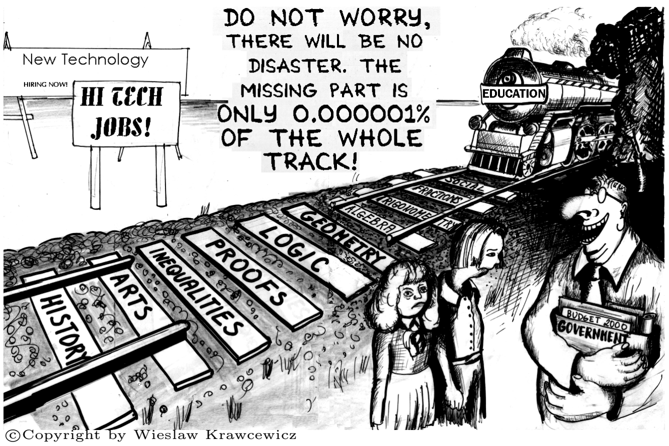
61 on the board. *"Sir,"* said another student, *" it should be sixty-nine."* *"Come on gentlemen, it can't be both,"* Kummer exclaimed, *"It must be one or the other."*



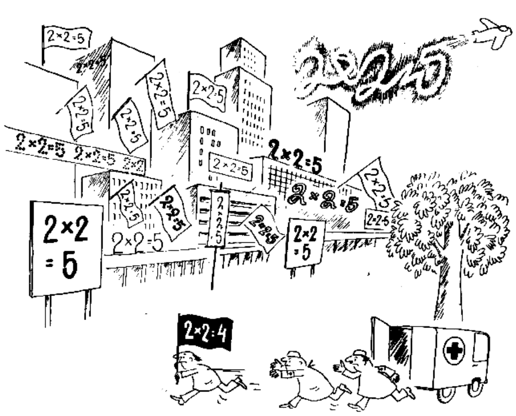
My geometry teacher was sometimes acute, and sometimes obtuse, but always, he was right.

"I do not feel obliged to believe that the same God who has endowed us with sense, reason, and intellect has intended us to forego their use." Galileo Galilei

"It is a miracle that curiosity survives formal education." Albert Einstein

A retired mathematician took up gardening, and is now growing carrots with square roots. (Zdislav V. Kovarik)

The retired mathematicians house was called aftermath. (Brian Skinner)



Einstein was feeling gloomy. A friend asked him, "*What's the matter?*" Einstein replied: *"My wife just doesn't understand me!"* (David Seppala)

"How do I love thee? Let me count the ways!" by Laurent W. Marcoux

Remember when you were young, and you fell in love for the first time? There you were, double original burger combo in hand, wiping the mustard stains from your sweetheart's chin, when the object of your affections (hereafter referred to as the OOYA) uttered the magic words:

"I kinda love you, I guess."

You immediately replied, "I love you more."

Not to be outdone, the OOYA countered with, "No, I love you twice as much."

Predictably, you responded "No, no, I love you ten times as much as that." (**Editor's note:** *Could happen!*)

Triumphantly, the OOYA slammed a fist on the table and exclaimed: "No way, I love you infinity times more than that!"

Oh oh. You started to sweat. Beads of perspiration forming on your brow were somewhat less than cool. How do you top that? What's bigger than infinity? A nervous smile betrayed your anxiety, as your teeth chattered and an uncontrollable twitch set your whole body in break dance motion. Maybe you shouldn't have said "Ok, let me get back to you on this," but who could blame you? What could you have done?

Enter mathematics laughing, stage right.

Of course, you always knew math was great as a breakfast supplement, but who would have thought that it could come to the rescue in your most desperate hour? (Ok, other than maybe Euler, Leibniz and Weierstraß.) Why is this? Well, because with the grecian formula of mathematics gently massaging the grey regions of your cerebral cortex, you *could say*: "But *which infinity* do you mean? *Countably infinite*?" Of course, you'd need to know that there was more than one infinity. And to know that, you'd need to know how to compare two different infinities. And that, you'll remember, is why you came to me in the first place.

