The following fact was conjectured in our Saturday review session. Though (convincingly, it seems) pronounced FALSE by your instructor, it is actually TRUE. What you learn from this: Everybody makes mistakes, and your instructor is no exception. Sincere apologies nevertheless.

Theorem Ω. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$ exist then f is uniformly continuous.

Proof. For convenience, let $\alpha := \lim_{x\to-\infty} f(x)$ and $\beta := \lim_{x\to+\infty} f(x)$. Given $\varepsilon > 0$, there exists $R > 0$ such that $|f(x) - \alpha| < \frac{1}{2}\varepsilon$ for all $x \le -R$, and $|f(x) - \beta| < \frac{1}{2}\varepsilon$ for all $x \ge R$. Whenever $x, y \ge R$, therefore,

$$|f(x) - f(y)| \le |f(x) - \beta| + |\beta - f(y)| < \varepsilon\,,$$

and, analogously, $|f(x) - f(y)| < \varepsilon$ whenever $x, y \le -R$. On the other hand, the interval $[-2R, 2R]$ is compact, and f restricted to it is uniformly continuous. Thus, there exists $\delta_1 > 0$ such that $x, y \in [-2R, 2R]$ and $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \varepsilon$. Overal, with $\delta := \min(\delta_1, R)$, for any $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. In other words, f is uniformly continuous. ☐

Here is another look at Theorem Ω, which may make its content even more plausible. Consider the continuous, one-to-one and onto function

$$h : \left\{ \begin{array}{ccc} (-1,1) & \to & \mathbb{R} \\ t & \mapsto & \tan(\frac{1}{2}\pi t) \end{array} \right.$$

and observe that

$$\lim\nolimits_{t\downarrow -1} f\circ h(t) = \lim\nolimits_{x\to-\infty} f(x)\,, \quad \lim\nolimits_{t\uparrow 1} f\circ h(t) = \lim\nolimits_{x\to+\infty} f(x)\,.$$

Thus with $F(t) := f \circ h(t)$ for all $|t| < 1$ and

$$F(\pm 1) := \lim\nolimits_{x\to\pm\infty} f(x)\,,$$

the function $F : [-1, 1] \to \mathbb{R}$ is continuous, hence uniformly continuous. Given $\varepsilon > 0$, therefore, there exists $\delta > 0$ such that $t, s \in [-1, 1]$ and $|t - s| < \delta$ implies $|F(t) - F(s)| < \varepsilon$. Note that if $x, y \in \mathbb{R}$ and $|x - y| < \delta$ then

$$|h^{-1}(x) - h^{-1}(y)| = \left|\frac{2}{\pi}\arctan x - \frac{2}{\pi}\arctan y\right| = \frac{2}{\pi}\int_{\min(x,y)}^{\max(x,y)} \frac{\mathrm{d}t}{1+t^2} \le \frac{2}{\pi}|x-y| < \delta\,,$$

and consequently

$$|f(x) - f(y)| = \big|f\circ h\big(h^{-1}(x)\big) - f\circ h\big(h^{-1}(y)\big)\big| = \big|F\big(h^{-1}(x)\big) - F\big(h^{-1}(y)\big)\big| < \varepsilon\,.$$

This, again, shows that f is uniformly continuous.