and $w = u + iv$, so that

$$zw = (xu - yv) + i(xv + yu)$$

and thus

$$\overline{zw} = (xu - yv) - i(xv + yu).$$

On the other hand, we have $\bar{z} = x - iy$ and $\bar{w} = u - iv$, which yields

$$\begin{aligned}\bar{z}\bar{w} &= (xu - (-y)(-v)) + i(x(-v) + (-y)u) \\ &= (xu - yv) - i(xv + yu) \\ &= \overline{zw},\end{aligned}$$

as claimed.

(iv): By (iii), we have

$$\overline{z^{-1}}\bar{z} = \overline{z^{-1}z} = \bar{1} = 1,$$

which yields the claim. □

For any $z = x + iy \in \mathbb{C}$, we note that $z\bar{z} = x^2 + y^2 \geq 0$. This provides us with a natural generalization of the absolute value function to $\mathbb{C}$.

Definition. For $z \in \mathbb{C}$, set $|z| := \sqrt{z\bar{z}}$.

Proposition 1.3. $|\cdot|$ *is the Euclidean norm on* $\mathbb{R}^2$*. In particular, the following hold:*

(i) $|z| \geq 0$ *with* $|z| = 0$ *if and only if* $z = 0$*;*

(ii) $|z + w| \leq |z| + |w|$ *for* $z, w \in \mathbb{C}$.

Moreover, we have $|zw| = |z||w|$ *for* $z, w \in \mathbb{C}$ *and* $z^{-1} = \frac{\bar{z}}{|z|^2}$ *for* $z \in \mathbb{C} \setminus \{0\}$.

Proof. Noting that $|x + iy| = \sqrt{x^2 + y^2}$, we see that $|\cdot|$ is the Euclidean norm, which entails (i) and (ii). Letting $z, w \in \mathbb{C}$, we see that

$$|zw|^2 = zw\overline{zw} = (z\bar{z})(w\bar{w}) = |z|^2|w|^2.$$

Also, since $|z|^2 = z\bar{z}$, we have $1 = z\frac{\bar{z}}{|z|^2}$ for $z \neq 0$ and thus $z^{-1} = \frac{\bar{z}}{|z|^2}$. □

There is a remarkable similarity between the complex multiplication rule

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu)$$

and the trigonometric angle sum formulae. Notice that

$$\begin{aligned}(\cos\theta, \sin\theta)\cdot(\cos\phi, \sin\phi) &= (\cos\theta\cos\phi - \sin\theta\sin\phi, \cos\theta\sin\phi + \sin\theta\cos\phi)\\ &= (\cos(\theta+\phi), \sin(\theta+\phi)).\end{aligned}$$

That is, multiplication of 2 complex numbers on the unit circle $x^2+y^2=1$ corresponds to addition of their angles of inclination to the x axis. In particular, the mapping $f(z)=z^2$ doubles the angle of $z=(x,y)$ and $f(z)=z^n$ multiplies the angle of z by n. These statements hold even if z lies on a circle of radius $r\neq 1$:

$$(r\cos\theta, r\sin\theta)^n = r^n(\cos n\theta, \sin n\theta);$$

this is known as *deMoivre's Theorem*.

Chapter 2

Complex Differentiation

Definition. Let $D \subset \mathbb{C}$, and let z_0 be an interior point of D, i.e. there exists $\epsilon > 0$ such that $B_\epsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\} \subset D$. A function $f : D \to \mathbb{C}$ is called *complex differentiable* at z_0 if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Proposition 2.1. *Let $D \subset \mathbb{C}$, let $z_0 \in \operatorname{int} D$, and let $f : D \to \mathbb{C}$ be complex differentiable at z_0. Then f is continuous at z_0.*

Proof. Since $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists, we have

$$0 = \lim_{z \to z_0} (z - z_0) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} (f(z) - f(z_0)),$$

so that $f(z_0) = \lim_{z \to z_0} f(z)$. $\square$

Proposition 2.2. *Let $D \subset \mathbb{C}$, and let $f, g : D \to \mathbb{C}$ be complex differentiable at $z_0 \in \operatorname{int} D$. Then the following functions are complex differentiable at z_0: $f + g$, fg, and, if $g(z_0) \neq 0$, $\frac{f}{g}$. Moreover, we have:*

$$\begin{aligned} (f+g)'(z_0) &= f'(z_0) + g'(z_0), \\ (fg)'(z_0) &= f'(z_0)g(z_0) + f(z_0)g'(z_0), \end{aligned}$$

and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof. As over $\mathbb{R}$. $\square$

Proposition 2.3. *Let $D, E \subset \mathbb{C}$, let $g : D \to \mathbb{C}$ and $f : E \to \mathbb{C}$ be such that $g(D) \subset E$, and let $z_0 \in \operatorname{int} D$ be such that $w_0 := g(z_0) \in \operatorname{int} E$. Further, suppose that g is complex differentiable at z_0 and f is complex differentiable at w_0. Then $f \circ g$ is complex differentiable at z_0 with*

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

Proof. As over $\mathbb{R}$. $\square$

Examples.

1. All constant functions are (on all of $\mathbb{C}$) complex differentiable, as is $z \mapsto z$ on $\mathbb{C}$. Consequently, all complex polynomials are complex differentiable on all of $\mathbb{C}$, and rational functions are complex differentiable wherever they are defined.

2. Let
$$f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \bar{z},$$
and let $z_0 = x_0 + iy_0 \in \mathbb{C}$. Assume that f is complex differentiable at z_0. Then we have
$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\ &= \lim_{y \to y_0} \frac{(x_0 - iy) - (x_0 - iy_0)}{(x_0 + iy) - (x_0 + iy_0)} \\ &= \lim_{y \to y_0} \frac{i(y_0 - y)}{i(y - y_0)} \\ &= -1 \end{aligned}$$
as well as
$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{\bar{z} - \bar{z}_0}{z - z_0} \\ &= \lim_{x \to x_0} \frac{(x - iy_0) - (x_0 - iy_0)}{(x + iy_0) - (x_0 + iy_0)} \\ &= \lim_{x \to x_0} \frac{x - x_0}{x - x_0} \\ &= 1, \end{aligned}$$
which is impossible. Hence, f is *not* complex differentiable at *any* $z_0 \in \mathbb{C}$. (On the other hand, f is continuously partially differentiable—as a function of two real variables—on all of $\mathbb{C}$.)

Lemma 2.1. *The following are equivalent for an $\mathbb{R}$-linear map $T : \mathbb{C} \to \mathbb{C}$:*

(i) *there exists* $c \in \mathbb{C}$ *such that* $T(z) = cz$ *for all* $z \in \mathbb{C}$*;*

(ii) T *is* $\mathbb{C}$*-linear;*

(iii) $T(i) = iT(1)$*;*

(iv) *the real* 2×2 *matrix representing* T *with respect to the standard basis of* $\mathbb{R}^2$ *may be written as*

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

for some real $a, b \in \mathbb{R}$.

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i): Set $c := T(1)$. For $z = x + iy \in \mathbb{C}$, this means that

$$\begin{aligned} T(x + iy) &= T(x) + T(iy) \\ &= xT(1) + yT(i) \\ &= xT(1) + iyT(1) \\ &= zT(1) \\ &= cz. \end{aligned}$$

(iv) $\Longleftrightarrow$ (iii): Let $a, b, c, d \in \mathbb{R}$ be such that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

represents T with respect to the standard basis of $\mathbb{R}^2$. Note that

$$T(1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = a + ic,$$

and

$$T(i) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = b + id.$$

Since

$$iT(1) = -c + ia,$$

we see that

$$T(i) = iT(1) \iff c = -b \text{ and } d = a.$$

$\square$

Theorem 2.1 (Cauchy–Riemann Equations)**.** *Let* $D \subset \mathbb{C}$ *be open, and let* $z_0 \in D$*. Let* $f \colon D \to \mathbb{C}$ *and denote* $u := \operatorname{Re} f$*,* $v := \operatorname{Im} f$*. Then the following are equivalent:*

(i) f *is complex differentiable at* z_0*;*

(ii) *f is totally differentiable at z_0 (in the sense of multivariable calculus), and the* Cauchy–Riemann differential equations

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0) \qquad \textit{and} \qquad \frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$$

hold.

Proof. (i) $\Longrightarrow$ (ii): Define

$$T\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto f'(z_0)z,$$

and note that

$$\frac{|f(z) - f(z_0) - T(z - z_0)|}{|z - z_0|} = \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \to 0$$

as $z \to z_0$. Therefore, f is totally differentiable at z_0. From multivariable calculus, it follows that the matrix representation of T with respect to the standard basis of $\mathbb{R}$ is the Jacobian of f, i.e.

$$J_f(z_0) = \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix}.$$

Since T is $\mathbb{C}$-linear, Lemma 2.1 yields that

$$u_x(z_0) = v_y(z_0) \qquad \text{and} \qquad u_y(z_0) = -v_x(z_0).$$

(ii) $\Longrightarrow$ (i): Since f is totally differentiable at z_0, we have a unique $\mathbb{R}$-linear map $T\colon \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0) - T(z - z_0)|}{|z - z_0|} = 0.$$

As we know from multivariable calculus, T is represented by $J_f(z_0)$ with respect to the standard basis of $\mathbb{R}^2$. Since the Cauchy–Riemann differential equations are supposed to hold, $J_f(z_0)$ is of the form described in Lemma 2.1(iv). By Lemma 2.1, there thus exists $c \in \mathbb{C}$ such that $T(z) = cz$ for all $z \in \mathbb{C}$. It follows that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - c \right| = \frac{|f(z) - f(z_0) - c(z - z_0)|}{|z - z_0|} \to 0$$

as $z \to z_0$. Hence, f is complex differentiable at z_0. $\square$

Remark. In the situation of Theorem 2.1, we have

$$f'(z_0)1 = \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_x(z_0) + iv_x(z_0)$$

as well as

$$f'(z_0)i = \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = u_y(z_0) + iv_y(z_0),$$

so that

$$f'(z_0) = u_x(z_0) + iv_x(z_0) = v_y(z_0) - iu_y(z_0).$$

Example. Let

$$f\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto |z|^2.$$

Then f is totally differentiable, with

$$u_x = 2x, \qquad u_y = 2y, \qquad v_x = v_y = 0,$$

noting that $v = 0$. The Cauchy–Riemann equations

$$u_x(z_0) = v_y(z_0) \qquad \text{and} \qquad u_y(z_0) = -v_x(z_0)$$

thus hold if and only if $z_0 = 0$. By Theorem 2.1, this means that f is complex differentiable at z_0 if and only if $z_0 = 0$.

Corollary 2.1.1. Let $D \subset \mathbb{C}$ be open and connected, and let $f\colon D \to \mathbb{C}$ be complex differentiable. Then f is constant on D if and only if $f' \equiv 0$.

Proof. Suppose that $f' \equiv 0$. From the remark after Theorem 2.1, it follows that

$$u_x = v_x = u_y = v_y \equiv 0.$$

Multivariable calculus then yields that f is constant. $\square$

Chapter 3

Power Series

Definition. A (complex) *power series* is an infinite series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ with $z, z_0, a_0, a_1, a_2, \ldots \in \mathbb{C}$. The point z_0 is called the *point of expansion* for the series.

Examples.

1. For $m \in \mathbb{N}$, we have

$$\sum_{n=0}^{m} z^n = \frac{1 - z^{m+1}}{1 - z}$$

   if $z \neq 1$. For $|z| < 1$, we obtain (letting $m \to \infty$)

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.$$

2. For $z \in \mathbb{C}$, define

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

   Let $z \neq 0$, and note that

$$\left|\frac{z^{n+1}}{(n+1)!}\right| \Bigg/ \left|\frac{z^n}{n!}\right| = \frac{|z|}{n+1} \to 0$$

   as $n \to \infty$. As the ratio test holds for series with summands in $\mathbb{C}$ as well as for series over $\mathbb{R}$, we conclude that $\exp(z)$ converges absolutely.

   Let $z, w \in \mathbb{C}$, and note that the Cauchy product formula for series over $\mathbb{R}$ also holds over $\mathbb{C}$. We obtain:

$$\begin{aligned}
\exp(z)\exp(w) &= \left(\sum_{j=0}^{\infty}\frac{z^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{w^k}{k!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{z^{n-k}}{(n-k)!}\frac{w^k}{k!} \qquad \text{by the Cauchy product formula, letting } n = j+k, \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}z^{n-k}w^k \\
&= \sum_{n=0}^{n}\frac{(z+w)^n}{n!} \\
&= \exp(z+w).
\end{aligned}$$

We call $\exp : \mathbb{C} \to \mathbb{C}$ the *exponential function*. The above property suggests using the shorthand e^z for $\exp(z)$. An interactive three-dimensional graph of $\exp(z)$ is shown in Figure 3.1.

3. The *sine* and *cosine* functions on $\mathbb{C}$ are defined as

$$\sin(z) := \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

and

$$\cos(z) := \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$$

for $z \in \mathbb{C}$. As for $\exp(z)$, we see that both $\sin(z)$ and $\cos(z)$ converge absolutely for all $z \in \mathbb{C}$. Moreover, we have for $z \in \mathbb{C}$:

$$\begin{aligned}
e^{iz} &= \sum_{n=0}^{\infty}\frac{(iz)^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(iz)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(iz)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} \\
&= \cos(z) + i\sin(z).
\end{aligned}$$

Interactive three-dimensional graphs of the complex cosine and sine functions are shown in Figures 3.2, and 3.3.

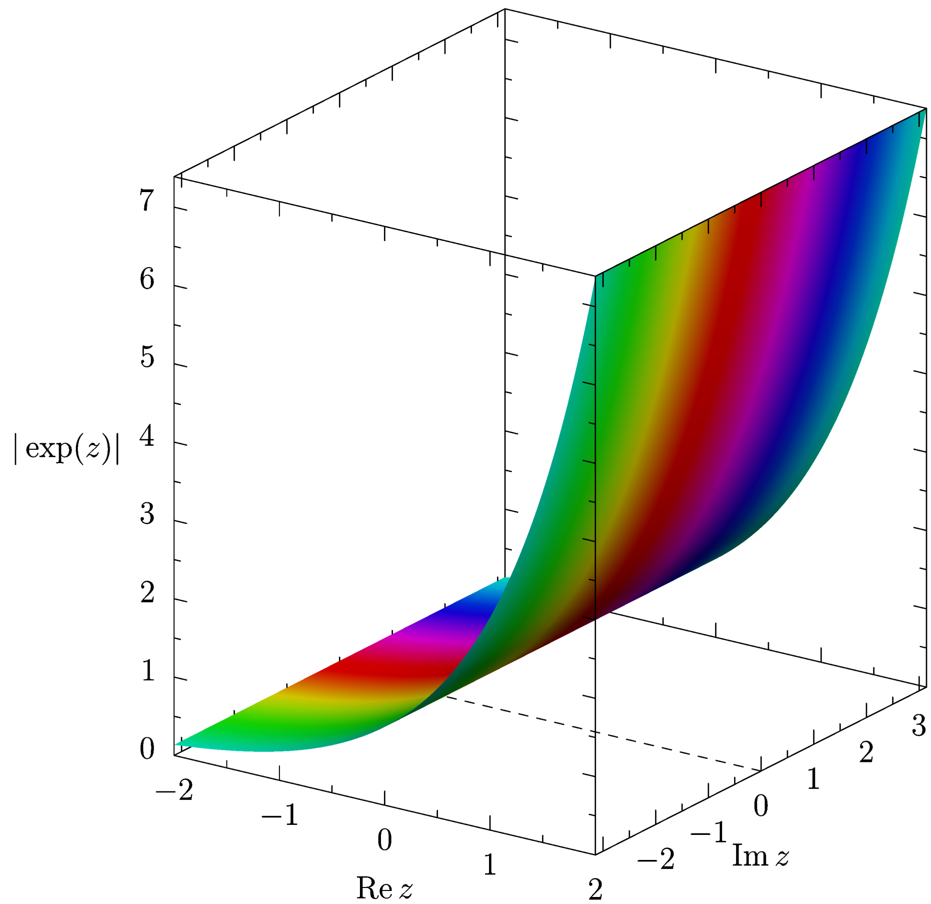


Figure 3.1: Surface plot of $\exp(z)$ in the complex plane, using an RGB color wheel to represent the phase. Red indicates real positive values.

Theorem 3.1 (Radius of Convergence)**.** *Let* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *be a complex power series. Then there exists a unique* $R \in [0,\infty]$ *with the following properties:*

- $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges absolutely at each* $z \in B_R(z_0)$*;*
- *for each* $r \in [0, R)$*, the series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *converges uniformly on* $B_r[z_0] := \{z \in \mathbb{C} : |z-z_0| \leq r\}$*;*
- $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *diverges for each* $z \notin B_R[z_0]$*.*

Moreover, R *can be computed via the Cauchy–Hadamard formula:*

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

It is called the radius of convergence for $\sum_{n=0}^{\infty} a_n(z-z_0)^n$*.*

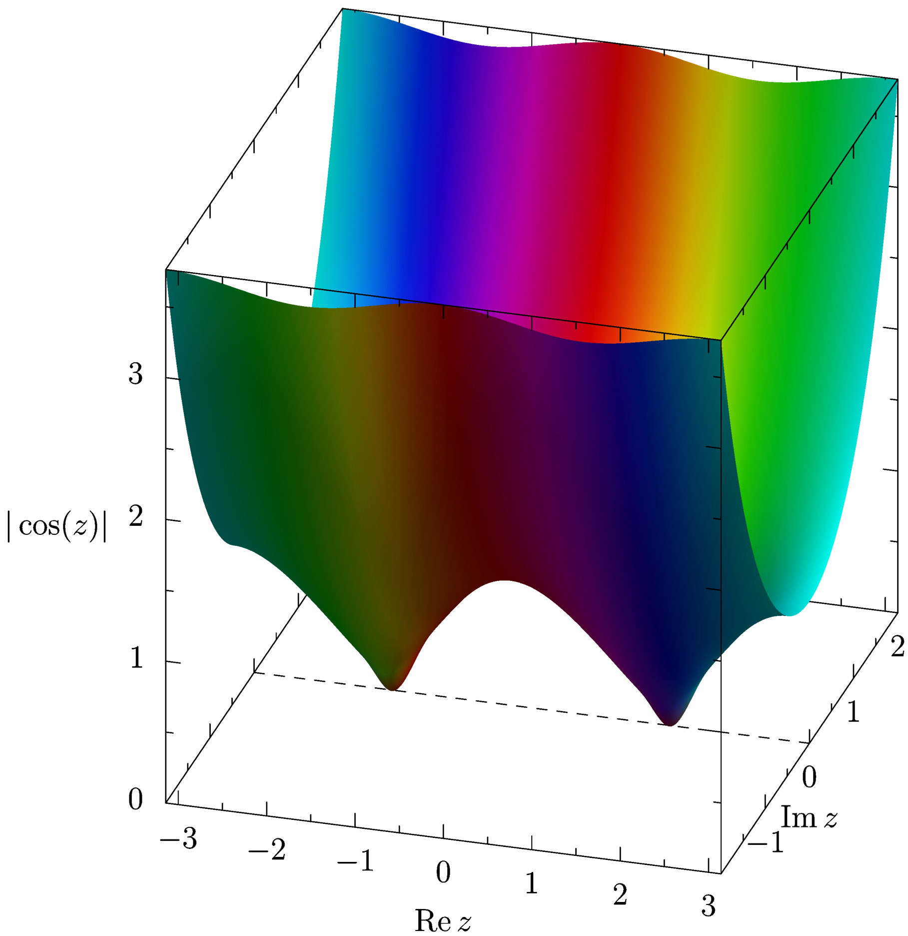


Figure 3.2: Surface plot of $\cos(z)$ in the complex plane, using an RGB color wheel to represent the phase. Red indicates real positive values.

Proof. The uniqueness of R follows from the first and the last property.

Let $R \in [0, \infty]$ be *defined* by the Cauchy–Hadamard formula (we set $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$).

Let $r \in [0, R)$, and choose $r' \in (r, R)$. It follows that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R} < \frac{1}{r'},$$

so that there exists $n_0 \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < \frac{1}{r'}$ whenever $n \geq n_0$, i.e.

$$|a_n| < \left(\frac{1}{r'}\right)^n$$

for all $n \geq n_0$. For $n \geq n_0$ and $z \in B_r[z_0]$, we then have

$$|a_n(z - z_0)^n| \leq \left(\frac{r}{r'}\right)^n.$$

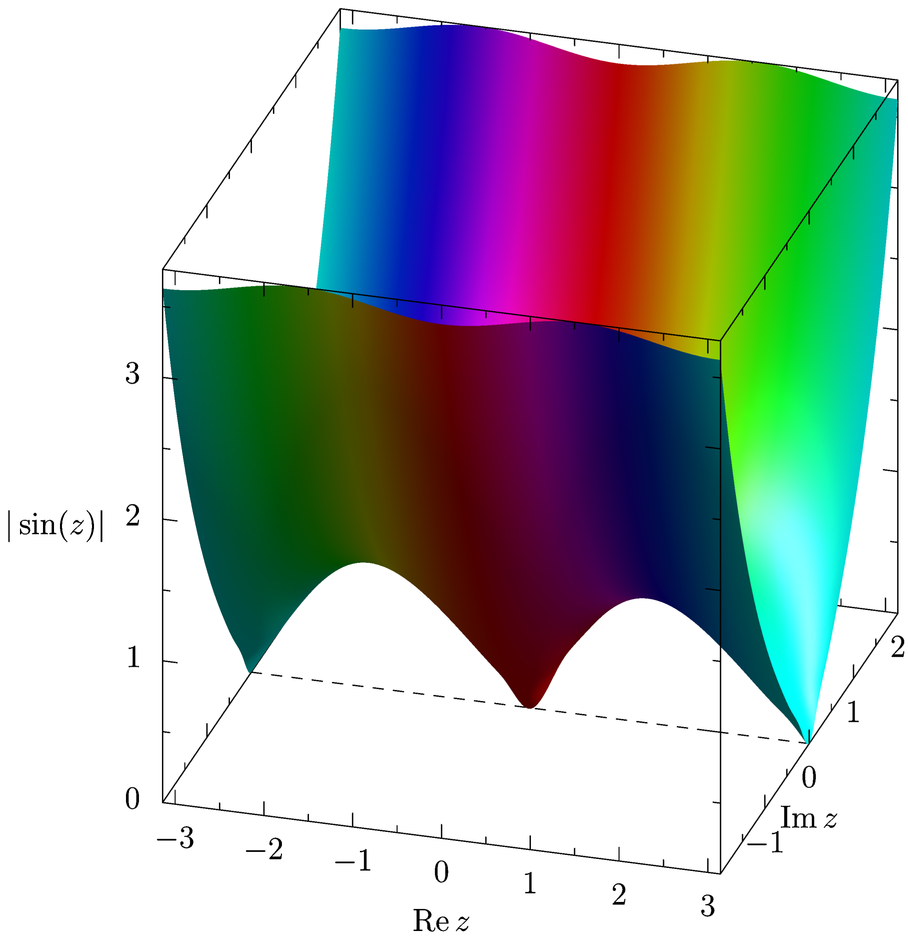


Figure 3.3: Surface plot of $\sin(z)$ in the complex plane, using an RGB color wheel to represent the phase. Red indicates real positive values.

Since $\frac{r}{r'} < 1$, we have $\sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n < \infty$. The Weierstraß M-test thus yields that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely and uniformly on $B_r[z_0]$.

Since every $z \in B_R(z_0)$ is contained in $B_r[z_0]$ for some $r \in [0, R)$, it follows that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely for each such z.

Let $z \notin B_R[z_0]$, i.e. $|z-z_0| > R$, so that

$$\frac{1}{|z-z_0|} < \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

and thus, for infinitely many $n \in \mathbb{N}$,

$$\frac{1}{|z-z_0|} < \sqrt[n]{|a_n|}$$

or, equivalently,

$$1 < |a_n(z-z_0)^n|.$$

It follows that $\{a_n(z-z_0)^n\}_{n=1}^\infty$ does not converge to zero. Consequently, $\sum_{n=0}^\infty a_n(z-z_0)^n$ diverges. $\square$

Examples.

1. $\sum_{n=0}^\infty z^n$: $R = 1$.
2. $\sum_{n=0}^\infty \frac{z^n}{n!}$: $R = \infty$.
3. $\sum_{n=0}^\infty n!\, z^n$: $R = 0$.
4. $\sum_{n=0}^\infty (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^\infty (-1)^n \frac{z^{2n}}{(2n)!}$: $R = \infty$.

Theorem 3.2 (Term-by-Term Differentiation)**.** *Let $\sum_{n=0}^\infty a_n(z-z_0)^n$ be a complex power series with radius of convergence R. Then*

$$f\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^\infty a_n(z-z_0)^n$$

is complex differentiable at each point $z \in B_R(z_0)$ with

$$f'(z) = \sum_{n=1}^\infty n a_n (z-z_0)^{n-1}.$$

Proof. Without loss of generality, suppose that $z_0 = 0$.

We first show that $\sum_{n=1}^\infty n a_n z^{n-1}$ converges absolutely for each $z \in B_R(0)$.

Let $z \in B_R(0)$, and choose r such that $|z| < r < R$. Since $\dfrac{1}{r} > \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$, there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \left(\frac{1}{r}\right)^n$ for $n \geq n_0$ and thus

$$|n a_n z^{n-1}| < \frac{n}{r}\left(\frac{|z|}{r}\right)^{n-1}$$

for $n \geq n_0$. Since $\frac{|z|}{r} < 1$, we know from the Ratio Test that $\sum_{n=1}^\infty \frac{n}{r}\left(\frac{|z|}{r}\right)^{n-1} < \infty$; the Comparison Test then yields that $\sum_{n=1}^\infty n a_n z^{n-1}$ converges absolutely.

In view of the foregoing, we may define

$$g\colon B_R(0) \to \mathbb{C}, \quad z \mapsto \sum_{n=1}^\infty n a_n z^{n-1}.$$

We shall devote the rest of the proof to showing that f is complex differentiable on $B_R(0)$ with $f' = g$.

To this end, fix $\epsilon > 0$, and define, for $z \in B_R(0)$ and $n \in \mathbb{N}$,

$$S_n(z) := \sum_{k=0}^n a_k z^k \qquad \text{and} \qquad R_n(z) := \sum_{k=n+1}^\infty a_k z^k.$$

Fix $z \in B_R(0)$ and let $r \in (0, R)$ be such that $z \in B_r(0)$. Note that

$$\frac{f(w)-f(z)}{w-z} - g(z) = \left(\frac{S_n(w)-S_n(z)}{w-z} - S_n'(z)\right) + (S_n'(z)-g(z)) + \frac{R_n(w)-R_n(z)}{w-z}$$

for all $w \in B_R(0) \setminus \{z\}$. We shall see that each of the three summands on the right-hand side of this equation has modulus less than $\frac{\epsilon}{3}$, provided that n is sufficiently large and w is sufficiently close to z.

We start with the last summand. First, note that

$$\frac{R_n(w)-R_n(z)}{w-z} = \sum_{k=n+1}^{\infty} a_k \frac{w^k - z^k}{w-z}$$

for all $w \in B_R(0) \setminus \{z\}$ and also that

$$\left|\frac{w^k - z^k}{w-z}\right| = \left|\sum_{j=1}^{k} w^{k-j} z^{j-1}\right| \leq \sum_{j=1}^{k} |w|^{k-j} |z|^{j-1} \leq k r^{k-1}$$

for all $w \in B_r(0) \setminus \{z\}$. Since $r < R$, we have $\sum_{k=1}^{\infty} k|a_k| r^{k-1} < \infty$. Consequently, there exists $n_1 \in \mathbb{N}$ such that $\sum_{k=n+1}^{\infty} k|a_k| r^{k-1} < \frac{\epsilon}{3}$ for all $n \geq n_1$ and therefore

$$\left|\frac{R_n(w)-R_n(z)}{w-z}\right| = \left|\sum_{k=n+1}^{\infty} a_k \frac{w^k - z^k}{w-z}\right| \leq \sum_{k=n+1}^{\infty} k|a_k| r^{k-1} < \frac{\epsilon}{3}$$

for all $n \geq n_1$ and all $w \in B_r(0) \setminus \{z\}$.

For the second summand, just note that $\lim_{n \to \infty} S_n'(z) = g(z)$; consequently, there exists $n_2 \in \mathbb{N}$ such that $|S_n'(z) - g(z)| < \frac{\epsilon}{3}$ for all $n \geq n_2$.

For the first summand, fix $n \geq \max\{n_1, n_2\}$. Since

$$\lim_{w \to z} \frac{S_n(w)-S_n(z)}{w-z} = S_n'(z),$$

there exists $\delta \in (0, r)$ such that

$$\left|\frac{S_n(w)-S_n(z)}{w-z} - S_n'(z)\right| < \frac{\epsilon}{3}$$

for all $w \in B_\delta(z) \subset B_r(0) \setminus \{z\}$. Consequently, we obtain for all $w \in B_\delta(z) \setminus \{z\}$ that

$$\left|\frac{f(w)-f(z)}{w-z} - g(z)\right| \leq \left|\frac{S_n(w)-S_n(z)}{w-z} - S_n'(z)\right| + |S_n'(z)-g(z)| + \left|\frac{R_n(w)-R_n(z)}{w-z}\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we see that $f'(z)$ exists and equals $g(z)$. $\square$

Problem 3.1. Show in Theorem 3.2 that the power series for f' and f have the same radius of convergence.

Examples.

1. $\exp'(z) = \exp(z)$.
2. $\sin'(z) = \cos(z)$.
3. $\cos'(z) = -\sin(z)$.

Corollary 3.2.1 (Higher Derivatives of Power Series)**.** Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series with radius of convergence R. Then

$$f\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is infinitely often complex differentiable on $B_R(z_0)$ with

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}.$$

for $z \in B_R(z_0)$ and $k \in \mathbb{N}$. In particular, when $z = z_0$ we see that

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

holds for each $n \in \mathbb{N}_0$.

Corollary 3.2.2 (Integration of Power Series)**.** Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a complex power series with radius of convergence R. Then

$$F\colon B_R(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$$

is complex differentiable on $B_R(z_0)$ with

$$F'(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for $z \in B_R(z_0)$.

Chapter 4

Complex Line Integrals

We call a function $f\colon [a,b] \to \mathbb{C}$ *integrable* if $\operatorname{Re} f, \operatorname{Im} f\colon [a,b] \to \mathbb{R}$ are integrable in the sense of real variables. (The Riemann integral will do.) In this case, we define

$$\int_a^b f(t)\,dt := \int_a^b \operatorname{Re} f(t)\,dt + i\int_a^b \operatorname{Im} f(t)\,dt.$$

Definition. A *curve* (or *path*) in $\mathbb{C}$ is a continuous map $\gamma\colon [a,b] \to \mathbb{C}$. We call

- $\gamma(a)$ the *initial point* of γ,
- $\gamma(b)$ the *endpoint* (or *terminal point*) of γ, and
- $\{\gamma\} := \gamma([a,b])$ the *trajectory* of γ.

Collectively, we call $\gamma(a)$ and $\gamma(b)$ the *endpoints* of γ.

Examples.

1. Let $z, w \in \mathbb{C}$. Then

$$\gamma\colon [0,1] \to \mathbb{C}, \quad t \mapsto z_0 + t(z - z_0)$$

   has the initial point z_0 and the endpoint z, and $\{\gamma\}$ is the line segment connecting z_0 with z.

2. For $k \in \mathbb{Z}$, let

$$\gamma_k\colon [0, 2\pi] \to \mathbb{C}, \quad \theta \mapsto e^{ik\theta}.$$

   Then $\gamma_k(0) = 1 = \gamma_k(2\pi)$ holds, and for $k \neq 0$, we have $\{\gamma_k\} = \{z \in \mathbb{C} : |z| = 1\}$.

Definition. A curve $\gamma\colon [a,b] \to \mathbb{C}$ is called *piecewise smooth* if there exists a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that $\gamma|_{[a_{j-1},a_j]}$ is continuously differentiable for $j = 1, \ldots, n$.

Definition. The *length* of a piecewise smooth curve $\gamma\colon [a,b] \to \mathbb{C}$ is defined as

$$\ell(\gamma) := \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} |\gamma'(t)|\,dt,$$

where $a = a_0 < a_1 < \cdots < a_n = b$ is a partition such that $\gamma|_{[a_{j-1},a_j]}$ is continuously differentiable for $j = 1, \ldots, n$.

Definition. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise smooth curve, let $a = a_0 < a_1 < \cdots < a_n = b$ be a partition such that $\gamma|_{[a_{j-1},a_j]}$ is continuously differentiable for $j = 1, \ldots, n$, and let $f : \{\gamma\} \to \mathbb{C}$ be continuous. Then the *line integral* (or *contour integral*) of f along γ is defined as

$$\int_\gamma f := \int_\gamma f(\zeta)\,d\zeta = \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} f(\gamma(t))\gamma'(t)\,dt.$$

Properties of the Line Integral.

