Solar Eclipses: Geometry, Frequency, Cycles

by
Hermann Koenig†

Total solar eclipses are spectacular shows in the sky, in particular, if they occur on a bright day around noon. In a narrow band on Earth, the moon completely obscures the sun and the solar corona becomes visible. Sun and moon both appear to the observer on Earth to subtend an angle of roughly $\theta = \frac{1}{2}^\circ$, even though the radius of the sun is 400 times larger than that of the moon. By pure chance, the sun is also about 400 times further away from the Earth than the moon. The values are

$$\theta_S \approx 2\sin\frac{\theta_S}{2} = \frac{2R}{D}, \quad \theta_M \approx 2\sin\frac{\theta_M}{2} = \frac{2r}{d} \quad \text{(in radians)}.$$

Notation	Value	Meaning
R	696 000 km	Radius of the sun
r	1 738 km	Radius of the moon
D	149 600 000 km	Distance from the sun to the Earth (mean value)
d	384 400 km	Distance from the moon to the center of the Earth (mean value)
$\theta_S (\theta_M)$		Apparent angle of the sun (moon), as seen from the surface of the Earth

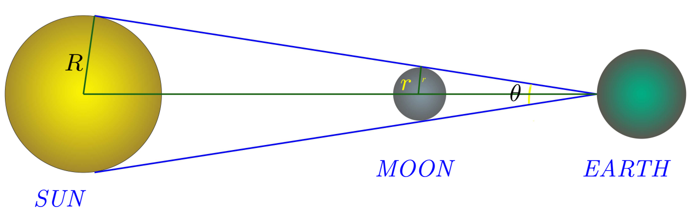

Figure 1

Angles of the sun and the moon (not to scale).

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Since $\frac{R}{r} \approx 400 \approx \frac{D}{d}$, we see that $\theta_S \approx \theta_M$. The sun is at one focus of the counterclockwise elliptical orbit of the Earth around the sun. Thus, the distance D between the sun and the Earth actually varies between 147 100 000 km (perihelion, which occurs each year around January 3) and 152 100 000 km (aphelion, which occurs around July 4). The moon's distance from the center of the Earth varies even more percentage-wise, between 357 300 km and 406 500 km (in the new moon position). Thus, the actual values of the angles θ_S and θ_M range as follows:

$$0.524^\circ \le \theta_S \le 0.542^\circ, \qquad 0.497^\circ \le \theta_M \le 0.567^\circ,$$

with mean values over time of $\theta_S = 0.533^\circ$ and $\theta_M = 0.527^\circ$. If the moon does not cover the sun completely, an annular or partial eclipse may result.

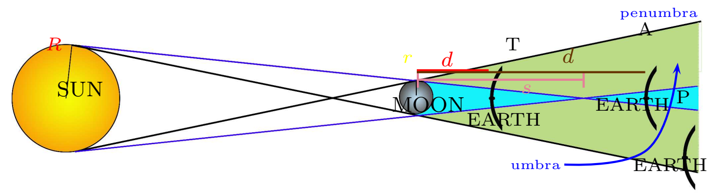


Figure 2

Three positions of Earth: Total (T), Annular (A), and Partial (P) eclipses.

Figure 2 illustrates the positions of the sun, moon, and Earth during solar eclipses, though not to scale. They are in line, with the moon in the new moon position. The very narrow ($\frac{1}{2}^\circ$ wide) shadow cone of the moon, the umbra, has its vertex at a distance s from the moon in the general direction of the Earth. We calculate, looking at Figure 2, that

$$\frac{r}{s} = \frac{R}{s + D - d}\ , \quad s = \frac{(D-d)r}{R-r} \approx \frac{Dr}{R} \approx \frac{D}{400.5};$$

hence

$$367\ 300 \text{ km} \le s \le 379\ 800 \text{ km}.$$

Thus, depending on the moon's distance, the Earth's surface can be on either side of the umbral vertex ("shadow boundary"). In position T, for $d < s$, a total solar eclipse occurs. If d is larger, $d > s$, the Earth is in position A, in the inverted umbral cone and the observer on Earth sees an annular eclipse; the moon obscures the central part of the sun but not the fringes. If the Earth is in the penumbra, a partial eclipse occurs.

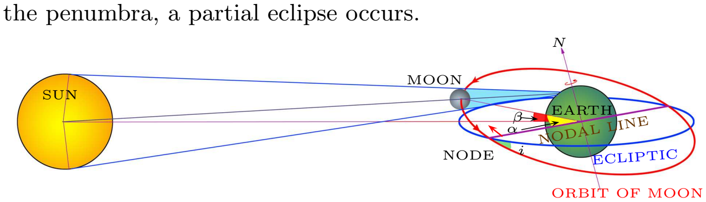


Figure 3

Sun, Earth, and moon: Angles α and β have to be small in the case of an eclipse (section of the Earth perpendicular to the ecliptic).

