NAME: ______________________

Long response problem. Show all your work

16 marks

1. Find the first three nonzero coefficients of the power series expansion about $x_0 = 0$ for the solution of the initial value problem

$$5y'' + (x+1)y = e^x, \qquad y(0) = 1, y'(0) = 0$$

Hint: Recall the Maclauren series $f(x) = \sum_{n=0}^{\infty} \frac{d^n f}{dx^n}(0)x^n/n!$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad , \quad y(0)=1 \Rightarrow a_0 = \boxed{1}$$

$$y'(0) = 0 \Rightarrow a_1 = 0$$

$$5\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=2}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$k = n-2 \qquad k = n+1 \qquad k = n$

$$\sum_{k=0}^{\infty} 5a_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} a_{k-1}x^k + \sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{\infty} \frac{x^k}{k!} = 0$$

$$\Rightarrow 10a_2 + a_0 - 1 + \sum_{k=1}^{\infty}\left(5a_{k+2}(k+2)(k+1) + a_{k-1} + a_k - \frac{1}{k!}\right)x^k = 0$$

$$\Rightarrow a_2 = \frac{1-a_0}{10} = 0$$

$$a_{k+2} = \left(\frac{1}{k!} - a_k - a_{k-1}\right) / 5(k+2)(k+1) \quad , \quad k = 1, 2, \ldots$$

$$a_3 = \frac{1-0-1}{30} = 0$$

$$a_4 = \left(\frac{1}{2} - 0 - 0\right)/60 = \boxed{\frac{1}{120}}$$

$$a_5 = \left(\frac{1}{6} - 0 - 0\right)/100 = \boxed{\frac{1}{600}}$$

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2. Use the method of separation of variables to find the formal solution of the problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - 2e^{-x}, \quad 0 < x < \pi, t > 0$$

$$u(0,t) = 0, \quad u(\pi,t) = -2 + 2e^{-\pi}, \quad t > 0$$

$$u(x,0) = 2e^{-x}, \quad 0 < x < \pi$$

Note: You must show the calculation for obtaining the eigenvalues and eigenfunctions for the related boundary value problem.

$u(x,t) = v(x) + w(x,t)$, $w(x,t), \partial_{xx} w(x,t) \xrightarrow[t \to \infty]{} 0$

$\Rightarrow v''(x) - 2e^{-x} = 0$, $v(0) = 0$, $v(\pi) = -2 + 2e^{-\pi}$

$v(x) = 2e^{-x} + Ax + B$

$v(0) = 0 \Rightarrow B = -2$

$v(\pi) = \cancel{-2} + \cancel{2e^{-\pi}} = \cancel{2e^{-\pi}} + A\pi - \cancel{2} \Rightarrow A = 0$

$\Rightarrow v(x) = 2e^{-x} - 2$

For $w(x,t)$ we solve

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}, & 0 < x < \pi, \ t > 0 \\ w(0,t) = w(\pi,t) = 0 \\ w(x,0) = 2e^{-x} - v(x) = \cancel{2e^{-x}} - \cancel{2e^{-x}} + 2 \end{cases}$$

$$w(x,t) = X(x)\,T(t)$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{T'}{T} = k$$

$$\begin{cases} X'' - kX = 0 \\ X(0) = X(\pi) = 0 \end{cases} \qquad T' - kT = 0$$

$$T = a e^{kt}$$

a) $k > 0$

$$X(x) = c_1 e^{\sqrt{k}\,x} + c_2 e^{-\sqrt{k}\,x}$$

$$X(0) = c_1 + c_2 = 0$$

$$X(\pi) = c_1 e^{\sqrt{k}\,\pi} + c_2 e^{-\sqrt{k}\,\pi} = 0$$

$$\Rightarrow c_1 = c_2 = 0$$

b) $k = 0$

$$X(x) = c_1 x + c_2$$

$$X(0) = c_2 = 0$$

$$X(\pi) = c_1 \pi = 0 \Rightarrow c_1 = 0$$

c) $k < 0$

$$X(x) = c_1 \cos(\sqrt{-k}\,x) + c_2 \sin(\sqrt{-k}\,x)$$

$$X(0) = c_1 = 0$$

$$X(\pi) = c_2 \sin(\sqrt{-k}\,\pi) = 0$$

$$\Rightarrow \sqrt{-k} = n \quad , \quad n = 1, 2, \ldots$$

$$k = -n^2$$

$$X_n(x) = c_n \sin(nx) \quad , \quad T_n = a_n e^{-n^2 t}$$

$$\Rightarrow w(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$$

$$w(x,0) = 2 = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$$

$$\Rightarrow b_n = \frac{4}{\pi} \int_0^{\pi} \sin(nx)\, dx =$$

$$= -\frac{4}{\pi n} \cos(nx) \Big|_0^{\pi} =$$

$$= -\frac{4}{n\pi}\left((-1)^n - 1 \right)$$

$$\Rightarrow u(x,t) = -2 + 2e^{-x} - \sum_{n=1}^{\infty} \frac{4}{n\pi} \left[(-1)^n - 1 \right] e^{-n^2 t} \sin(nx)$$

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1. The solution y of

$$(y + ye^{(x-2)y})dx + (x + (x-2)e^{(x-2)y} + 1)dy = 0, \quad y(1) = 1$$

at $x = 2$ equals

(a) $e^{-1}/2$ (b) $(1 + 2e^{-1})/5$ (c) $(2 + e^{-1})/3$ (d) $(1 + 2e)/5$ (e) $(1 + e^{-1})/3$.