1. Let γ be a piecewise smooth curve, let $\lambda, \mu \in \mathbb{C}$, and let $f, g\colon \{\gamma\} \to \mathbb{C}$ be continuous. Then we have

$$\int_\gamma (\lambda f + \mu g) = \lambda \int_\gamma f + \mu \int_\gamma g.$$

2. Let γ be a piecewise smooth curve, let $f : \{\gamma\} \to \mathbb{C}$ be continuous, and let $C \geq 0$ be such that $|f(\zeta)| \leq C$ for $\zeta \in \{\gamma\}$. Then

$$\left| \int_\gamma f \right| \leq C\,\ell(\gamma)$$

holds.

3. Let $\gamma : [c,d] \to \mathbb{C}$ be a piecewise smooth curve, let $\phi : [a,b] \to [c,d]$ be a continuously differentiable function with $\phi(a) = c$ and $\phi(b) = d$, and let $f : \{\gamma\} \to \mathbb{C}$ be continuous. Then we have

$$\int_\gamma f = \int_{\gamma\circ\phi} f.$$

4. Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be continuous with antiderivative $F\colon D \to \mathbb{C}$; i.e. F is complex differentiable at each $z \in D$, with $F'(z) = f(z)$. Then

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a))$$

holds for every piecewise smooth curve $\gamma\colon [a,b] \to D$.

Definition. A curve $\gamma\colon [a,b] \to \mathbb{C}$ is called *closed* if $\gamma(a) = \gamma(b)$.

Proposition 4.1. *Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be continuous with an antiderivative. Then $\int_\gamma f = 0$ holds for each closed, piecewise smooth curve γ in D.*

Example. Let $z_0 \in \mathbb{C}$, let $r > 0$, and let

$$\gamma\colon [0, 2\pi] \to \mathbb{C}, \quad \theta \mapsto re^{i\theta} + z_0,$$

i.e. γ is a counterclockwise-oriented circle centered at z_0 with radius r.

Let $n \in \mathbb{Z}$, and consider $\int_\gamma (\zeta - z_0)^n \, d\zeta$.

For $n \neq -1$, let

$$F\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{(z - z_0)^{n+1}}{n+1},$$

so that $F'(z) = (z - z_0)^n$ for all $z \in \mathbb{C}$. It follows that $\int_\gamma (\zeta - z_0)^n \, d\zeta = 0$.

On the other hand, we have

$$\int_\gamma (\zeta - z_0)^{-1} \, d\zeta = \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}} \, d\theta = \int_0^{2\pi} i \, dt = 2\pi i.$$

Consequently,

$$\mathbb{C} \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{z - z_0}$$

has *no* antiderivative.

Recall the following definition from multivariable calculus:

Definition. A subset $D \subset \mathbb{C}$ is called *connected* if there are *no* open sets $U, V \subset \mathbb{C}$ with

- $U \cap D \neq \varnothing \neq V \cap D$;
- $U \cup V \supset D$;
- $U \cap V \subset \mathbb{C} \setminus D$.

In other words, there are no open sets U and V, each containing points of D, such that every point of D lies in exactly one of the sets U and V.

Definition. Let $D \subset \mathbb{C}$ be open. A function $f\colon D \to \mathbb{C}$ is called *locally constant* if, for each $z_0 \in D$, there exists $\epsilon > 0$ such that $B_\epsilon(z_0) \subset D$ and f is constant on $B_\epsilon(z_0)$.

The following curve constructions will be useful in understanding the relation between locally constant functions and connectivity.

1. Given $a < b < c$ and two curves $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(b)$, the *concatenation* of γ_1 and γ_2 is the curve
$$\gamma_1 \oplus \gamma_2 \colon [a,c] \to \mathbb{C}, \quad t \mapsto \begin{cases} \gamma_1(t), & t \in [a,b], \\ \gamma_2(t), & t \in [b,c]. \end{cases}$$
If γ_1 and γ_2 are piecewise smooth, then so is $\gamma_1 \oplus \gamma_2$, and we have
$$\int_{\gamma_1 \oplus \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$
for each continuous $f \colon \{\gamma_1\} \cup \{\gamma_2\} \to \mathbb{C}$.

2. For any curve $\gamma \colon [a,b] \to \mathbb{C}$, the *reversed curve* is defined as
$$\gamma^- \colon [a,b] \to \mathbb{C}, \quad t \mapsto \gamma(a+b-t).$$
If γ is piecewise smooth, then so is γ^-, and we have
$$\int_{\gamma^-} f = -\int_\gamma f$$
for each continuous $f \colon \{\gamma\} \to \mathbb{C}$.

3. We denote the *straight line segment* $\{z_0 + t(z - z_0) : t \in [0,1]\}$ by $[z_0, z]$.

Proposition 4.2 (Locally Constant vs. Connectivity)**.** *Let $D \subset \mathbb{C}$ be open. Then the following are equivalent:*

(i) *D is connected;*

(ii) *every locally constant function $f \colon D \to \mathbb{C}$ is constant;*

(iii) *for any $z, w \in D$, there exists a piecewise smooth curve $\gamma \colon [a,b] \to D$ such that $\gamma(a) = z$ and $\gamma(b) = w$.*

Proof. (iii) $\Longrightarrow$ (ii): Let $f \colon D \to \mathbb{C}$ be a locally constant function, and let $z, w \in D$. Let $\gamma \colon [a,b] \to D$ be a piecewise smooth curve with $\gamma(a) = z$ and $\gamma(b) = w$. Since f is locally constant, the function
$$[a,b] \to \mathbb{C}, \quad t \mapsto f(\gamma(t))$$
is differentiable with zero derivative and therefore constant. It follows that $f(z) = f(\gamma(a)) = f(\gamma(b)) = f(w)$.

(ii) $\Longrightarrow$ (i): Suppose that D is *not* connected. Then there exist non-empty open sets $U, V \subset \mathbb{C}$ with $U \cap V = \varnothing$ and $U \cup V = D$. Define
$$f \colon D \to \mathbb{C}, \quad z \mapsto \begin{cases} 0, & z \in U, \\ 1, & z \in V. \end{cases}$$

Then f is locally constant, but not constant.

(i) $\Longrightarrow$ (iii): Let $z \in D$, and set

$$U := \{w \in D : \exists \text{ a piecewise smooth curve } \gamma\colon [a,b] \to D \text{ with } \gamma(a) = z,\ \gamma(b) = w\}.$$

Obviously, $U \neq \varnothing$ (because $z \in U$).

We claim that U is open. To see this, let $w_0 \in U$, so that there exists a piecewise smooth curve $\gamma\colon [a,b] \to D$ with $\gamma(a) = z$ and $\gamma(b) = w_0$. Choose $\epsilon > 0$ such that $B_\epsilon(w_0) \subset D$. For $w \in B_\epsilon(w_0)$, we know that $[w_0, w]$ is a curve in $B_\epsilon(w_0) \subset D$ with initial point w_0 and endpoint w. Consequently, $\gamma \oplus [w_0, w]$ is a piecewise smooth curve in D with initial point z and endpoint w. It follows that $w \in U$, and since $w \in B_\epsilon(w_0)$ was arbitrary, we have $B_\epsilon(w_0) \subset U$. This proves the openness of U.

Next, we claim that $D \setminus U$ is also open. To see this, let $w_0 \in D \setminus U$, and let $\epsilon > 0$ be so small that $B_\epsilon(w_0) \subset D$. Assume towards a contradiction that there exists $w \in B_\epsilon(w_0) \cap U$. Let $\gamma\colon [a,b] \to D$ be a piecewise smooth curve with $\gamma(a) = z$ and $\gamma(b) = w$. Then $\gamma \oplus [w, w_0]$ is a piecewise smooth curve in D with initial point z and endpoint w_0, so that $w_0 \in U$. This contradicts the choice of $w_0 \in D \setminus U$. It follows that $B_\epsilon(w_0) \cap U = \varnothing$, i.e. $B_\epsilon(w_0) \subset D \setminus U$.

Since U and $D \setminus U$ are both open with $U \cup (D \setminus U) = D$ and $U \cap (D \setminus U) = \varnothing$, the connectedness of D yields that $D \setminus U = \varnothing$, i.e. $D = U$.

□

Lemma 4.1. *Suppose $D \subset \mathbb{C}$ is an open connected set (a region) and $f\colon D \to \mathbb{C}$ is continuous. Let $z_0 \in D$. For each $z \in D$, let $\gamma_z : [a,b] \to D$ be a piecewise smooth curve in D such that $\gamma_z(a) = z_0$ and $\gamma_z(b) = z$. Consider the function*

$$F\colon D \to \mathbb{C}, \quad z \mapsto \int_{\gamma_z} f(\zeta)\, d\zeta.$$

For each z, let $\delta > 0$ such that $B_\delta(z) \subset D$. If the condition

$$F(w) - F(z) = \int_{[z,w]} f(\zeta)\, d\zeta$$

holds for each z and all $w \in B_\delta(z)$, then F is an antiderivative for f.

Proof. Let $z \in D$. Given $\epsilon > 0$, choose $\delta > 0$ small enough such that $B_\delta(z) \subset D$ and

$$|\zeta - z| < \delta \Rightarrow |f(\zeta) - f(z)| < \epsilon.$$

For all $w \in B_\delta(z)$, we find

$$\begin{aligned}
\left| \frac{F(w) - F(z)}{w - z} - f(z) \right| &= \frac{1}{|w - z|} \left| \int_{[z,w]} f - \int_{[z,w]} f(z) \right| \\
&= \frac{1}{|w - z|} \left| \int_{[z,w]} (f - f(z)) \right| \\
&\leq \frac{|w - z|}{|w - z|} \sup\{|f(\zeta) - f(z)| \colon \zeta \in \{[z,w]\}\} \\
&= \sup\{|f(\zeta) - f(z)| \colon \zeta \in \{[z,w]\}\} < \epsilon.
\end{aligned}$$

That is, $F'(z) = f(z)$. $\square$

Theorem 4.1 (Antiderivative Theorem)**.** *Let $D \subset \mathbb{C}$ be open and connected and let $f : D \to \mathbb{C}$ be continuous. Then the following are equivalent:*

(i) *f has an antiderivative;*

(ii) *$\int_\gamma f(\zeta)\, d\zeta = 0$ for any closed, piecewise smooth curve γ in D;*

(iii) *for any piecewise smooth curve γ in D, the value of $\int_\gamma f$ depends only on the inital point and the endpoint of γ.*

Proof. (i) $\Longrightarrow$ (ii) is Proposition 4.1.

(ii) $\Longrightarrow$ (iii): Let $\gamma, \Gamma : [a, b] \to D$ be piecewise smooth curves with $\gamma(a) = \Gamma(a)$ and $\gamma(b) = \Gamma(b)$. Then $\gamma \oplus \Gamma^-$ is a closed, piecewise smooth curve, so that

$$0 = \int_{\gamma\oplus\Gamma^-} f = \int_\gamma f + \int_{\Gamma^-} f = \int_\gamma f - \int_\Gamma f.$$

(iii) $\Longrightarrow$ (i): Fix $z_0 \in D$. For each $z \in D$, choose a piecewise smooth curve $\gamma_z : [a, b] \to D$ with $\gamma_z(a) = z_0$ and $\gamma_z(b) = z$ and let

$$F : D \to \mathbb{C}, \quad z \mapsto \int_{\gamma_z} f(\zeta)\, d\zeta.$$

For each $z \in D$, choose $\delta > 0$ such that $B_\delta(z) \subset D$ and note for $w \in B_\delta(z)$ that

$$\begin{aligned} F(w) &= \int_{\gamma_w} f(\zeta)\, d\zeta \\ &= \int_{\gamma_z\oplus[z,w]} f(\zeta)\, d\zeta \quad \text{by (iii)} \\ &= \int_{\gamma_z} f(\zeta)\, d\zeta + \int_{[z,w]} f(\zeta)\, d\zeta \\ &= F(z) + \int_{[z,w]} f(\zeta)\, d\zeta. \end{aligned}$$

Lemma 4.1 then implies that F is an antiderivative for f. $\square$

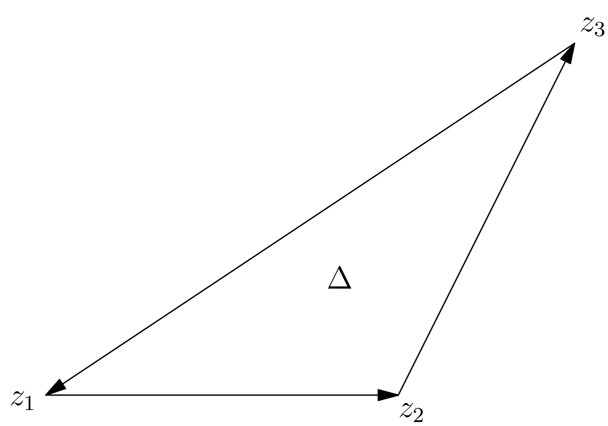
From now on, we shall use the word *curve* as shorthand for *piecewise smooth curve*.

Chapter 5

Cauchy's Integral Theorem and Formula

Definition. Let $D \subset \mathbb{C}$ be open. If $f : D \to \mathbb{C}$ is complex differentiable at each $z \in D$, then we call f *holomorphic* (or *analytic*) on D.

Let z_1, z_2, and z_3 be three different points in $\mathbb{C}$. They span a triangle Δ. Its boundary can be parametrized as a curve with counterclockwise orientation.

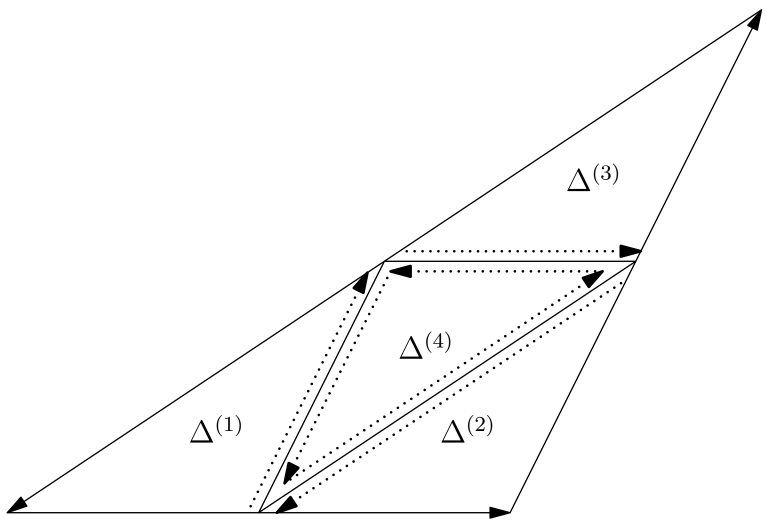


We denote this curve by $\partial\Delta$.

Theorem 5.1 (Goursat's Lemma)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $\Delta \subset D$ be a triangle. Then we have*

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = 0.$$

Proof. First, we note that the result holds trivially whenever z_1, z_2, and z_3 are colinear. Otherwise, we can split Δ at its medians into four subtriangles $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$, and $\Delta^{(4)}$ as shown in the following figure:



As the line segments in the interior of Δ also occur as their reversed paths, we have

$$\int_{\partial\Delta} f = \sum_{j=1}^{4} \int_{\partial\Delta^{(j)}} f,$$

so that

$$\left| \int_{\partial\Delta} f \right| \leq \sum_{j=1}^{4} \left| \int_{\partial\Delta^{(j)}} f \right|.$$

Choose $j \in \{1,2,3,4\}$ such that $\left| \int_{\partial\Delta^{(j)}} f \right|$ is largest, and set $\Delta_1 := \Delta^{(j)}$. It follows that

$$\left| \int_{\partial\Delta} f \right| \leq 4 \left| \int_{\partial\Delta_1} f \right|;$$

also, note that $\ell(\partial\Delta_1) = \frac{1}{2}\ell(\partial\Delta)$.

Repeat this argument with Δ_1 in place of Δ, and obtain a triangle $\Delta_2 \subset \Delta_1$ with

$$\ell(\partial\Delta_2) = \frac{1}{2}\ell(\partial\Delta_1) = \frac{1}{4}\ell(\partial\Delta)$$

and

$$\left| \int_{\partial\Delta_1} f \right| \leq 4 \left| \int_{\partial\Delta_2} f \right|,$$

so that

$$\left| \int_{\partial\Delta} f \right| \leq 4 \left| \int_{\partial\Delta_1} f \right| \leq 16 \left| \int_{\partial\Delta_2} f \right|.$$

Inductively, we obtain triangles

$$\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$$

with

$$\ell(\partial\Delta_n) = \frac{1}{2^n}\ell(\partial\Delta)$$

and

$$\left|\int_{\partial\Delta} f\right| \leq 4^n \left|\int_{\partial\Delta_n} f\right|$$

for $n \in \mathbb{N}$.

Let $z_0 \in \bigcap_{n=1}^\infty \Delta_n$, and define

$$r\colon D \to \mathbb{C}, \quad z \mapsto f(z) - f(z_0) - f'(z_0)(z - z_0),$$

so that $\lim_{z\to z_0} \frac{|r(z)|}{|z-z_0|} = 0$ and $\int_\gamma (r - f) = \int_\gamma [-f(z_0) - f'(z_0)(z - z_0)]\, dz = 0$ for each closed curve γ in D (noting that the integrand has an antiderivative). Consequently,

$$\left|\int_{\partial\Delta} f\right| \leq 4^n \left|\int_{\partial\Delta_n} r\right|$$

for $n \in \mathbb{N}$. Let $\epsilon > 0$ and choose $\delta > 0$ such that

$$\left|\frac{r(z)}{z - z_0}\right| \leq \frac{\epsilon}{[\ell(\partial\Delta)]^2}$$

for all $z \in D$ with $|z - z_0| < \delta$. Choose $n \in \mathbb{N}$ such that $\Delta_n \subset B_\delta(z_0)$. For $z \in \Delta_n$, this means that

$$|z - z_0| \leq \ell(\partial\Delta_n) = \frac{1}{2^n}\ell(\partial\Delta).$$

We thus obtain:

$$\begin{aligned}
\left|\int_{\partial\Delta} f\right| &\leq 4^n \left|\int_{\partial\Delta_n} r\right| \\
&\leq 4^n \ell(\partial\Delta_n) \sup_{\zeta\in\partial\Delta_n} |r(\zeta)| = 2^n \ell(\partial\Delta) \sup_{\zeta\in\partial\Delta_n} |r(\zeta)| \\
&\leq 2^n \ell(\partial\Delta) \sup_{\zeta\in\partial\Delta_n} \frac{\epsilon}{[\ell(\partial\Delta)]^2} \underbrace{|\zeta - z_0|}_{\leq \frac{1}{2^n}\ell(\partial\Delta)} \\
&\leq \epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, this proves the claim. □

Definition. A set $D \subset \mathbb{C}$ is called *star shaped* if there exists $z_0 \in D$ such that $[z_0, z] \subset D$ for each $z \in D$. The point z_0 is called a *center* for D.

Theorem 5.2. *Let $D \subset \mathbb{C}$ be open and star shaped with center z_0, and let $f\colon D \to \mathbb{C}$ be continuous such that*

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = 0$$

for each triangle $\Delta \subset D$ with z_0 as a vertex. Then f has an antiderivative.

Proof. Let $z_0 \in D$ be a center for D. Define

$$F\colon D \to \mathbb{C}, \quad z \mapsto \int_{[z_0,z]} f.$$

Let $z \in D$, and let $\delta > 0$ be such that $B_\delta(z) \subset D$. Let $w \in B_\delta(z)$. Since $[z_0, z] \oplus [z, w] \oplus [w, z_0]$ is the boundary of a triangle $\Delta \subset D$,

$$\int_{[z_0,z]\oplus[z,w]\oplus[w,z_0]} f = 0,$$

so that

$$F(w) = \int_{[z_0,w]} f = -\int_{[w,z_0]} f = \int_{[z_0,z]} f + \int_{[z,w]} f = F(z) + \int_{[z,w]} f.$$

Lemma 4.1 then implies that F is an antiderivative for f. $\square$

Corollary 5.2.1. Let $D \subset \mathbb{C}$ be open and star shaped, and let $f : D \to \mathbb{C}$ be holomorphic. Then f has an antiderivative.

Proof. Apply Goursat's Lemma and Theorem 5.2. $\square$

Corollary 5.2.2. Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be holomorphic. Then, for each $z_0 \in D$, there exists a neighbourhood $U \subset D$ of z_0 such that $f|_U$ has an antiderivative.

Corollary 5.2.3 (Cauchy's Integral Theorem for Star-Shaped Domains)**.** Let $D \subset \mathbb{C}$ be open and star shaped, and let $f\colon D \to \mathbb{C}$ be holomorphic. Then $\int_\gamma f(\zeta)\, d\zeta = 0$ for each closed curve γ in D.

Proof. This follows from Corollary 5.2.1 and Theorem 4.1. $\square$

Example. The *sliced plane* is defined as

$$\mathbb{C}_- := \{z \in \mathbb{C} : z \notin (-\infty, 0]\}.$$

Then $\mathbb{C}_-$ is star shaped (1 is a center, for instance). As seen in the proof of Theorem 5.2, the function

$$\operatorname{Log}\colon \mathbb{C}_- \to \mathbb{C}, \quad z \mapsto \int_{[1,z]} \frac{1}{\zeta}\, d\zeta$$

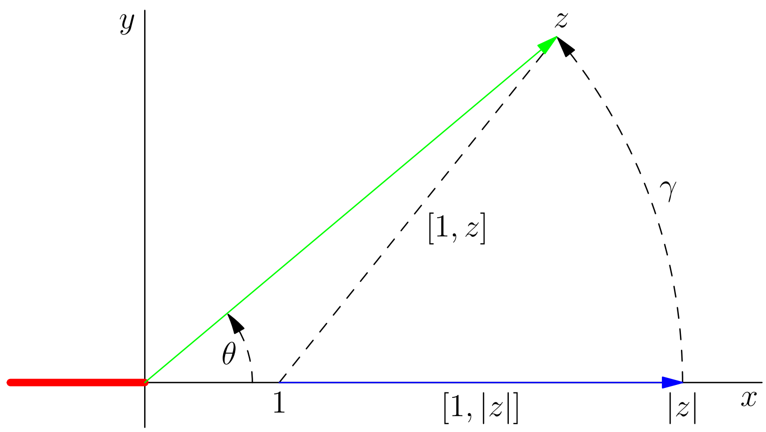
is an antiderivative of $z \mapsto \frac{1}{z}$ on $\mathbb{C}_-$; it is called the *principal branch of the logarithm*.

Let $z \in \mathbb{C}_-$, and let γ_z be any curve in $\mathbb{C}_-$ with initial point 1 and endpoint z. From Theorem 4.1, we conclude that $\int_{\gamma_z} \frac{1}{\zeta}\, d\zeta = \operatorname{Log} z$.

For any $z \in \mathbb{C}_-$, there exists a unique $\theta \in (-\pi, \pi)$—the *principal argument* $\operatorname{Arg} z$ of z—such that $z = |z|e^{i\theta}$. For $z \in \mathbb{C}_-$, the curve

$$\gamma\colon [0, \theta] \to \mathbb{C}, \quad t \mapsto |z|e^{it}.$$

has the initial point $|z|$ and the endpoint z. It follows that $[1,|z|] \oplus \gamma$ is curve with initial point 1 and endpoint z as shown:

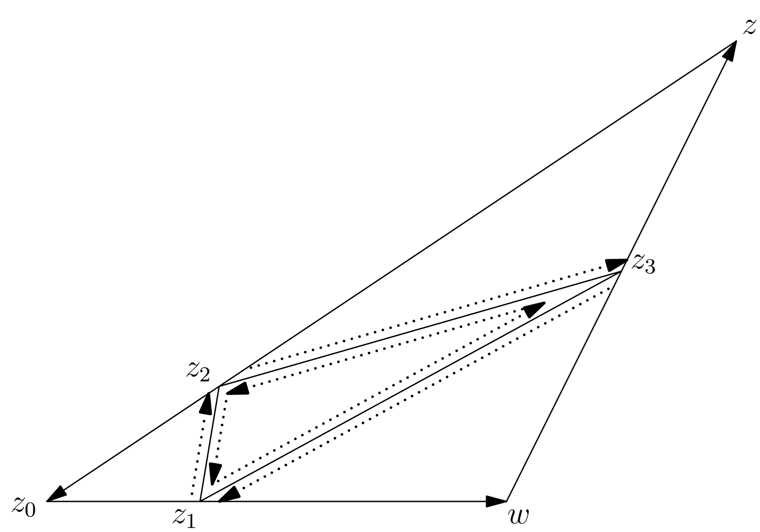


It follows that

$$\operatorname{Log} z := \int_{[1,|z|]} \frac{1}{\zeta}\, d\zeta + \int_{\gamma} \frac{1}{\zeta}\, d\zeta = \log|z| + i \int_0^{\theta} \frac{|z|e^{it}}{|z|e^{it}}\, dt = \log|z| + i \operatorname{Arg} z.$$

Lemma 5.1. *Let $D \subset \mathbb{C}$ be open and star shaped with center z_0, and let $f : D \to \mathbb{C}$ be continuous such that $f|_{D \setminus \{z_0\}}$ is holomorphic. Then f has an antiderivative on D.*

Proof. Let Δ be a triangle in D having a vertex z_0:



Let z_1 be an interior point of $[z_0, w]$, let z_2 be an interior point of $[z_0, z]$, and let z_3 be an interior point of $[w, z]$. As shown above, we use these points to split Δ into four subtriangles, denoted by $\Delta(z_0, z_1, z_2)$, $\Delta(z_1, z_3, z_2)$, $\Delta(z_1, w, z_3)$, and $\Delta(z_2, z_3, z)$. As in the proof of Goursat's Lemma, we have

$$\int_{\partial\Delta} f = \int_{\partial\Delta(z_0,z_1,z_2)} f + \int_{\partial\Delta(z_1,z_3,z_2)} f + \int_{\partial\Delta(z_1,w,z_3)} f + \int_{\partial\Delta(z_2,z_3,z)} f.$$

Since $\Delta(z_1, z_3, z_2), \Delta(z_1, w, z_3), \Delta(z_2, z_3, z) \subset D \setminus \{z_0\}$, and since f is holomorphic on $D \setminus \{z_0\}$, Goursat's Lemma yields

$$\int_{\partial\Delta(z_1,z_3,z_2)} f = \int_{\partial\Delta(z_1,w,z_3)} f = \int_{\partial\Delta(z_2,z_3,z)} f = 0,$$

so that

$$\int_{\partial\Delta} f = \int_{\partial\Delta(z_0,z_1,z_2)} f.$$

It follows that

$$\left|\int_{\partial\Delta} f\right| = \left|\int_{\partial\Delta(z_0,z_1,z_2)} f\right| \leq \ell(\partial\Delta(z_0, z_1, z_2)) \sup_{\zeta\in\partial\Delta(z_0,z_1,z_2)} |f(\zeta)|.$$

Since $|f|$ is continuous on Δ, it is bounded above by some $M > 0$. By placing z_1 and z_2 sufficiently close to z_0, we see that $\ell(\partial\Delta(z_0, z_1, z_2))$ can be made smaller than every $\epsilon/M > 0$. We deduce that $\int_{\partial\Delta} f = 0$. The result then follows from Theorem 5.2. ☐

Let $z_0 \in \mathbb{C}$, and let $r > 0$. Slightly abusing notation, we use $\partial B_r(z_0)$ to denote the boundary of $B_r(z_0)$ oriented counterclockwise.

Lemma 5.2. *Let $D \subset \mathbb{C}$ be open, let $z_0 \in D$, and let $r > 0$ be such that $B_r[z_0] \subset D$. Then*

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta = 2\pi i$$

for all $z \in B_r(z_0)$.

Proof. Through direct computation, we saw on pg. 24 that

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z_0}\, d\zeta = 2\pi i.$$

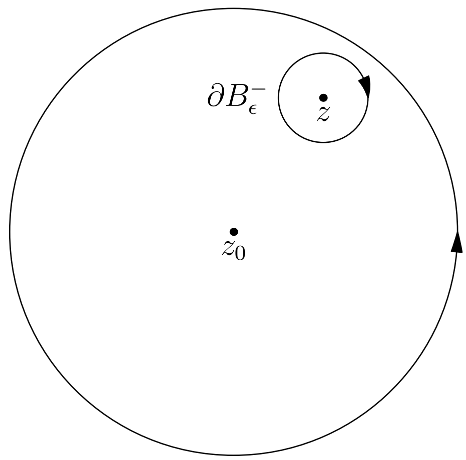
Let $z \in B_r(z_0)$, and choose $\epsilon > 0$ such that $B_\epsilon[z] \subset B_r(z_0)$, so that

$$\int_{\partial B_\epsilon(z)} \frac{1}{\zeta - z}\, d\zeta = 2\pi i.$$

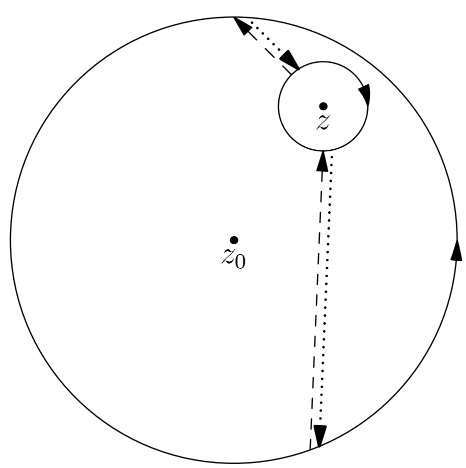
We need to show that

$$\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta = \int_{\partial B_\epsilon(z)} \frac{1}{\zeta - z}\, d\zeta = -\int_{\partial B_\epsilon(z)^-} \frac{1}{\zeta - z}\, d\zeta.$$

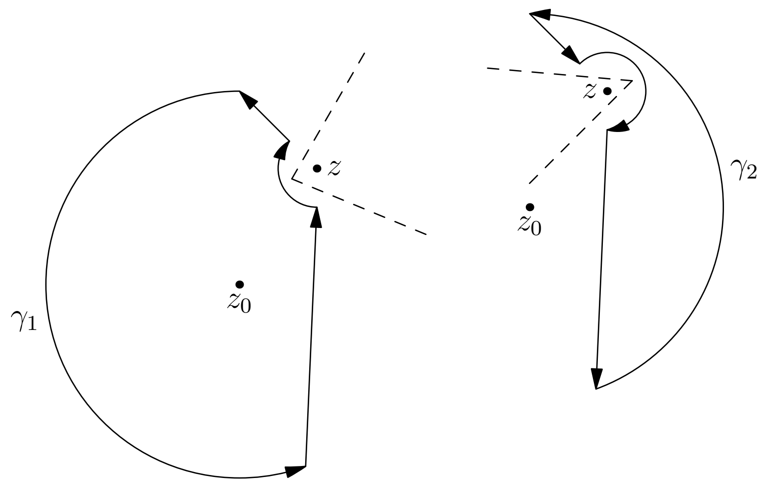
This is how the situation looks like:



We connect the boundaries of $B_r(z_0)$ and $B_\epsilon(z)$ through line segments:



Consider the following two closed curves γ_1 and γ_2:



Then it is clear that

$$\int_{\gamma_1} \frac{1}{\zeta - z}\, d\zeta + \int_{\gamma_2} \frac{1}{\zeta - z}\, d\zeta = \int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta + \int_{\partial B_\epsilon(z)^-} \frac{1}{\zeta - z}\, d\zeta.$$

The sketches also show that there exist star-shaped open set $D_j \subset \mathbb{C} \setminus \{z\}$ with $\{\gamma_j\} \subset D_j$ for $j = 1, 2$. Cauchy's integral theorem for star-shaped domains thus yields that

$$\int_{\gamma_j} \frac{1}{\zeta - z}\, d\zeta = 0$$

for $j = 1, 2$, which proves the claim. □

Theorem 5.3 (Cauchy's Integral Formula for Circles)**.** *Let $D \subset \mathbb{C}$ be open, let $f \colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in B_r(z_0)$.

Remark. A consequence of Theorem 5.3 is that the values of f on all of $B_r[z_0]$ are pre-determined by those on $\partial B_r(z_0)$!

Proof of Theorem 5.3. Since $B_r[z_0]$ is compact, we may choose $R > 0$ be such that $B_r[z_0] \subset B_R(z_0) \subset D$. Let $z \in B_r(z_0)$, and note that z is a center for the star-shaped domain $B_R(z_0)$.

Define

$$g \colon D \to \mathbb{C}, \quad \zeta \mapsto \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z}, & \zeta \neq z, \\ f'(z), & \zeta = z. \end{cases}$$

Then g is continuous on D and holomorphic on $D \setminus \{z\}$. By Lemma 5.1, g has an antiderivative on $B_R(z_0)$. Since $\partial B_r(z_0)$ is closed, this means that

$$0 = \int_{\partial B_r(z_0)} g(\zeta)\, d\zeta = \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) \underbrace{\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta}_{=2\pi i},$$

using the identity $\int_{\partial B_r(z_0)} \frac{1}{\zeta - z}\, d\zeta = 2\pi i$ established by Lemma 5.2. Division by $2\pi i$ yields the claim. □

Corollary 5.3.1 (Mean Value Equation)**.** Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

Proof. Parametrize $\partial B_r(z_0)$ as

$$\gamma \colon [0, 2\pi] \to \mathbb{C}, \quad \theta \mapsto z_0 + re^{i\theta}.$$

The Cauchy Integral Formula then yields

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \end{aligned}$$

□

Lemma 5.3. *Let $D \subset \mathbb{R}^N$ be open, and let $f \colon [a, b] \times D \to \mathbb{R}$ be continuous such that $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \colon [a, b] \times D \to \mathbb{R}$ all exist and are continuous. Define*

$$g \colon D \to \mathbb{R}, \quad x \mapsto \int_a^b f(t, x)\, dt.$$

Then g is continuously partially differentiable with

$$\frac{\partial g}{\partial x_j}(x) = \int_a^b \frac{\partial f}{\partial x_j}(t, x)\, dt$$

for $x \in D$ and $j = 1, \ldots, N$.