Shouldn't there then be a solar eclipse at every new moon, one every lunar month, after $T_{\text{lun}} = 29.531$ days (on average)? This would be the case if the moon were to orbit the Earth in the ecliptic (the plane of Earth's elliptical movement around the sun). However, the orbital planes are inclined to each another by $i = 5.14^\circ$. They intersect along the nodal line (see Figure 3). Solar eclipses occur when the moon crosses this line in the decreasing or increasing node (north to south or south to north), or is close to the node when the nodal line points toward the sun: eclipses require the moon to be in or near the ecliptic (thus the name!). If the moon is not close to either node, it will be too far north or south of the ecliptic; its narrow shadow cone will miss the Earth. Let β be the angle between the moon–Earth and sun–Earth lines in the new-moon position. For $\beta > 0.95°$, the center of the moon is more than

$$0.95° \times \frac{\pi}{180} \times 384\,000 \text{ km} \approx 6370 \text{ km} = R_0$$

away from the ecliptic. Because of the sun's large distance from the moon and the Earth, the central line of the moon's shadow is practically parallel to the ecliptic and will miss the Earth, which has the radius R_0. Therefore, no central (i.e., either total or annular) eclipse will occur. The angle β is directly related to the angle α between the moon–Earth line and the nodal line in the moon's orbital plane. Figure 4 and a little bit of spherical trigonometry give the formula

$$\sin \alpha = \frac{\sin \beta}{\sin i}.$$

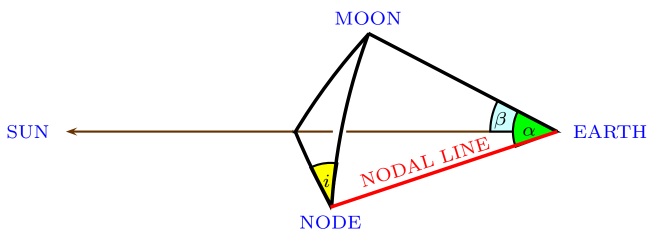


Figure 4

The spherical triangle: node, moon, sun (positions and angles as in Figure 3).

Therefore, if $|\alpha| \leq 10.5°$, then $|\beta|$ is $\leq 0.95°$, and a central eclipse will occur (slightly different values are possible since D and d vary). Similarly, for $|\beta| \leq 1.4°$ and $|\alpha| \leq 16°$, at least partial eclipses will occur. In this case, in the above argument the Earth's radius R_0 has to be replaced by R_0 plus the fairly large radius (more than 3000 km) of the penumbra, giving the larger bound for beta.

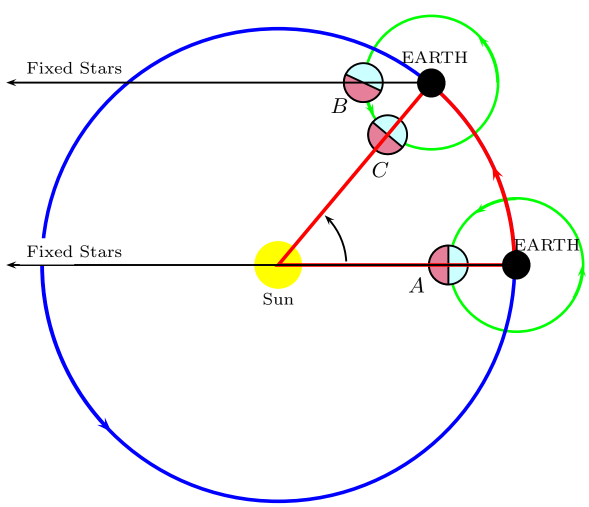


Figure 5

Lunar and sidereal months: (A) new moon, (B) one sidereal month later, and (C) one lunar month later (distances and angles are not to scale).

It takes the moon only $T_{\text{sid}} = 27.322$ days on average to orbit the Earth once with respect to the fixed stars. This is the sidereal month. During this time, the Earth progresses in its orbit around the sun. Hence, the moon needs more time to move from one new moon position to the next; this is the previously mentioned lunar month T_{lun} (see Figure 5). The sidereal and lunar months are related as follows:

$$\frac{1}{T_{\text{sid}}} = \frac{1}{T_{\text{lun}}} + \frac{1}{J},$$

where $J = 365.242$ days, the length of the (tropical) year.