2. The solution y of

$$x^2y'' - xy' + 2y = 2, \qquad y(1) = 0, y'(1) = 0$$

equals at $x = e^{\pi}$

(a) $1 + e^{\pi}$ (b) $1 + e^{-\pi}$ (c) $1 - e^{-\pi}$ (d) $1 - e^{\pi}$ (e) $1 - \pi$.

3. If w solves the problem

$$w'' + 6w' + 5w = e^{-t}\delta(t-2), \qquad w(0) = w'(0) = 0$$

where δ is the Dirac Delta function, then at $t = 3$, w equals

(a) $(1 - e^{-4})/4$ (b) $(e^{-3} - e^{-7})/4$ (c) $(3e^{-3} + e^{-7})/16$ (d) 0 (e) None of these.

4. If y solves

$$y + \int_0^t y(v)e^{t-v}dv = 1$$

then y at $t = 1$ equals

(a) 0 (b) 1 (c) 2 (d) -1 (e) $(1 + e^{-2})/2$

5. Let x, y solve the system

$$\begin{aligned} x' - y &= 1 \\ x + y' &= 0 \end{aligned}$$

subject to the initial conditions $x(0) = y(0) = 0$. Then at $t = \pi$, (x, y) equals

(a) $(0, -1)$ (b) $(0, 0)$ (c) $(-1, 0)$ (d) $(0, 2)$ (e) $(0, -2)$.

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6. If

$$f(x) = \begin{cases} 1, & -1 \le x < 0 \\ -2, & 0 \le x < 1. \end{cases}$$

and f is expanded in a Fourier series

$$a_0/2 + \sum_{n=1}^{\infty} [a_n \cos(n\pi x/T) + b_n \sin(n\pi x/T)]$$

then a_0, a_3, b_3 are (in this order)

(a) $(0, 0, -2/\pi)$ (b) $(-1, 0, 2/\pi)$ (c) $(1, 0, 2/\pi)$ (d) $(-1, 0, -2/\pi)$ (e) $(0, 0, 0)$.

7. The eigenvalues λ for the initial value problem

$$y'' - 2y' + \lambda y = 0, \qquad y(0) = 0, \quad y'(\pi) + y(\pi) = 0$$

satisfy the equation

(a) $\tan(\sqrt{\lambda + 1}\pi) + \sqrt{\lambda + 1}/2 = 0$ (b) $\sin(\sqrt{\lambda - 1}\pi) = 0$ (c) $\cos(\sqrt{\lambda - 1}\pi) = 0$
(d) $\tan(\sqrt{\lambda - 1}\pi) + \sqrt{\lambda - 1}/2 = 0$ (e) None of these.

1. This is an exact equation

$$\frac{\partial F}{\partial x} = y + y e^{(x-2)y}$$

$$\Rightarrow F(x,y) = yx + e^{(x-2)y} + g(y)$$

$$\frac{\partial F}{\partial y} = x + (x-2) e^{(x-2)y} + g'(y)$$

$$= x + (x-2) e^{(x-2)y} + 1$$

$$\Rightarrow g'(y) = 1 \Rightarrow g(y) = y$$

$\Rightarrow$ the solution is

$$xy + e^{(x-2)y} + y = C$$

At $x = 1$

$$1 + e^{-1} + 1 = C \Rightarrow C = 2 + e^{-1}$$

At $x = 2$

$$2y(2) + y(2) = 1 + e^{-1}$$

$$\Rightarrow y(2) = \frac{1 + e^{-1}}{3}$$

2. $x = e^t$

$\Rightarrow y''(t) - 2y'(t) + 2y(t) = 2 \quad , \quad y(0) = 0, \quad y'(0) = 0$

$r^2 - 2r + 2 = 0$

$r_{1,2} = 1 \pm \sqrt{-1} = 1 \pm i$

$\Rightarrow y_h(t) = c_1 e^t \cos t + c_2 e^t \sin t$

$y_p(t) = A$ (The right-hand-side is 2)

$\Rightarrow 2A = 2 \Rightarrow A = 1$

$\Rightarrow y(t) = c_1 e^t \cos t + c_2 e^t \sin t + 1$

$y(0) = c_1 + 1 = 0 \Rightarrow c_1 = -1$

$y'(0) = -(e^t \cos t - e^t \sin t)\big|_{t=0} + c_2 (e^t \sin t + e^t \cos t)\big|_{t=0}$