Proof. There is no loss of generality supposing that $N = 1$.

Let $x_0, x \in D$ be such that every number between them is also in D. By the Mean Value Theorem of single variable calculus, there exists, for each $t \in [a, b]$, a real number ξ_t between x_0 and x such that

$$\frac{f(t,x) - f(t,x_0)}{x - x_0} = \frac{\partial f}{\partial x}(t, \xi_t).$$

Let $\epsilon > 0$. By uniform continuity, there exists a $\delta > 0$ such that

$$\left| \frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2) \right| < \frac{\epsilon}{b - a}$$

for any $t \in [a, b]$ and all x_1, x_2 between x_0 and x with $|x_1 - x_2| < \delta$.

Suppose that $|x_0 - x| < \delta$. Then we have:

$$\begin{aligned}
\left| \frac{g(x) - g(x_0)}{x - x_0} - \int_a^b \frac{\partial f}{\partial x}(t, x_0)\, dt \right| &= \left| \int_a^b \left(\frac{f(t,x) - f(t,x_0)}{x - x_0} - \frac{\partial f}{\partial x}(t, x_0) \right) dt \right| \\
&\leq \int_a^b \left| \frac{f(t,x) - f(t,x_0)}{x - x_0} - \frac{\partial f}{\partial x}(t, x_0) \right| dt \\
&= \int_a^b \left| \frac{\partial f}{\partial x}(t, \xi_t) - \frac{\partial f}{\partial x}(t, x_0) \right| dt \\
&\leq \int_a^b \frac{\epsilon}{b - a} dt, \qquad \text{because } |\xi_t - x_0| < \delta, \\
&= \epsilon.
\end{aligned}$$

This proves that g is differentiable at x_0 with $g'(x_0) = \int_a^b \frac{\partial f}{\partial x}(t, x_0)\, dt$.

A similar (but easier) argument shows that g' is continuous. □

Lemma 5.4. *Let $D \subset \mathbb{C}$ be open, and let $f\colon [a, b] \times D \to \mathbb{C}$ be continuous such that $\frac{\partial f}{\partial z}\colon [a, b] \times D \to \mathbb{C}$ exists and is continuous. Define*

$$g\colon D \to \mathbb{R}, \quad z \mapsto \int_a^b f(t, z)\, dt.$$

Then g is holomorphic with

$$g'(z) = \int_a^b \frac{\partial f}{\partial z}(t, z)\, dt$$

for $z \in D$.

Proof. Apply Lemma 5.3 to $\operatorname{Re} f$ and $\operatorname{Im} f$ and note that the Cauchy Riemann differential equations are satisfied. □

Theorem 5.4 (Higher Derivatives)**.** *Let $D \subset \mathbb{C}$ be open, let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$, and let $f \colon D \to \mathbb{C}$ be continuous such that*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

holds for all $z \in B_r(z_0)$. Then f is infinitely often complex differentiable on $B_r(z_0)$ and satisfies

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \tag{$*$}$$

holds for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

Proof. We prove by induction on $n \in \mathbb{N}_0$: f is n-times complex differentiable and ($*$) holds.

For $n = 0$, the claim is clear, so suppose that it is true for some $n \in \mathbb{N}_0$. Define

$$F \colon [0, 2\pi] \times B_r(z_0) \to \mathbb{C}, \quad (\theta, z) \mapsto \frac{n!}{2\pi} \frac{f(z_0 + re^{i\theta}) re^{i\theta}}{(z_0 + re^{i\theta} - z)^{n+1}}.$$

Then F is continuous and, by the induction hypothesis, satisfies

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta = \int_0^{2\pi} F(\theta, z) \, d\theta$$

for all $z \in B_r(z_0)$. Furthermore,

$$\frac{\partial F}{\partial z}(\theta, z) = \frac{(n+1)!}{2\pi} \frac{f(z_0 + re^{i\theta}) re^{i\theta}}{(z_0 + re^{i\theta} - z)^{n+2}}$$

is continuous on $[0, 2\pi] \times B_r(z_0)$. From Lemma 5.4, we thus conclude that $f^{(n)}$ is holomorphic on D with

$$f^{(n+1)}(z) = \int_0^{2\pi} \frac{\partial F}{\partial z}(\theta, z) \, d\theta = \frac{(n+1)!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+2}} \, d\zeta.$$

□

Corollary 5.4.1 (Generalized Cauchy Integral Formula)**.** Let $D \subset \mathbb{C}$ be open, and let $f \colon D \to \mathbb{C}$ be holomorphic. Then f is infinitely often complex differentiable on D. Moreover, for any $z_0 \in D$ and $r > 0$ such that $B_r[z_0] \subset D$, the *generalized Cauchy integral formula*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

holds for all $z \in B_r(z_0)$ and $n \in \mathbb{N}_0$.

Proof. Apply Theorem 5.3 and 5.4. □

Example. We shall use Cauchy's integral theorem and formula to evaluate the line integral

$$\int_\gamma \frac{e^\zeta}{\zeta(\zeta-1)}\,d\zeta$$

for various curves γ:

(a) $\gamma = \partial B_\pi(-3)$: The function

$$f\colon \mathbb{C}\setminus\{1\} \to \mathbb{C}, \quad z \mapsto \frac{e^z}{z-1}$$

is holomorphic, and we have $B_\pi[-3] \subset \mathbb{C}\setminus\{1\}$. Cauchy's integral formula thus yields:

$$\int_{\partial B_\pi(-3)} \frac{e^\zeta}{\zeta(\zeta-1)}\,d\zeta = \int_{\partial B_\pi(-3)} \frac{f(\zeta)}{\zeta}\,d\zeta = 2\pi i\,f(0) = -2\pi i.$$

(b) $\gamma = \partial B_{\frac{1}{2}}(i)$: As the integrand is holomorphic in the star-shaped domain $B_{\frac{3}{4}}(i)$, Cauchy's integral theorem yields that

$$\int_{\partial B_{\frac{1}{2}}(i)} \frac{e^\zeta}{\zeta(\zeta-1)}\,d\zeta = 0.$$

(c) $\gamma = \partial B_2(0)$: the method of partial fractions yields

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}.$$

Since $0, 1 \in B_2(0)$, we obtain with the help of Cauchy's integral formula:

$$\int_{\partial B_2(0)} \frac{e^\zeta}{\zeta(\zeta-1)}\,d\zeta = \int_{\partial B_2(0)} \frac{e^\zeta}{\zeta-1}\,d\zeta - \int_{\partial B_2(0)} \frac{e^\zeta}{\zeta}\,d\zeta = 2\pi i\,(e-1).$$

Theorem 5.5 (Characterizations of Holomorphic Functions)**.** *Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be continuous. Then the following are equivalent:*

(i) *f is holomorphic;*

(ii) *the Morera condition holds, i.e. $\int_{\partial\Delta} f(\zeta)\,d\zeta = 0$ for each triangle $\Delta \subset D$;*

(iii) *for each $z_0 \in D$ and $r > 0$ with $B_r[z_0] \subset D$, we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for $z \in B_r(z_0)$;

(iv) *for each* $z_0 \in D$, *there exists* $r > 0$ *with* $B_r[z_0] \subset D$ *and*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in B_r(z_0)$;

(v) f *is infinitely often complex differentiable on* D;

(vi) *for each* $z_0 \in D$, *there exists a neighbourhood* $U \subset D$ *of* z_0 *such that* f *has an antiderivative on* U.

Proof. (i) $\Longrightarrow$ (ii) is Goursat's Lemma.

(i) $\Longrightarrow$ (iii) is the Cauchy Integral Formula for circles, and (iii) $\Longrightarrow$ (iv) is trivial.

(iv) $\Longrightarrow$ (v) follows immediately from Theorem 5.4, and (v) $\Longrightarrow$ (i) is again trivial.

(ii) $\Longrightarrow$ (vi) follows from Theorem 5.2 because every $z_0 \in D$ has an open, star-shaped neighbourhood contained in D.

(vi) $\Longrightarrow$ (v): Let $z_0 \in D$, and let $U \subset D$ be a neighbourhood of z_0 such that f has an antiderivative, say F, on U. Then F is holomorphic on U. Applying (i) $\Longrightarrow$ (v) to F, we see that F is infinitely often complex differentiable on U. Consequently, $f = F'$ is infinitely complex differentiable on U. Since $z_0 \in D$ was arbitrary, we conclude that f is infinitely often complex differentiable on D. $\square$

We conclude this chapter with Liouville's Theorem and its application to the Fundamental Theorem of Algebra.

Definition. A holomorphic function defined on all of $\mathbb{C}$ is called *entire*.

Theorem 5.6 (Liouville's Theorem)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a bounded entire function. Then* f *is constant.*

Proof. We will show that $f' \equiv 0$.

Let $C \geq 0$ be such that $|f(z)| \leq C$ for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be arbitrary, and let $r > 0$. By the generalized Cauchy integral formula, we have

$$|f'(z)| = \frac{1}{2\pi} \left| \int_{\partial B_r(z)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \leq \frac{1}{2\pi}\, \ell(\partial B_r(z)) \sup_{\zeta \in \partial B_r(z)} \frac{|f(\zeta)|}{|\zeta - z|^2} \leq \frac{1}{2\pi} 2\pi r \frac{C}{r^2} = \frac{C}{r}.$$

Letting $r \to \infty$, we obtain $f'(z) = 0$. This completes the proof. $\square$

Corollary 5.6.1 (Fundamental Theorem of Algebra)**.** Let p be a non-constant polynomial with complex coefficients. Then p has a zero.

Proof. Assume that p has *no* zero. Then the function

$$f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto \frac{1}{p(z)}$$

is entire. Since p is a nonconstant polynomial, we have $\lim_{|z|\to\infty}|p(z)|= \infty$ and thus $\lim_{|z|\to\infty}|f(z)|= 0$. Let $R > 0$ be such that $|f(z)|\leq 1$ for $|z|> R$. Since f is continuous it is bounded on $B_R[0]$, and by the choice of R, it is bounded on $\mathbb{C} \setminus B_R[0]$, too, and thus bounded on all of $\mathbb{C}$. By Liouville's Theorem, f is thus constant, and so is therefore p, which is a contradiction. □

Problem 5.1. Let $D \subset \mathbb{C}$ be open.

(a) Suppose $f : D \to \mathbb{C}$ has an antiderivative on D and $0 \notin f(D)$. Show that there exists a holomorphic function $g : D \to \mathbb{C}$ such that $f = \exp \circ g$.
Hint: consider $\frac{f'(z)}{f(z)}$.

(b) If D is star shaped, and $f : D \to \mathbb{C}$ is holomorphic such that $0 \notin f(D)$, show that there exists a holomorphic function $g : D \to \mathbb{C}$ such that $f = \exp \circ g$.

Problem 5.2. Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $f : B_r[z_0] \to \mathbb{C}$ be continuous such that $f|_{B_r(z_0)}$ is holomorphic. Show that

$$f(z) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in B_r(z_0)$. Hint: For $\theta \in (0, 1)$, apply the Cauchy integral formula to $z \mapsto f(\theta(z - z_0) + z_0)$ on $\partial B_{\frac{r}{\theta}}(z_0)$; then let $\theta \to 1^-$.

Chapter 6

Convergence of Holomorphic Functions

Recall the following definition:

Definition. Let $D \subset \mathbb{C}$ be open. A sequence $(f_n)_{n=1}^{\infty}$ of $\mathbb{C}$-valued functions on D is said to converge *uniformly* on D to $f\colon D \to \mathbb{C}$ if, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq N$ and all $z \in D$.

We recall the following theorem from analysis:

Theorem 6.1 (Uniform Convergence Preserves Continuity)**.** *Let $D \subset \mathbb{C}$ be open, and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous, $\mathbb{C}$-valued functions on D converging uniformly on D to $f\colon D \to \mathbb{C}$. Then f is continuous.*

We now "localize" the notion of uniform convergence:

Definition. Let $D \subset \mathbb{C}$ be open. Then a sequence $(f_n)_{n=1}^{\infty}$ of $\mathbb{C}$-valued functions on D is said to converge *locally uniformly* on D to $f\colon D \to \mathbb{C}$ if, for each $z_0 \in D$, there exists a neighbourhood $U \subset D$ of z_0 such that $(f_n|_U)_{n=1}^{\infty}$ converges to $f|_U$ uniformly on U.

Proposition 6.1 (Local Uniform Convergence)**.** *Let $D \subset \mathbb{C}$ be open, and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous, $\mathbb{C}$-valued functions on D converging locally uniformly on D to $f\colon D \to \mathbb{C}$. Then f is continuous.*

Proof. Let $z_0 \in D$, and let $U \subset D$ be a neighbourhood of z_0 such that $f_n|_U \to f|_U$ uniformly on U By Theorem 6.1, $f|_U$ is continuous. Hence, f is continuous at z_0. □

Proposition 6.2 (Compact Convergence)**.** *Let $D \subset \mathbb{C}$ be open, and let $f, f_1, f_2, \ldots\colon D \to \mathbb{C}$ be functions. Then the following are equivalent:*

(i) $(f_n)_{n=1}^{\infty}$ *converges to f locally uniformly on D;*

(ii) *for each compact $K \subset D$, the sequence $(f_n|_K)_{n=1}^\infty$ converges to $f|_K$ uniformly on K.*

Proof. (i) $\Longrightarrow$ (ii): Let $K \subset D$ be compact. For each $z \in K$, there exists a neighbourhood $U_z \subset D$ of z such that $f_n|_{U_z} \to f|_{U_z}$ uniformly on U_z. Since K is compact, there exist $z_1, \ldots, z_m \in K$ such that

$$K \subset U_{z_1} \cup \cdots \cup U_{z_m}.$$

Let $\epsilon > 0$. For each $j = 1, \ldots, m$, there exists $n_j \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon$ for all $n \geq n_j$ and all $z \in U_{z_j}$. Set $N := \max\{n_1, \ldots, n_m\}$. Then $|f_n(z) - f(z)| < \epsilon$ holds for all $n \geq N$ and $z \in K$.

(ii) $\Longrightarrow$ (i): Let $z_0 \in D$, and let $r > 0$ be such that $B_r[z_0] \subset D$. Since $B_r[z_0]$ is compact, $(f_n|_{B_r[z_0]})_{n=1}^\infty$ converges uniformly on $B_r[z_0]$ to $f|_{B_r[z_0]}$. Trivially, $(f_n|_{B_r(z_0)})_{n=1}^\infty$ thus converges uniformly on $B_r(z_0)$ to $f|_{B_r(z_0)}$. $\square$

Instead of locally uniform convergence, we therefore often speak of *compact convergence*.

Lemma 6.1. *Let $D \subset \mathbb{C}$ be open, let γ be a curve in D, and let $f, f_1, f_2, \ldots \colon D \to \mathbb{C}$ be continuous functions such that $(f_n|_{\{\gamma\}})_{n=1}^\infty$ converges to $f|_{\{\gamma\}}$ uniformly on $\{\gamma\}$. Then we have*

$$\int_\gamma f(\zeta)\, d\zeta = \lim_{n\to\infty} \int_\gamma f_n(\zeta)\, d\zeta.$$

Proof. Let $\epsilon > 0$, and choose $N \in \mathbb{N}$ such that

$$|f_n(\zeta) - f(\zeta)| < \frac{\epsilon}{\ell(\gamma) + 1}$$

for all $n \geq N$ and $z \in \{\gamma\}$. For $n \geq N$, we thus obtain:

$$\begin{aligned} \left| \int_\gamma f_n - \int_\gamma f \right| &= \left| \int_\gamma (f_n - f) \right| \\ &\leq \ell(\gamma) \sup\{|f_n(\zeta) - f(\zeta)| \colon \zeta \in \{\gamma\}\} \\ &\leq \frac{\epsilon\, \ell(\gamma)}{\ell(\gamma) + 1} \\ &< \epsilon. \end{aligned}$$

$\square$

Lemma 6.2. *Let $D \subset \mathbb{C}$ be open, let γ be a curve in D, and let $f_1, f_2, \ldots \colon D \to \mathbb{C}$ be continuous functions converging compactly to $f \colon D \to \mathbb{C}$. Then f is continuous, and we have*

$$\int_\gamma f(\zeta)\, d\zeta = \lim_{n\to\infty} \int_\gamma f_n(\zeta)\, d\zeta.$$

Proof. If $\gamma : [a, b] \to \mathbb{C}$ is a curve (and thus continuous) then $\{\gamma\} = \gamma([a, b])$ is compact. Hence, Lemma 6.1 applies. $\square$

Theorem 6.2 (Weierstraß Theorem)**.** *Let $D \subset \mathbb{C}$ be open, let $f_1, f_2, \ldots : D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^\infty$ converges to $f : D \to \mathbb{C}$ compactly. Then f is holomorphic, and $(f_n^{(k)})_{n=1}^\infty$ converges compactly to $f^{(k)}$ for each $k \in \mathbb{N}$.*

Proof. By Theorem 6.1, f is continuous.

To see that f is holomorphic, let $\Delta \subset D$ be a triangle. By Goursat's Lemma, $\int_{\partial\Delta} f_n(\zeta)\, d\zeta = 0$ holds for all $n \in \mathbb{N}$. From Lemma 6.2, we conclude that

$$\int_{\partial\Delta} f(\zeta)\, d\zeta = \lim_{n\to\infty} \int_{\partial\Delta} f_n(\zeta)\, d\zeta = 0,$$

i.e. f satisfies the Morera condition and thus is holomorphic.

Let $z_0 \in D$, and let $0 < r < R$ be such that $B_r[z_0] \subset B_R(z_0) \subset B_R[z_0] \subset D$. For any $z \in B_r(z_0)$, we have

$$\begin{aligned} |f_n'(z) - f'(z)| &= \frac{1}{2\pi} \left| \int_{\partial B_R(z_0)} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \\ &\leq \frac{1}{2\pi} \ell(\partial B_R(z_0)) \sup_{\zeta \in \partial B_R(z_0)} \left| \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^2} \right| \\ &\leq \frac{R}{(R-r)^2} \sup_{\zeta \in \partial B_R(z_0)} |f_n(\zeta) - f(\zeta)|. \end{aligned}$$

Let $\epsilon > 0$, and choose $N \in \mathbb{N}$ such that

$$|f_n(\zeta) - f(\zeta)| < \epsilon \frac{(R-r)^2}{R}$$

for all $n \geq N$ and $\zeta \in \partial B_R(z_0)$. Then it follows from the above estimates that $|f_n'(z) - f'(z)| \leq \epsilon$ for all $n \geq N$ and $z \in B_r(z_0)$. Consequently, $(f_n'|_{B_r(z_0)})_{n=1}^\infty$ converges to $f'|_{B_r(z_0)}$ uniformly on $B_r(z_0)$. As $z_0 \in D$ is arbitrary, this means that $(f_n')_{n=1}^\infty$ converges to f locally uniformly, i.e. compactly, on D.

For higher derivatives, the claim now follows by induction. $\square$

Lemma 6.3. *Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $z \in B_r(z_0)$. Then*

$$\frac{1}{\zeta - z} = \sum_{n=0}^\infty \frac{1}{(\zeta - z_0)^{n+1}} (z - z_0)^n$$

converges absolutely and uniformly on $\partial B_r(z_0)$.

Proof. Let $\zeta \in \partial B_r(z_0)$, and note that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}.$$

Since $|z - z_0| < r$ and $|\zeta - z_0| = r$, we have $\left|\frac{z-z_0}{\zeta - z_0}\right| < 1$, so that

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(\zeta - z_0)^{n+1}} (z - z_0)^n.$$

Since $\left|\frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}}\right| = \frac{|z-z_0|^n}{r^{n+1}}$ and $\sum_{n=0}^{\infty} \frac{|z-z_0|^n}{r^{n+1}} < \infty$, the Weierstraß M-test yields absolute and uniform convergence on $\partial B_r(z_0)$. $\square$

Theorem 6.3 (Power Series for Holomorphic Functions)**.** *Let $D \subset \mathbb{C}$ be open. Then the following are equivalent for $f : D \to \mathbb{C}$:*

(i) *f is holomorphic;*

(ii) *for each $z_0 \in D$, there exists $r > 0$ with $B_r(z_0) \subset D$ and $a_0, a_1, a_2, \ldots \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all $z \in B_r(z_0)$;*

(iii) *for each $z_0 \in D$ and $r > 0$ with $B_r(z_0) \subset D$, we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

for all $z \in B_r(z_0)$.

Proof. (iii) $\Longrightarrow$ (ii) is trivial; (ii) $\Longrightarrow$ (i) follows from Theorem 3.2.

(i) $\Longrightarrow$ (iii): Let $z_0 \in D$, and let $r > 0$ be such that $B_r(z_0) \subset D$. Let $z \in B_r(z_0)$ and choose $\rho \in (0, r)$ such that $z \in B_\rho(z_0)$. Then we have:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \\
&= \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n \, d\zeta, \qquad \text{by Lemma 6.3}, \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right) (z - z_0)^n, \qquad \text{by Lemma 6.1}, \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.
\end{aligned}$$

$\square$

Chapter 7

Elementary Properties of Holomorphic Functions

Theorem 7.1 (Identity Theorem). *Let $D \subset \mathbb{C}$ be open and connected, and let $f, g \colon D \to \mathbb{C}$ be holomorphic. Then the following are equivalent:*

(i) *$f = g$;*

(ii) *the set $\{z \in D : f(z) = g(z)\}$ has a cluster point in D;*

(iii) *there exists $z_0 \in D$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)$ for all $n \in \mathbb{N}_0$.*

Proof. Without loss of generality, it suffices to prove the case where $g = 0$.

(i) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (ii): Let $z_0 \in D$ be as in (iii), and let $r > 0$ be such that $B_r(z_0) \subset D$. Then we have by Theorem 6.3 that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = 0$$

for all $z \in B_r(z_0)$, so that

$$B_r(z_0) \subset \mathbf{Z}(f) := \{z \in D : f(z) = 0\}.$$

Every point in $B_r(z_0)$ is therefore a cluster point of $\mathbf{Z}(f)$.

(ii) $\Longrightarrow$ (i): Let

$$V := \{z \in D : z \text{ is a cluster point of } \mathbf{Z}(f)\},$$

so that $V \neq \varnothing$ by (ii).

We claim that V is open. Let $z_0 \in V$, and let $r > 0$ be such that $B_r(z_0) \subset D$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for some $a_0, a_1, a_2, \ldots \in \mathbb{C}$ and all $z \in B_r(z_0)$. As z_0 is a cluster point of $\mathbf{Z}(f)$, there exists a sequence $(z_k)_{k=1}^{\infty}$ in $\mathbf{Z}(f) \setminus \{z_0\}$ such that $z_0 = \lim\limits_{k\to\infty} z_k$. Inductively, we find that $a_n = 0$ for all $n \in \mathbb{N}_0$: given that $a_n = 0$ for $n = 0, 1, 2, m-1$ we see that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \frac{f(z_k)}{(z_k - z_0)^m} \\ &= \lim_{k\to\infty} \sum_{n=m}^{\infty} a_n (z_k - z_0)^{n-m} = a_m, \end{aligned}$$

on noting, in view of Theorems 3.1 and 6.3, that the uniform convergence of the power series on, say, $B_{r/2}[z_0]$ justifies the interchange of limits. Hence $f(z) = 0$ for all $z \in B_r(z_0)$. That is, $B_r(z_0) \subset \mathbf{Z}(f)$ so that $B_r(z_0) \subset V$.

We now claim that $D \setminus V$ is also open. Let $z_0 \in D \setminus V$, and $\epsilon > 0$ be such that $B_\epsilon(z_0) \subset D$ and $(B_\epsilon(z_0) \setminus \{z_0\}) \cap \mathbf{Z}(f) = \varnothing$. For any $z \in B_\epsilon(z_0) \setminus \{z_0\}$ and $\delta > 0$ such that $B_\delta(z) \subset B_\epsilon(z_0) \setminus \{z_0\}$, we thus have $(B_\delta(z) \setminus \{z\}) \cap \mathbf{Z}(f) = \varnothing$. It follows that $z \notin V$. Consequently, $B_\epsilon(z_0) \subset D \setminus V$.

In summary, V and $D \setminus V$ are both open and clearly satisfy $D = V \cup (D \setminus V)$ and $V \cap (D \setminus V) = \varnothing$. The connectedness of D yields $D = V$ and thus $\mathbf{Z}(f) = D$. ☐

Examples.

1. There is no non-zero entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f\left(\frac{1}{n}\right) = 0$ for $n \in \mathbb{N}$. For any entire function f with this property, 0 is a cluster point of $\mathbf{Z}(f)$. By the Identity Theorem, this means $f \equiv 0$.

2. Since $\mathbb{R}$ has cluster points in $\mathbb{C}$, the holomorphic extensions of exp, cos, and sin from $\mathbb{R}$ to $\mathbb{C}$ are unique. For analogous reasons, Log is the only holomorphic extension of log to $\mathbb{C}_-$.

3. There is no entire function f such that

$$f\left(\frac{1}{n}\right) = \frac{1}{n} \qquad \text{and} \qquad f\left(-\frac{1}{n}\right) = \frac{1}{n^2}$$

for $n \in \mathbb{N}$. In the view of the identity theorem, the first condition necessitates that $f(z) = z$ whereas the second one implies that $f(z) = (-z)^2$ for all $z \in \mathbb{C}$.

Lemma 7.1. *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Suppose that*

$$|f(z_0)| < \inf_{z \in \partial B_r(z_0)} |f(z)|.$$

Then f has a zero in $B_r(z_0)$.

Proof. Assume otherwise, i.e. f has no zero in $B_r(z_0)$. The hypothesis implies that f has no zero on $\partial B_r(z_0)$, so that f has no zero in $B_r[z_0]$. Assume however, that for each $R > 0$ such that $B_r[z_0] \subset B_R(z_0) \subset D$, there is a zero of f in $B_R(z_0)$. Then we have a sequence $(R_n)_{n=1}^\infty$ in (r, ∞) with $r = \lim_{n\to\infty} R_n$ such that $B_r[z_0] \subset B_{R_n}(z_0) \subset D$ and $\mathbf{Z}(f) \cap B_{R_n}(z_0) \neq \varnothing$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, pick $z_n \in \mathbf{Z}(f) \cap B_{R_n}(z_0)$. Then $(z_n)_{n=1}^\infty$ is bounded, and thus has a convergent subsequence $(z_{n_k})_{k=1}^\infty$ with limit z'. Clearly, $z' \in \mathbf{Z}(f)$, and since $\lim_{k\to\infty} R_{n_k} = r$, we have $z' \in B_r[z_0]$, which is impossible. Consequently, f has no zero on some $B_R(z_0)$ with $B_r[z_0] \subset B_R(z_0) \subset D$.

From the Cauchy Integral Formula, we obtain

$$\begin{aligned}\frac{1}{|f(z_0)|} &= \left|\frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{1}{f(\zeta)}\frac{1}{\zeta - z_0}\,d\zeta\right| \\ &\leq \frac{1}{2\pi} 2\pi r \sup_{\zeta\in\partial B_r(z_0)} \frac{1}{|f(\zeta)|r} = \frac{1}{\inf_{\zeta\in\partial B_r(z_0)}|f(\zeta)|}\end{aligned}$$

and thus

$$|f(z_0)| \geq \inf_{\zeta\in\partial B_r(z_0)} |f(\zeta)|,$$

which is a contradiction. $\square$

Theorem 7.2 (Open Mapping Theorem)**.** *Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be holomorphic and not constant. Then $f(D) \subset \mathbb{C}$ is open and connected.*

Proof. By the continuity of f, it is clear that $f(D)$ is connected.

Let $w_0 \in f(D)$, and let $z_0 \in D$ be such that $w_0 = f(z_0)$. Choose $r > 0$ such that $B_r[z_0] \subset D$ and such that $\{z \in B_r[z_0] : f(z) = w_0\} = \{z_0\}$. (This can be accomplished with the help of the Identity Theorem.) Let $\epsilon = \frac{1}{2} \inf_{\partial B_r(z_0)} |f(z) - w_0| > 0$. We claim that $B_\epsilon(w_0) \subset f(D)$. Let $w \in B_\epsilon(w_0)$. For $z \in \partial B_r(z_0)$, we have

$$|f(z) - w| \geq |f(z) - w_0| - |w - w_0| > 2\epsilon - \epsilon = \epsilon.$$

It follows that

$$|f(z_0) - w| = |w - w_0| < \epsilon \leq \inf_{z\in\partial B_r(z_0)} |f(z) - w|.$$

By Lemma 7.1, this means that

$$D \to \mathbb{C}, \quad z \mapsto f(z) - w$$

has a zero in $B_r(z_0)$. It follows that $w \in f(D)$. $\square$

Theorem 7.3 (Maximum Modulus Principle)**.** *Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be holomorphic such that the function*

$$|f| : D \to \mathbb{C}, \quad z \mapsto |f(z)|$$

attains a local maximum on D. Then f is constant.

Proof. Let $z_0 \in D$ be such that $|f|$ attains a local maximum at z_0, i.e. there exists $\epsilon > 0$ such that $B_\epsilon(z_0) \subset D$ and $|f(z_0)| \geq |f(z)|$ for all $z \in B_\epsilon(z_0)$. Then $f(z_0)$ is not an interior point of $f(B_\epsilon(z_0))$, so that $f|_{B_\epsilon(z_0)}$ is constant by the Open Mapping Theorem. The Identity Theorem then yields that f is constant. □

Corollary 7.3.1. Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be holomorphic such that $|f|$ attains a local minimum on D. Then f is constant or f has a zero.

Proof. Suppose that f has no zero. Applying the Maximum Modulus Principle to $1/f$ yields that f is constant. □

Corollary 7.3.2 (Maximum Modulus Principle for Bounded Domains)**.** Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f : \overline{D} \to \mathbb{C}$ be continuous such that $f|_D$ is holomorphic. Then $|f|$ attains its maximum over $\overline{D}$ on ∂D.

Proof. The claim is trivial if f is constant, so suppose that f is not constant.

Since f is continuous and $\overline{D}$ is compact, there exists a point $z_0 \in \overline{D}$ with $|f(z_0)| = \max\left\{|f(z)| : z \in \overline{D}\right\}$. If $z_0 \in D$, then $|f|$ would attain a local maximum at z_0, which is impossible by the Maximum Modulus Principle. Therefore $z_0 \in \partial D$ must hold. □

From now on, we shall use $\mathbb{D}$ to denote the open unit disc $B_1(0)$.

Theorem 7.4 (Schwarz's Lemma)**.** *Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be holomorphic such that $f(0) = 0$. Then one has*

$$|f(z)| \leq |z| \quad \text{for } z \in \mathbb{D} \qquad \text{and} \qquad |f'(0)| \leq 1.$$

Moreover, if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$, then there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for $z \in \mathbb{D}$.

Proof. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be the power series expansion of f. Since $f(0) = 0$, we have $a_0 = 0$. Define

$$g : \mathbb{D} \to \mathbb{C}, \quad z \mapsto \sum_{n=1}^\infty a_n z^{n-1}.$$

Then g is holomorphic with $g(0) = a_1 = f'(0)$ and $f(z) = zg(z)$ for $z \in \mathbb{D}$. Let $r \in (0, 1)$. Then we have

$$|g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r}$$

for $z \in \partial B_r(0)$ and thus for all $z \in B_r[0]$ by the Maximum Modulus Principle. Letting $r \to 1$, we deduce that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$ and thus $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$ as well as $|f'(0)| = |g(0)| \leq 1$.

Suppose that there exists $z_0 \in \mathbb{D} \setminus \{0\}$ such that $|f(z_0)| = |z_0|$ or $|f'(0)| = |g(0)| = 1$. Then $|g|$ has a maximum at z_0 or 0, respectively, so that g is constant. Hence, there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = zg(z) = cz$ for $z \in \mathbb{D}$. □

Definition. Let $D_1, D_2 \subset \mathbb{C}$ be open. Then $f\colon D_1 \to D_2$ is called *biholomorphic* (or *conformal*) if

(a) f is bijective and

(b) both f and f^{-1} are holomorphic.

Corollary 7.4.1. Let $f\colon \mathbb{D} \to \mathbb{D}$ be biholomorphic such that $f(0) = 0$. Then there exists $c \in \mathbb{C}$ with $|c|= 1$ such that $f(z) = cz$ for $z \in \mathbb{D}$.

