The Box Principle
Dragos Hrimiuc

There are different versions of the Box Principle (or Pigeonhole Principle). Essentially it says:

If \( mn + 1 \) balls are distributed in \( n \) boxes, then at least one box has more than \( m \) balls.

We can reformulate this principle into a slightly more general form:

If \( mn + 1 \) balls are distributed in \( n \) boxes, then at least one box has more than one ball.

We consider the following integers:

\[
\begin{align*}
1 &= a_1, \\
2 &= a_1 + a_2, \\
3 &= a_1 + a_2 + a_3, \\
& \vdots \\
mn &= a_1 + \ldots + a_{mn}.
\end{align*}
\]

If any of these integers is divisible by \( n \), then the proof is done. Otherwise, if we divide these integers by \( n \) the remainders can be 1, 2, \ldots, \( n-1 \).

Take \( n \) boxes labeled from 1 to \( n-1 \). Now divide \( s_k \) by \( n \) for each \( k = 1, 2, \ldots, n \), find the remainder, say \( r \), and then put \( s_k \) in the box \( r \). Since there are only \( n-1 \) such boxes and \( n \) numbers, at least two of the sums, say \( s_p \) and \( s_q \) with \( p < q \), will be in the same box.

In this case \( s_q - s_p = a_{p+1} + a_{p+2} + \ldots + a_q \) is divisible by \( n \).

Let us now solve some typical questions by using the Box Principle. In this way you will be familiarized with the method and this will be useful for solving future problems.

**Problem 1**

Prove that in any group of five people there are two who have an identical number of friends within the group.

**Solution:**

Take 5 boxes labeled 0, 1, 2, 3, 4. If a person has 0 friends put that person in box 0, if he has 1 friend put that person in box 1 and so on. Remark that the box 0 and 4 cannot be simultaneously occupied. Thus we have 5 persons and at most 4 occupied boxes. Therefore in at least one box there are two persons.

**Problem 2**

Given 2000 integers show that two of them can be chosen such that their difference is divisible by 1999.

**Solution:**

When we divide a number by 1999 then the remainder can be 0, 1, \ldots, 1998. Take 1999 boxes numbered 0, 1, \ldots, 1998. Pick up a number from those 2000 and divide it by 1999. If the remainder is \( i \) put it in the box \( i \).

Since we have 2000 numbers and only 1999 boxes at least one box contains two numbers. Thus we get at least two numbers that provide the same remainder if we divide them by 1999. Then the difference of these two numbers is divisible by 1999.

**Problem 3**

Let \( a_1, a_2, \ldots, a_n \) be \( n \) integers. Prove that we can choose a subset of these numbers such that their sum is divisible by \( n \).

**Solution:**

We consider the following integers:

\[
\begin{align*}
1 &= a_1, \\
2 &= a_1 + a_2, \\
3 &= a_1 + a_2 + a_3, \\
& \vdots \\
n &= a_1 + \ldots + a_n.
\end{align*}
\]

If any of these integers is divisible by \( n \), then the proof is done. Otherwise, if we divide these integers by \( n \) the remainders can be 0, \ldots, \( n-1 \).

Take 1999 boxes numbered from 0 to \( n-1 \). Now divide \( s_k \) by \( n \) for each \( k = 1, 2, \ldots, n \), find the remainder, say \( r \), and then put \( s_k \) in the box \( r \). Since there are only \( n-1 \) such boxes and \( n \) numbers, at least two of the sums, say \( s_p \) and \( s_q \) with \( p < q \), will be in the same box.

In this case \( s_q - s_p = a_{p+1} + a_{p+2} + \ldots + a_q \) is divisible by \( n \).

**Problem 4**

Prove that there is an integer whose decimal representation consists entirely of 1’s and that is divisible by 1999.

**Solution:**

Consider the integers

\[
\begin{align*}
1,11,\ldots, \underbrace{1_{1999} \ldots 1}_{1999 \text{ digits}}.
\end{align*}
\]

If we divide these integers by 1999, we get the remainders 0, 1, \ldots, 1998. If 0 occurs, the proof is finished. If not, at least two numbers have the same remainder (Box Principle) when they are divided by 1999. Therefore, their difference \( 111 \ldots 10 \ldots 00 \) will be divisible by 1999. Cancelling all zeros from the end, we get a number consisting of ones and divisible by \( n \).

**Problem 5**

201 points are selected inside a square of side 1. Prove
that there exists a disk of radius $1/14$ that covers at least three points.

**Solution:**

If we subdivide the square into 100 small squares of side $1/10$, by the Box Principle there will be a square that contains at least 3 points. The smallest circle containing this square has radius $\sqrt{2}/20$. Since $\sqrt{2}/20 < 1/14$, the circle with radius $1/14$ centered at the center of the square will cover the entire square.

Now, try yourself to solve the following problems by using the Box Principle:

1. The Empire State Building has 102 floors. Suppose that an elevator stops 52 times as it descends from the top floor. Show that it stops at two floors whose sum is 102.

2. Twenty-eight points are selected inside a cube with edge 1. Show that there are at least two points that are separated by a distance not greater than $\sqrt{3}/3$.

**Math Jokes**

The guy gets on a bus and starts threatening everybody: “I’ll integrate you! I’ll differentiate you!!!” So everybody gets scared and runs away. Only one person stays. The guy comes up to him and says: “Aren’t you scared, I’ll integrate you, I’ll differentiate you!!” And the other guy says; “No, I’m not scared, I’m e$^x$.”

A shoeseller meets a mathematician and complains that he does not know what size shoes to buy. “No problem,” says the mathematician, “there is a simple equation for that,” and he shows him the Gaussian normal distribution. The shoeseller stares some time at the equation and asks, “What is that symbol?”

“That is the Greek letter pi.”

“What is pi?” “That is the ratio between the circumference and the diameter of a circle.”

Upon this the shoeseller cries out: “What does a circle have to do with shoes?!?”

**Diploma Exams Around the World**

Wieslaw Krawcewicz

It is interesting to examine the standards of high school math education in countries other than Canada and the U.S. See if you can solve these mathematics exam problems from other parts of the world. All the problems discussed in this section come from actual Diploma Exams that were given in some European or Asian countries in recent years.

**Problem 1:** Solve the following equation

\[
x^3 + 4x^2 + 8x + \frac{1}{x^3} + \frac{4}{x^2} + \frac{8}{x} = 70.
\]

**Solution:** Before attempting to solve this problem it is appropriate to rearrange the expression (1) in the following form

\[
\left(x^3 + \frac{1}{x^3}\right) + 4 \left(x^2 + \frac{1}{x^2}\right) + 8 \left(x + \frac{1}{x}\right) = 70.
\]

Notice that the terms containing the second and third powers can be written using the standard procedure of completing the square and cube. More precisely, it is possible to use the well-known formulae

\[(a + b)^2 = x^2 + 2ab + b^2\]

and

\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]

to rewrite Eq. (2) as

\[
\left(x + \frac{1}{x}\right)^3 - 3 \left(x + \frac{1}{x}\right) + 4 \left(x + \frac{1}{x}\right)^2 - 2 \]

\[+ 8 \left(x + \frac{1}{x}\right) = 70,
\]

so Eq. (1) is equivalent to

\[
\left(x + \frac{1}{x}\right)^3 + 4 \left(x + \frac{1}{x}\right)^2 + 5 \left(x + \frac{1}{x}\right) = 78.
\]

Next, we apply the substitution

\[X = x + \frac{1}{x},\]