Alright then. Let's not be overambitious in our first steps into the great big world of infinite sets. Being humble, gentle souls, let's try to decide how to compare two regular numbers, shall we? For instance, how do we know that 700 is bigger than 400? (**Editor's note:** 700 *is* bigger than 400, by the way.) A mathematician's tools are:

- their off-beat good looks
- logic

Since your own good looks are what got you into this mess, let's try to appeal to logic to get you out.

Let's suppose that you are at a dance. There are Guys, there are Girls, there's Alanis Morrissette blaring on the stereo. Not her CD, mind you, the *real* Alanis Morrisette. Just your luck. You're a Will Smith fan, and you're jiggy with that. But we digress. There are Guys, lots of them. There are Girls, lots of them. They're dancing around, and moving, and they're impossible to count, mostly because you keep forgetting if you've counted those two goofy looking Guys standing in the corner. But maybe you want to know if there are more Guys than Girls. What to do? What to do?

It's time for one of those breathtaking moments of mathematical inspiration. Here it comes. It's a good one. Wait for it! Ok, how about this? You start to **pair them up**. That's it. C'est tout. Das ist alles. Eso es todo. Who'da thunk it? You tell each Girl to choose 1 (and ONLY 1) Guy to dance with. If those two goofy Guys, or heck, if any other Guys are left debating the finer merits of Pepsi vs. Coke, there must have been more Guys than Girls. On the other hand, if there are still Girls standing about talking about how the Atlanta Falcons blew it bigtime in the Superbowl, there must have been more Girls. Somehow, this is too simple, n'est-ce pas? But it works. In fact, it works so well, we're going to milk the living Begeezuz out of it. What we're going to do next is to steal...make that BORROW this idea to help us compare two *infinite sets*.

Consider the *natural numbers* $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Being very considerate, consider also the *even* natural numbers, $\mathbb{E} = \{2, 4, 6, 8, \ldots\}$. Most of us would agree that there are infinitely many elements in each set. But which is bigger? To a mathematician, the trick is to take both sets to a dance. Indeed, we are mean, lean dancing machines. You may have noticed what snappy dressers we tend to be. Then again, maybe not.

So, suppose we have infinitely many Guys, each wearing a T-shirt with a natural number on it. Snazzy, eh? No two Guys are allowed to have the same number. Suppose we have infinitely many Girls, each wearing a T-shirt with an *even* natural number on it. Sounds like a great party already, you're thinking? No two Girls are allowed to have the same number.

The Question is:

Is it *possible* to have every Guy dancing with *exactly* one Girl, and every Girl dancing with *exactly* one Guy, with no Guys or Girls sitting alone by the coke machine?

Here's one way: Suppose, just suppose that we pair them up like this:

Guy No.	1	2	3	4	5	6	7	8	9	...
Girl No.	2	4	6	8	10	12	14	16	18	...

Of course, Guy 1 and Guy 2 could switch partners. That would still work. No one asked you if there is only *one* way of pairing them up. Nope. The question was, is it *possible* to pair them up?

BUT, you say, wait a second! What if we pair them up like THIS????

Guy No.	2	4	6	8	10	12	14	16	18	...
Girl No.	2	4	6	8	10	12	14	16	18	...

Then, oh sure, all of the Girls are dancing, but there's lots of Guys left over!!

Well, do I look particularly nervous to you? No. Is there a reason? Yes. I never asked if *every* pairing would work. I only asked if it is *possible* to *find* a pairing! This isn't one of them, but we found one, in fact two, different pairings above. That's more than enough.

For a mathematician, the fact that we can pair off all of the even numbers with all of the natural numbers means that the two sets, even though they are infinite, must be the same size. But this doesn't sound too literate, so instead we impress the media types by saying that the sets $\mathbb{N}$ and $\mathbb{E}$ have the same *cardinality*. We call the cardinality of $\mathbb{N}$ (or of $\mathbb{E}$, for that matter) ALEPH NOUGHT, and we write $|\mathbb{N}| = \aleph_0$. Unless I've been lied to all of my life by my teachers and colleagues, $\aleph$ is the first letter of the Hebrew alphabet. Whenever we can pair up the elements of a set $\mathbb{X}$ with $\mathbb{N}$ like we just did with $\mathbb{E}$, we say that $\mathbb{X}$ is *denumerable* or *countable*.[2] That is because we can use $\mathbb{N}$ to help us "count" the elements of $\mathbb{X}$. Still with me?