Proof. Let $z \in \mathbb{D}$. Then $|f(z)|\leq |z|$ holds by Schwarz's Lemma, as does

$$|z|= |f^{-1}(f(z))|\leq |f(z)|.$$

The result then follows from Schwarz's Lemma. □

Lemma 7.2. *Let $w \in \mathbb{D}$, and define*

$$\phi_w\colon \mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{w - z}{1 - \bar{w}z}.$$

Then:

(i) *ϕ_w maps $\mathbb{D}$ bijectively onto $\mathbb{D}$;*

(ii) *$\phi_w(w) = 0$;*

(iii) *$\phi_w(0) = w$;*

(iv) *$\phi_w^{-1} = \phi_w$.*

Proof. Obviously, ϕ_w is holomorphic and extends continuously to $\overline{\mathbb{D}}$.

Then for $|z|= 1$ we may express

$$|\phi_w(z)|= \left|\frac{w - z}{1 - \bar{w}z}\right| \frac{1}{|\bar{z}|} = \left|\frac{w - z}{\bar{z} - \bar{w}}\right| = 1.$$

By the Maximum Modulus Principle, $\phi_w(\mathbb{D}) \subset \overline{\mathbb{D}}$ holds. Since ϕ_w is not constant, $\phi_w(\mathbb{D})$ is open and thus contained in the interior of $\overline{\mathbb{D}}$, i.e. in $\mathbb{D}$.

It is obvious that $\phi_w(w) = 0$ and $\phi_w(0) = w$.

Moreover, we have for $z \in \mathbb{D}$:

$$\begin{aligned}
(\phi_w \circ \phi_w)(z) &= \frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\frac{w-z}{1-\bar{w}z}} \\
&= \frac{w(1 - \bar{w}z) - (w - z)}{(1 - \bar{w}z) - \bar{w}(w - z)} \\
&= \frac{-|w|^2 z + z}{1 - |w|^2} \\
&= z.
\end{aligned}$$

Hence, ϕ_w is bijective with $\phi_w^{-1} = \phi_w$. □

Theorem 7.5 (Biholomorphisms of $\mathbb{D}$)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be biholomorphic. Then there exist $w \in \mathbb{D}$ and $c \in \partial\mathbb{D}$ with $f(z) = c\phi_w(z)$ for $z \in \mathbb{D}$.*

Proof. Set $w := f^{-1}(0)$. Then $f \circ \phi_w : \mathbb{D} \to \mathbb{D}$ is biholomorphic with $(f \circ \phi_w)(0) = 0$. By Corollary 7.4.1, there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(\phi_w(z)) = cz$ for $z \in \mathbb{D}$, so that

$$f(z) = f(\phi_w(\phi_w(z))) = c\,\phi_w(z)$$

for $z \in \mathbb{D}$. $\square$

Chapter 8

The Singularities of a Holomorphic Function

Definition. Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ be holomorphic. We call $z_0 \in \mathbb{C} \setminus D$ an *isolated singularity* of f if there exists $\epsilon > 0$ such that $B_\epsilon(z_0) \setminus \{z_0\} \subset D$. We say that the isolated singularity z_0 is *removable* if there exists a holomorphic function $g \colon D \cup \{z_0\} \to \mathbb{C}$ such that $g|_D = f$.

Theorem 8.1 (Riemann's Removability Condition)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity of f. Then the following are equivalent:*

(i) *z_0 is removable;*

(ii) *there is a continuous function $g \colon D \cup \{z_0\} \to \mathbb{C}$ such that $g|_D = f$;*

(iii) *there exists $\epsilon > 0$ with $B_\epsilon(z_0) \setminus \{z_0\} \subset D$ such that f is bounded on $B_\epsilon(z_0) \setminus \{z_0\}$.*

Proof. (i) $\Longrightarrow$ (ii) follows from the continuity of a differentiable function.

(ii) $\Longrightarrow$ (iii) follows from the boundedness of g on a compact set $B_\epsilon[z_0] \subset D$.

(iii) $\Longrightarrow$ (i): Let $C \geq 0$ be such that $|f(z)| \leq C$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$. Define

$$h \colon D \cup \{z_0\} \to \mathbb{C}, \quad z \mapsto \begin{cases} (z - z_0)^2 f(z), & z \neq z_0, \\ 0, & z = z_0. \end{cases}$$

Then we have for $z \in B_\epsilon(z_0) \setminus \{z_0\}$ that

$$\left| \frac{h(z) - h(z_0)}{z - z_0} \right| = |(z - z_0) f(z)| \leq C|z - z_0|.$$

Hence, h is holomorphic with $h'(z_0) = h(z_0) = 0$. Let $h(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$ be the power series representation of h on $B_\epsilon(z_0)$. Then $h'(z_0) = h(z_0) = 0$ means that $a_0 = a_1 = 0$, so that $h(z) = \sum_{n=2}^\infty a_n (z - z_0)^n$ for $z \in B_\epsilon(z_0)$ and thus $f(z) = \sum_{n=0}^\infty a_{n+2} (z - z_0)^n$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$.

Define

$$g\colon D\cup\{z_0\}\to\mathbb{C},\quad z\mapsto\begin{cases}\displaystyle\sum_{n=0}^{\infty}a_{n+2}(z-z_0)^n, & z\in B_\epsilon(z_0),\\ f(z), & z\in D\setminus B_\epsilon(z_0).\end{cases}$$

Then g is a holomorphic function extending f. □

Definition. Let $D\subset\mathbb{C}$ be open, let $f\colon D\to\mathbb{C}$ be holomorphic, and let $z_0\in\mathbb{C}\setminus D$ be an isolated singularity of f. Then z_0 is called a *pole* of f if $\lim\limits_{z\to z_0}|f(z)|=\infty$.

Example. For $n\in\mathbb{N}$, the function

$$\mathbb{C}\setminus\{0\}\to\mathbb{C},\quad z\mapsto\frac{1}{z^n}$$

has a pole at 0.

Theorem 8.2 (Poles)**.** *Let $D\subset\mathbb{C}$ be open, let $f\colon D\to\mathbb{C}$ be holomorphic, and let $z_0\in\mathbb{C}\setminus D$ be an isolated singularity of f. Then z_0 is a pole of f $\iff$ there exist a unique $k\in\mathbb{N}$ and a holomorphic function $g\colon D\cup\{z_0\}\to\mathbb{C}$ such that $g(z_0)\neq 0$ and*

$$f(z)=\frac{g(z)}{(z-z_0)^k}$$

for $z\in D$.

Proof.
"$\Leftarrow$" This follows directly from the definition of a pole.
"$\Rightarrow$" Let us prove the uniqueness first. Suppose that there exist natural numbers $k_1\leq k_2$ and holomorphic functions $g_1,g_2\colon D\cup\{z_0\}\to\mathbb{C}$ such that $g_j(z_0)\neq 0$ and

$$f(z)=\frac{g_j(z)}{(z-z_0)^{k_j}}$$

for $z\in D$ and $j=1,2$. If $k_2>k_1$, we then find for all $z\in D$ that

$$g_2(z_0)=\lim_{z\to z_0}g_2(z)=\lim_{z\to z_0}(z-z_0)^{k_2-k_1}g_1(z)=0\cdot g_1(z_0)=0,$$

which is a contradiction. Hence $k_1=k_2$ and thus $g_1=g_2$ on D and, by continuity, on $D\cup\{z_0\}$.

To establish the existence of k and g, choose $r>0$ such that $B_r(z_0)\setminus\{z_0\}\subset D$ and $|f(z)|\geq 1$ for all $z\in B_r(z_0)\setminus\{z_0\}$. Then

$$B_r(z_0)\setminus\{z_0\}\to\mathbb{C},\quad z\mapsto\frac{1}{f(z)}$$

is holomorphic and bounded and thus, by Riemann's Removability Criterion, has a holomorphic extension $h\colon B_r(z_0) \to \mathbb{C}$ with $h(z_0) = \lim_{z \to z_0} 1/f(z) = 0$. Note that z_0 is the *only* zero of h. Let

$$h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in B_r(z_0)$ be the power series representation of h. Set $k := \min\{n \in \mathbb{N}_0 : a_n \neq 0\}$. Since $a_0 = h(z_0) = 0$, we have $k \geq 1$. Define

$$\tilde{h}\colon B_r(z_0) \to \mathbb{C}, \quad z \mapsto \sum_{n=k}^{\infty} a_n (z - z_0)^{n-k}.$$

Then $\tilde{h}$ is holomorphic, has no zeros, and satisfies $h(z) = (z - z_0)^k \tilde{h}(z)$ for $z \in B_r(z_0)$. For $z \in B_r(z_0) \setminus \{z_0\}$, we thus have

$$f(z) = \frac{1}{(z - z_0)^k \tilde{h}(z)},$$

so that we can construct the holomorphic function

$$g\colon D \cup \{z_0\} \quad \mathbb{C}, \quad z \mapsto \begin{cases} (z - z_0)^k f(z), & z \neq z_0, \\ \frac{1}{\tilde{h}(z_0)}, & z = z_0. \end{cases}$$

□

Definition. Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be a pole of f. Then the positive integer k in Theorem 8.2 is called the *order* of z_0 and denoted by $\operatorname{ord}(f, z_0)$. If $\operatorname{ord}(f, z_0) = 1$, we call z_0 a *simple pole* of f.

Example. For $m \in \mathbb{N}$, consider

$$f_m\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{\sin z}{z^m}.$$

We claim that f_1 has a removable singularity at 0 whereas f_m has a pole of order $m - 1$ at 0 for $m \geq 2$.

Recall that

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

for $z \in \mathbb{C}$. For $z \neq 0$, we thus have

$$f_1(z) = \frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Define

$$g\colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}.$$

Then g is holomorphic and extends f_1. Hence, f_1 has a removable singularity at 0.

For $m \geq 2$ and $z \neq 0$, note that $f_m(z) = \frac{g(z)}{z^{m-1}}$. Since $g(0) \neq 0$, we see that f_m has a pole of order $m-1$ at 0.

Example. Consider

$$f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto e^{\frac{1}{z}}.$$

Then 0 is not removable because $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{n\to\infty} e^n = \infty$. But 0 is not a pole for f either: for $n \in \mathbb{N}$, we have

$$\left| f\left(\frac{i}{n}\right)\right| = |e^{-in}| = 1.$$

Definition. Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity of f. Then z_0 is called *essential* if it is neither removable nor a pole.

Theorem 8.3 (Casorati–Weierstraß Theorem)**.** *Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity of f. Then z_0 is essential $\iff$ $\overline{f(B_\epsilon(z_0) \cap D)} = \mathbb{C}$ for each $\epsilon > 0$.*

Proof. "$\Leftarrow$" For each $n \in \mathbb{N}$ choose $z_n \in B_{\frac{1}{n}}(z_0) \cap D$ such that $|f(z_n) - n| < \frac{1}{n}$. It follows that $\lim_{n\to\infty} |f(z_n)| = \infty$. Hence, z_0 cannot be removable.

For each $n \in \mathbb{N}$, choose $z'_n \in B_{\frac{1}{n}}(z_0) \cap D$ such that $|f(z'_n)| < \frac{1}{n}$. This means that $\lim_{n\to\infty} f(z'_n) = 0$, so that z_0 is not a pole either.

"$\Rightarrow$" Assume for some $\epsilon_0 > 0$ that $\overline{f(B_{\epsilon_0}(z_0) \cap D)} \neq \mathbb{C}$. Without loss of generality, suppose that $B_{\epsilon_0}(z_0) \setminus \{z_0\} \subset D$. Let $w_0 \in \mathbb{C}$ and $\delta > 0$ be such that $B_\delta(w_0) \subset \mathbb{C} \setminus f(B_{\epsilon_0}(z_0) \setminus \{z_0\})$. Consider

$$g\colon B_{\epsilon_0}(z_0) \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{f(z) - w_0}.$$

Then g is holomorphic with

$$|g(z)| = \frac{1}{|f(z) - w_0|} \leq \frac{1}{\delta}$$

for $z \in B_{\epsilon_0}(z_0) \setminus \{z_0\}$. Hence, z_0 is a removable singularity of g. Let $\tilde{g}\colon B_{\epsilon_0}(z_0) \to \mathbb{C}$ be a holomorphic extension of g.

Case 1: $\tilde{g}(z_0) \neq 0$. Since $f(z) = \frac{1}{\tilde{g}(z)} + w_0$ for $z \in B_{\epsilon_0}(z_0) \setminus \{z_0\}$, this means that z_0 is a removable singularity of f, contradicting the fact that z_0 is an essential singularity.

Case 2: $\tilde{g}(z_0) = 0$. For $z \in B_{\epsilon_0}(z_0)$, we have

$$|f(z)| = \left| \frac{1}{\tilde{g}(z)} + w_0 \right| \geq \frac{1}{|\tilde{g}(z)|} - |w_0| \underset{z \to z_0}{\to} \infty.$$

Hence, z_0 is a pole of f, again contradicting the fact that z_0 is an essential singularity. $\square$

Problem 8.1.

Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$. Show that the following are equivalent for $n \in \mathbb{N}$:

(i) $f^{(k)}(z_0) = 0$ for $k = 0, \ldots, n-1$ and $f^{(n)}(z_0) \neq 0$;

(ii) there exists a holomorphic function $g \colon D \to \mathbb{C}$ with $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^n g(z)$ for $z \in D$.

If either condition holds, we say that z_0 is a *zero* of f of *order* n.

Problem 8.2.

Let $D \subset \mathbb{C}$ be open, let $f, g : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ be a zero of order n for f and of order $m \geq 1$ for g. Show the singularity z_0 of $\frac{f}{g}$ is

(i) removable if $m \leq n$ and

(ii) a pole of order $m - n$ otherwise.

Problem 8.3.

Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be an isolated singularity of f.

(a) Show that, if z_0 is a pole of order k of f, then it is a pole of order $k + 1$ of f'.

(b) Show that $\exp \circ f$ has either a removable or an essential singularity at z_0.

Chapter 9

Holomorphic Functions on Annuli

Definition. Let $z_0 \in \mathbb{C}$, and let $r, R \in [0, \infty]$ be such that $r < R$. Then the *annulus* centered at z_0 with inner radius r and outer radius R is defined as

$$A_{r,R}(z_0) := \{z \in \mathbb{C} : r < |z - z_0| < R\}.$$

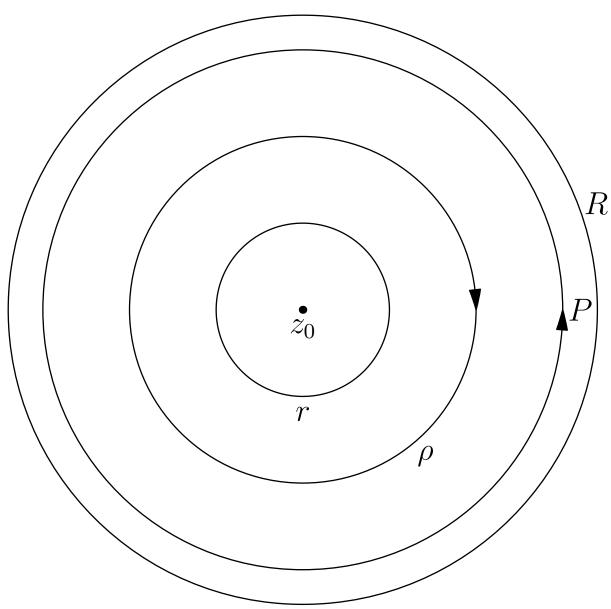
Theorem 9.1 (Cauchy's Integral Theorem for Annuli)**.** *Let $z_0 \in \mathbb{C}$, let $r, \rho, P, R \in [0, \infty]$ be such that $r < \rho < P < R$, and let $f : A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then we have*

$$\int_{\partial B_P(z_0)} f(\zeta)\, d\zeta = \int_{\partial B_\rho(z_0)} f(\zeta)\, d\zeta.$$

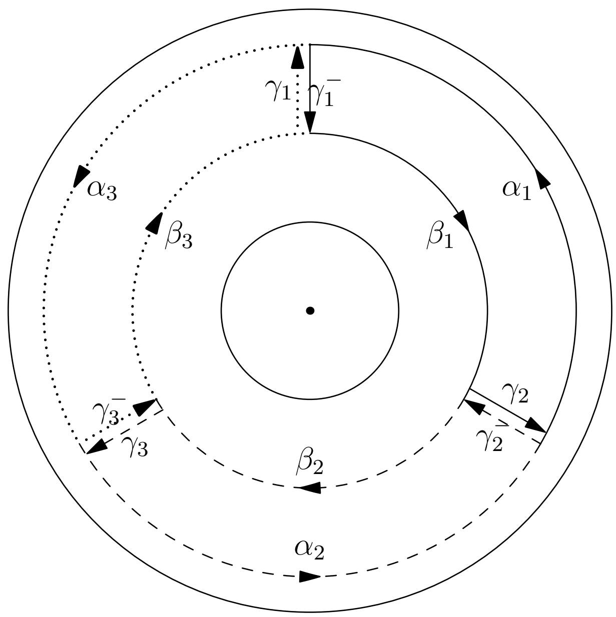
Proof. The claim is equivalent to

$$\int_{\partial B_P(z_0)} f(\zeta)\, d\zeta + \int_{\partial B_\rho(z_0)^-} f(\zeta)\, d\zeta = 0.$$

Consider



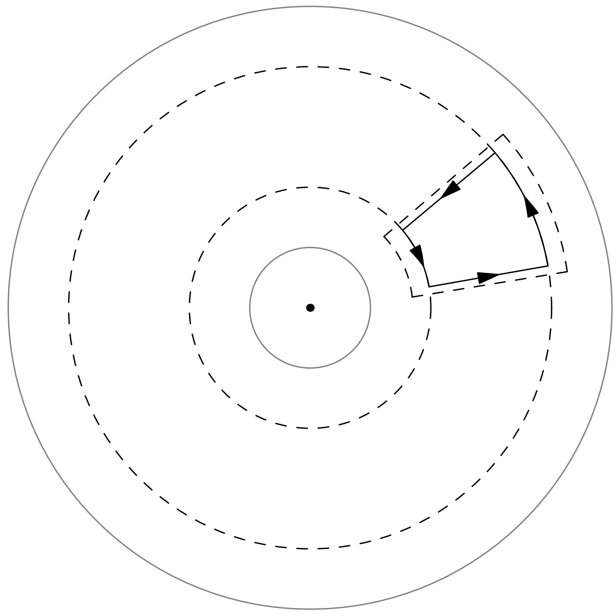
Split $\partial B_P(z_0)$ and $\partial B_\rho(z_0)^-$ into finitely many arc segments $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively, and connect them with line segments $\gamma_1, \ldots, \gamma_n$ as shown below for $n = 3$:



We thus obtain

$$\begin{aligned}\int_{\partial B_P(z_0)} f(\zeta)\,d\zeta + \int_{\partial B_\rho(z_0)^-} f(\zeta)\,d\zeta &= \sum_{j=1}^{n} \int_{\alpha_j} f(\zeta)\,d\zeta + \sum_{j=1}^{n} \int_{\beta_j} f(\zeta)\,d\zeta \\ &= \sum_{j=1}^{n} \int_{\alpha_j \oplus \gamma_j^- \oplus \beta_j \oplus \gamma_{(j+1) \bmod n}} f(\zeta)\,d\zeta\end{aligned}$$

By making the arc segments $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ sufficiently small, we can ensure that each of the closed curves $\alpha_1 \oplus \gamma_1^- \oplus \beta_1 \oplus \gamma_2, \ldots, \alpha_{n-1} \oplus \gamma_{n-1}^- \oplus \beta_{n-1} \oplus \gamma_n, \alpha_n \oplus \gamma_n^- \oplus \beta_n \oplus \gamma_1$ lies inside a star-shaped open subset of $A_{r,R}(z_0)$:



It follows that

$$\int_{\partial B_P(z_0)} f(\zeta)\,d\zeta + \int_{\partial B_\rho(z_0)^-} f(\zeta)\,d\zeta$$
$$= \sum_{j=1}^{n} \int_{\alpha_j \oplus \gamma_j^- \oplus \beta_j \oplus \gamma_{(j+1) \bmod n}} f(\zeta)\,d\zeta = 0$$

as claimed. □

Theorem 9.2 (Laurent Decomposition). *Let $z_0 \in \mathbb{C}$, let $r, R \in [0, \infty]$ be such that $r < R$, and let $f : A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then there exists a holomorphic function*

$$g \colon B_R(z_0) \to \mathbb{C} \qquad \text{and} \qquad h \colon \mathbb{C} \setminus B_r[z_0] \to \mathbb{C}$$

with $f = g + h$ on $A_{r,R}(z_0)$. Moreover, h can be chosen such that $\lim_{|z| \to \infty} h(z) = 0$, in which case g and h are uniquely determined.

Proof. We prove the uniqueness assertion first.

Let $g_1, g_2 : B_R(z_0) \to \mathbb{C}$ and $h_1, h_2 : \mathbb{C} \setminus B_r[z_0] \to \mathbb{C}$ be holomorphic such that $\lim_{|z| \to \infty} h_j(z) = 0$ for $j = 1, 2$ and

$$f = g_1 + h_1 = g_2 + h_2.$$

It follows that $g_1 - g_2 = h_2 - h_1$ on $A_{r,R}(z_0)$. Define

$$F \colon \mathbb{C} \to \mathbb{C}, \quad z \mapsto \begin{cases} g_1(z) - g_2(z), & z \in B_R(z_0), \\ h_2(z) - h_1(z), & z \in \mathbb{C} \setminus B_r[z_0]. \end{cases}$$

Then F is entire with $\lim_{|z|\to\infty} |F(z)| = \lim_{|z|\to\infty} |h_2(z) - h_1(z)| = 0$. Hence, F is bounded and entire and thus constant by Liouville's theorem. Since $\lim_{|z|\to\infty} |F(z)| = 0$, this means that $F \equiv 0$, so that $g_1 = g_2$ and $h_1 = h_2$.

To show that g and h exists, for $z \in A_{r,R}(z_0)$ choose ρ and P such that

$$r < \rho < |z - z_0| < P < R.$$

Define

$$G\colon A_{r,R}(z_0) \to \mathbb{C}, \quad \zeta \mapsto \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \neq z, \\ f'(z), & \zeta = z. \end{cases}$$

Then G is certainly holomorphic on $A_{r,R}(z_0)\backslash\{z\}$. Because it is continuous on $A_{r,R}(z_0)$, Riemann's Removability Criterion implies that G is in fact holomorphic on all of $A_{r,R}(z_0)$. It follows from Cauchy's Integral Theorem for Annuli that

$$\int_{\partial B_\rho(z_0)} G(\zeta)\, d\zeta = \int_{\partial B_P(z_0)} G(\zeta)\, d\zeta,$$

i.e.

$$\int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) \underbrace{\int_{\partial B_\rho(z_0)} \frac{1}{\zeta - z}\, d\zeta}_{=0} = \int_{\partial B_P(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) \underbrace{\int_{\partial B_P(z_0)} \frac{1}{\zeta - z}\, d\zeta}_{=2\pi i},$$

Let us define the holomorphic functions (cf. Lemma 5.4)

$$h(z) := \frac{-1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

on $\mathbb{C} \setminus B_\rho[z_0]$ and

$$g(z) := \frac{1}{2\pi i} \int_{\partial B_P(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

on $B_P(z_0)$, noting from Cauchy's Integral Theorem for Annuli that these definitions are independent of the precise choices of $\rho \in (r, |z - z_0|)$ and $P \in (|z - z_0|, R)$. Then the above result may be expressed as

$$-2\pi i\, h(z) = 2\pi i\, g(z) - 2\pi i\, f(z)$$

and thus

$$f(z) = g(z) + h(z).$$

Finally, we note that h satisfies

$$|h(z)| \leq \rho \sup_{\zeta \in \partial B_\rho(z_0)} \left| \frac{f(\zeta)}{\zeta - z} \right| \leq \rho \frac{\sup_{\zeta \in \partial B_\rho(z_0)} |f(\zeta)|}{\operatorname{dist}(z, \partial B_\rho(z_0))} \xrightarrow[|z|\to\infty]{} 0.$$

$\square$

Definition. The function h in Theorem 9.2 is called the *principal part* and g is called the *secondary part* of the *Laurent decomposition* $f = g + h$.

Theorem 9.3 (Laurent Coefficients)**.** *Let $z_0 \in \mathbb{C}$, let $r, R \in [0, \infty]$ be such that $r < R$, and let $f \colon A_{r,R}(z_0) \to \mathbb{C}$ be holomorphic. Then f has a representation*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in A_{r,R}(z_0)$ as a Laurent series, which converges uniformly and absolutely on compact subsets of $A_{r,R}(z_0)$. Moreover, for every $n \in \mathbb{Z}$ and $\rho \in (r, R)$, the coefficients a_n are uniquely determined as

$$a_n = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta.$$

Proof. Let g and h be as in Theorem 9.2 (in particular, with $\lim\limits_{|z|\to\infty} h(z) = 0$).

For $z \in B_R(z_0)$, we have the Taylor series

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

which converges uniformly and absolutely on compact subsets of $B_R(z_0)$.

Define

$$\tilde{h} \colon A_{0,\frac{1}{r}}(0) \to \mathbb{C}, \quad z \mapsto h\left(z_0 + \frac{1}{z}\right),$$

so that $\tilde{h}$ is holomorphic with $\lim\limits_{z\to 0} \tilde{h}(z) = 0$. Hence, $\tilde{h}$ has a removable singularity at 0 and thus extends to $B_{\frac{1}{r}}(0)$ as a holomorphic function. This holomorphic function, which we also denote by $\tilde{h}$, can then be expanded in a Taylor series for $z \in B_{\frac{1}{r}}(0)$:

$$\tilde{h}(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=1}^{\infty} b_n z^n,$$

so that

$$h(z) = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

converges uniformly and absolutely on compact subsets of $\mathbb{C} \setminus B_r[z_0]$.

Set $a_n := b_{-n}$ for $n < 0$. For $z \in A_{r,R}(z_0)$, we obtain

$$f(z) = g(z) + h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Finally, pick $m \in \mathbb{Z}$ and $\rho \in (r, R)$. Note that

$$\frac{f(z)}{(z-z_0)^{m+1}} = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^{n-m-1} = \sum_{n=-\infty}^{\infty} a_{n+m+1} (z-z_0)^n$$

converges uniformly on $\partial B_\rho(z_0)$. Hence, we find

$$\int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{m+1}}\, d\zeta = \sum_{n=-\infty}^{\infty} a_{n+m+1} \int_{\partial B_\rho(z_0)} (\zeta - z_0)^n \, d\zeta = a_m \int_{\partial B_\rho(z_0)} \frac{1}{\zeta - z_0}\, d\zeta = 2\pi i \, a_m,$$

noting that $(\zeta - z_0)^n$ has an antiderivative for all $n \neq -1$. Thus

$$a_m = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{m+1}}\, d\zeta$$

$\square$

Corollary 9.3.1. Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $f \colon B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent representation $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$. Then the singularity z_0 of f is

(i) removable if and only if $a_n = 0$ for $n < 0$;

(ii) a pole of order $k \in \mathbb{N}$ if and only if $a_{-k} \neq 0$ and $a_n = 0$ for all $n < -k$;

(iii) essential if and only if $a_n \neq 0$ for infinitely many $n < 0$.

Proof.

(i) The "if" part follows from Theorem 6.3.

Conversely, suppose that z_0 is a removable singularity, and let $\tilde{f} \colon B_r(z_0) \to \mathbb{C}$ be a holomorphic extension of f with Taylor expansion $\tilde{f}(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$ for $z \in B_r(z_0)$. The uniqueness of the Laurent representation yields $a_n = b_n$ for $n \in \mathbb{N}_0$ and $a_n = 0$ for $n < 0$.

(ii) For the "if" part, set

$$g(z) := (z-z_0)^k f(z) = \sum_{n=-k}^{\infty} a_n (z-z_0)^{n+k}$$

for $z \in B_r(z_0) \setminus \{z_0\}$. Then g extends holomorphically to $B_r(z_0)$ with $g(z_0) = a_{-k} \neq 0$. By definition, we have $f(z) = \frac{g(z)}{(z-z_0)^k}$ for $z \in B_r(z_0) \setminus \{z_0\}$. Hence, f has a pole of order k at z_0.

For the converse, let $g\colon B_r(z_0) \to \mathbb{C}$ be holomorphic such that $g(z_0) \neq 0$ and $f(z) = \frac{g(z)}{(z-z_0)^k}$ for $z \in B_r(z_0) \setminus \{z_0\}$. Let $g(z) = \sum_{n=0}^\infty b_n (z-z_0)^n$ for $z \in B_r(z_0)$ be the Taylor series of g, so that

$$f(z) = \sum_{n=0}^\infty b_n (z-z_0)^{n-k}$$

for $z \in B_r(z_0) \setminus \{z_0\}$. The uniqueness of the Laurent representation yields that $a_n = b_{n+k}$ for $n \geq -k$ and $a_n = 0$ for $n < -k$.

(iii) This follows from (i) and (ii) by elimination.

□

Examples.

1. Let
$$f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto e^{-\frac{1}{z^2}}.$$
Then f has the Laurent representation
$$f(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z^{2n}}$$
for $z \in \mathbb{C} \setminus \{0\}$ and thus has an essential singularity at 0.

2. Let
$$f\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{e^z - 1}{z^3},$$
so that
$$f(z) = \frac{1}{z^3} \sum_{n=1}^\infty \frac{z^n}{n!} = \sum_{n=1}^\infty \frac{z^{n-3}}{n!}$$
for $z \in \mathbb{C} \setminus \{0\}$. Hence, f has a pole of order two at 0.

Remark. The Laurent representation of a holomorphic function on an annulus $A_{r,R}(z_0)$ depends not only on z_0, but also on r and R.

Example. Consider the function

$$f\colon \mathbb{C} \setminus \{1, 3\} \to \mathbb{C}, \quad z \mapsto \frac{2}{z^2 - 4z + 3},$$

and note that
$$f(z) = \frac{1}{1-z} - \frac{1}{3-z}.$$
Then f has the following Laurent representations:

(a) On $A_{0,1}(0)$: For $|z|< 1$, we have

$$\frac{1}{1-z} = \sum_{n=0}^{n} z^n$$

and, for $|z|< 3$,

$$\frac{1}{3-z} = \frac{1}{3\left(1-\frac{z}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n.$$

We thus have for $z \in A_{0,1}(0)$ that

$$f(z) = \sum_{n=0}^{\infty}\left(1-\frac{1}{3^{n+1}}\right) z^n.$$

(b) On $A_{1,3}(0)$: For $|z|> 1$, we have

$$\frac{1}{1-z} = -\frac{1}{z-1} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}},$$

so that, for $z \in A_{1,3}(0)$:

$$f(z) = -\left(\sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}\right).$$

(c) On $A_{3,\infty}(0)$: For $|z|> 3$, we have

$$-\frac{1}{3-z} = \frac{1}{z-3} = \frac{1}{z\left(1-\frac{3}{z}\right)} = \sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}}$$

and thus, for $z \in A_{3,\infty}(0)$:

$$f(z) = \sum_{n=1}^{\infty}(3^{n-1}-1)\frac{1}{z^n}.$$

Chapter 10

The Winding Number of a Curve

Definition. Let γ be a closed curve in $\mathbb{C}$, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then the *winding number* of γ with respect to z is defined as

$$\nu(\gamma, z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \, d\zeta.$$

Remark. Geometrically, $\nu(\gamma, z)$ is the number of times γ winds around z in the counterclockwise direction.

Lemma 10.1. *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a curve, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then there exist open discs $D_1, \ldots, D_n \subset \mathbb{C} \setminus \{z\}$ and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_{j-1}, t_j]) \subset D_j$ for $j = 1, \ldots, n$.*

Proof. Let $\epsilon := \operatorname{dist}(z, \{\gamma\}) > 0$. Since γ is uniformly continuous, there exists $\delta > 0$ such that $|\gamma(t) - \gamma(t')| < \epsilon$ for all $t, t' \in [0,1]$ such that $|t - t'| < \delta$. Choose $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $|t_{j-1} - t_j| < \delta$ for $j = 1, \ldots, n$, and set $D_j := B_\epsilon(\gamma(t_j))$ for $j = 1, \ldots, n$. By the choice of ϵ, it is clear that $D_1, \ldots, D_n \subset \mathbb{C} \setminus \{z\}$. For $j = 1, \ldots, n$, let $t \in [t_{j-1}, t_j]$, and note that $|t - t_j| \leq |t_{j-1} - t_j| < \delta$, so that $|\gamma(t) - \gamma(t_j)| < \epsilon$, i.e. $\gamma(t) \in D_j$; consequently, $\gamma([t_{j-1}, t_j]) \subset D_j$ holds. $\square$

Proposition 10.1. *Let γ be a closed curve in $\mathbb{C}$, and let $z \in \mathbb{C} \setminus \{\gamma\}$. Then $\nu(\gamma, z) \in \mathbb{Z}$.*

Proof. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and open discs $D_1, \ldots, D_n$ such that $\gamma([t_{j-1}, t_j]) \subset D_j$ for $j = 1, \ldots, n$.

