This is the promised continuation of our mystery series “The Rose and the Nautilus.” The last time we met, we saw the connection of Fibonacci’s down-to-earth sequence of “rabbit numbers”:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . . ,

with the elusive and legendary divine proportion, incarnated by rectangles with a strange property: when you cut a square away from one of them, you are left with a rectangle of the same shape—so you can continue cutting off squares indefinitely. So far, we don’t know for sure that such rectangles exist—they may be a pipe dream or a pie in the sky. All we have our hands on, so far, are number sequences like the one above, where two successive numbers (high enough in the sequence) give us an approximately golden rectangle.

Today we’ll make the journey from the golden rectangle to the golden triangle, the major building block of the pentagram (the five-cornered star on U.S. military vehicles), and hence the Pentagon (home of U.S. military brass). How fitting that the emblem of this remarkable institution shares its basic symmetry with the rose and many other flowers, from buttercups to petunias!

On the right side of Figure 1 you see a rose. The underlying pentagram (shown in light blue) is made up of golden triangles, each of which is obtained from a golden rectangle by collapsing one of the shorter sides to a point, like so:

And if we do not have a perfect golden triangle handy, we’ll take an approximate one made of rabbit numbers.

If you were given the large, two-tone yellowish triangle standing in Figure 2, you would not have much trouble filling in the rest of the diagram, would you? The proportions of this triangle are determined by \( a = 144 \) (base) and \( a + b = 233 \) (side) units of length—two successive rabbit numbers.

And the pentagram connection? If we knew that the base angle of the large two-tone triangle is equal to twice its vertex angle, the latter would be \( 1/5 \) of 180 degrees—see? Now, \( a/b \) is very nearly equal to \( (a+b)/a \), which means that the lower pale yellow triangle is well-nigh isosceles, and the distance from the lower left red dot to the blue dot is nearly equal to \( a \). Hence, the upper deep yellow triangle is well-nigh isosceles, and its exterior angle at the blue dot is roughly twice the vertex angle—Q.E.D.

Do I hear you grumbling? You are not happy with all this uncontrolled fuzziness? You’d like crisp, clean equality instead of all those “well-nighs”? Well, last year we tried to work with so-called “real” numbers, and you hated them when you were asked to produce crisp, clean proofs. And then somebody said that it all came down to Cauchy sequences and/or Dedekind cuts plus continuity, and when we tried that approach you all freaked out. Let’s face it: if you want numerical proportions, you must choose between honest imperfection (e.g., rabbits) or dishonest perfection (e.g., root five). This is not rocket science: Eudoxos realized it way back around 400 B.C. So, let’s get back to geometry, shall we? I admit that we have not yet constructed a perfect golden rectangle, but I promise you it isn’t hard. For today, let us suppose it done and continue from there.

Our task is to repeat the argument pertaining to the pentagram connection (Figure 2) without any “nearly,” “roughly,” or “well nigh.” Just go over it again, and you’ll see that everything will be perfectly exact, as long as the pale yellow triangle in the lower right of Figure 2 is truly isosceles.

Into Geometry

We have defined what a golden rectangle should be, but never actually shown how to get one. Let’s fill that gap now, using only a compass and a straight-edge (i.e., the notions of a circle and a straight line).

Starting with the yellow square (see Figure 3), we draw a circle centered at the midpoint of its base and going through the top corner \( C \). It will meet the base line \( FB \) at two points \( A \) and \( E \), and the triangle \( AEC \) has a right angle at \( C \)—okay? (If you doubt it, just cut the triangle by a line joining \( C \) to the center of the circle, and tally the angles in the two isosceles pieces you get.) Since the purple rectangle is just the gray one shifted over, its diagonal is also perpendicular to \( AC \). If you turned it through 90 degrees, this diagonal would line up with \( AC \). Thus, the rectangle \( ABCD \) is similar to the purple one and is therefore golden. Any questions?

In Figure 4, we are looking at a golden rectangle. We remove the square at the bottom, shown in two shades of yellow. The residual rectangle at the top is gray, and a copy of it is shown (left side) inside the yellow square. This gray rectangle as well as its copy is similar to the original big rectangle, so the two red diagonals are parallel.

