RECREATIONAL MATHEMATICS

ON THE DISTRIBUTION OF FIRST DIGITS OF POWERS

By Leo Moser and Nathaniel Macon

It is clear that the last digit (to the right) of 2^n is 2, 4, 6, or 8. Further, E. P. Starke [1][1] has recently shown that given any integer m, there exists an n such that 2^n has only 1's and 2's as its last m digits. If, however, we consider the first digits of 2^n, a simple machine computation reveals that already for $n \leqq 100$, 68 of the combination of digits 11, 12, ..., 99 appear.[2] In this note, we will prove that every finite sequence of digits appears as the first digits of some power of 2, and will generalize this result in several directions.

We first prove the following theorem:

THEOREM I. *Let $a > 1$ and $b > 1$ be integers such that $a^r \neq b^s$ with r and s positive integers. Let β be any positive real number. Then given any $\eta > 0$, there exist integers n and p such that*

$$|a^n/b^p - \beta| < \eta. \tag{1}$$

Proof. Since b^x is a continuous function for all real x, it is sufficient to make

$$|\log_b (a^n/b^p) - \log_b \beta| < \epsilon, \qquad \epsilon = \epsilon(\eta). \tag{2}$$

Let $\log_b a = \theta$ and $\log_b \beta = \alpha$. Then we must make

$$|n\theta - p - \alpha| < \epsilon. \tag{3}$$

Now by Kronecker's theorem (first proved by Jacobi) [3, p. 363], (3) can be satisfied provided that θ is irrational. Here θ is irrational, since $\theta = \log_b a = s/r$ implies $a^r = b^s$, which is contrary to hypothesis. Thus the theorem is proved.

Let k be any integer expanded in the decimal system, and let a be an integer such that $a^r \neq 10^s$. Since 10 is not a perfect square or higher power, this is equivalent to the statement that a is not a power of 10. To find a power of a which starts with the digits of k, we have merely to satisfy (1) with $b = 10$, $\beta = k + \frac{1}{2}$, and $\eta < \frac{1}{2}$. Theorem I guarantees that this can be done.

For example, if we wish to find a power of 2 which has 65 as its first digits, we solve $|2^n/10^p - 65.5| < \frac{1}{2}$. We know that a solution exists. Actually the smallest solution here is given by $|2^{16}/10^3 - 65.5| = 0.036$, and $2^{16} = 65,536$ starts with the digits 65. We also know that there exists a power of 3 which starts with the same digits, 65. Indeed, $3^8 = 6561$.

We next consider the following question: What is the probability P_k that the first digits of a^n will be the digits of k? To put it more precisely, denote

[1] Boldface numbers in brackets refer to the references at the end of the paper.
[2] For $n \leq 50$, cf. [2].

by $r(k, n)$ the number of elements of $\{a^1, a^2, \ldots, a^n\}$ which begin with the digits of k. We seek $P_k = \lim_{n \to \infty} [r(k, n)/n]$. The answer is given by the theorem which follows.

THEOREM II. *If a is not a power of 10, and k is any integer, the probability that the first digits of a^n will be those of k is given by*

$$P_k = \log_{10} [(k + 1)/k]. \tag{4}$$

Proof. In order for a^n to begin with the digits of k, it is necessary and sufficient that the mantissa of $\log_{10} a^n$ lie between the mantissa of k and that of $k + 1$. More exactly, we need

$$(\log_{10} k) \leqq (n \log_{10} a) < (\log_{10} [k + 1]), \tag{5}$$

where here (x) denotes the fractional part of x. It follows from Wehl's theorem on the uniform distribution of $(n\theta)$ [3, p. 378] that $(n \log_{10} a)$ is uniformly distributed on the interval $(0, 1)$. From (5), $(n \log_{10} a)$ must lie on the subinterval of length

$$(\log_{10} [k + 1]) - (\log_{10} k) = (\log_{10} [k + 1] - \log_{10} k) = \log_{10} [(k + 1)/k].$$

Thus the proof is complete.

It is curious that P_k is independent of a. We have

$$\log_{10} [(k + 1)/k] = \log_{10} e \cdot \log_e (1 + 1/k) \doteq 0.4343/k.$$

Thus P_k is very nearly inversely proportional to k. It is interesting to note that although 100 has three digits and 99 only two, P_{99} and P_{100} differ only slightly.