Let $j \in \{1, \ldots, n\}$. Since $z \notin D_j$, there exists a holomorphic function $L_j\colon D_j \to \mathbb{C}$ such that

$$e^{L_j(w)} = w - z \quad \text{for } w \in D_j.$$

On noting that $\gamma(t_n) = \gamma(t_0)$, it is convenient to denote $D_{n+1} := D_1$ and $L_{n+1} := L_1$. For $j = 1, \ldots, n$ we then see that $\gamma(t_j) \in D_j \cap D_{j+1}$ and hence

$$\exp(L_j(\gamma(t_j))) = \gamma(t_j) - z = \exp(L_{j+1}(\gamma(t_j))),$$

so that

$$\exp(L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j))) = 1$$

and thus

$$L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j)) \in 2\pi i\,\mathbb{Z}.$$

On differentiating $e^{L_j(w)} = w - z$, we find that $e^{L_j(w)} L_j'(w) = 1$. Thus

$$L_j'(w) = \frac{1}{w-z} \quad \text{for } w \in D_j;$$

this allows us to express

$$\begin{aligned}
\int_\gamma \frac{1}{\zeta - z}\, d\zeta &= \sum_{j=1}^{n} \int_{\gamma|_{[t_{j-1},t_j]}} \frac{1}{\zeta - z}\, d\zeta \\
&= \sum_{j=1}^{n} [L_j(\gamma(t_j)) - L_j(\gamma(t_{j-1}))] \\
&= \sum_{j=1}^{n} L_j(\gamma(t_j)) - \sum_{j=0}^{n-1} L_{j+1}(\gamma(t_j)) \\
&= L_n(\gamma(t_n)) - L_1(\gamma(t_0)) + \sum_{j=1}^{n-1} [L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j))]. \\
&= L_n(\gamma(t_n)) - L_{n+1}(\gamma(t_n)) + \sum_{j=1}^{n-1} [L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j))]. \\
&= \sum_{j=1}^{n} [L_j(\gamma(t_j)) - L_{j+1}(\gamma(t_j))].
\end{aligned}$$

We thus see that $\displaystyle\int_\gamma \frac{1}{\zeta - z}\, d\zeta \in 2\pi i\,\mathbb{Z}$. □

Definition. Let γ be a closed curve in $\mathbb{C}$. We define the *interior* and *exterior* of γ to be

$$\operatorname{int}\gamma := \{z \in \mathbb{C} \setminus \{\gamma\} : \nu(\gamma, z) \neq 0\}$$

and

$$\operatorname{ext}\gamma := \{z \in \mathbb{C} \setminus \{\gamma\} : \nu(\gamma, z) = 0\}.$$

Proposition 10.2 (Winding Numbers Are Locally Constant). *Let γ be a closed curve in $\mathbb{C}$. Then:*

(i) *the map*
$$\mathbb{C} \setminus \{\gamma\} \to \mathbb{C}, \quad z \mapsto \nu(\gamma, z)$$
is locally constant;

(ii) *there exists $R > 0$ such that $\mathbb{C} \setminus B_R[0] \subset \operatorname{ext} \gamma$.*

Proof. (i): Let $z_0 \in \mathbb{C} \setminus \{\gamma\}$ and choose $R > r > 0$ such that $B_R(z_0) \subset \mathbb{C} \setminus \{\gamma\}$.

Consider the function
$$F \colon \{\gamma\} \times B_r[z_0] \to \mathbb{C}, \quad (\zeta, z) \mapsto \frac{1}{\zeta - z}.$$

Then F is continuous and thus uniformly continuous. Choose $\delta \in (0, r)$ such that
$$z \in B_\delta(z_0), \quad \zeta \in \{\gamma\} \Rightarrow |F(\zeta, z) - F(\zeta, z_0)| < \frac{\pi}{\ell(\gamma) + 1}.$$

Then
$$\begin{aligned} |\nu(\gamma, z) - \nu(\gamma, z_0)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) d\zeta \right| \\ &\leq \frac{\ell(\gamma)}{2\pi} \sup_{\zeta \in \{\gamma\}} |F(\zeta, z) - F(\zeta, z_0)| \\ &\leq \frac{\ell(\gamma)}{2\pi} \frac{\pi}{\ell(\gamma) + 1} \\ &< \frac{1}{2}. \end{aligned}$$

Since $\nu(\gamma, z) - \nu(\gamma, z_0) \in \mathbb{Z}$, this means that $\nu(\gamma, z) = \nu(\gamma, z_0)$.

(ii): For any $z \in \mathbb{C} \setminus \{\gamma\}$, we have
$$|\nu(\gamma, z)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z} \, d\zeta \right| \leq \frac{\ell(\gamma)}{2\pi} \frac{1}{\operatorname{dist}(z, \{\gamma\})}.$$

Since $\lim_{|z| \to \infty} \operatorname{dist}(z, \{\gamma\}) = \infty$, there exists $R > 0$ such that $|\nu(\gamma, z)| \leq \frac{\ell(\gamma)}{2\pi} \frac{1}{\operatorname{dist}(z, \{\gamma\})} < 1$ for all $z \in \mathbb{C}$ such that $|z| > R$. Since $\nu(\gamma, z) \in \mathbb{Z}$ for all $z \in \mathbb{C} \setminus \{\gamma\}$, this implies that $\nu(\gamma, z) = 0$ for all $z \in \mathbb{C}$ with $|z| > R$. $\square$

Chapter 11

The General Cauchy Integral Theorem

Definition. Let $D \subset \mathbb{C}$ be open. We call a closed curve γ in D *homologous to zero* if $\nu(\gamma, z) = 0$ for each $z \in \mathbb{C} \setminus D$. That is, the interior of γ is a subset of D.

Definition. An open connected subset D of $\mathbb{C}$ is *simply connected* if every closed curve in D is homologous to zero. Equivalently, the interior of every closed curve in D is a subset of D.

Theorem 11.1 (Cauchy's Integral Formula)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let γ be a closed curve in D that is homologous to zero. Then, for $n \in \mathbb{N}_0$ and $z \in D \setminus \{\gamma\}$, we have*

$$\nu(\gamma, z) f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

Proof. It is enough to prove the claim for $n = 0$: for $n \geq 1$, differentiate the integral with respect to z and use induction.

Define

$$g \colon D \times D \to \mathbb{C}, \quad (w, z) \mapsto \begin{cases} \frac{f(w)-f(z)}{w-z}, & w \neq z, \\ f'(z), & w = z. \end{cases}$$

We claim that g is continuous. To see this, let $(w_0, z_0) \in D \times D$. As g is clearly continuous at (w_0, z_0) if $w_0 \neq z_0$, we need only show that g is continuous at (z_0, z_0). Given $\epsilon > 0$, choose $\delta > 0$ small enough that $B_\delta[z_0] \subset D$ and $|f'(z) - f'(z_0)| < \epsilon$ for all $z \in B_\delta[z_0]$. For $(w, z) \in B_\delta(z_0) \times B_\delta(z_0)$ we find:

- *if* $w = z$:

$$|g(w, z) - g(z_0, z_0)| = |f'(z) - f'(z_0)| < \epsilon;$$

- *if* $w \neq z$:

$$
\begin{aligned}
|g(w,z) - g(z_0,z_0)| &= \left| \frac{f(w) - f(z)}{w - z} - f'(z_0) \right| \\
&= \left| \frac{1}{w - z} \int_{[z,w]} [f'(\zeta) - f'(z_0)]\, d\zeta \right| \leq \sup_{\zeta \in \{[z,w]\}} |f'(\zeta) - f'(z_0)| \leq \epsilon.
\end{aligned}
$$

Thus, g is continuous.

Next, define

$$
h_0 \colon D \to \mathbb{C}, \quad z \mapsto \int_\gamma g(\zeta, z)\, d\zeta.
$$

We claim that h_0 is holomorphic. It is easy to see that h_0 is continuous. To see that it is indeed holomorphic, we shall show that it satisfies the Morera condition. Let $\Delta \subset D$ be a triangle. For fixed $\zeta \in \{\gamma\}$, the function

$$
D \to \mathbb{C}, \quad z \mapsto g(\zeta, z)
$$

is holomorphic as a consequence of Riemann's Removability Condition. Goursat's Lemma thus yields

$$
\int_{\partial\Delta} g(\zeta, z)\, dz = 0
$$

for each $\zeta \in \{\gamma\}$. As a consequence, we find

$$
\begin{aligned}
0 &= \int_\gamma \left(\int_{\partial\Delta} g(\zeta, z)\, dz \right) d\zeta \\
&= \int_{\partial\Delta} \left(\int_\gamma g(\zeta, z)\, d\zeta \right) dz \\
&= \int_{\partial\Delta} h_0(z)\, dz,
\end{aligned}
$$

so that h_0 is holomorphic as claimed.

Define

$$
h_1 \colon \operatorname{ext} \gamma \to \mathbb{C}, \quad z \mapsto \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.
$$

Then h_1 is holomorphic. For $z \in D \cap \operatorname{ext} \gamma$, we note that

$$\begin{aligned} h_0(z) &= \int_\gamma g(\zeta, z)\, d\zeta \\ &= \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta \\ &= \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) \underbrace{\int_\gamma \frac{1}{\zeta - z}\, d\zeta}_{=0} \\ &= \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= h_1(z). \end{aligned}$$

Define

$$h \colon D \cup \operatorname{ext} \gamma, \quad z \mapsto \begin{cases} h_0(z), & z \in D, \\ h_1(z), & z \in \operatorname{ext} \gamma. \end{cases}$$

Then h is holomorphic. Since γ is homologous to zero, we have $\mathbb{C} \setminus D \subset \operatorname{ext} \gamma$. Hence, h is entire.

For any $z \in \operatorname{ext} \gamma$, we have the estimate

$$|h(z)| = |h_1(z)| \leq \frac{\ell(\gamma)}{\operatorname{dist}(z, \{\gamma\})} \sup_{\zeta \in \{\gamma\}} |f(\zeta)|. \tag{$*$}$$

Let $R > 0$ be such that $\mathbb{C} \setminus B_R(0) \subset \operatorname{ext} \gamma$. Since $(*)$ implies that h is bounded on $\mathbb{C} \setminus B_R(0)$ and h is trivially bounded by continuity on $B_R[0]$, we see that h is bounded on $\mathbb{C}$ and hence constant by Liouville's Theorem. From $(*)$ again, we see that $\lim_{|z| \to \infty} |h(z)| = 0$. Hence, $h \equiv 0$.

In summary, we have for $z \in D \setminus \{\gamma\}$ that

$$0 = h(z) = h_0(z) = \int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi i \nu(\gamma, z) f(z).$$

□

Theorem 11.2 (Cauchy's Integral Theorem)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and let γ be a closed curve in D that is homologous to zero. Then $\int_\gamma f(\zeta)\, d\zeta = 0$.*

Proof. Let $z_0 \in D \setminus \{\gamma\}$ be arbitrary, and define

$$g \colon D \to \mathbb{C}, \quad z \mapsto (z - z_0) f(z),$$

so that

$$0 = 2\pi i \nu(\gamma, z_0) g(z_0) = \int_\gamma \frac{g(\zeta)}{\zeta - z_0}\, d\zeta = \int_\gamma f(\zeta)\, d\zeta.$$

□

Corollary 11.2.1. Let D be an open, connected subset of $\mathbb{C}$. Then D is simply connected $\iff$ every holomorphic function on D has an antiderivative.

Problem 11.1. Let $D \subset \mathbb{C}$ be open and connected such that, for each holomorphic $f : D \to \mathbb{C}$, there is a sequence $(p_n)_{n=1}^{\infty}$ of polynomials converging to f compactly on D. Show that D is simply connected.

Definition. Let $D \subset \mathbb{C}$ be open and connected and $n \in \mathbb{N}$. We say that D *admits*

(a) *holomorphic logarithms* if, for every holomorphic function $f : D \to \mathbb{C}$ with $\mathbf{Z}(f) = \varnothing$, there exists a holomorphic function $g \colon D \to \mathbb{C}$ with $f = \exp \circ g$;

(b) *holomorphic nth roots* if for every holomorphic function $f \colon D \to \mathbb{C}$ with $\mathbf{Z}(f) = \varnothing$, there exists a holomorphic function $h_n \colon D \to \mathbb{C}$ with $f(z) = [h_n(z)]^n$ for $z \in D$;

(c) *holomorphic roots* if D admits holomorphic nth roots for each $n \in \mathbb{N}$.

Corollary 11.2.2 (Holomorphic Logarithms)**.** A simply connected domain admits holomorphic logarithms.

Proof. This follows from Corollary 11.2.1 and Problem 5.1(a). $\square$

Corollary 11.2.3 (Holomorphic Roots)**.** A simply connected domain admits holomorphic roots.

Proof. Let g be such that $f = \exp \circ g$, and set $h_n := \exp \circ \left(\frac{g}{n}\right)$ for $n \in \mathbb{N}$. $\square$

Chapter 12

The Residue Theorem and Applications

Definition. Let $z_0 \in \mathbb{C}$, let $r > 0$, and let $f\colon B_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for $z \in B_r(z_0) \setminus \{z_0\}$. Then a_{-1} is called the *residue* of f at z_0 and denoted by $\operatorname{res}(f, z_0)$.

Remarks.

1. By Theorem 9.3, we have

$$\operatorname{res}(f, z_0) = \frac{1}{2\pi i} \int_{\partial B_\rho(z_0)} f(\zeta)\, d\zeta$$

   for any $\rho \in (0, r)$.

2. If f has a removable singularity at z_0, then $\operatorname{res}(f, z_0) = 0$.

3. Suppose that f has a simple pole at z_0, i.e.

$$f(z) = \sum_{n=-1}^{\infty} a_n (z - z_0)^n$$

   with $a_{-1} \neq 0$, then

$$\operatorname{res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

4. Suppose that $f(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^n$ has a pole of order k at z_0. On letting $g(z) = (z - z_0)^k f(z)$, we see that $\operatorname{res}(f, z_0)$ is the coefficient in the Taylor series of $g(z) = \sum_{n=-k}^{\infty} a_n (z - z_0)^{n+k} = \sum_{n=0}^{\infty} a_{n-k} (z - z_0)^n$ corresponding to $n = k - 1$:

$$\operatorname{res}(f, z_0) = \frac{g^{(k-1)}(z_0)}{(k-1)!} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right]_{z=z_0}.$$

Examples.

1. Let
$$f(z) = \frac{e^{iz}}{z^2+1},$$
so that f has a simple pole at $z_0 = i$. It follows that
$$\operatorname{res}(f,i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{e^{iz}}{z+i} = -\frac{i}{2e}.$$

2. Let
$$f(z) = \frac{\cos(\pi z)}{\sin(\pi z)},$$
so that f has a simple pole at each $n \in \mathbb{Z}$. For $n \in \mathbb{Z}$, we thus have:
$$\begin{aligned}\operatorname{res}(f,n) &= \lim_{z\to n}(z-n)\frac{\cos(\pi z)}{\sin(\pi z)}\\ &= \lim_{z\to n}(z-n)\frac{\cos(\pi z)}{\sin(\pi z)-\sin(\pi n)}\\ &= \frac{1}{\pi}\lim_{z\to n}\frac{\pi z-\pi n}{\sin(\pi z)-\sin(\pi n)}\cos(\pi z)\\ &= \frac{1}{\pi}.\end{aligned}$$

3. Let
$$f(z) = \frac{1}{(z^2+1)^3};$$
then f has a pole of order 3 at $z_0 = i$. With
$$g(z) = (z-i)^3 f(z) = \frac{1}{(z+i)^3},$$
we have
$$g'(z) = -\frac{3}{(z+i)^4} \qquad \text{and} \qquad g''(z) = \frac{12}{(z+i)^5},$$
so that
$$\operatorname{res}(f,i) = \frac{1}{2}\frac{12}{(2i)^5} = -\frac{3i}{16}.$$

Theorem 12.1 (Residue Theorem)**.** *Let $D \subset \mathbb{C}$ be open and simply connected, $z_1, \ldots, z_n \in D$ be such that $z_j \neq z_k$ for $j \neq k$, $f : D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}$ be holomorphic, and γ be a closed curve in $D \setminus \{z_1, \ldots, z_n\}$. Then we have*
$$\int_\gamma f(\zeta)\,d\zeta = 2\pi i \sum_{j=1}^n \nu(\gamma, z_j)\ \operatorname{res}(f, z_j).$$

Proof. Let $\epsilon > 0$ be such that $B_\epsilon(z_j) \subset D$ for $j = 1, \ldots, n$, with $z_k \notin B_\epsilon(z_j)$ for $k \neq j$. For $j = 1, \ldots, n$, we have Laurent representations

$$f(z) = \sum_{k=-\infty}^{\infty} a_k^{(j)} (z - z_j)^k$$

for $z \in B_\epsilon(z_j) \setminus \{z_j\}$, so that $\operatorname{res}(f, z_j) = a_{-1}^{(j)}$. For $j = 1, \ldots, n$, define

$$h_j \colon \mathbb{C} \setminus \{z_j\} \to \mathbb{C}, \quad z \mapsto \sum_{k=-\infty}^{-1} a_k^{(j)} (z - z_j)^k,$$

so that h_j is holomorphic on $\mathbb{C} \setminus \{z_j\}$. Define

$$g \colon D \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}, \quad z \mapsto f(z) - \sum_{j=1}^{n} h_j(z),$$

and note that $z_1, \ldots, z_n$ are removable singularities for g.

Since D is simply connected, Cauchy's Integral Theorem yields:

$$\begin{aligned}
0 &= \int_\gamma g(\gamma)\, d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} \int_\gamma h_j(\zeta)\, d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} \int_\gamma \left(\sum_{k=-\infty}^{-1} a_k^{(j)} (\zeta - z_j)^k \right) d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} \sum_{k=-\infty}^{-1} a_k^{(j)} \int_\gamma (\zeta - z_j)^k\, d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} a_{-1}^{(j)} \int_\gamma \frac{1}{\zeta - z_j}\, d\zeta \\
&= \int_\gamma f(\zeta)\, d\zeta - \sum_{j=1}^{n} \operatorname{res}(f, z_j)\, 2\pi i\, \nu(\gamma, z_j).
\end{aligned}$$

□

Corollary 12.1.1. Let $D \subset \mathbb{C}$ be open and simply connected, $f : D \to \mathbb{C}$ be holomorphic, and γ be a closed curve in D. Then we have

$$\nu(\gamma, z)\, f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for $z \in D \setminus \{\gamma\}$.

Proof. Fix $z \in D \setminus \{\gamma\}$, and define

$$g\colon D \setminus \{z\} \to \mathbb{C}, \quad w \mapsto \frac{f(w)}{w-z}.$$

Then g is holomorphic with an isolated singularity at z. Let

$$f(w) = \sum_{n=0}^{\infty} a_n (w-z)^n$$

be the Taylor series expansion of f near z, so that

$$g(w) = \sum_{n=-1}^{\infty} a_{n+1} (w-z)^n,$$

and thus $\operatorname{res}(g,z) = a_0 = f(z)$. The Residue Theorem then yields:

$$2\pi i\, \nu(\gamma, z)\, f(z) = 2\pi i\, \nu(\gamma, z)\ \operatorname{res}(g,z) = \int_\gamma g(\zeta)\, d\zeta = \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

□

12.1 Applications of the Residue Theorem to Real Integrals

Proposition 12.1 (Rational Trigonometric Polynomials)**.** *Let p and q be polynomials of two real variables such that $q(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ with $x^2 + y^2 = 1$. Then we have*

$$\int_0^{2\pi} \frac{p(\cos t, \sin t)}{q(\cos t, \sin t)}\, dt = 2\pi i \sum_{z \in \mathbb{D}} \operatorname{res}(f, z),$$

where

$$f(z) = \frac{1}{iz} \cdot \frac{p\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}{q\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}.$$

Proof. Just note that, by the Residue Theorem,

$$\begin{aligned} 2\pi i \sum_{z \in \mathbb{D}} \operatorname{res}(f, z) &= \int_{\partial \mathbb{D}} f(\zeta)\, d\zeta \\ &= \int_0^{2\pi} f(e^{i\theta}) i\, e^{i\theta}\, d\theta \\ &= \int_0^{2\pi} \frac{p(\cos\theta, \sin\theta)}{q(\cos\theta, \sin\theta)}\, d\theta. \end{aligned}$$

□

Examples.

1. Let $a > 1$. What is $\displaystyle\int_0^{\pi} \frac{dt}{a + \cos t}$?

   First, note that

$$\int_0^{\pi} \frac{dt}{a+\cos t} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{dt}{a+\cos t} = \frac{1}{2}\int_0^{2\pi} \frac{dt}{a+\cos t}.$$

   Let

$$p(x,y) = 1 \qquad \text{and} \qquad q(x,y) = a + x,$$

   so that

$$\begin{aligned} f(z) &= \frac{1}{iz} \cdot \frac{1}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)} \\ &= \frac{-i}{az + \frac{z^2}{2} + \frac{1}{2}} \\ &= \frac{-2i}{z^2 + 2az + 1} \\ &= \frac{-2i}{(z-z_1)(z-z_2)}, \end{aligned}$$

   where

$$z_1 = -a + \sqrt{a^2-1} \in \mathbb{D} \qquad \text{and} \qquad z_2 = -a - \sqrt{a^2-1} \notin \mathbb{D}, \tag{12.1}$$

   on noting that $1 + \sqrt{a^2-1} > a$ implies that $z_1 > -1$.

   By Proposition 12.1, we thus obtain

$$\begin{aligned} \int_0^{\pi} \frac{dt}{a+\cos t} &= \frac{1}{2}\int_0^{2\pi} \frac{dt}{a+\cos t} \\ &= \pi i \operatorname{res}(f, z_1) \\ &= \pi i \cdot \frac{-2i}{z_1 - z_2} \\ &= \frac{2\pi}{2\sqrt{a^2-1}} \\ &= \frac{\pi}{\sqrt{a^2-1}}. \end{aligned}$$

2. Let $a > 0$. What is $\displaystyle\int_0^{2\pi} \frac{dt}{(a+\cos t)^2}$?

   Let

$$p(x,y) = 1 \qquad \text{and} \qquad q(x,y) = (a+x)^2,$$

so that

$$\begin{aligned} f(z) &= \frac{1}{iz} \cdot \frac{1}{\left(a + \frac{1}{2}\left(z + \frac{1}{z}\right)\right)^2} \\ &= \frac{-4iz}{\left(z^2 + 2az + 1\right)^2} \\ &= \frac{-4iz}{(z - z_1)^2 (z - z_2)^2}, \end{aligned}$$

where z_1 and z_2 are again given by Eq. 12.1.

At z_1, the function f has a pole of order two. In order to calculate $\operatorname{res}(f, z_1)$, set

$$g(z) := (z - z_1)^2 f(z) = \frac{-4iz}{(z - z_2)^2},$$

so that

$$\begin{aligned} g'(z) &= -4i \left[\frac{1}{(z - z_2)^2} - \frac{2z}{(z - z_2)^3} \right] \\ &= \frac{-4i}{(z - z_2)^3} [(z - z_2) - 2z] \\ &= \frac{4i(z + z_2)}{(z - z_2)^3}; \end{aligned}$$

it follows that

$$\begin{aligned} \operatorname{res}(f, z_1) &= g'(z_1) \\ &= \frac{-4i(-2a)}{8\left(\sqrt{a^2 - 1}\right)^3} \\ &= \frac{-ai}{\left(\sqrt{a^2 - 1}\right)^3}. \end{aligned}$$

From Proposition 12.1, we conclude that

$$\int_0^{2\pi} \frac{dt}{(a + \cos t)^2} = 2\pi i \operatorname{res}(f, z_1) = \frac{2\pi a}{\left(\sqrt{a^2 - 1}\right)^3}.$$

Problem 12.1.

Let $D \subset \mathbb{C}$ be open, let $f \colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in D$ be a zero of order one of f. Show that

$$\operatorname{res}\left(\frac{1}{f}; z_0\right) = \frac{1}{f'(z_0)}.$$

Problem 12.2.

Let $D \subset \mathbb{C}$ be open, let $f\colon D \to \mathbb{C}$ be holomorphic, and let $z_0 \in \mathbb{C} \setminus D$ be a simple pole of f. Show that

$$\operatorname{res}(gf; z_0) = g(z_0) \operatorname{res}(f; z_0)$$

for every holomorphic function $g\colon D \cup \{z_0\} \to \mathbb{C}$.

Proposition 12.2 (Rational Functions)**.** *Let p and q be polynomials of one real variable with $\deg q \geq \deg p + 2$ and such that $q(x) \neq 0$ for $x \in \mathbb{R}$. Then we have*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{z \in \mathbb{H}} \operatorname{res}\left(\frac{p}{q}, z\right),$$

where

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Proof. Since $\deg q \geq \deg p + 2$, the Comparison Test yields that the indefinite integral exists.

For $r > 0$ consider the semicircle

$$\gamma_r\colon [0, \pi] \to \mathbb{C}, \quad \theta \mapsto re^{i\theta}.$$

Let $\epsilon > 0$ be such that, for $D := \{z \in \mathbb{C} : \operatorname{Im} z > -\epsilon\}$, we have

$$\{z \in \mathbb{H} : q(z) = 0\} = \{z \in D : q(z) = 0\}.$$

Then D is simply connected and $\frac{p}{q}$ is holomorphic on D except at the zeros of q in $\mathbb{H}$. For r large enough so that all zeros of q in D lie in the interior of $[-r, r] \oplus \gamma_r$, we see by the Residue Theorem that

$$\int_{[-r,r]\oplus\gamma_r} \frac{p(\zeta)}{q(\zeta)}\, d\zeta = 2\pi i \sum_{z \in D} \operatorname{res}\left(\frac{p}{q}, z\right) = 2\pi i \sum_{z \in \mathbb{H}} \operatorname{res}\left(\frac{p}{q}, z\right).$$

Since $\deg q \geq \deg p + 2$, there exist numbers $R > 0$ and $C \geq 0$ such that

$$\left|\frac{p(z)}{q(z)}\right| \leq \frac{C}{|z|^2}$$

for all $z \in \mathbb{C}$ with $|z| \geq R$. It follows that

$$\left|\int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)}\, d\zeta\right| \leq \pi r \sup_{\zeta \in \{\gamma_r\}} \frac{C}{|\zeta|^2} \leq \frac{\pi C}{r}$$

for $r \geq R$ and thus

$$\lim_{r \to \infty} \int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)}\, d\zeta = 0.$$

We then find that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx &= \lim_{r\to\infty} \int_{[-r,r]} \frac{p(\zeta)}{q(\zeta)}\,d\zeta \\
&= \lim_{r\to\infty} \int_{[-r,r]} \frac{p(\zeta)}{q(\zeta)}\,d\zeta + \lim_{r\to\infty} \int_{\gamma_r} \frac{p(\zeta)}{q(\zeta)}\,d\zeta \\
&= \lim_{r\to\infty} \int_{[-r,r]\oplus\gamma_r} \frac{p(\zeta)}{q(\zeta)}\,d\zeta \\
&= 2\pi i \sum_{z\in\mathbb{H}} \operatorname{res}\left(\frac{p}{q}, z\right).
\end{aligned}$$

□

Examples.

1. What is $\displaystyle\int_0^{\infty} \frac{1}{1+x^6}\,dx$?

   The zeros of $q(z) = 1 + z^6$ are of the form $e^{i\theta}$ where $\theta \in [0, 2\pi)$ is such that $e^{i6\theta} = -1 = e^{i\pi}$, i.e. $6\theta - \pi \in 2\pi\,\mathbb{Z}$, so that $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$. For $k = 1, \ldots, 6$, let

   $$z_k = e^{i(2k-1)\frac{\pi}{6}}.$$

   Then $\frac{1}{q}$ has a simple pole at z_k for $k = 1, \ldots, 6$.

   By Problem 12.1, we have

   $$\operatorname{res}\left(\frac{1}{q}, z_k\right) = \frac{1}{q'(z_k)} = \frac{1}{6z_k^5} = -\frac{z_k}{6},$$

   so that by, Proposition 12.2,

   $$\begin{aligned}
   \int_0^{\infty} \frac{1}{1+x^6}\,dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+x^6}\,dx \\
   &= \pi i \sum_{k=1}^{3} \operatorname{res}\left(\frac{1}{q}, z_k\right) \\
   &= -\frac{\pi i}{6}\left(e^{i\frac{\pi}{6}} + e^{i\frac{\pi}{2}} + e^{i\frac{5\pi}{6}}\right) \\
   &= -\frac{\pi i}{6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} + i + \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \\
   &= \frac{\pi}{6}\left(2\sin\frac{\pi}{6} + 1\right) \\
   &= \frac{\pi}{3}.
   \end{aligned}$$

2. What is $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n}\,dx$, where $n \in \mathbb{N}$?

The polynomial $q(z) := (z^2+1)^n$ has zeros of order n at $\pm i$. Define

$$g(z) = (z-i)^n \frac{1}{q(z)} = (z+i)^{-n},$$

so that

$$g^{(n-1)}(z) = (-n)\cdots(-2n+2)(z+i)^{-2n+1}$$

and thus

$$\begin{aligned}\operatorname{res}\left(\frac{1}{q}, i\right) &= \frac{g^{(n-1)}(i)}{(n-1)!} \\ &= \frac{1}{(n-1)!\,2^{2n-1}i} \cdot n \cdots (2n-2) \\ &= \frac{(2n-2)!}{i 2^{2n-1}(n-1)!^2}.\end{aligned}$$

It follows that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n}\,dx = 2\pi i \operatorname{res}\left(\frac{1}{q}, i\right) = \frac{\pi}{2^{2n-2}} \frac{(2n-2)!}{(n-1)!^2};$$

in particular, we have

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi,$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{2}, \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\,dx = \frac{3\pi}{8}.$$

Problem 12.3.

(a) Prove that $\sin\theta \geq \frac{2}{\pi}\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$.

(b) Use part (a) to show that for $R > 0$ that

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}.$$

(c) Let C_R be the semicircular contour $\{Re^{i\theta} : 0 \leq \theta \leq \pi\}$, with $R > 0$. Use part (b) to establish *Jordan's Lemma:*

$$\left|\int_{C_R} e^{iz}\,dz\right| < \pi.$$

Problem 12.4.

Let D be an open set. If $f\colon D \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, where f has a simple pole at z_0, and $C_r = \{z_0 + re^{i\theta} : \alpha \leq \theta \leq \beta\}$, prove the *Fractional Residue Theorem*:

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = (\beta - \alpha) i \operatorname{res}(f, z_0).$$

12.2 The Gamma Function

For $\mathrm{Re}(z) > 0$, define

$$\Gamma_+(z) := \int_{0^+}^{\infty} e^{-t} t^{z-1}\, dt,$$

where the integration is performed along the positive real axis. Then Γ_+ is holomorphic in the right half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. A single integration by parts yields the following recurrence relation

$$\begin{aligned} \Gamma_+(z+1) &= \int_{0^+}^{\infty} e^{-t} t^{z}\, dt = -e^{-t} t^{z}\Big|_0^{\infty} + z \int_{0^+}^{\infty} e^{-t} t^{z-1}\, dt \\ &= z\Gamma_+(z). \end{aligned} \tag{12.2}$$

Since $\Gamma_+(1) = \int_0^\infty e^{-t}\, dt = 1 = 0!$, we see that $\Gamma_+(n+1) = n!$ for $n \in \mathbb{N}_0$. Continuing in this manner we find that $\Gamma_+(z+n) = (z+n-1)\ldots(z+1)z\Gamma_+(z)$. On rearranging this formula,

$$\Gamma_+(z) = \frac{\Gamma_+(z+n)}{z(z+1)\ldots(z+n-1)},$$

it is possible to analytically continue the function to the left-half plane:

$$\Gamma(z) := \begin{cases} \Gamma_+(z) & \mathrm{Re}(z) > 0, \\ \dfrac{\Gamma_+(z+n)}{z(z+1)\ldots(z+n-1)} & -n < \mathrm{Re}(z) \le -n+1, z \ne -n+1, n = 1, 2, 3, \ldots \end{cases}$$

The resulting function $\Gamma(z)$ is holomorphic in the complex plane except at $z = 0, -1, -2, \ldots$, where it has simple poles. The graph of $\Gamma(x)$ for $x \in \mathbb{R}$ is shown in Figure 12.1 and an interactive three-dimensional plot of the surface $\Gamma(z)$ for $z \in \mathbb{C}$ is shown in Figure 12.2.

We proceed to derive a few useful relationships involving the Γ function.