The period of time the moon needs to move from one descending node to the next, the so-called draconic month $T_{\text{dr}} = 27.212$ days, is even shorter then the sidereal month since the nodal line rotates clockwise once every 18.62 years around the Earth, in a gyroscopic effect caused by the sun and Earth. So how often do eclipses occur? In one draconic month's orbit, four angular sectors of 10.5°, two on each side of the two nodes, are favourable for central eclipses if the moon is positioned there. Hence, on average, a total or annular solar eclipse will happen every

$$\frac{360°}{4 \times 10.5°} \times 27.21 \text{ days} = 233 \text{ days}$$

somewhere on Earth, which means 156 per century. As for (at least) partial eclipses, the frequency is one every

$$\frac{360°}{4 \times 16°} \times 27.21 \text{ days} = 153 \text{ days},$$

or 238 per century. These numbers agree with long-time statistics of solar eclipses. Since $\theta_M < \theta_S$ holds on average, annular eclipses slightly outnumber total eclipses: of those 156 central eclipses, about 65 are total, 78 are annular, and 13 are mixed. Total eclipses are more likely to occur in summer (June to August) since the sun is close to its aphelion and appears to us at the smallest possible angle. Annular eclipses dominate during the northern-hemisphere winter. Since the Earth's axis is tilted toward the sun during the summer, total eclipses are slightly more frequent in the northern hemisphere of the Earth than in the southern; the opposite holds true for annular eclipses.

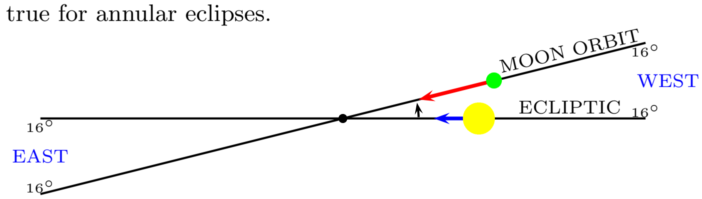


Figure 6

The "danger zone" of an eclipse near a node: the moon passing the sun near the descending node (view fixing the node).

From the Earth, let us look toward a node of the moon's orbit and the ecliptic, when the sun and the moon are in the ±16° sectors around the node. The (say) descending node points every

$$J_{\text{ecl}} = \frac{365.242}{1 + 1/18.62} \text{ days} = 346.62 \text{ days}$$

toward the sun: this is the ecliptic year. Here 365.242 days is the length of the (tropical) year. Therefore, the sun needs

$$\frac{2 \times 16°}{360°} \times 346.62 \text{ days} = 30.8 \text{ days}$$

to pass the "danger zone" of an eclipse. Since this is more than a lunar month, the moon will overtake the sun at least once, maybe twice, during this time. This results in one or two solar eclipses every half ecliptic year. We conclude, therefore, that every year there are at least two and at most five solar eclipses (total, annular or partial) per year *somewhere on Earth*. The fifth eclipse may occur in the "leftover" 18.62 days $(365.24 - 346.62)$, although this is a rather rare event, happening the next time in the year 2206. Typically, one total and one annular eclipse or two or four partial eclipses occur in a given year; this being the case, for example, in 2002, 2004, and 2000, respectively.

At a *specific location*, say Edmonton, Calgary, or Vancouver, a total solar eclipse is quite rare, with one happening about every 390 years *on average*. This figure, however, is subject to large variations. For example, a coastal strip in Angola is the scene of two total solar eclipses in 2001 and 2002, whereas London did not experience any total solar eclipses between 878 A.D. and 1715 (when Halley produced the first eclipse map).

Total solar eclipses are favoured if the moon's distance d is small (close to its perigee) and the sun's distance D is large (close to its aphelion). A small sun is covered by a large moon. The time between two successive perigees of the moon (closest distance points) is the anomalistic month $T_{\text{an}} = 27.555$ days. It is larger than the draconic month since the perigee moves slowly in a counterclockwise direction under the influence of the sun, completing one rotation in 8.85 years. Interestingly, there are good rational approximations of the ratios of these different types of months: 223 lunar months are almost the same as 242 draconic months, 19 ecliptic years, or 239 anomalistic months. This is the Saros period S:

$$\begin{aligned} S = 223\ T_{\text{lun}} &= 6585.32 \text{ days}, \\ 242\ T_{\text{dr}} &= 6585.36 \text{ days}, \\ 19\ J_{\text{ecl}} &= 6585.78 \text{ days}, \\ 239\ T_{\text{an}} &= 6585.54 \text{ days}. \end{aligned}$$