The following question naturally arises: Given a set of integers $k_1, k_2, \ldots, k_t$ and a set of integers $a_1, a_2, \ldots, a_t$, under what conditions does there exist a single integer n such that a_i^n have the digits of k_i for their first digits ($i = 1, 2, \ldots, t$)? For example, is it possible to find an n such that 2^n starts with the digits 20 and 3^n with the digits 17? We prove the following theorem:

THEOREM III. *Let $a_1, a_2, \ldots, a_t$ be a set of integers, each greater than 1 and such that $\log_b a_i$ ($i = 1, 2, \ldots, t$) and 1 are linearly independent over the rational integers; and let $\beta_1, \beta_2, \ldots, \beta_t$ be any set of real numbers. Then there exists an integer n and integers p_i such that for any $\eta > 0$, we have*

$$|a_i^n/b^{p_i} - \beta_i| < \eta, \qquad i = 1, 2, \ldots, t. \tag{6}$$

Proof. As in the proof of Theorem I, it suffices to make

$$|n\theta_i - p_i - \alpha_i| < \epsilon, \qquad \epsilon = \epsilon(\eta), \qquad i = 1, 2, \ldots, t, \tag{7}$$

where $\theta_i = \log_b a_i$ and $\alpha_i = \log_b \beta_i$. It follows from the t dimensional case of Kronecker's theorem [3, p. 370] that this is possible, since $\theta_1, \theta_2, \ldots, \theta_t$ and 1 are linearly independent.

Now let $k_1, k_2, \ldots, k_t$ be any set of integers, not necessarily distinct, and let $a_1, a_2, \ldots, a_t$ be a set of integers satisfying the conditions of Theorem III with $b = 10$. In order to find an integer n such that $a_1^n, a_2^n, \ldots, a_t^n$ have the digits of $k_1, k_2, \ldots, k_t$, respectively, as first digits we have only to satisfy (6) with

$$\beta_i = k_i + \frac{1}{2}, (i = 1, 2, \ldots, t) \text{ and } \eta < \frac{1}{2}.$$

Thus for the example discussed earlier with $a_1 = 2$ and $a_3 = 3$, the linear independence of $\log_{10} 2$, $\log_{10} 3$, and 1 follows from the fact that 10 contains the factor 5, not contained in $2 \cdot 3$; and hence it is possible to find an n satisfying the required condition. The smallest n here is 11, and $2^{11} = 2048$ while $3^{11} = 177,147$.

H. Weyl [4] has generalized Kronecker's t dimensional theorem and has proved that if $\theta_1, \ldots, \theta_t$ are linearly independent with 1, i. e., if $\sum_{i=1}^{t} p_i \theta_i = q$ has no solution with p_i and q integers, then $(n\theta_1, n\theta_2, \ldots, n\theta_t)$ is equidistributed over the t dimensional unit hypercube. From this a theorem analogous to Theorem II can be shown to hold for sets of powers. We leave the exact formulation and proof to the reader.

REFERENCES

1. E. P. Starke, "Solution to Problem 4327," *Am. Math. Monthly*, v. 57 (1950), p. 350.
2. *British Association Math. Tables*, v. 9, Cambridge (1940).
3. Hardy and Wright, *An Introduction to the Theory of Numbers*, 2nd ed., Oxford (1945).
4. H. Weyl, "Über die Gleichverteilung von Zahlen mod. Eins," *Math. Annalen*, v. 77 (1916), p. 313–352.

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CURIOSA

245. A Humorous Proof of Euclid's Postulate. In the *Revue Scientifique*, Ser. 4, v. 21, 1903, p. 215, we find an interesting note by Gaston Tarry, entitled "Demonstration humoristique du postulatum d'Euclide." It deals with Euclid's famous Fifth Postulate, or rather with one of its substitutes which is given by Heath (*The Thirteen Books of Euclid's Elements*, v. 1, Cambridge, 1908), in the following form: *Through a given point only one parallel can be drawn to a given straight line* or: *Two straight lines which intersect one another cannot both be parallel to the same straight line.*

The following is an abridged statement of Tarry's proof.