- For $\alpha \in (0, 1)$ we have

$$\Gamma(\alpha) = \int_{0^+}^{\infty} e^{-t} t^{\alpha-1}\, dt = 2\int_{0^+}^{\infty} e^{-y^2} y^{2\alpha-1}\, dy \qquad (\text{letting } t = y^2),$$

which leads to

$$\begin{aligned} \Gamma(\alpha)\Gamma(1-\alpha) &= \left(2\int_{0^+}^{\infty} e^{-y^2} y^{2\alpha-1}\, dy\right)\left(2\int_{0^+}^{\infty} e^{-x^2} x^{1-2\alpha}\, dx\right) \\ &= 4\int_{0^+}^{\infty}\int_{0^+}^{\infty} e^{-(x^2+y^2)} \left(\frac{y}{x}\right)^{2\alpha-1} dx\, dy \\ &= 4\int_0^{\pi/2} \tan^{2\alpha-1}\theta \int_0^{\infty} e^{-r^2} r\, dr\, d\theta \\ &= 2\int_0^{\pi/2} \tan^{2\alpha-1}\theta\, d\theta. \end{aligned}$$

In particular, we see for $\alpha = 1/2$ that $\Gamma^2(1/2) = 2\int_0^{\pi/2} d\theta = \pi$ and

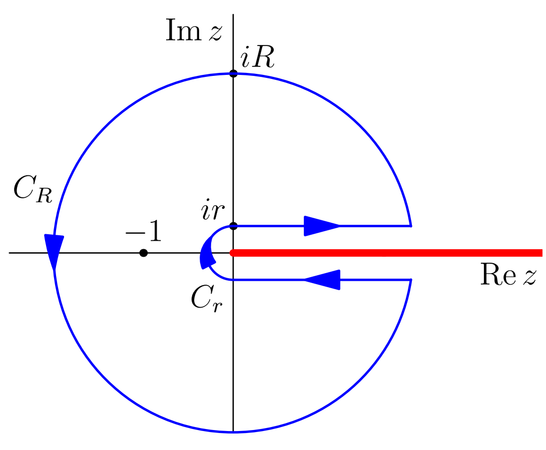
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

A substitution then leads to the important result $\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\pi/a}$ for $a > 0$.

For arbitrary $\alpha \in (0,1)$, we find, on substituting $z = \tan^2\theta$,

$$I(\alpha) := \Gamma(\alpha)\Gamma(1-\alpha) = 2\int_{0^+}^{\pi/2} \tan^{2\alpha-1}\theta\,d\theta = \int_{0^+}^{\infty} \frac{z^{\alpha-1}}{1+z}\,dz.$$

The integral here can be evaluated by a contour integration in the complex plane, noting that the function $z^{\alpha-1} = e^{(\alpha-1)\log z}$ is holomorphic on the star-shaped domain obtained by slicing the complex plane along the positive real axis. This *branch cut* is shown in red in the following figure. In other words we choose the antiderivative $\log z = \log|z| + i\arg z$ of the function $z \mapsto 1/z$, where $\arg z \in [0, 2\pi)$.



Here the large circular contour C_R is chosen to have radius $R \geq 2$, so that $|1+z| \geq R/2$ on C_R, and the small semicircular contour C_r is chosen to have radius $r \leq 1/2$, so that $|1+z| \geq 1/2$ on C_r. On denoting

$$f(z) := \frac{z^{\alpha-1}}{1+z} = \frac{e^{(\alpha-1)\log z}}{1+z},$$

we then see, accounting for the residue from the pole of f at $z = -1$, that

$$2\pi i e^{(\alpha-1)i\pi} = \int_{ir}^{R+ir} f + \int_{C_R} f + \int_{R-ir}^{-ir} f + \int_{C_r} f.$$

Since $\alpha < 1$, we see that the contribution from the circular arc C_R is

$$\left|\int_{C_R} f\right| \le \frac{R^{\alpha-1}}{\frac{R}{2}} \cdot 2\pi R = 4\pi R^{\alpha-1} \underset{R\to\infty}{\to} 0.$$

Likewise, since $\alpha > 0$, the contribution from the semicircular contour C_r is

$$\left|\int_{C_r} f\right| \le \frac{r^{\alpha-1}}{\frac{1}{2}} \cdot \pi r = 2\pi r^{\alpha} \underset{r\to 0}{\to} 0.$$

We thus deduce that

$$\begin{aligned}2\pi i e^{(\alpha-1)i\pi} &= \lim_{\substack{r\to 0\\ R\to\infty}} \left[\int_{ir}^{R+ir} f - \int_{-ir}^{R-ir} f\right]\\ &= \int_{0^+}^{\infty} \frac{e^{(\alpha-1)\log|z|}}{1+z}\, dz - \int_{0^+}^{\infty} \frac{e^{(\alpha-1)(\log|z|+i2\pi)}}{1+z}\, dz\\ &= I(\alpha)\big(1 - e^{(\alpha-1)2\pi i}\big).\end{aligned}$$

Thus

$$\pi = I(\alpha) \cdot \frac{e^{-(\alpha-1)\pi i} - e^{(\alpha-1)\pi i}}{2i} = I(\alpha) \cdot \frac{-e^{-\alpha\pi i} + e^{\alpha\pi i}}{2i},$$

from which we see that

$$I(\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha},$$

On extending this result by analytic continuation, one finds for all $z \in \mathbb{C} \setminus \mathbb{Z}$ that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

- For $\alpha \ge 1$ and positive x and λ, another frequently encountered integral can be expressed in terms of Γ using the substitution $u = xt^{\lambda}$:

$$\int_0^{\infty} e^{-xt^{\lambda}} t^{\alpha-1}\, dt = \frac{1}{\lambda x^{\frac{\alpha}{\lambda}}} \int_0^{\infty} e^{-u} u^{\frac{\alpha}{\lambda}-1}\, du = \frac{\Gamma\left(\frac{\alpha}{\lambda}\right)}{\lambda x^{\frac{\alpha}{\lambda}}}. \tag{12.3}$$

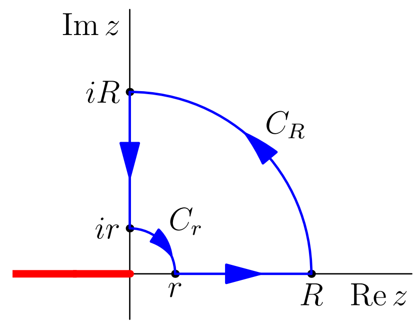
For the special case $\alpha = x = 1$, this result simplifies to

$$\int_0^{\infty} e^{-t^{\lambda}}\, dt = \frac{1}{\lambda}\Gamma\left(\frac{1}{\lambda}\right) = \Gamma\left(1 + \frac{1}{\lambda}\right).$$

For $0 < \alpha < 1$ and $x > 0$, a related integral is

$$\int_0^{\infty} e^{ixt} t^{\alpha-1}\, dt = \frac{i^{\alpha}\Gamma(\alpha)}{x^{\alpha}}. \tag{12.4}$$

To establish this result, it is convenient to introduce a branch cut, shown in red, along the negative real axis:



We note that $f(z) = e^{ixz} z^{\alpha-1}$ is holomorphic inside the blue contour. Cauchy's Integral Theorem thus implies that

$$0 = \int_r^R f(t)\,dt + \int_{C_R} f + i\int_R^r f(it)\;dt + \int_{C_r} f.$$

Since $\alpha < 1$, we see on using Problem 12.3 (a) that

$$\begin{aligned}\left|\int_{C_R} f\right| &\le \int_0^{\pi/2} e^{-xR\sin\theta} R^{\alpha-1} R\;d\theta \\ &\le R^{\alpha-1}\int_0^{\pi/2} e^{-2xR\theta/\pi} R\;d\theta = R^{\alpha-1}\frac{\pi}{2x}\left(1-e^{-xR}\right) \underset{R\to\infty}{\to} 0.\end{aligned}$$

Likewise, since $\alpha > 0$, we see that

$$\left|\int_{C_r} f\right| \le r^\alpha \frac{\pi}{2x}\left(\frac{1-e^{-xr}}{r}\right) \underset{r\to 0}{\to} 0.$$

Hence

$$\int_0^\infty f(t)\,dt = -\int_\infty^0 f(it)\;idt = i^\alpha \int_0^\infty e^{-xt} t^{\alpha-1}\;dt = \frac{i^\alpha \Gamma(\alpha)}{x^\alpha},$$

as claimed.

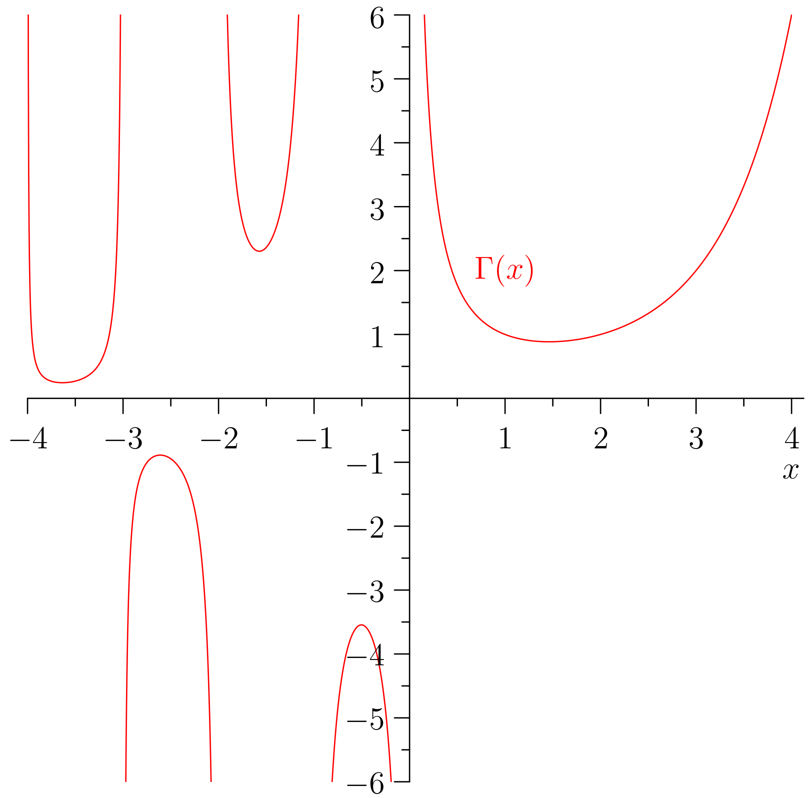


Figure 12.1: Graph of $\Gamma(x)$ on the real line.

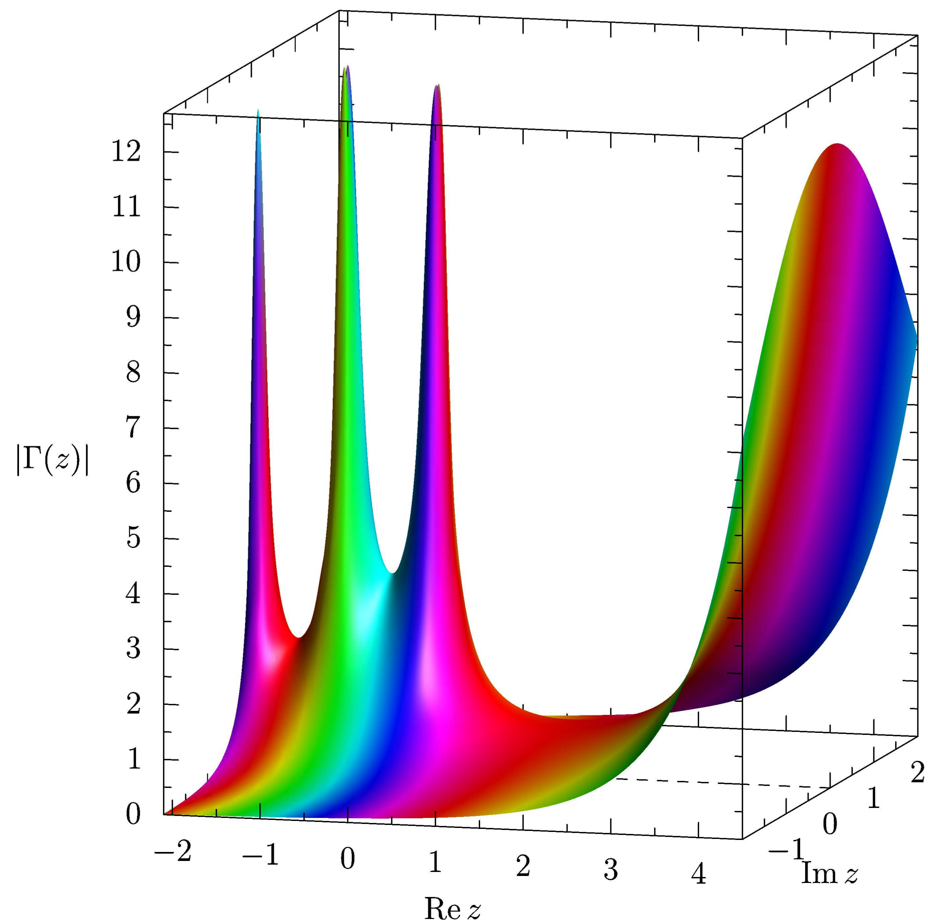


Figure 12.2: Surface plot of $\Gamma(z)$ in the complex plane, using an RGB color wheel to represent the phase. Red indicates real positive values. The poles at the negative integers and 0 are evident.

Chapter 13

Function Theoretic Consequences of the Residue Theorem

Definition. Let $D \subset \mathbb{C}$ be open. We call $S \subset D$ *discrete* in D if it has no cluster points in D.

Example. If D is open and connected, and $f\colon D \to \mathbb{C}$ is holomorphic and not identically zero, then $\mathbf{Z}(f)$ is discrete.

Remark. Let $S \subset D$ be discrete, and let $K \subset D$ be compact. If $K \cap S$ were infinite, then $K \cap S$ would have cluster points, which would lie in $K \subset D$. Thus, $K \cap S$ must be finite.

Remark. Let S be a discrete subset of an open set D. For suitably small radii $r(z) > 0$, we note that D can be expressed as a countable union of compact sets:

$$D = \bigcup_{z \in D \cap \mathbb{Q}^2} B_{r(z)}[z].$$

On denoting these compact sets as $\{K_n\}_{n=1}^{\infty}$, we see that each $K_n \cap S$ is finite. Thus

$$S = \bigcup_{n=1}^{\infty} (K_n \cap S).$$

is either finite or countably infinite.

Example. If D is open and connected, and $f\colon D \to \mathbb{C}$ is holomorphic and not identically zero, then $\mathbf{Z}(f)$ is at most countably infinite.

Proposition 13.1. *Let $D \subset \mathbb{C}$ be open, let γ be a closed curve in D, and let $S \subset D$ be discrete. Then $S \cap \operatorname{int}\gamma$ is finite.*

Proof. By Proposition 10.2(ii), there exists $R > 0$ such that $\operatorname{int}\gamma \subset B_R[0]$. $\square$

Definition. Denote the set of poles of a holomorphic function f by $\mathbf{P}(f)$.

Definition. Let $D \subset \mathbb{C}$ be open. A *meromorphic function* on D is a holomorphic function $f \colon D \setminus \mathbf{P}(f) \to \mathbb{C}$ such that $\mathbf{P}(f) \subset D$ is discrete in D.

Remark. If D is open and connected, and $f, g \colon D \to \mathbb{C}$ are holomorphic, then $\dfrac{f}{g}$ is meromorphic if g is not identically zero.

Proposition 13.2. *Let $D \subset \mathbb{C}$ be open, and let f be meromorphic on D. Then, for each $z_0 \in D$, there exist $\epsilon > 0$ with $B_\epsilon(z_0) \subset D$ and holomorphic functions $g, h \colon B_\epsilon(z_0) \to \mathbb{C}$ such that $f(z) = \frac{g(z)}{h(z)}$ for $z \in B_\epsilon(z_0) \setminus \{z_0\}$.*

Proof. If z_0 is not a pole of f, the claim is clear.

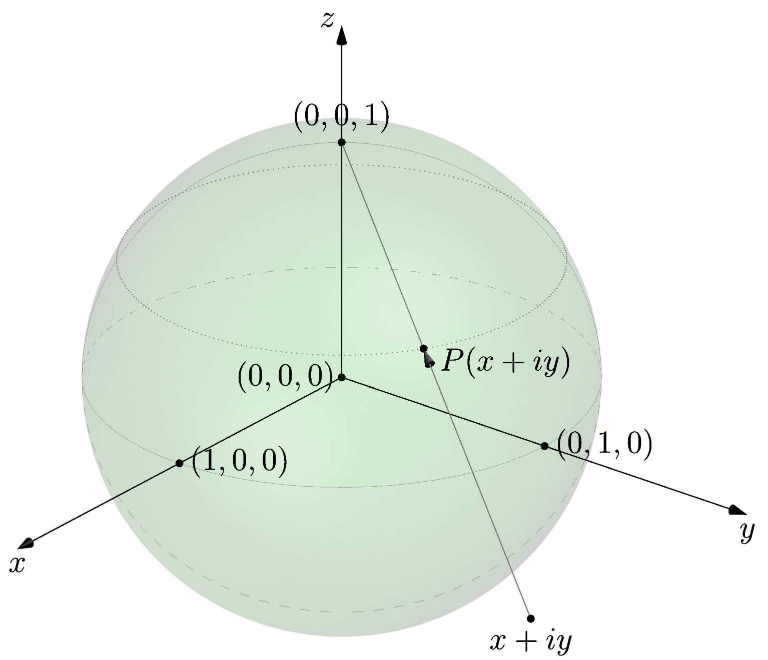
Otherwise, choose $\epsilon > 0$ so small that $B_\epsilon(z_0) \subset D$ and $\mathbf{P}(f) \cap B_\epsilon(z_0) = \{z_0\}$. We can then find a holomorphic function $g \colon B_\epsilon(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$ and $k \in \mathbb{N}$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^k}$$

for $z \in B_\epsilon(z_0) \setminus \{z_0\}$. Setting $h(z) := (z - z_0)^k$ yields the claim. □

Definition. It is convenient to define the set of *extended complex numbers* $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ and extend the domain of meromorphic functions to include their poles, assigning the function value ∞ at the poles. Here $\omega \cdot \infty$ is identified with ∞ for all $\omega \in \partial\mathbb{D}$. Furthermore, we define $1/0 = \infty$ and $1/\infty = 0$.

Definition. We can reinterpret $\mathbb{C}_\infty$ as the unit sphere S^2 in $\mathbb{R}^3$ by connecting every point $x + iy$ in the xy plane (which we identify with $\mathbb{C}$) to the north pole $(0, 0, 1)$ with a straight line that intersects S^2 at a point $P(x + iy)$. This defines an injective map $P \colon \mathbb{C}_\infty \to S^2$. Under this mapping, 0 maps to the south pole, ∞ maps to the north pole, and the unit circle $\partial\mathbb{D}$ maps to the equator. One can readily show that P is continuous, and that the inverse $P^{-1} \colon S^2 \to \mathbb{C}_\infty$ is also continuous; this allows us to identify $\mathbb{C}_\infty$ with the *Riemann sphere* S^2.



Lemma 13.1. *Let $D \subset \mathbb{C}$ be open and connected, and let $S \subset D$ be discrete. Then $D \setminus S$ is open and connected.*

Proof. Let $z \in D \setminus S$. Since S is discrete in D, there exists $\epsilon_1 > 0$ such that $B_{\epsilon_1}(z) \cap S = \varnothing$. Also, since D is open, there exists $\epsilon_2 > 0$ with $B_{\epsilon_2}(z) \subset D$. Setting $\epsilon := \min\{\epsilon_1, \epsilon_2\}$, we get $B_\epsilon(z)$. This proves the openness of $D \setminus S$.

Assume that $D \setminus S$ is not connected. Then there exist open sets $U \neq \varnothing \neq V$ with $U \cap V = \varnothing$ and $U \cup V = D \setminus S$. Let $s \in S$, and choose $\epsilon > 0$ such that $B_\epsilon(s) \subset D$ and $B_\epsilon(s) \cap S = \{s\}$. Set $W := B_\epsilon(s) \setminus \{s\}$, and note that W is open and connected. Since $(U \cap W) \cap (V \cap W) = \varnothing$ and $(U \cap W) \cup (V \cup W) = W$, the connectedness of W yields that either $U \cap W = \varnothing$ or $V \cap W = \varnothing$ and thus $W \subset U$ or $W \subset V$.

Set

$$S_U := \{s \in S : \text{there exists } \epsilon > 0 \text{ such that } B_\epsilon(s) \setminus \{s\} \subset U\}$$

and

$$S_V := \{s \in S : \text{there exists } \epsilon > 0 \text{ such that } B_\epsilon(s) \setminus \{s\} \subset V\}.$$

By the foregoing, we have $S = S_U \cup S_V$, and trivially, $S_U \cap S_V = \varnothing$ holds. Set

$$\tilde{U} := U \cup S_U \qquad \text{and} \qquad \tilde{V} := V \cup S_V.$$

Then $\tilde{U} \neq \varnothing \neq \tilde{V}$ are easily seen to be open and clearly satisfy $\tilde{U} \cap \tilde{V} = \varnothing$ and $\tilde{U} \cup \tilde{V} = D$, which contradicts the connectedness of D. $\square$

Theorem 13.1 (Meromorphic Functions Form a Field)**.** *Let $D \subset \mathbb{C}$ be open and connected. Then the meromorphic functions on D, where we define $(f+g)(z) = \lim_{w \to z}[f(w) + g(w)]$ and $(fg)(z) = \lim_{w \to z}[f(w)g(w)]$, form a field.*

Proof. It is easily checked that the meromorphic functions do indeed form a commutative ring. For each meromorphic function $f \not\equiv 0$ on D define

$$\tilde{f} \colon D \to \mathbb{C}, \quad z \mapsto \frac{1}{f(z)}.$$

As $\mathbf{P}(f)$ is discrete, $D \setminus \mathbf{P}(f)$ is connected by Lemma 13.1. From the Identity Theorem, we then conclude that $\mathbf{Z}(f)$ is discrete, too. Thus $\tilde{f}$ is meromorphic and $(f\tilde{f})(z) = 1$ (the multiplicative identity) for $z \in D$. ☐

Definition. Let $z_0 \in \mathbf{Z}(f)$. If $f(z) = (z - z_0)^k g(z)$, where g is a holomorphic function with $g(z_0) \neq 0$, we say that $k := \operatorname{ord}(f, z_0)$.

Theorem 13.2 (Argument Principle)**.** *Let $D \subset \mathbb{C}$ be open and simply connected, let f be meromorphic on D, and let γ be a closed curve in $D \setminus (\mathbf{P}(f) \cup \mathbf{Z}(f))$. Then we have*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{z \in \mathbf{Z}(f)} \nu(\gamma, z) \operatorname{ord}(f, z) - \sum_{z \in \mathbf{P}(f)} \nu(\gamma, z) \operatorname{ord}(f, z).$$

Proof. By the Residue Theorem, we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{z \in \mathbf{Z}(f)} \nu(\gamma, z) \operatorname{res}\left(\frac{f'}{f}, z\right) + \sum_{z \in \mathbf{P}(f)} \nu(\gamma, z) \operatorname{res}\left(\frac{f'}{f}, z\right).$$

Let $z_0 \in \mathbf{Z}(f)$, and let $k := \operatorname{ord}(f, z)$. Then there is a holomorphic function g with $g(z_0) \neq 0$ such that $f(z) = (z - z_0)^k g(z)$ and thus

$$f'(z) = k\,(z - z_0)^{k-1} g(z) + (z - z_0)^k g'(z).$$

It follows that

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \frac{g'(z)}{g(z)}$$

for z near z_0, so that

$$\operatorname{res}\left(\frac{f'}{f}, z\right) = k.$$

Let $z_0 \in \mathbf{P}(f)$, and let $k := \operatorname{ord}(f, z_0)$. Then $f(z) = \dfrac{g(z)}{(z - z_0)^k}$ holds with g holomorphic such that $g(z_0) \neq 0$ and, consequently,

$$f'(z) = -k(z - z_0)^{-(k+1)} g(z) + (z - z_0)^{-k} g'(z).$$

It follows that

$$\frac{f'(z)}{f(z)} = \frac{-k}{z - z_0} + \frac{g'(z)}{g(z)}$$

for $z \neq z_0$ near z_0, so that

$$\operatorname{res}\left(\frac{f'}{f}, z\right) = -k.$$

☐

Definition. Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be holomorphic. We say that f *attains* $w_0 \in \mathbb{C}$ *with multiplicity* $k \in \mathbb{N}$ *at* $z_0 \in D$ if the function

$$D \to \mathbb{C}, \quad z \mapsto f(z) - w_0$$

has a zero of order k at z_0.

Theorem 13.3 (Bifurcation Theorem)**.** *Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{C}$ be holomorphic, and suppose that, at $z_0 \in D$, the function f attains w_0 with multiplicity $k \in \mathbb{N}$. Then there exist neighbourhoods $V \subset D$ of z_0 and $W \subset f(V)$ of w_0 such that, for each $w \in W \setminus \{w_0\}$, there exist distinct $z_1, \ldots, z_k \in V$ with $f(z_1) = \cdots = f(z_k) = w$, where f attains w at each z_j with multiplicity one.*

Proof. In view of the Identity Theorem, we may choose $\epsilon > 0$ with $B_\epsilon[z_0] \subset D$ such that $f(z) \neq w_0$ and $f'(z) \neq 0$ for all z in the open connected set $B_\epsilon(z_0) \setminus \{z_0\}$.

Set $V := B_\epsilon(z_0)$ and $\gamma := \partial B_\epsilon(z_0)$. Choose $\delta > 0$ such that $B_\delta(w_0) \subset \mathbb{C} \setminus \{f \circ \gamma\}$, and set $W := B_\delta(w_0)$. Let $w \in W$. By the Argument Principle, the number of times w is attained in V (counting multiplicity) is

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(\zeta)}{f(\zeta) - w}\, d\zeta = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{d\zeta}{\zeta - w} = \nu(f \circ \gamma, w).$$

As $\nu(f \circ \gamma, \cdot)$ is constant on W, the number of times w is attained in V is the same as the number of times w_0 is attained in V, i.e. k. Since $f'(z) \neq 0$ for all $z \in V \setminus \{z_0\}$, for $w \neq w_0$ there exist distinct $z_1, \ldots, z_k \in V \setminus \{z_0\}$ such that $f(z_1) = \cdots = f(z_k) = w$; necessarily, f attains w at each z_j with multiplicity one. $\square$

Theorem 13.4 (Hurwitz's Theorem)**.** *Let $D \subset \mathbb{C}$ be open and connected, let $f_1, f_2, \ldots\colon D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^\infty$ converges to f compactly on D, and suppose that $\mathbf{Z}(f_n) = \varnothing$ for $n \in \mathbb{N}$. Then $f \equiv 0$ or $\mathbf{Z}(f) = \varnothing$.*

Proof. In view of Theorem 6.2, we note that f itself is holomorphic. Suppose that $f \not\equiv 0$, but that there exists $z_0 \in \mathbf{Z}(f)$. Choose $\epsilon > 0$ such that $B_\epsilon[z_0] \subset D$ and $f(z) \neq 0$ for all $z \in B_\epsilon[z_0] \setminus \{z_0\}$, and note that

$$0 = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f_n'(\zeta)}{f_n(\zeta)}\, d\zeta = \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \operatorname{ord}(f, z_0),$$

which is a contradiction. $\square$

Corollary 13.4.1. Let $D \subset \mathbb{C}$ be open and connected, let $f_1, f_2, \ldots : D \to \mathbb{C}$ be holomorphic such that $(f_n)_{n=1}^\infty$ converges to f compactly on D, and suppose that f_n is injective for $n \in \mathbb{N}$. Then f is constant or injective.

Proof. Suppose that f is not constant. Let $z_0 \in D$ be arbitrary, and define

$$g_n : D \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto f_n(z) - f_n(z_0)$$

for $n \in \mathbb{N}$. Then $g_1, g_2, \ldots$ have no zeros. Since f is not constant, the function

$$D \setminus \{z_0\} \to \mathbb{C}, \quad z \mapsto f(z) - f(z_0)$$

is not zero, so that it has no zeros by Hurwitz's theorem, i.e. $f(z) \neq f(z_0)$ for all $z \in D$, $z \neq z_0$. $\square$

Theorem 13.5 (Rouché's Theorem). *Let $D \subset \mathbb{C}$ be open and simply connected, and let $f, g : D \to \mathbb{C}$ be holomorphic. Suppose that γ is a closed curve in D such that* $\operatorname{int}\gamma = \{z \in D \setminus \{\gamma\} : \nu(\gamma, z) = 1\}$ *and that*

$$|f(\zeta) - g(\zeta)| < |f(\zeta)|$$

for $\zeta \in \{\gamma\}$. Then f and g have the same number of zeros in $\operatorname{int}\gamma$ *(counting multiplicity).*

Proof. For $t \in [0,1]$, define $h_t := f + t(g - f)$, so that $h_0 = f$ and $h_1 = g$. Also, since

$$|t(g - f)| \leq |g - f| < |f|$$

for any $t \in [0,1]$ on $\{\gamma\}$, the functions h_t have no zeros on $\{\gamma\}$. For $t \in [0,1]$, let $n(t) \in \mathbb{N}_0$ denote the number of zeros of h_t in $\operatorname{int}\gamma$. Since the functions h_t have no poles in D, we know from the Argument Principle that

$$n(t) = \frac{1}{2\pi i}\int_\gamma \frac{h_t'(\zeta)}{h_t(\zeta)}\, d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta) + t(g'(\zeta) - f'(\zeta))}{f(\zeta) + t(g(\zeta) - f(\zeta))}\, d\zeta.$$

We thus see that $n(t)$ is a continuous function of t. But since $n(t)$ can only taken on integer values, it must be constant on $[0,1]$; in particular, $n(0) = n(1)$. $\square$

Example. How many zeros does $z^4 - 4z + 2$ have in $\mathbb{D}$?

Set

$$g(z) := z^4 - 4z + 2 \qquad \text{and} \qquad f(z) = -4z + 2.$$

For $\zeta \in \partial\mathbb{D}$, we have $|f(\zeta)| \geq |-4z| - 2 = 4 - 2 = 2$, so that

$$|f(\zeta) - g(\zeta)| = |\zeta^4| = 1 < 2 \leq |f(\zeta)|.$$

Since f has precisely one zero in $\mathbb{D}$, so does g.

Corollary 13.5.1 (Fundamental Theorem of Algebra). Let p be a polynomial with $n := \deg p \geq 1$. Then p has n zeros (counting multiplicity).

Proof. Let
$$p(z) = a_n z^n + \cdots + a_1 z + a_0$$
with $a_n \neq 0$, and let $g(z) := a_n z^n$, so that $\lim_{|z|\to\infty} \left| \frac{p(z) - g(z)}{g(z)} \right| = 0$. Choose $R > 0$ such that
$$\left| \frac{p(z) - g(z)}{g(z)} \right| < 1$$
for $z \in \mathbb{C}$ with $|z| \geq R$. Consequently, if $z \in \partial B_R(0)$, we have $|p(z) - g(z)| < |g(z)|$. By Rouché's Theorem, p thus has as many zeros in $B_R(0)$ as g, namely n. Since p has at most n zeros, these are all of the zeros of p. $\square$

Chapter 14

Harmonic Functions

Definition. Let $D \subset \mathbb{R}^N$ be open, and let $u \colon D \to \mathbb{R}$ be twice continuously partially differentiable. Then u is called *harmonic* if

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2} \equiv 0.$$

We will only be concerned with harmonic functions on $\mathbb{R}^2$, i.e. on $\mathbb{C}$.

Proposition 14.1 (Harmonic Components)**.** *Let $D \subset \mathbb{C}$ be open, and let $f \colon D \to \mathbb{C}$ be holomorphic. Then* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *are harmonic.*

Proof. Clearly, $\operatorname{Re} f$ and $\operatorname{Im} f$ are twice continuously differentiable.

We have

$$\begin{aligned}
\frac{\partial^2(\operatorname{Re} f)}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial}{\partial x}\operatorname{Re} f \\
&= \frac{\partial}{\partial x}\frac{\partial}{\partial y}\operatorname{Im} f, \qquad \text{by Cauchy–Riemann}, \\
&= \frac{\partial}{\partial y}\frac{\partial}{\partial x}\operatorname{Im} f \\
&= -\frac{\partial^2(\operatorname{Re} f)}{\partial y^2}, \qquad \text{by Cauchy–Riemann again},
\end{aligned}$$

so that $\Delta \operatorname{Re} f \equiv 0$, i.e. $\operatorname{Re} f$ is harmonic. Similarly, one sees that $\operatorname{Im} f$ is harmonic. □

Remark. The converse of Proposition 14.1 is not true: a harmonic function need not be the real part of some holomorphic function. Consider

$$u \colon \mathbb{C} \setminus \{0\} \to \mathbb{R}, \quad z \mapsto \log|z|,$$

so that

$$u(x, y) = \log\sqrt{x^2 + y^2} = \frac{1}{2}\log(x^2 + y^2)$$

for $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. The partial derivatives of u with respect to x are

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}.$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

Moreover, since u is symmetric in x and y, we find

$$\frac{\partial^2 u}{\partial y^2} = \frac{-y^2 + x^2}{(x^2 + y^2)^2}.$$

Consequently, u is harmonic.

Now suppose that there is a holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $\operatorname{Re} f = u$. On $\mathbb{C}_-$, we then have that $\operatorname{Re} f = \log|z| = \operatorname{Re} \operatorname{Log}$. The Cauchy–Riemann Equations thus yield

$$\frac{\partial (\operatorname{Im} f)}{\partial x}(z) = -\frac{\partial \operatorname{Re} f}{\partial y}(z) = -\frac{\partial (\operatorname{Re} \operatorname{Log})}{\partial y}(z) = \frac{\partial (\operatorname{Im} \operatorname{Log})}{\partial x}(z),$$

so that

$$f'(z) = \frac{\partial \operatorname{Re} f}{\partial x}(z) + i\frac{\partial (\operatorname{Im} f)}{\partial x}(z) = \frac{\partial \operatorname{Re} \operatorname{Log}}{\partial x}(z) + i\frac{\partial (\operatorname{Im} \operatorname{Log})}{\partial x}(z) = \operatorname{Log}' z = \frac{1}{z}$$

for $z \in \mathbb{C}_-$. By continuity, it follows that $f'(z) = \frac{1}{z}$ for all $z \in \mathbb{C} \setminus \{0\}$, so that f is an antiderivative of $z \mapsto \frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$. This is impossible (cf. page 24).

