Multiple Choice Questions

Mark your answers ***on the official answer sheet on the last page*** and ***detach it***. It will be collected ***after 60 minutes***.

1. For which value of k is the equation

$$x^2 \sin(y^k)dx + x^3y^2 \cos(y^k)dy = 0$$

exact?

(a) $k = 0$

(b) $k = 1$

(c) $k = 2$

(d) None of these.

K = 3

2. Consider the non-homogeneous linear equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^x$. A particular solution to this equation can be obtained:

(a) only by the method of undetermined coefficients;

(b) only by the method of variation of parameters;

(c) by both, the method of undetermined coefficients, and method of variation of parameters;

(d) by neither of these methods.

3. The mass-spring system described by

$$y'' + by + 4y = 0, \quad y(0) = 1, y'(0) = 0$$

is in the state of *underdamped* motion if

(a) $b > 4$

(b) $0 < b < 4$

(c) $b = 4$

(d) Any real value of b.

4. The mass-spring system described by

$$y'' + ky = \cos 5t, \quad y(0) = 1, y'(0) = 0$$

would be in the state of *resonant* motion if

(a) $k < 5$
(b) $k = 0$
(c) $k = 25$
(d) None of these.

5. The minimum value of the radius of convergence ρ of the power series expansion about $x_0 = 1$ for the solution of the equation:

$$(x^2 + 1)y'' + \frac{1}{x+1}y = 0$$

is:

(a) $\rho = 0$
(b) $\rho = 1$
(c) $\rho = 2$
(d) None of these.

$\rho \geq \sqrt{2}$

Long Answer Questions

You must show all your work.

1. Solve the initial-value problem:

$$\frac{dy}{dx} = \frac{y^2 + x\sqrt{x^2+y^2}}{xy}, \quad y(1) = 1.$$

Homogeneous equation

$\frac{y}{x} = u \quad \Rightarrow \quad u + x\frac{du}{dx} = \frac{dy}{dx}$

$u + x\frac{du}{dx} = \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^{-2} + 1} = u + \sqrt{u^{-2}+1}$

$\Rightarrow \quad \frac{u\,du}{\sqrt{u^2+1}} = \frac{dx}{x} \quad$ or $\quad \sqrt{u^2+1} = \ln|x| + C$

$\sqrt{\frac{y^2}{x^2}+1} = \ln|x| + C$

Since $y(1) = 1 \Rightarrow \sqrt{2} = \ln 1 + C = C$

$\Rightarrow \quad y^2 = x^2\left[(\ln|x| + \sqrt{2})^2 - 1\right]$

Since $y(1) = 1 > 0$:

$y(x) = +\sqrt{x^2\left[(\ln|x| + \sqrt{2})^2 - 1\right]}$

2. Find the general solution of the equation

$$2y'' - 16y' + 32y = t^{-3}e^{4t}$$

The solution to the homogeneous equation is: $y_h(x) = c_1 e^{4t} + c_2 t e^{4t} = c_1 y_1 + c_2 y_2$

Then $y_p(t) = v_1 y_1 + v_2 y_2$ is a particular solution and:

(1) $v_1' e^{4t} + v_2' t e^{4t} = 0$

(2) $4v_1' e^{4t} + v_2'(e^{4t} + 4te^{4t}) = \frac{t^{-3}e^{4t}}{2}$

$\Rightarrow v_2' = \frac{1}{2}t^{-3}$ or $v_2 = -\frac{1}{4}t^{-2}$

From (1) $\Rightarrow v_1' = -\frac{1}{2}t^{-2}$ or $v_1 = \frac{1}{2}t^{-1}$

Then $y(t) = \left(c_1 + c_2 t + \frac{1}{4}t^{-1}\right)e^{4t}$ is the general solution.

3. Consider a power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ for the equation

$$(1-x)y'' + xy' - y = 0.$$

(a) Find the recurrence relation for the a_n's.

(b) Find the first 5 coefficients a_n.

(c) Find the general formula for the a_n's as a function of n.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=2}^{\infty} a_n n(n-1) x^{n-1} +$$

$$\sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Shifting indices we obtain:

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{k=1}^{\infty} a_{k+1}(k+1) k x^k +$$

$$\sum_{k=1}^{\infty} a_k k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

or

$$2a_2 - a_0 + \sum_{k=1}^{\infty} \Big[a_{k+2}(k+2)(k+1) - a_{k+1}(k+1)k$$

$$+ a_k (k-1) \Big] x^k = 0$$

(a) $$a_{k+2} = \frac{k}{k+2} a_{k+1} - \frac{k-1}{(k+2)(k+1)} a_k, \quad k = 1, \ldots$$

(b) $a_0, a_1, a_2 = \frac{a_0}{2}$,

$$a_3 = \frac{1}{3} a_2 = \frac{1}{6} a_0$$

$$a_4 = \frac{2}{4} a_3 - \frac{1}{12} a_2 = \frac{1}{12} a_0 - \frac{1}{24} a_0 = \frac{1}{24} a_0$$

$$a_5 = \frac{3}{5} \frac{1}{24} a_0 - \frac{1}{10} \cdot \frac{1}{6} a_0 = \frac{1}{120} a_0$$

(c)

$$\Rightarrow y(x) = a_0\left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) + a_1 x$$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + (a_1 - a_0)x = c_1 e^x + c_2 x$$

where $c_1 = a_0$, $c_2 = a_1 - a_0$