Imagine them as rubber bands, and the dot in the lower left corner as a hinge. Of course, we are not dealing with rubber bands and hinges, but with geometric
entities in our minds. Yet this metaphor allows us to avoid tedious descriptions, and suggests that we speak of the brown vertical and horizontal sides as bars attached to that hinge. In Figure 5, see how we make a big isosceles triangle by simply swinging the vertical bar through a small angle. The rubber bands will remain parallel. That is important, because it means that the smaller triangle in the lower left has the same base angles as the big one. Do you agree? You are not quite convinced?

How perceptive of you—and right you are: mechanics is not geometry. I promise to come back to those rubber bands. But first let us see how to get a pentagram from the triangle in Figure 5, supposing that the parallelism holds up. The two-tone yellowish triangle in Figure 2 is, of course, our double triangle from Figure 5, with the smaller copy laid on its side. Does everything fit as it should?

First the angles: since big and small have the same base angles, small fits snugly into the lower right corner of big. Then the sides: the one labelled “a” in Figure 5 was the side of the square in Figure 4, hence fits exactly along the base. So it works! But remember: now that we have moved into geometry, “a” and “b” are line segments, not numbers.

Desargues

This might seem a bit lengthy, but if you mull it over, it’s quite straightforward—except for the parallelism of those rubber bands, which seems so obvious mechanically, but hard to prove geometrically. In fact, it is false in pure plane geometry—some diligent geometers have found counterexamples—but becomes inevitably true if the plane sits in a three-dimensional space. This was discovered by Monsieur Girard Desargues about 350 years ago.

Here he is, on the left, staring at his theorem (its “affine” version), shown on the right in Figure 6. What does it look like to you? A leaning tower with two platforms in it? Good! What it says is this: if two triangles ABC and A′B′C′ are lined up so that the lines AA′, BB′, and CC′ meet at a single point, and if AB is parallel to A′B′ and BC parallel to B′C′, it follows that AC is also parallel to A′C′. Obvious?

It almost would be if we were looking at a spatial setup. Then the hypothesis would quickly imply that the entire two “platforms” are parallel. But this is meant as a flat, planar diagram, and, as I have said, the theorem is impossible to prove within purely planar geometry.

You think you can prove it by using the equations of those lines? Goodness, now we are really turning around in circles! You want coordinates again—but not all planes have coordinates. Now that you’ve brought it up, let’s get this straight: yes, you can prove the theorem by linear equations in any Cartesian plane, even if only rational coordinates are allowed, but the result is useless for golden triangles unless you allow irrational ones. But that’s precisely what we wanted to avoid this semester, isn’t it?

We have gone way over time—let’s prove Girard’s theorem at our next meeting. You will see that the proof is clever but not hard to follow. We pretend that his diagram is the planar projection of a spatial gizmo, then transfer the hypothesis to the latter, draw our conclusion up there in space and finally project down again. For today, let me just show you how it helps to keep our “rubber bands” parallel.

Figure 7 shows the bare bones of Figures 4 and 5 superimposed. Do you see the rubber bands? In their original position (attached to the vertical bar) they are parallel. In Figure 8, we added two short red lines—which are also parallel, being the bases of two isosceles triangles with the same vertex angle. Hence, the rubber bands remain parallel in their new position (attached to the slanted bar). Et voilà!

More about Girard Desargues? He was an engineer and mathematician who lived, worked, and died in Lyons, France. He did a lot of fine work and is known as the “father of projective geometry.” That’s the geometry of perspective. Please, now, let me go: I barely have time to get to my concert. More next time...yes, yes, I promise.

A mathematician and a stockbroker go to the races to bet on horses. The stockbroker suggests a bet of $10 000. That’s too much for the mathematician’s taste—first, he wants to understand the rules, have a look at the horse, etc.

“Don’t worry,” the stockbroker says. “I know an empirical algorithm that allows me to find the number of the winning horse with absolute certainty.”

This does not convince the mathematician.

“You are too theoretical!” the stockbroker exclaims, and puts his $10 000 on a horse.

The horse comes in first—making the stockbroker even richer than he already is. The mathematician is baffled.

“What is your algorithm?” he wants to know.

“It’s rather easy. I have two children, three and five years old. I add up their ages and bet on that number.”

“But three plus five is eight—and that horse had number nine!”

“I told you that you’re too theoretical! Didn’t I just experimentally prove that my calculation is correct?!”