Definition. Let $D \subset \mathbb{C}$ be open, and let $u \colon D \to \mathbb{R}$ be harmonic. We call a harmonic function $v \colon D \to \mathbb{R}$ a *harmonic conjugate* of u if $u + i\,v$ is holomorphic.

Theorem 14.1 (Harmonic Conjugates)**.** *Let $D \subset \mathbb{C}$ be open and suppose that there exists $(x_0, y_0) \in D$ with the following property: for each $(x, y) \in D$, we have*

- *$(x, t) \in D$ for each t between y and y_0 and*
- *$(s, y_0) \in D$ for each s between x and x_0.*

Then every harmonic function on D has a harmonic conjugate.

Proof. Let $u \colon D \to \mathbb{R}$ be harmonic. We will find a harmonic $v \colon D \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{$*$}$$

For $(x, y) \in D$, define

$$v(x, y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x, t)\, dt + \phi(x),$$

where ϕ will be specified later. First, note that

$$\begin{aligned}\frac{\partial v}{\partial x}(x,y) &= \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2}(x,t)\,dt + \phi'(x), \qquad \text{by Lemma 5.3,}\\ &= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2}(x,t)\,dt + \phi'(x)\\ &= -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,y_0) + \phi'(x).\end{aligned}$$

Hence, if we want the Cauchy–Riemann differential equations to hold for $u + iv$, we require that $\phi'(x) = -\frac{\partial u}{\partial y}(x, y_0)$. We thus set

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)\,ds.$$

Then $(*)$ holds, so that

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = -\frac{\partial^2 v}{\partial y^2},$$

i.e. v is harmonic. □

Example. Let

$$u\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto xy.$$

Then u is harmonic and

$$v(x,y) = \int_0^y t\,dt - \int_0^x s\,ds = \frac{y^2}{2} - \frac{x^2}{2}$$

is a harmonic conjugate for u.

Corollary 14.1.1. Let $D \subset \mathbb{C}$ be open, and let $u\colon D \to \mathbb{R}$ be harmonic. Then, for each $z_0 \in D$, there is a neighbourhood $U \subset D$ of z_0 such that $u|_U$ has a harmonic conjugate.

Corollary 14.1.2. Let $D \subset \mathbb{C}$ be open, and let $u\colon D \to \mathbb{R}$ be harmonic. Then u is infinitely often partially differentiable.

Corollary 14.1.3. Let $D \subset \mathbb{C}$ be open and connected, and let $u : D \to \mathbb{R}$ be harmonic. Then the following are equivalent:

(i) $u \equiv 0$;

(ii) there exists a nonempty open set $U \subset D$ with $u|_U \equiv 0$.

Proof. Of course, only (ii) $\Longrightarrow$ (i) needs proof.

Given a nonempty open set $U \subset D$ with $u|_U \equiv 0$, let $z_0 \in U$. Corollary 14.1.1 implies that there exists a holomorphic function $f = u + iv$ on an open ball $B_\epsilon(z_0) \subset U$ of z_0. But then

$$f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \equiv 0$$

on $B_\epsilon(z_0)$. Consequently, $f = u + iv$ is constant on $B_\epsilon(z_0) \subset D$. The Identity Theorem then implies that f is constant throughout the open connected set D. $\square$

Corollary 14.1.4. Let $D \subset \mathbb{C}$ be open, let $u \colon D \to \mathbb{R}$ be harmonic, and let $z_0 \in D$ and $r > 0$ be such that $B_r[z_0] \subset D$. Then we have

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

Corollary 14.1.5. Let $D \subset \mathbb{C}$ be open and connected, and let $u : D \to \mathbb{R}$ be harmonic with a local maximum or minimum on D. Then u is constant.

Proof. It is enough to consider the case of a local maximum: otherwise, replace u by $-u$.

Let $z_0 \in D$ be a point where u attains a local maximum. Let $\epsilon > 0$ be such that $B_\epsilon(z_0) \subset D$ and $u(z) \leq u(z_0)$ for all $z \in B_\epsilon(z_0)$. Let v be a harmonic conjugate of u on $B_\epsilon(z_0)$. Hence, $f := u + i\,v : B_\epsilon(z_0) \to \mathbb{C}$ is holomorphic such that $\operatorname{Re} f$ has a local maximum at z_0. On considering the holomorphic function e^f, with modulus $e^{\operatorname{Re} f}$, we then see that $e^{\operatorname{Re} f}$ has a local maximum at z_0. The Maximum Modulus Principle then implies that e^f, and hence its modulus $e^{\operatorname{Re} f}$, must be constant on $B_\epsilon(z_0)$. On taking the real logarithm, we see that $u = \operatorname{Re} f$ also is constant on $B_\epsilon(z_0)$. On applying Corollary 14.1.3 to u minus this constant value, we then see that u is constant throughout D. $\square$

Corollary 14.1.6. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $u \colon \overline{D} \to \mathbb{R}$ be continuous such that $u|_D$ is harmonic. Then u attains its maximum and minimum over $\overline{D}$ on ∂D.

The Dirichlet Problem. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f \colon \partial D \to \mathbb{R}$ be continuous. Is there a continuous $g \colon \overline{D} \to \mathbb{R}$ such that $g|_{\partial D} = f$ and $g|_D$ is harmonic?

Remark. If the Dirichlet problem has a solution, it must be unique. To see this, let $g_1, g_2 \colon \overline{D} \to \mathbb{R}$ be such that $g_j|_{\partial D} = f$ and $g_j|_D$ is harmonic for $j = 1, 2$. Then $g_1 - g_2$ vanishes on ∂D. Since $g_1 - g_2$ attains both its maximum and minimum on ∂D, it follows that $g_1 - g_2 \equiv 0$ on $\overline{D}$.

Definition. Let $r > 0$. The *Poisson kernel* for $B_r(0)$ is defined as

$$P_r(\zeta, z) := \frac{r^2 - |z|^2}{2\pi|\zeta - z|^2}$$

for $z \in B_r(0)$ and $\zeta \in \partial B_r(0)$.

Lemma 14.1. *Let $D \subset \mathbb{C}$ be open, let $r > 0$ be such that $B_r[0] \subset D$, and let $f\colon D \to \mathbb{C}$ be holomorphic. Then we have*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\,d\theta = \int_0^{2\pi} f(re^{i\theta})P_r(re^{i\theta}, z)\,d\theta.$$

for $z \in B_r(0)$.

Proof. Fix $z \in B_r(0)$, and define $g(w) := \dfrac{f(w)}{r^2 - w\overline{z}}$, which is holomorphic for w in $B_{r+\epsilon}(0)$ for some $\epsilon > 0$. The Cauchy Integral Formula then yields

$$\frac{f(z)}{r^2 - |z|^2} = g(z) = \frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{g(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(re^{i\theta})ire^{i\theta}}{re^{i\theta} - z}\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\theta})re^{i\theta}}{(r^2 - re^{i\theta}\overline{z})(re^{i\theta} - z)}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{-i\theta} - \overline{z})(re^{i\theta} - z)}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(re^{i\theta})}{|re^{i\theta} - z|^2}\,d\theta,$$

so that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\,d\theta.$$

$\square$

Remark. If we apply Lemma 14.1 to the function $f = 1$ we see for all $z \in B_r(0)$ that

$$\int_0^{2\pi} P_r(re^{i\theta}, z)\,d\theta = 1.$$

Theorem 14.2 (Poisson's Integral Formula)**.** *Let $r > 0$, and let $u\colon B_r[0] \to \mathbb{R}$ be continuous such that $u|_{B_r(0)}$ is harmonic. Then*

$$u(z) = \int_0^{2\pi} u(re^{i\theta})P_r(re^{i\theta}, z)\,d\theta$$

holds for all $z \in B_r(0)$.

Proof. Suppose first that u extends to $B_R(0)$ for some $R > r$ as a harmonic function. Then u has a harmonic conjugate v on $B_R(0)$, so that $f := u + i\,v$ is holomorphic. By Lemma 14.1, we have, for $z \in B_r(0)$, that

$$u(z) + i\,v(z) = f(z)$$
$$= \int_0^{2\pi} f(re^{i\theta})P_r(re^{i\theta}, z)\,d\theta = \int_0^{2\pi} u(re^{i\theta})P_r(re^{i\theta}, z)\,d\theta + i\int_0^{2\pi} v(re^{i\theta})P_r(re^{i\theta}, z)\,d\theta,$$

so that

$$u(z) = \int_0^{2\pi} u(re^{i\theta}) P_r(re^{i\theta}, z)\, d\theta.$$

Suppose now that u is arbitrary. For $t \in (0,1)$, define

$$u_t \colon B_{\frac{r}{t}}(0) \to \mathbb{R}, \quad z \mapsto u(tz).$$

Then u_t is harmonic, and by the foregoing we have

$$u_t(z) = \int_0^{2\pi} u_t(re^{i\theta}) P_r(re^{i\theta}, z)\, d\theta$$

for $z \in B_r(0)$. Letting $t \to 1^-$ (cf. Problem 5.2), we obtain for $z \in B_r(0)$ that

$$u(z) = \lim_{t\to 1} u_t(z) = \lim_{t \to 1} \int_0^{2\pi} u_t(re^{i\theta}) P_r(re^{i\theta}, z)\, d\theta = \int_0^{2\pi} u(re^{i\theta}) P_r(re^{i\theta}, z)\, d\theta.$$

□

Theorem 14.3. *Let $r > 0$, and let $f \colon \partial B_r(0) \to \mathbb{R}$ be continuous. Define*

$$g \colon B_r[0] \to \mathbb{R}, \quad z \mapsto \begin{cases} f(z), & z \in \partial B_r(0), \\ \int_0^{2\pi} f(re^{i\theta}) P_r(re^{i\theta}, z)\, d\theta, & z \in B_r(0). \end{cases}$$

Then g is harmonic on $B_r(0)$ and continuous on $B_r[0]$.

Proof. There is no loss of generality if we suppose that $r = 1$.

For $z \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$, note that

$$\operatorname{Re} \frac{\zeta + z}{\zeta - z} = \operatorname{Re} \frac{(\zeta + z)(\bar{\zeta} - \bar{z})}{|\zeta - z|^2} = \frac{1}{|\zeta - z|^2} \operatorname{Re}(|\zeta|^2 - |z|^2 + z\bar{\zeta} - \zeta\bar{z}) = \frac{1 - |z|^2}{|\zeta - z|^2} = 2\pi P_1(\zeta, z).$$

As the real part of a holomorphic function,

$$\mathbb{D} \to \mathbb{R}, \quad z \mapsto P_1(\zeta, z)$$

is therefore harmonic for each $\zeta \in \partial\mathbb{D}$. We thus obtain for $z = x + iy \in \mathbb{D}$:

$$(\Delta g)(z) = \frac{\partial^2 g}{\partial x^2}(z) + \frac{\partial^2 g}{\partial y^2}(z) = \int_0^{2\pi} f(e^{i\theta}) \left(\frac{\partial^2}{\partial x^2} P_1(e^{i\theta}, z) + \frac{\partial^2}{\partial y^2} P_1(e^{i\theta}, z) \right) d\theta = 0.$$

Consequently, g is harmonic on $B_r(0)$.

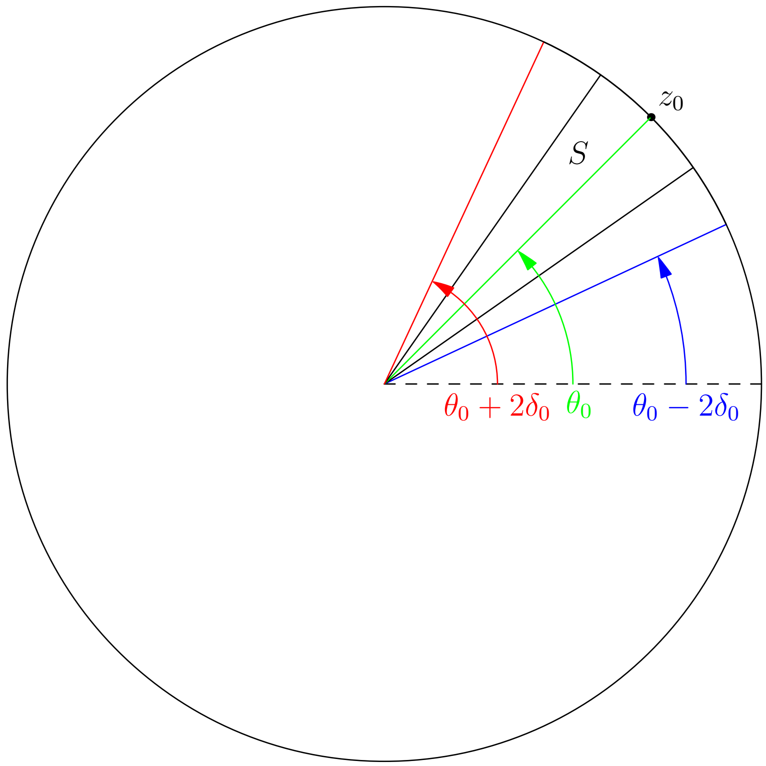
What remains to be shown is that g is continuous at any point $z_0 \in \partial\mathbb{D}$.

Let $z_0 = e^{i\theta_0}$, and suppose without loss of generality (if necessary considering instead $g(-z)$) that $\theta_0 \in (0, 2\pi)$. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $|g(z_0) - g(z)| < \epsilon$ for all $z \in \mathbb{D}$ with $|z_0 - z| < \delta$.

For $\delta_0 > 0$, let $J := [\theta_0 - 2\delta_0, \theta_0 + 2\delta_0]$. By making $\delta_0 > 0$ sufficiently small, we can ensure that $J \subset [0, 2\pi]$ and $|f(e^{i\theta}) - f(z_0)| < \frac{\epsilon}{2}$ for $\theta \in J$. Set

$$S := \{se^{i\theta} : s \in [0,1),\ \theta \in [\theta_0 - \delta_0, \theta_0 + \delta_0]\},$$

and note that $C := \inf\{|e^{i\theta} - z| \colon \theta \in [0, 2\pi] \setminus J,\ z \in S\} > 0$.



Since $\int_0^{2\pi} P_1(e^{i\theta}, z)\, d\theta = 1$ for all $z \in \mathbb{D}$, we have

$$\begin{aligned} g(z) - g(z_0) &= \int_0^{2\pi} (f(e^{i\theta}) - f(z_0)) P_1(e^{i\theta}, z)\, d\theta \\ &= \underbrace{\int_J (f(e^{i\theta}) - f(z_0)) P_1(e^{i\theta}, z)\, d\theta}_{I_1} + \underbrace{\int_{[0,2\pi]\setminus J} (f(e^{i\theta}) - f(z_0)) P_1(e^{i\theta}, z)\, d\theta}_{I_2}. \end{aligned}$$

Note that

$$|I_1| \leq \int_J \underbrace{|f(e^{i\theta}) - f(z_0)|}_{<\frac{\epsilon}{2}} P_1(e^{i\theta}, z)\, d\theta \leq \frac{\epsilon}{2} \int_0^{2\pi} P_1(e^{i\theta}, z)\, d\theta = \frac{\epsilon}{2}.$$

Set $K := \sup_{\zeta\in\partial\mathbb{D}}|f(\zeta)|$. For $z \in S$, we then have

$$\begin{aligned}
|I_2| &\leq \int_{[0,2\pi]\setminus J} (|f(e^{i\theta})|+|f(z_0)|)P_1(e^{i\theta}, z)\,d\theta \\
&= \frac{1}{2\pi}\int_{[0,2\pi]\setminus J} (|f(e^{i\theta})|+|f(z_0)|)\frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\theta \\
&\leq \frac{K}{\pi}\int_{[0,2\pi]\setminus J} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\,d\theta \\
&\leq \frac{K}{\pi C^2}\int_{[0,2\pi]\setminus J} (1-|z|^2)\,d\theta, \qquad \text{because } z \in S, \\
&\leq \frac{2K}{C^2}(1-|z|^2)
\end{aligned}$$

Choose $\delta \in (0, \delta_0)$ so small that $|z_0 - z| < \delta$ for $z \in \mathbb{D}$ implies

$$1 - |z|^2 < \frac{C^2}{2K}\frac{\epsilon}{2}.$$

For $z \in \mathbb{D}$ with $|z_0 - z| < \delta$, we then have $z \in S$ and hence $|I_2| < \frac{\epsilon}{2}$. On combining these results, we see that $|g(z_0) - g(z)| < \epsilon$. $\square$

Definition. Let $D \subset \mathbb{C}$ be open, and let $f\colon D \to \mathbb{C}$ be continuous. We say that f has the *mean value property* if, for every $z_0 \in D$, there exists $R > 0$ with $B_R[z_0] \subset D$ such that

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

for all $r \in [0, R]$.

Theorem 14.4. *Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{C}$ have the mean value property such that $|f|$ attains a local maximum at $z_0 \in D$. Then f is constant on a neighbourhood of z_0.*

Proof. Choose $R > 0$ with $B_R[z_0] \subset D$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in B_R[z_0]$ and $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$ for all $r \in [0, R]$. If $f(z_0) = 0$ then the result is trivial. Otherwise, let

$$h(z) = \frac{|f(z_0)|}{f(z_0)}f(z)$$

and set $g := \operatorname{Re} h - |h(z_0)|$. Then g has the mean value property and satisfies

$$g(z) \leq |h(z)| - |h(z_0)| \leq 0$$

for $z \in B_R[z_0]$. It follows that

$$0 = g(z_0) = \int_0^{2\pi} \underbrace{g(z_0 + re^{i\theta})}_{\leq 0}\,d\theta$$

for all $r \in [0, R]$. As the integrand is continuous, we conclude that $g(z_0 + re^{i\theta}) = 0$ for all $r \in [0, R]$ and $\theta \in [0, 2\pi]$, i.e. $g \equiv 0$ on $B_R[z_0]$. This means that, for $z \in B_R[z_0]$, we have

$$|h(z)| \leq |h(z_0)| = \operatorname{Re} h(z) \leq |h(z)|,$$

so that $\operatorname{Re} h(z) = |h(z)| = |h(z_0)|$ for $z \in B_R[z_0]$. That is, $h(z) = |h(z)| = |h(z_0)| = |f(z_0)|$, so that $f(z) = f(z_0)$ for $z \in B_R[z_0]$. □

Corollary 14.4.1. Let $D \subset \mathbb{C}$ be open, let $f : D \to \mathbb{R}$ be continuous and have the mean value property, and suppose that f has a local maximum or minimum at $z_0 \in D$. Then f is constant on a neighbourhood of z_0.

Proof. We only consider the case of a local maximum (for a local minimum, replace f by $-f$).

Let $R > 0$ be such that $B_R[z_0] \subset D$ and $f(z) \leq f(z_0)$ for all $z \in B_R[z_0]$. Choose C such that $f(z) + C \geq 0$ for all $z \in B_R[z_0]$. It follows that $|f + C|$ has a local maximum at z_0. Hence, $f + C$ is constant on a neighbourhood of z_0, as is f. □

Corollary 14.4.2. Let $D \subset \mathbb{C}$ be open, connected, and bounded, and let $f : \overline{D} \to \mathbb{R}$ be continuous such that $f|_D$ has the mean value property. Then f attains its maximum and minimum on ∂D.

Proof. Without loss of generality, suppose that f is not constant. Let $z_0 \in \overline{D}$ be such that $f(z_0)$ is maximal. Set

$$V := \{z \in D : f(z) < f(z_0)\}.$$

Then V is open and not empty. Let $z \in D \setminus V$, i.e. $f(z) = f(z_0)$. Then f has a local maximum at z, so that, by Corollary 14.4.1, $f(w) = f(z) = f(z_0)$ for w in a neighbourhood, say $W \subset D$, of z. Consequently, $W \subset D \setminus V$ holds, so that z is an interior point of $D \setminus V$. Since $z \in D \setminus V$ is arbitrary, this shows that $D \setminus V$ is open. Since D is connected, and $V \neq \varnothing$, we must have $D \setminus V = \varnothing$, i.e. $V = D$.

The case of a minimum is treated analogously. □

Corollary 14.4.3 (Equivalence of Harmonic and Mean-Value Properties)**.** Let $D \subset \mathbb{C}$ be open, and let $f : D \to \mathbb{R}$ be continuous. Then the following are equivalent:

(i) f is harmonic;

(ii) f has the mean value property.

Proof. Only (ii) $\Longrightarrow$ (i) needs proof (cf. Corollary 14.1.4).

Let $z_0 \in D$, and let $R > 0$ be such that $B_R[z_0] \subset D$. By Theorem 14.3, there is a continuous function $g : B_R[z_0] \to \mathbb{R}$ such that $g|_{\partial B_R[z_0]} = f|_{\partial B_R[z_0]}$ and $g|_{B_r(z_0)}$ is harmonic. Consequently, $(g - f)|_{B_R(z_0)}$ has the mean value property. By Corollary 14.4.2, this means that $g - f$ attains its maximum and minimum over $B_R[z_0]$ on $\partial B_R[z_0]$, so that $g = f$ on $B_R[z_0]$. Hence, $f|_{B_R(z_0)}$ is harmonic, i.e. $\Delta f \equiv 0$ on $B_R[z_0]$. Since $z_0 \in D$ was arbitrary, this means that $\Delta f \equiv 0$. □

Chapter 15

Analytic Continuation along a Curve

Example. Let

$$D_1 := \{z \in \mathbb{C} : \operatorname{Re} z > 0\},$$
$$D_2 := \{z \in \mathbb{C} : \operatorname{Im} z > \operatorname{Re} z\},$$

and

$$D_3 := \{z \in \mathbb{C} : \operatorname{Im} z < -\operatorname{Re} z\},$$

so that

$$D_1 \cup D_2 \cup D_3 = \mathbb{C} \setminus \{0\}.$$

Let

$$g\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad z \mapsto \frac{1}{z},$$

and let $f_1 = \operatorname{Log}|_{D_1}$, so that f_1 is an antiderivative of g on D_1. Since D_2 is simply connected, g also has an antiderivative on D_2; since $(f_1' - f_2')|_{D_1 \cap D_2} = (g-g)|_{D_1 \cap D_2} \equiv 0$, it follows that $(f_1 - f_2)|_{D_1 \cap D_2}$ is constant, and by altering f_2 by an additive constant, we can achieve that $f_1|_{D_1 \cap D_2} = f_2|_{D_1 \cap D_2}$. In the same fashion, we can find an antiderivative f_3 of g on D_3 such that $f_2|_{D_2 \cap D_3} = f_3|_{D_2 \cap D_3}$. However, $f_1|_{D_1 \cap D_3} \neq f_3|_{D_1 \cap D_3}$ because otherwise, we would have an antiderivative of g on all of $\mathbb{C} \setminus \{0\}$, which we know to be impossible.

Since $f_1' - f_3'|_{D_1 \cap D_3} = g - g_{D_1 \cap D_3} \equiv 0$, however, there exists $c \in \mathbb{C}$ such that $f_3(z) = f_1(z) + c$ for $z \in D_1 \cap D_2$. We claim that $c = 2\pi i$. To see this, let $z_1, z_2, z_3 \in \partial\mathbb{D}$ be such that $z_1 \in D_1 \cap D_3$, $z_2 \in D_2 \cap D_1$, and $z_3 \in D_3 \cap D_2$. Let γ_{z_1,z_2}, γ_{z_2,z_3}, and γ_{z_3,z_1} be the arc segments of $\partial\mathbb{D}$ from z_1 to z_2, from z_2 to z_3, and from z_3 to z_1, respectively.

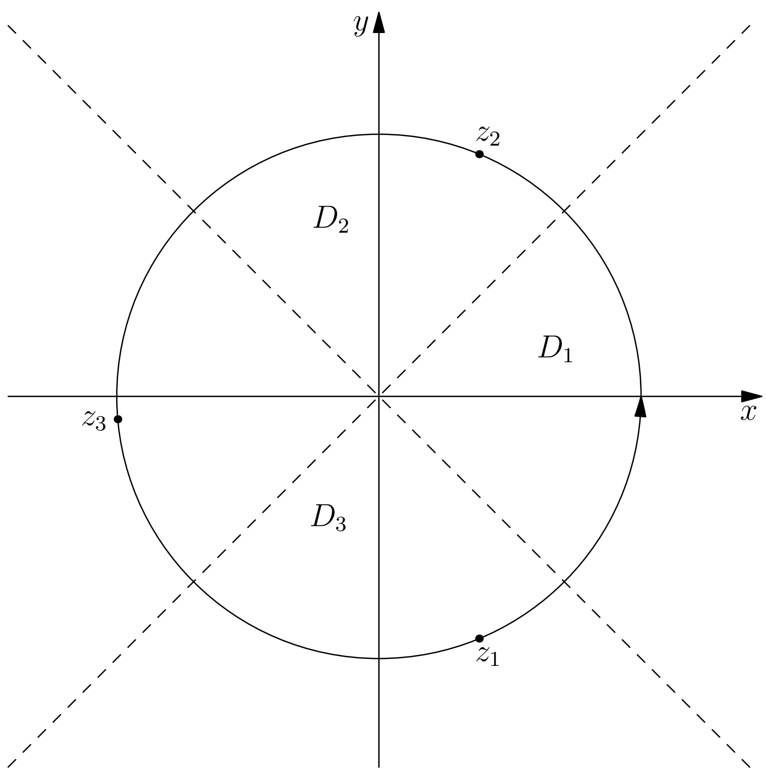
Since f_j is an antiderivative of g on D_j for $j = 1, 2, 3$, we obtain

$$\int_{\gamma_{z_1,z_2}} g = f_1(z_2) - f_1(z_1), \qquad \int_{\gamma_{z_2,z_3}} g = f_2(z_3) - f_2(z_2),$$

$$\text{and} \qquad \int_{\gamma_{z_3,z_1}} g = f_3(z_1) - f_3(z_3).$$

It follows that

$$\begin{aligned} c &= f_3(z_1) - f_1(z_1) \\ &= f_3(z_1) - f_3(z_3) + f_2(z_3) - f_2(z_2) + f_1(z_2) - f_1(z_1) \\ &= \int_{\gamma_{z_3,z_1}} g + \int_{\gamma_{z_2,z_3}} g + \int_{\gamma_{z_1,z_2}} g \\ &= \int_{\gamma_{z_1,z_2} \oplus \gamma_{z_2,z_3} \oplus \gamma_{z_3,z_1}} g \\ &= \int_{\partial \mathbb{D}} \frac{1}{\zeta}\, d\zeta \\ &= 2\pi i. \end{aligned}$$



Definition. A *function element* is a pair (D, f), where $D \subset \mathbb{C}$ is open and connected, and $f\colon D \to \mathbb{C}$ is a holomorphic function. For a given function element (D, f) and

$z_0 \in D$, the *germ* of f at z_0—denoted by $\langle f \rangle_{z_0}$—is the collection of all function elements (E, g) such that $z_0 \in E$ and there is a neighbourhood $U \subset D \cap E$ of z_0 such that $f(z) = g(z)$ for all $z \in U$.

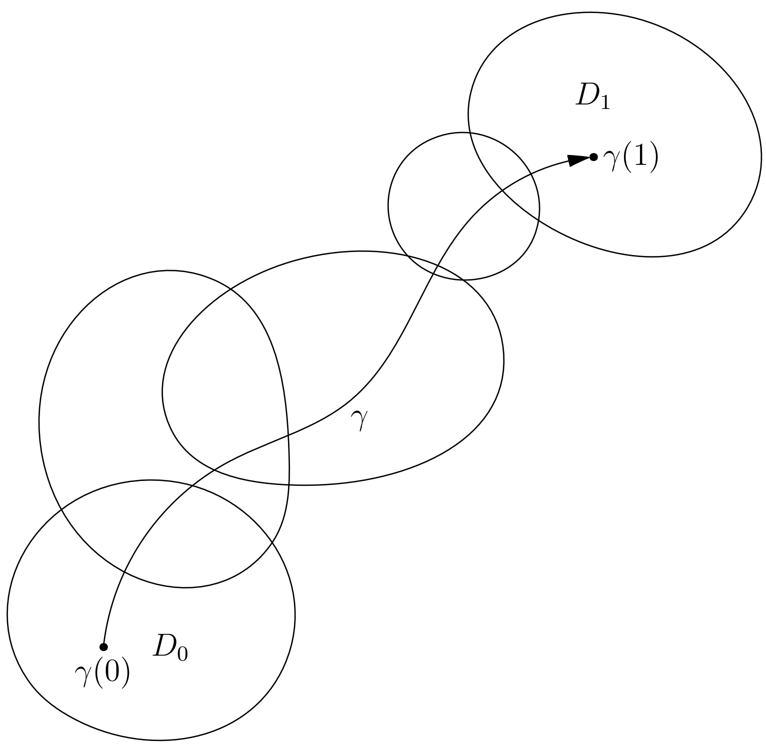
Definition. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a path, and suppose that, for each $t \in [0,1]$, there is a function element (D_t, f_t) such that:

(a) $\gamma(t) \in D_t$;

(b) there exists $\delta > 0$ such that, whenever $s \in [0,1]$ is such that $|s-t| < \delta$, then $\gamma(s) \in D_t$ and $\langle f_s \rangle_{\gamma(s)} = \langle f_t \rangle_{\gamma(s)}$.

Then we call $\{(D_t, f_t) : t \in [0,1]\}$ an *analytic continuation* along γ and say that (D_1, f_1) is obtained by analytic continuation of (D_0, f_0) along γ.

Remark. Since γ is continuous and D_t is open for each $t \in D_t$, it is clear that there exists $\delta > 0$ such that $\gamma(s) \in D_t$ for all $s \in [0,1]$ such that $|s-t| < \delta$. What is important about part (b) of the definition is that $\langle f_s \rangle_{\gamma(s)} = \langle f_t \rangle_{\gamma(s)}$, i.e. there is a neighbourhood $U_s \subset D_s \cap D_t$ of $\gamma(s)$ such that $f_s(z) = f_t(z)$ for $z \in U_s$.



Theorem 15.1 (Monodromy Theorem)**.** *Let $\gamma : [0,1] \to \mathbb{C}$ be a path, and let $\{(D_t, f_t) : t \in [0,1]\}$ and $\{(E_t, g_t) : t \in [0,1]\}$ be analytic continuations along γ such that $\langle f_0 \rangle_{\gamma(0)} = \langle g_0 \rangle_{\gamma(0)}$. Then we have $\langle f_1 \rangle_{\gamma(1)} = \langle g_1 \rangle_{\gamma(1)}$.*

Proof. Let

$$I = \{t \in [0,1] : \langle f_t \rangle_{\gamma(t)} = \langle g_t \rangle_{\gamma(t)}\},$$

so that $0 \in I$.

We first claim that I is closed. Let $t \in \overline{I}$, and let $\delta > 0$ be such that $\gamma(s) \in D_t \cap E_t$ and

$$\langle f_s \rangle_{\gamma(s)} = \langle f_t \rangle_{\gamma(s)} \qquad \text{and} \qquad \langle g_s \rangle_{\gamma(s)} = \langle g_t \rangle_{\gamma(s)}$$

for all $s \in [0,1]$ with $|s-t| < \delta$. Since $t \in \overline{I}$, there exists $s \in I$ with $|s-t| < \delta$. There is thus a neighbourhood $U \subset D_t \cap D_s \cap E_t \cap E_s$ of $\gamma(s)$ such that $f_s(z) = g_s(z)$ for all $z \in U$ by the definition of I. From the choice of δ, we also have—after possibly making U smaller—that $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for $z \in U$. It follows that $f_t(z) = g_t(z)$ for $z \in U$, so that $t \in I$.

Let $t_0 := \sup I$. Let $\delta > 0$ be such that $\gamma(s) \in D_{t_0} \cap E_{t_0}$ and

$$\langle f_s \rangle_{\gamma(s)} = \langle f_{t_0} \rangle_{\gamma(s)} \qquad \text{and} \qquad \langle g_s \rangle_{\gamma(s)} = \langle g_{t_0} \rangle_{\gamma(s)}$$

for all $s \in [0,1]$ with $|s-t| < \delta$. Since I is closed, we have $t_0 \in I$ and thus $f_{t_0}(z) = g_{t_0}(z)$ for all z in some neighbourhood V of $\gamma(t_0)$ contained in $D_{t_0} \cap E_{t_0}$. It follows that $\langle f_{t_0} \rangle_{\gamma(s)} = \langle g_{t_0} \rangle_{\gamma(s)}$ for all $s \in [0,1]$ such that $\gamma(s) \in V$. For $\delta > 0$ sufficiently small, we thus have $\langle f_s \rangle_{\gamma(s)} = \langle g_s \rangle_{\gamma(s)}$ for any $s \in [0,1]$ with $|s-t| < \delta$. It follows that $[0,1] \cap (t_0 - \delta, t_0 + \delta) \subset I$. Since $t_0 = \sup I$, this means that $t_0 = 1$, so that $I = [0,1]$. $\square$

Chapter 16

Montel's Theorem

Definition. Let $S \subset \mathbb{R}^N$. A family $\mathcal{F}$ of functions on S into $\mathbb{R}^M$ is called *equicontinuous* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$ and for all $x, y \in S$ such that $|x - y| < \delta$.