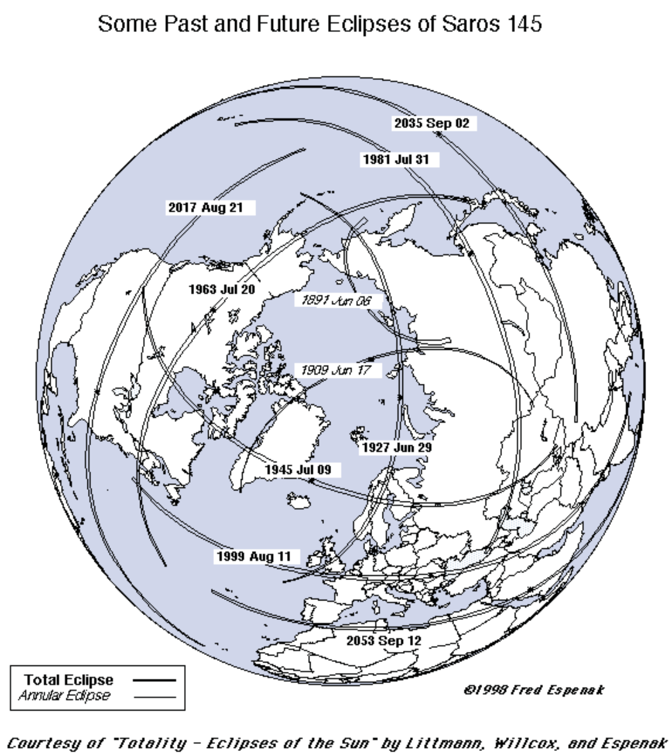
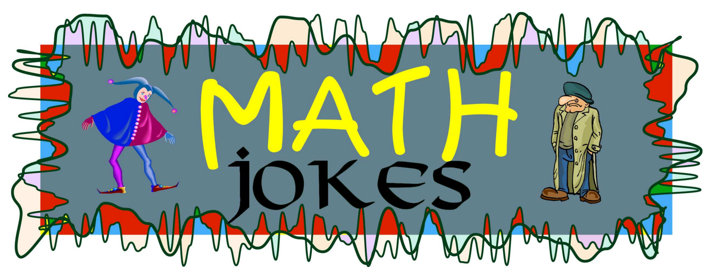


Figure 7

Paths of nine successive total solar eclipses in the Saros series 145.

In years, S is 18 years plus $10\frac{1}{3}$ or $11\frac{1}{3}$ days, depending on the number of leap years during this time. Solar eclipses thus tend to repeat after the period S: the moon is again in the new moon position and in the same type of node (decreasing/increasing) pointing toward the sun, the Earth–sun distance is about the same after almost 18 years and the Earth–moon distance is very similar after 239 anomalistic months. This means that the type of eclipse typically is the same (annular, total, or partial): the latitude of the tracks of central eclipses on Earth is only slightly shifted north/south, but the longitude of the next eclipse in a Saros cycle is $0.32 \times 360^\circ = 115^\circ$ further to the west. After three such periods, 54 years and one month, a very similar eclipse will reappear in almost the same longitude, latitude being somewhat shifted north or south. Since the above periods do not coincide perfectly, any such Saros series of eclipses eventually ends after about 72 eclipses, which move slowly in about 1 300 years from the south to the north pole or vice-versa. Figure 7 shows the paths of nine successive total solar eclipses in the Saros series 145; the calculations were done by Fred Espenak (NASA/GSFC).

So when is the next total solar eclipse in Western Canada? You will have to wait until the afternoon of August 22, 2044, when an eclipse will be experienced in Edmonton and Calgary, with the sun being at an altitude of 10° above the horizon. This eclipse will have predecessors in its Saros series in 2008 in Siberia/China (around the time of the Beijing Olympic Games) and 2026 in the North Atlantic and Spain. The last total eclipse in Edmonton was in 1433—a gap of more than 600 years, although in 1869 there was one visible just south of the city area. Banff, Calgary, Ohio, and Virginia will experience another total eclipse in September 2099. This one will have its Saros predecessors in 2045 in the U.S., tracking from Oregon to Florida, and in 2009 in Shanghai, China and in the western Pacific. The 2009 occurrence will be the most massive total eclipse of the 21$^{\text{st}}$ century, with a totality phase lasting up to $6\frac{2}{3}$ minutes. The 1999 eclipse in central Europe will be followed in its Saros series 145 by a total eclipse in August 2017 in the U.S., tracking from Oregon to South Carolina. This is the next total solar eclipse in the U.S.; its totality phase will last up to $2\frac{1}{2}$ minutes.



A mathematician has spent years trying to prove the Riemann hypothesis, without success. Finally, he decides to sell his soul to the devil in exchange for a proof. The devil promises to deliver a proof within four weeks.

Four weeks pass, but nothing happens. Half a year later, the devil shows up again—in a rather gloomy mood.

"I'm sorry," he says. "I couldn't prove the Riemann hypothesis either. " But"—and his face lightens up—"I think I found a really interesting lemma. . . "

"The number you have dialed is imaginary. Please, rotate your phone by 90 degrees and try again. . . "