$= 0$

$\Rightarrow -1 + c_2 = 0 \Rightarrow c_2 = 1$

If $x = e^{\pi} \Rightarrow t = \pi$

$\Rightarrow y(t = \pi) = -e^{\pi} \cos \pi + e^{\pi} \sin \pi + 1 =$

$= 1 + e^{\pi}$

3. Taking the LT we get

$$s^2W + 6sW + 5W = \int_0^\infty e^{-st} e^{-t} \delta(t-2)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-t(s+1)} \delta(t-2)\,dt =$$

$$= e^{-2(s+1)}$$

$$\Rightarrow W(s) = \frac{e^{-2} e^{-2s}}{(s+5)(s+1)}$$

$$\frac{1}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1} = \frac{As + A + Bs + 5B}{(s+5)(s+1)}$$

$$\Rightarrow A + B = 0$$

$$A + 5B = 1 \quad \Rightarrow \quad B = \frac{1}{4}, \quad A = -\frac{1}{4}$$

$$\Rightarrow w(t) = e^{-2}\left(-\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+5}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+1}\right\}\right)$$

$$= \frac{e^{-2}}{4}\left(e^{-(t-2)}u(t-2) - e^{-5(t-2)}u(t-2)\right)$$

$$w(3) = \frac{e^{-2}}{4}\left(e^{-1} - e^{-5}\right) = \frac{e^{-3} - e^{-7}}{4}$$

4. Taking the LT we obtain

$$Y + \frac{Y}{s-1} = \frac{1}{s}$$

$$Y(s) = \frac{s-1}{s^2} = \frac{1}{s} - \frac{1}{s^2}$$

$$\Rightarrow y(t) = 1 - t$$

$$\Rightarrow y(1) = 0$$

5. Taking the LT

$$\begin{cases} sX - Y = \frac{1}{s} \\ X + sY = 0 \end{cases} \Rightarrow (s^2+1)X = 1$$

$$X(s) = \frac{1}{s^2+1} \Rightarrow x(t) = \sin t$$

$$y(t) = x' - 1 = \cos t - 1$$

$$\left(x(\pi), y(\pi)\right) = (0, -2)$$

6.

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0}\cos(n\pi x)\,dx - 2\int_{0}^{1}\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} - 2\frac{1}{n\pi}\sin(n\pi x)\Big|_{0}^{1} =$$

$$= 0$$

$$a_0 = \int_{-1}^{0} dx - 2\int_{0}^{1} dx = -1$$

$$b_n = \int_{-1}^{0}\sin(n\pi x)\,dx - 2\int_{0}^{1}\sin(n\pi x)\,dx =$$

$$= -\frac{1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi}\cos(n\pi x)\Big|_{0}^{1} =$$

$$= -\frac{1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{2}{n\pi}\left((-1)^n - 1\right) =$$

$$= \frac{3}{n\pi}(-1)^n - \frac{3}{n\pi}$$

$$b_3 = -\frac{3}{3\pi} - \frac{3}{3\pi} = -\frac{2}{\pi}$$

$$\Rightarrow (a_0, a_3, b_3) = \left(-1, 0, -\frac{2}{\pi}\right)$$

7.

$r^2 - 2r + \lambda = 0$

$\Rightarrow r_{1,2} = 1 \pm \sqrt{1-\lambda}$

a) $1-\lambda > 0$

$y(x) = C_1 e^{(1+\sqrt{1-\lambda})x} + C_2 e^{(1-\sqrt{1-\lambda})x}$

It is easy to check that $C_1 = C_2 = 0$ from the boundary conditions.

b) $1-\lambda = 0$

$y(x) = C_1 e^x + C_2 x e^x$

$y(0) = C_1 = 0$

$y'(\pi) + y(\pi) = C_2(\pi e^\pi + e^\pi) + C_2 \pi e^\pi = 0$

$\Rightarrow C_2 = 0$

c) $1-\lambda < 0$

$y(x) = C_1 e^x \cos(\sqrt{\lambda-1}\, x) + C_2 e^x \sin(\sqrt{\lambda-1}\, x)$

$y(0) = C_1 = 0$

$\Rightarrow y'(\pi) + y(\pi) = \cancel{C_2}(\cancel{e^\pi} \sin(\sqrt{\lambda-1}\,\pi) + \cancel{e^\pi} \cos(\sqrt{\lambda-1}\,\pi)\sqrt{\lambda-1})$
$+ \cancel{C_2}\cancel{e^\pi} \sin(\sqrt{\lambda-1}\,\pi) = 0$

$\Rightarrow \boxed{\tan(\sqrt{\lambda-1}\,\pi) + \frac{\sqrt{\lambda-1}}{2} = 0}$