Lemma 16.1. *Let $S \subset \mathbb{R}^N$. Then S contains a countable dense subset.*

Proof. Let $\{x_1, x_2, x_3, \ldots\}$ be a dense, countable subset of $\mathbb{R}^N$, e.g., $\mathbb{Q}^N$. For $n, m \in \mathbb{N}$ with $S \cap B_{\frac{1}{m}}(x_n) \neq \varnothing$, choose $y_{n,m} \in S \cap B_{\frac{1}{m}}(x_n)$. Then

$$\left\{ y_{n,m} : n, m \in \mathbb{N}, S \cap B_{\frac{1}{m}}(x_n) \neq \varnothing \right\} \subset S$$

is countable.

Let $\epsilon > 0$ and $x \in S$. Choose $m \in \mathbb{N}$ so large that $\frac{1}{m} < \frac{\epsilon}{2}$. Since $\{x_1, x_2, x_3, \ldots\}$ is dense in $\mathbb{R}^N$, there exists $n \in \mathbb{N}$ such that $|x_n - x| < \frac{1}{m}$ and thus $x \in S \cap B_{\frac{1}{m}}(x_n) \neq \varnothing$. It follows that

$$|y_{n,m} - x| \leq |y_{n,m} - x_n| + |x_n - x| < \frac{2}{m} < \epsilon$$

□

Theorem 16.1 (Arzelà–Ascoli Theorem)**.** *Let $K \subset \mathbb{R}^N$ be compact, and let $\mathcal{F}$ be an equicontinuous and uniformly bounded family of functions from K to $\mathbb{R}^M$. Then every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on K.*

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}$, and let $\{x_1, x_2, x_3, \ldots\}$ be a countable dense subset of K.

Since $(f_n(x_1))_{n=1}^\infty$ is a bounded sequence in $\mathbb{R}^M$, there exists a subsequence $(f_{n,1})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that $(f_{n,1}(x_1))_{n=1}^\infty$ converges.

Since $(f_{n,1}(x_2))_{n=1}^\infty$ is a bounded sequence in $\mathbb{R}^M$, there exists a subsequence $(f_{n,2})_{n=1}^\infty$ of $(f_{n,1})_{n=1}^\infty$ such that $(f_{n,2}(x_2))_{n=1}^\infty$ converges.

Continuing inductively in this fashion, we obtain, for each $k \in \mathbb{N}$, a subsequence $(f_{n,k})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ such that, for each $k \in \mathbb{N}$,

- $(f_{n,k+1})_{n=1}^\infty$ is a subsequence of $(f_{n,k})_{n=1}^\infty$, and
- $(f_{n,k}(x_k))_{n=1}^\infty$ converges.

For $n \in \mathbb{N}$, set $g_n := f_{n,n}$. Then $(g_n)_{n=1}^\infty$ is a subsequence of $(f_n)_{n=1}^\infty$, and $(g_n(x_k))_{n=1}^\infty$ converges for each $k \in \mathbb{N}$.

We show that $(g_n)_{n=1}^\infty$ is a uniform Cauchy sequence on K (and thus convergent).

Let $\epsilon > 0$. Choose $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{3}$ for all $f \in \mathcal{F}$ and for all $x, y \in K$ with $|x - y| < \delta$. Since K is compact, there exist $y_1, \ldots, y_\nu \in K$ such that $K \subset \bigcup_{j=1}^\nu B_{\frac{\delta}{2}}(y_j)$. Since $\{x_1, x_2, x_3, \ldots\}$ is dense in K, there exist $k_1, \ldots, k_\nu \in \mathbb{N}$ such that $x_{k_j} \in B_{\frac{\delta}{2}}(y_j)$. It follows that $K \subset \bigcup_{j=1}^\nu B_\delta(x_{k_j})$.

By construction, $(g_n(x_k))_{n=1}^\infty$ is a Cauchy sequence for each $k \in \mathbb{N}$. Choose $N \in \mathbb{N}$ such that

$$|g_n(x_{k_j}) - g_m(x_{k_j})| < \frac{\epsilon}{3}$$

for $n, m \geq N$ and $j = 1, \ldots, \nu$. Let $x \in K$ be arbitrary, and let $n, m \geq N$. Choose $j \in \{1, \ldots, \nu\}$ such that $x \in B_\delta(x_{k_j})$, and note that

$$|g_n(x) - g_m(x)| \leq \underbrace{|g_n(x) - g_n(x_{k_j})|}_{<\frac{\epsilon}{3}} + \underbrace{|g_n(x_{k_j}) - g_m(x_{k_j})|}_{<\frac{\epsilon}{3}} + \underbrace{|g_m(x_{k_j}) - g_m(x)|}_{<\frac{\epsilon}{3}} < \epsilon.$$

Hence, $(g_n)_{n=1}^\infty$ is a uniform Cauchy sequence on K. □

Proposition 16.1. *Let $D \subset \mathbb{R}^N$ be open, and let $\mathcal{F}$ be a family of functions from D to $\mathbb{R}^M$ that is equicontinuous and uniformly bounded on compact subsets of D. Then every sequence in $\mathcal{F}$ has a compactly convergent subsequence.*

Proof. For each $k \in \mathbb{N}$, define $K_k := B_k[0]$ if $D = \mathbb{R}^N$ and $K_k := B_k[0] \cap \{x \in D : \operatorname{dist}(x, \partial D) \geq \frac{1}{k}\}$ if $D \neq \mathbb{R}^N$. Notice that

- $\bigcup_{k=1}^\infty K_k = D$ and
- $K_k \subset \operatorname{int} K_{k+1}$ for $n \in \mathbb{N}$.

Let $(f_n)_{n=1}^\infty$ be a sequence in $\mathcal{F}$. By the Arzelà–Ascoli Theorem, there exists a subsequence $(f_{n,1})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ and a function $g_1 \colon K_1 \to \mathbb{R}^M$ such that $f_{n,1}|_{K_1} \to g_1$ uniformly on K_1. Invoking the Arzelà–Ascoli Theorem again, we obtain a subsequence $(f_{n,2})_{n=1}^\infty$ of $(f_{n,1})_{n=1}^\infty$ and a function $g_2 \colon K_2 \to \mathbb{R}^M$ such that $f_{n,2}|_{K_2} \to g_2$ uniformly on K_2. Inductively, we thus obtain, for each $k \in \mathbb{N}$, a subsequence $(f_{n,k})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ and a function $g_k \colon K_k \to \mathbb{R}^M$ such that, for each $k \in \mathbb{N}$,

- $(f_{n,k+1})_{n=1}^\infty$ is a subsequence of $(f_{n,k})_{n=1}^\infty$, and
- $f_{n,k}|_{K_k} \to g_k$ uniformly on K_k.

Define $g\colon D \to \mathbb{R}^M$ as follows: for $x \in D$, let k be the smallest natural number such that $x \in K_k$, set $g(x) := g_k(x)$. Then $f_{n,n}|_{K_k} \to g|_{K_k}$ uniformly on K_k.

Let $K \subset D$ be compact. By the choices of $K_1, K_2, \ldots$, we have $K \subset D = \bigcup_{k=1}^{\infty} \operatorname{int} K_k$, so that $\{\operatorname{int} K_k : k \in \mathbb{N}\}$ is an open cover for K. Since K is compact, and since $\operatorname{int} K_k \subset \operatorname{int} K_{k+1}$ for $k \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $K \subset \operatorname{int} K_{k_0} \subset K_{k_0}$. Since $f_{n,n}|_{K_{k_0}} \to g|_{K_{k_0}}$ uniformly on K_{k_0}, it follows that $f_{n,n}|_K \to g|_K$ uniformly on K. ∎

Theorem 16.2 (Montel's Theorem)**.** *Let $D \subset \mathbb{C}$ be open, and let $\mathcal{F}$ be a family of holomorphic functions on D that is uniformly bounded on compact subsets of D. Then every sequence in $\mathcal{F}$ has a subsequence that converges compactly to a holomorphic function on D.*

Proof. In view of Proposition 16.1, we only need to show that $\mathcal{F}$ is equicontinuous on compact subsets of D.

Let $z_0 \in D$, and let $r > 0$ be such that $B_{2r}[z_0] \subset D$. There exists $C > 0$ such that $|f(\zeta)| \leq C$ for all $f \in \mathcal{F}$ and all $\zeta \in \partial B_{2r}(z_0)$.

Let $f \in \mathcal{F}$, and let $z, w \in B_r(z_0)$. Then we have:

$$\begin{aligned}
|f(z) - f(w)| &= \frac{1}{2\pi}\left|\int_{\partial B_{2r}(z_0)} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w}\right) d\zeta\right| \\
&= \frac{1}{2\pi}\left|\int_{\partial B_{2r}(z_0)} \frac{f(\zeta)(\zeta - w + z - \zeta)}{(\zeta - z)(\zeta - w)}\, d\zeta\right| \\
&= \frac{|z-w|}{2\pi}\left|\int_{\partial B_{2r}(z_0)} \frac{f(\zeta)}{(\zeta - z)(\zeta - w)}\, d\zeta\right| \\
&\leq \frac{|z-w|}{2\pi} 4\pi r \frac{C}{r^2} \\
&= \frac{2C}{r}|z - w|.
\end{aligned}$$

For $\epsilon > 0$, choose $\delta := \dfrac{r\epsilon}{2C}$, so that $|f(z) - f(w)| < \epsilon$ for all $z, w \in B_r(z_0)$ with $|z - w| < \delta$. ∎

Chapter 17

The Riemann Mapping Theorem

Definition. Let $D_1, D_2 \subset \mathbb{C}$ be open and connected. We say that D_1 and D_2 are *biholomorphically equivalent* if there is a biholomorphic map from D_1 onto D_2.

Examples.

1. Let $z_1, z_2 \in \mathbb{C}$, and let $r_1, r_2 > 0$. Then $B_{r_1}(z_1)$ and $B_{r_2}(z_2)$ are biholomorphically equivalent because

$$B_{r_1}(z_1) \to B_{r_2}(z_2), \quad z \mapsto \frac{r_2}{r_1}(z - z_1) + z_2$$

is biholomorphic.

2. Consider the *Cayley transform*

$$f \colon \mathbb{H} \to \mathbb{C}, \quad z \mapsto \frac{z-i}{z+i}.$$

Let $x, y \in \mathbb{R}$ with $y > 0$, and let $z = x + iy$. Then

$$\begin{aligned} |z-i|^2 &= |x + i(y-1)|^2 \\ &= x^2 + y^2 - 2y + 1 \\ &< x^2 + y^2 + 2y + 1 \\ &= |x + i(y+1)|^2 \\ &= |z+i|^2 \end{aligned}$$

holds, so that $|f(z)| < 1$. Consequently, we have $f(\mathbb{H}) \subset \mathbb{D}$. Consider

$$g \colon \mathbb{D} \to \mathbb{C}, \quad z \mapsto i\frac{1+z}{1-z},$$

and note that

$$\begin{aligned} g(f(z)) &= i\frac{1+\frac{z-i}{z+i}}{1-\frac{z-i}{z+i}} \\ &= i\frac{z+i+z-i}{z+i-z+i} \\ &= i\frac{2z}{2i} \\ &= z \end{aligned}$$

for $z \in \mathbb{H}$. Hence, f is injective. Let $x^2 + y^2 < 1$, and note that

$$\begin{aligned} g(x+iy) &= i\frac{(1+x)+iy}{(1-x)-iy} \\ &= i\frac{[(1+x)+iy][(1-x)+iy]}{(1-x)^2+y^2} \\ &= -\frac{2y}{(1-x)^2+y^2} + i\underbrace{\frac{1-(x^2+y^2)}{(1-x)^2+y^2}}_{>0} \in \mathbb{H}. \end{aligned}$$

For $z \in \mathbb{D}$, we can thus evaluate

$$\begin{aligned} f(g(z)) &= \frac{i\frac{1+z}{1-z} - i}{i\frac{1+z}{1-z} + i} \\ &= \frac{1+z-1+z}{1+z+1-z} \\ &= \frac{2z}{2} \\ &= z. \end{aligned}$$

Hence, f is also surjective and thus bijective with inverse g. Since f and g are obviously holomorphic, this means that $\mathbb{D}$ and $\mathbb{H}$ are biholomorphically equivalent.

3. There is no biholomorphic map $f : \mathbb{C} \to \mathbb{D}$ because any holomorphic map from $\mathbb{C}$ to $\mathbb{D}$ is bounded and thus constant by Liouville's theorem. Hence, $\mathbb{C}$ and $\mathbb{D}$ are not biholomorphically equivalent.

Proposition 17.1. *Let $D_1, D_2 \subset \mathbb{C}$ be open and connected such that D_1 is simply connected, and suppose that D_1 and D_2 are biholomorphically equivalent. Then D_2 is simply connected.*

Proof. Let $f\colon D_1 \to D_2$ be biholomorphic. Let $g\colon D_2 \to \mathbb{C}$ be holomorphic, and let γ be a closed curve in D_2. Since $f^{-1} \circ \gamma$ is a closed curve in the simply connected set D_1, we see that

$$\int_\gamma g(\zeta)\,d\zeta = \int_{f\circ f^{-1}\circ\gamma} g(\zeta)\,d\zeta = \int_{f^{-1}\circ\gamma} g(f(\zeta))f'(\zeta)\,d\zeta = 0.$$

$\square$

Example. Let $r, R \in [0,\infty]$ be such that $r < R$. Then $A_{r,R}(0)$ is not biholomorphically equivalent to $\mathbb{D}$ or $\mathbb{C}$.

Biholomorphic maps have a very interesting geometric property.

Given and open set $D \subset \mathbb{R}^N$ and curves $\gamma_1, \gamma_2\colon [0,1] \to D$, suppose there exist $t_1, t_2 \in (0,1)$ such that $\gamma_1(t_1) = \gamma_2(t_2) = x_0$. In order to define the *angle between* γ_1 *and* γ_2 *at* x_0, we further suppose that there exists $\epsilon > 0$ such that γ_j is differentiable on $(t_j - \epsilon, t_j + \epsilon)$ for $j = 1, 2$ with $\gamma_j'(t_j) \neq 0$. The angle is then defined to be the unique $\theta \in [0,\pi]$ such that

$$\cos\theta = \frac{\gamma_1'(t_1)\cdot\gamma_2'(t_2)}{|\gamma_1'(t_1)||\gamma_2'(t_2)|}.$$

Given two open sets $D_1, D_2 \subset \mathbb{R}^N$, a differentiable map $f\colon D_1 \to D_2$ is called *angle preserving* at $x_0 \in D_1$ if, for any two curves γ_1 and γ_2 in D_1, the angle between $f\circ\gamma_1$ and $f\circ\gamma_2$ at $f(x_0)$ is the same between γ_1 and γ_2 at x_0.

Recall that a real $N \times N$ matrix A is called *orthogonal* if it is invertible with $A^{-1} = A^t$.

Lemma 17.1. *Let $D_1, D_2 \subset \mathbb{R}^N$ be open, let $x_0 \in D_1$, and let $f : D_1 \to D_2$ be differentiable such that $J_f(x_0)$ is orthogonal. Then f is angle preserving at x_0.*

Proof. Let γ_1 and γ_2 be two curves in D_1 satisfying the necessary requirements, and note that

$$\begin{aligned}
&\text{cosine of the angle between } f\circ\gamma_1 \text{ and } f\circ\gamma_2 \text{ at } f(x_0)\\
&= \frac{(f\circ\gamma_1)'(t_1)\cdot(f\circ\gamma_2)'(t_2)}{|(f\circ\gamma_1)'(t_1)||(f\circ\gamma_2)'(t_2)|}\\
&= \frac{J_f(x_0)\gamma_1'(t_1)\cdot J_f(x_0)\gamma_2'(t_2)}{|J_f(x_0)\gamma_1'(t_1)||J_f(x_0)\gamma_2'(t_2)|}, \qquad \text{by the chain rule,}\\
&= \frac{J_f(x_0)^t J_f(x_0)\gamma_1'(t_1)\cdot\gamma_2'(t_2)}{|J_f(x_0)\gamma_1'(t_1)||J_f(x_0)\gamma_2'(t_2)|}\\
&= \frac{\gamma_1'(t_1)\cdot\gamma_2'(t_2)}{|\gamma_1'(t_1)||\gamma_2'(t_2)|}\\
&= \text{cosine of the angle between } \gamma_1 \text{ and } \gamma_2 \text{ at } x_0.
\end{aligned}$$

$\square$

Example. Let z be a complex number. Then multiplication by z is a $\mathbb{R}$-linear map from $\mathbb{C} = \mathbb{R}^2$ into itself and thus uniquely represented by a real 2×2 matrix A of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a = \operatorname{Re} z$ and $b = \operatorname{Im} z$. It follows that A^t is the matrix representing $\bar{z}$. Hence, A is orthogonal if and only if $|z| = 1$.

Theorem 17.1 (Conformality at Nondegenerate Points)**.** *Let $D_1, D_2 \subset \mathbb{C}$ be open, and let $f \colon D_1 \to D_2$ be holomorphic. Then f is angle preserving at $z_0 \in D_1$ whenever $f'(z_0) \neq 0$.*

Proof. Let $z_0 \in D_1$ be such that $f'(z_0) \neq 0$. In view of Lemma 17.1 and the example following it, the claim is clear if $|f'(z_0)| = 1$.

For the general case, let

$$\frac{1}{|f'(z_0)|} D_2 := \left\{ \frac{z}{|f'(z_0)|} : z \in D_2 \right\},$$

and define

$$g \colon D_1 \to \frac{1}{|f'(z_0)|} D_2, \quad z \mapsto \frac{f(z)}{|f'(z_0)|}$$

and

$$h \colon \frac{1}{|f'(z_0)|} D_2 \to D_2, \quad z \mapsto |f'(z_0)| z.$$

Then g is angle preserving at z_0 because $|g'(z_0)| = 1$, and it is easily seen that h is angle preserving at $g(z_0)$. Consequently, $f = h \circ g$ is angle preserving at z_0. $\square$

Corollary 17.1.1 (Conformality of Biholomorphic Maps)**.** Let $D_1, D_2 \subset \mathbb{C}$ be open and connected, and let $f \colon D_1 \to D_2$ be biholomorphic. Then f is angle preserving at every point of D_1.

Theorem 17.2 (Holomorphic Inverses)**.** *Let $D_1, D_2 \subset \mathbb{C}$ be open and connected, and let $f : D_1 \to D_2$ be holomorphic and bijective. Then f is biholomorphic and $\mathbf{Z}(f') = \varnothing$.*

Proof. We first show that f^{-1} is continuous.

Let $w_0 \in D_2$, and let $\epsilon > 0$ be such that $B_\epsilon(f^{-1}(w_0)) \subset D_1$. By the Open Mapping Theorem, $f(B_\epsilon(f^{-1}(w_0)))$ is open. Hence, there exists $\delta > 0$ such that $B_\delta(w_0) \subset f(B_\epsilon(f^{-1}(w_0)))$. Hence, if $w \in B_\delta(w_0)$, then $f^{-1}(w) \in B_\epsilon(f^{-1}(w_0))$, That is, f^{-1} is continuous at w_0.

Let $z_0 \in D_1$ and let $w_0 := f(z_0)$. We see that f' is differentiable at z_0 only if $f'(z_0) \neq 0$: for $w \in D_2 \setminus \{w_0\}$,

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{f^{-1}(w) - f^{-1}(w_0)}{f(f^{-1}(w)) - f(f^{-1}(w_0))} \underset{w \to w_0}{\longrightarrow} \frac{1}{f'(z_0)}.$$

Since f is not constant, f' is not identically zero and thus $\mathbf{Z}(f')$ is discrete. We claim that $f(\mathbf{Z}(f'))$ is also discrete. Assume that $f(\mathbf{Z}(f'))$ is not discrete. Then there exist $w_0 \in D_2$ and a sequence $(z_n)_{n=1}^{\infty}$ in $\mathbf{Z}(f')$ such that $w_0 \neq f(z_n)$ for $n \in \mathbb{N}$, but $w_0 = \lim_{n\to\infty} f(z_n)$. By the bijectivity and continuity of f^{-1}, we have $f^{-1}(w_0) \neq z_n$ for $n \in \mathbb{N}$ and $f^{-1}(w_0) = \lim_{n\to\infty} z_n$. Hence, $f^{-1}(w_0)$ is a cluster point of $\mathbf{Z}(f')$, which is impossible.

Hence, f^{-1} is holomorphic on $D_2 \setminus f(\mathbf{Z}(f'))$. Since f^{-1} is continuous and $f(\mathbf{Z}(f'))$ is discrete, Riemann's Removability Criterion then yields the holomorphy of f^{-1} on all of D_2. Thus $\mathbf{Z}(f') = \varnothing$. □

Corollary 17.2.1. Let $D \subset \mathbb{C}$ be open and connected, and let $f : D \to \mathbb{C}$ be holomorphic and injective. Then $\mathbf{Z}(f') = \varnothing$.

Proof. If f is injective, it is not constant. By the Open Mapping Theorem, $f(D)$ is therefore open and connected. Apply Theorem 17.2 with $D_1 = D$ and $D_2 = f(D)$. □

Theorem 17.3 (Riemann Mapping Theorem)**.** *Let $D \subsetneq \mathbb{C}$ be open and connected and admit holomorphic square roots, and let $z_0 \in D$. Then there is a unique biholomorphic function $f \colon D \to \mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$.*

Proof. Uniqueness: Let $g \colon D \to \mathbb{D}$ be another such function. Then $f \circ g^{-1} \colon \mathbb{D} \to \mathbb{D}$ is biholomorphic with $(f \circ g^{-1})(0) = f(z_0) = 0$. By Corollary 7.4.1, there exists $c \in \mathbb{C}$ with $|c|= 1$ such that

$$f(g^{-1}(z)) = cz$$

for $z \in \mathbb{D}$ and thus

$$f(z) = f(g^{-1}(g(z))) = c\,g(z)$$

for $z \in D$. Differentiation yields $f'(z) = c\,g'(z)$ for $z \in D$. Since $|c|= 1$ and both $f'(z_0)$ and $g'(z_0)$ are real and positive, we conclude that $c = 1$.

Existence: Let

$$\mathcal{F} := \{f \colon D \to \mathbb{D} : f \text{ is injective and holomorphic with } f(z_0) = 0 \text{ and } f'(z_0) > 0\}.$$

Claim 1. $\mathcal{F} \neq \varnothing$.

Since $D \neq \mathbb{C}$, there exists $w \in \mathbb{C} \setminus D$. Since D admits holomorphic square roots, there is a holomorphic function $g \colon D \to \mathbb{C}$ such that $[g(z)]^2 = z - w$ for $z \in D$. Note for $z_1, z_2 \in D$ that $g(z_1) = \pm g(z_2) \Rightarrow z_1 = z_2$. In particular, this means that g is injective and thus not constant.

By the Open Mapping Theorem, there exists $r > 0$ with $B_r(g(z_0)) \subset g(D)$.

Assume that there exists a point $z \in D$ with $g(z) \in B_r(-g(z_0))$:

$$r > |g(z) + g(z_0)| = |-g(z) - g(z_0)|;$$

this means that $-g(z) \in B_r(g(z_0)) \subset g(D)$. Hence, there exists $\tilde{z} \in D$ with $g(\tilde{z}) = -g(z)$ and thus $\tilde{z} = z$, which in turn yields that $0 = [g(z)]^2 = z - w$. This contradicts $w \notin D$. Hence, $g(D) \cap B_r(-g(z_0)) = \varnothing$ must hold.

Define

$$\tilde{g}\colon D \to \mathbb{D}, \quad z \mapsto \frac{r}{g(z) + g(z_0)},$$

Then $\tilde{g} : D \to \mathbb{D}$ is holomorphic and injective. Let $a := \tilde{g}(z_0)$. Then $\phi_a \circ \tilde{g} : D \to \mathbb{D}$ is holomorphic and injective with $(\phi_a \circ \tilde{g})(z_0) = 0$ and $(\phi_a \circ \tilde{g})'(z_0) \neq 0$ by Corollary 17.2.1. Let $c \in \mathbb{C}$ with $|c| = 1$ be such that $c\,(\phi_a \circ \tilde{g})'(z_0) > 0$. Then $c\,(\phi_a \circ \tilde{g}) \in \mathcal{F}$, so that indeed $\mathcal{F} \neq \varnothing$.

Claim 2. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$ converging compactly to $f\colon D \to \mathbb{C}$. Then either $f \equiv 0$ or $f \in \mathcal{F}$.

It is straightforward that $f(z_0) = 0$, $f'(z_0) \geq 0$, and $f(D) \subset \overline{\mathbb{D}}$. By Corollary 13.4.1, $f \equiv 0$ or f is injective. If f is injective, then $f'(z_0) \neq 0$ must hold by Corollary 17.2.1, so that $f'(z_0) > 0$. Also, since $f(D) \subset \overline{\mathbb{D}}$ is open, we have $f(D) \subset \mathbb{D}$, and hence $f \in \mathcal{F}$.

Claim 3. There exists $f \in \mathcal{F}$ such that $f(D) = \mathbb{D}$.

Choose a sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that

$$\lim_{n\to\infty} f_n'(z_0) = \sup\{\tilde{f}'(z_0) : \tilde{f} \in \mathcal{F}\} \in (0, \infty].$$

Notice that $|f_n(z)| \leq 1$ for all $z \in D$. By Montel's Theorem, there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ that converges compactly to some $f\colon D \to \mathbb{D}$. In particular,

$$f'(z_0) = \sup\{\tilde{f}'(z_0) : \tilde{f} \in \mathcal{F}\} > 0 \tag{$*$}$$

holds, so that $f \in \mathcal{F}$ by Claim 2.

Assume that there exists $w \in \mathbb{D} \setminus f(D)$. Since D admits holomorphic square roots, there is a holomorphic function $h\colon D \to \mathbb{C}$ such that

$$[h(z)]^2 = -(\phi_w \circ f)(z) = \frac{f(z) - w}{1 - \bar{w} f(z)}.$$

for $z \in D$. In particular, $h(D) \subset \mathbb{D}$, h is injective and hence $h'(z_0) \neq 0$. We then evaluate the derivative of each side of the above equation at $z = z_0$, noting that $f(z_0) = 0$:

$$2h(z_0)h'(z_0) = f'(z_0) + \bar{w} f'(z_0)(-w) = (1 - |w|^2) f'(z_0).$$

We also note that $|h(z_0)|^2 = |w|$ and define

$$g\colon D \to \mathbb{C}, \quad z \mapsto -\frac{|h'(z_0)|}{h'(z_0)}(\phi_{h(z_0)} \circ h)(z).$$

Then g is injective with $g(D) \subset \mathbb{D}$ and $g(z_0) = 0$. Since $\phi_a'(a) = -1/(1-|a|^2)$,

$$\begin{aligned} g'(z_0) &= -\frac{|h'(z_0)|}{h'(z_0)} \cdot \phi'_{h(z_0)}(h(z_0))h'(z_0) \\ &= \frac{|h'(z_0)|}{1-|h(z_0)|^2} \\ &= \frac{(1-|w|^2)f'(z_0)}{2\sqrt{|w|}(1-|w|)} \\ &= \frac{1+|w|}{2\sqrt{|w|}} \cdot f'(z_0) > f'(z_0) > 0, \end{aligned}$$

so that $g \in \mathcal{F}$ and $g'(z_0) > f'(z_0)$. This contradicts $(*)$. ☐

Theorem 17.4 (Simply Connected Domains). *The following are equivalent for an open and connected set $D \subset \mathbb{C}$:*

(i) *D is simply connected;*

(ii) *D admits holomorphic logarithms;*

(iii) *D admits holomorphic roots;*

(iv) *D admits holomorphic square roots;*

(v) *D is all of $\mathbb{C}$ or biholomorphically equivalent to $\mathbb{D}$;*

(vi) *every holomorphic function $f\colon D \to \mathbb{C}$ has an antiderivative;*

(vii) *$\int_\gamma f(\zeta)\,d\zeta = 0$ for each holomorphic function $f\colon D \to \mathbb{C}$ and each closed curve γ in D;*

(viii) *for every holomorphic function $f\colon D \to \mathbb{C}$, we have*

$$\nu(\gamma, z)f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for each closed curve γ in D and all $z \in D \setminus \{\gamma\}$;

(ix) *every harmonic function $u\colon D \to \mathbb{R}$ has a harmonic conjugate.*

Proof. (i) $\Longrightarrow$ (ii) is Corollary 11.2.2, (ii) $\Longrightarrow$ (iii) is shown in the proof of Corollary 11.2.3, (iii) $\Longrightarrow$ (iv) is trivial, (iv) $\Longrightarrow$ (v) follows from Theorem 17.3, and (v) $\Longrightarrow$ (i) is implied by Proposition 17.1.

(i) $\Longleftrightarrow$ (vi) is Corollary 11.2.1 and (vi) $\Longleftrightarrow$ (vii) follows from Theorem 4.1.

(i) $\Longrightarrow$ (viii) follows from Theorem 11.1, and (viii) $\Longrightarrow$ (vii) is established in the proof of Theorem 11.2.

(v) $\Longrightarrow$ (ix): Let $u\colon D \to \mathbb{R}$ be harmonic. If $D = \mathbb{C}$, the existence of a harmonic conjugate is immediate by Theorem 14.1. So suppose that $D \neq \mathbb{C}$. Hence, there is a biholomorphic map $f\colon D \to \mathbb{D}$. It is easily seen that $\tilde{u} := u \circ f^{-1}\colon \mathbb{D} \to \mathbb{R}$ is harmonic and by Theorem 14.1 has a harmonic conjugate $\tilde{v}\colon \mathbb{D} \to \mathbb{R}$. Then $v := \tilde{v} \circ f\colon D \to \mathbb{R}$ is a harmonic conjugate of u.

(ix) $\Longrightarrow$ (ii): Let $f\colon D \to \mathbb{C}$ be holomorphic such that $\mathbf{Z}(f) = \varnothing$. Then $u := \log|f|$ is harmonic and thus has a harmonic conjugate $v : D \to \mathbb{R}$ so that $g := u + i\,v$ is holomorphic. On D we have

$$|\exp g| = |\exp(u + i\,v)| = \exp u = |f|,$$

so that

$$D \to \mathbb{C}, \quad z \mapsto \frac{f(z)}{\exp(g(z))}$$

is a holomorphic function whose range lies on $\partial\mathbb{D}$ and therefore isn't open. By the Open Mapping Theorem, this means that there exists $c \in \partial\mathbb{D}$ such that $f(z) = c\exp(g(z))$ for $z \in D$. Choose $\theta \in \mathbb{R}$ with $\exp(i\theta) = c$, and note that $f(z) = \exp(g(z) + i\theta)$ for $z \in D$. $\square$

Definition. Two (not necessarily piecewise smooth) closed curves $\gamma_1, \gamma_2\colon [0,1] \to D$ with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ are called *path homotopic* if there is a continuous function $\Gamma\colon [0,1] \times [0,1] \to D$ such that,

$$\Gamma(0,t) = \gamma_1(t) \qquad \text{and} \qquad \Gamma(1,t) = \gamma_2(t)$$

for $t \in [0,1]$ and

$$\Gamma(s,0) = \gamma_1(0) \qquad \text{and} \qquad \Gamma(s,1) = \gamma_1(1)$$

for all $s \in [0,1]$.

Definition. A closed curve γ is called *homotopic to zero* if γ and the constant curve $\gamma(0)$ are path homotopic.

Further Characterizations of Simply Connected Domains. There are further conditions that characterize simply connected domains. We will only state them, without giving proofs. Simple connectedness is also equivalent to:

(x) *every (not necessarily smooth) curve in D is homotopic to zero*;

(xi) *D is homeomorphic to $\mathbb{D}$.*

Condition (xi) makes no reference to holomorphic functions and is entirely topological in nature. It means that there is a bijective, continuous map $f\colon D \to \mathbb{D}$ with a continuous inverse. Since (x) is preserved under homeomorphisms, we see that (xi) implies (x). For the converse, it is sufficient to show that $\mathbb{C}$ is homeomorphic to $\mathbb{D}$ (for $D \neq \mathbb{C}$, this is clear by Theorem 17.3). Since

$$\mathbb{C} \to \mathbb{D}, \quad z \mapsto \frac{z}{1+|z|}$$

and

$$\mathbb{D} \to \mathbb{C}, \quad z \mapsto \frac{z}{1-|z|}$$

are continuous and inverse to each other, this is indeed the case.

The converse to Problem 11.1 states that the property

(xii) *for every holomorphic function $f\colon D \to \mathbb{C}$, there exists a sequence of polynomials converging to f compactly on D*

always holds for a simply connected domain. The proof relies on *Runge's Approximation Theorem.*

There is also an equivalent condition for simple connectedness involving the extended complex plane $\mathbb{C}_\infty$:

(xiii) $\mathbb{C}_\infty \setminus D$ *is connected.*

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