Let n be the number of points on a straight line. If at every point of the line a perpendicular to it be drawn, there will be n points on each of the n perpendiculars. Hence there are n^2 points in the plane.

Now let x denote the number of lines passing through a given point. On each of these lines there are n points, or $n - 1$ if the given point is not counted. Hence the number of points on all of these lines is $x(n - 1)$, if the common point is not counted. Hence on the plane there are $x(n - 1) + 1$ points.

If the above reasoning is correct we must have

$$x(n - 1) + 1 = n^2, \text{ or } x(n - 1) = n^2 - 1,$$
$$\therefore x = n + 1$$

that is, through a given point pass $n + 1$ lines.

Consider now a given point P with the $n + 1$ lines passing through it and a given line L with its n points.

If every one of the $n + 1$ lines passing through P would intersect L there would be $n + 1$ points of intersection. But the line L supposedly contains only n points. Hence there must be one line passing through P that does not intersect L.

In the same volume (p. 373) there is a note by Cadenat claiming that this procedure is not at all humorous. To show its merits he uses it to prove that in Riemannian Geometry there are no parallel lines and that in Lobachevsky-Bolyai Geometry there are infinitely many lines passing through a point and parallel to a given line.

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246. Two Timely Problems. *Problem 1.* Write down any number from the square array of numbers given below, and cross out the row and column in which that number is located. Now write down another number from the array, not yet crossed out, and again cross out the row and column in which that number is located. Continue in this way until all numbers in the array have been crossed out. The problem is to carry out the above process in such a way that the sum of the numbers written down is *1951*.

213	356	152	246	363	139
459	602	398	492	609	385
245	388	184	278	395	171
303	446	242	336	453	229
382	525	321	415	532	308
158	301	97	191	308	84

Problem 2. Place the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in the ten blank spaces, in such an order that the indicated division will have a remainder of *1951*.

$$3168 \overline{)71\text{---}543\text{---}2\text{---}985\text{---}2\text{---}356\text{---}8\text{---}7\text{---}836\text{---}7\text{---}39,967.}$$

Solutions: Both of these problems are as easy as can be, for one has only to make an attempt and he will be successful (barring, of course, errors in arithmetic). In the first problem one can obtain 6! sets of numbers, each of which will total to 1951, while in the second problem there are 10! ways of placing the digits, but again each of these will yield the remainder 1951.

Of course the real problem is to discover the principles used in the construction of these problems, and this we leave to the reader.

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247. "Proof" that $2 \cdot 2 = 8$.

$$\cos^2 x = 1 - \sin^2 x$$
$$(\cos^2 x)^{3/2} = \cos^3 x = (1 - \sin^2 x)^{3/2}$$
$$\cos^3 x + 3 = (1 - \sin^2 x)^{3/2} + 3$$
$$2[\cos^3 x + 3] = 2[(1 - \sin^2 x)^{3/2} + 3.]$$

For $x = \pi/2$, this becomes $2 \cdot 3 = 2 \cdot 3$. However, for $x = \pi$ we get

$$2(-1 + 3) = 2(1 + 3),$$

or

$$2 \cdot 2 = 2 \cdot 4 = 8.$$

A. SPILBERG in *Revue Scientifique*, v. 19, 1903

248. Equiareal Triangles. The following areas are common to three or four Pythagorean triangles of which two are primitive. The sides of the triangles are $(x^2 + y^2)z$, $(x^2 - y^2)z$, $2xyz$. If $z = 1$ the triangle is primitive.

		y	x	z	y	x	z	y	x	z	y	x	z
1	341,880	55	56	1	37	40	1	15	22	2	2	35	2
2	2,042,040	40	51	1	3	88	1	21	34	2			
3	17,957,940	77	92	1	4	165	1	12	23	13	5	28	13
4	116,396,280	88	133	1	35	152	1	10	143	2	34	99	2
5	352,173,360	560	561	1	119	176	1	204	209	2			
6	1,071,572,040	155	232	1	40	301	1	11	290	2	9	310	2
7	1,728,483,120	144	259	1	35	368	1	52	207	2	20	279	2

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