Let's look at the *integers*, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$. How big is $\mathbb{Z}$? Here we go again. I'll pair them up this way:

Guy No.	1	2	3	4	5	6	7	8	9	...
Girl No.	0	1	-1	2	-2	3	-3	4	-4	...

Hopefully you'll agree that each Girl has one Guy, each Guy has one Girl, and everyone is happily dancing to the Best of Billy Idol (ask your grandpa who he was). Cool. We've just seen that $\mathbb{Z}$ is countable. It has cardinality $\aleph_0$.

Oooh, ooooh, I've got one! The *rationals*, $\mathbb{Q}$!!!! $\mathbb{Q}$ is just the set of all fractions, so

$$\mathbb{Q} = \left\{ \frac{p}{q} : \ p, q \in \mathbb{Z}, \ q \neq 0 \right\}.$$

[2]Technically, *countable* means either *denumerable* or *finite*.

There are *infinitely many* rationals between each pair of integers, right? For example, between 1 and 2 you have er... hold on... oh yeah, you have: $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}$, etc.

Again: between two SINGLE integers, there are INFINITELY MANY rationals! Certainly I'm not going to tell you that $\mathbb{Q}$ and $\mathbb{N}$ are the same size! Well, to make a long story short, *yes I am*!

So, I've got to go dancing again. This time, as you might imagine, choosing the pairs is more delicate. (Like, maybe it's a slow dance and certain Girls prefer certain Guys or something. Hmm, maybe not.) Here's how I'll write out the combinations this time. I'll write them as ordered pairs: the first coordinate of my ordered pair will be the "Guy Number" (the Guys are playing the role of $\mathbb{N}$ in this scenario), and the second coordinate is the "Girl Number" (the Girls are playing the role of $\mathbb{Q}$, and doing a fine job of it, I might add). So, for example, the ordered pair $(14, \frac{3}{2})$ means that Guy 14 is dancing with Girl $\frac{3}{2}$. Keeping this in mind, here is my pairing:

$$
\begin{array}{llllll}
(1,0) & & & & & \\
(2,\frac{1}{1}) & (3,\frac{2}{1}) & (7,\frac{3}{1}) & (8,\frac{4}{1}) & (16,\frac{5}{1}) & \dots \\
(4,\frac{-1}{1}) & (6,\frac{-2}{1}) & (9,\frac{-3}{1}) & (15,\frac{-4}{1}) & (18,\frac{-5}{1}) & \dots \\
(5,\frac{1}{2}) & (10,\frac{2}{2}) & (14,\frac{3}{2}) & (19,\frac{4}{2}) & (27,\frac{5}{2}) & \dots \\
(11,\frac{-1}{2}) & (13,\frac{-2}{2}) & (20,\frac{-3}{2}) & (26,\frac{-4}{2}) & (33,\frac{-5}{2}) & \dots \\
(12,\frac{1}{3}) & (21,\frac{2}{3}) & (24,\frac{3}{3}) & (34,\frac{4}{3}) & (42,\frac{5}{3}) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

By drawing a line from Guy 1 to Guy 2 to Guy 3 to Guy 4 and so on, you should be able to see the pattern for choosing pairs. Of course, it's harder than before, but it is still "doable" (—that's a word, isn't it?), and after all, the rationals are a pretty complicated set of numbers. So, $\mathbb{Q}$ *is* countable. Every natural number is dancing with some rational and vice-versa, and no one is left over.

Hmm. One thing you may have noticed, if you are really perspicacious (and there are treatments for this nowadays). You may have noticed that the rational number $\frac{2}{2}$ is the same as the rational number $\frac{1}{1}$, or $\frac{3}{3}$, or $\frac{4}{4}$ for that matter. In other words, Girl $\frac{1}{1}$ is dancing with Guys 2, 10, 14 and a whole lot more! That's a technicality we can get around by simply not writing down "$\frac{2}{2}$" if it already comes up before as "$\frac{1}{1}$." Similarly, we wouldn't write down "$\frac{3}{3}$," or "$\frac{-12}{-12}$." I *would have* done that, honest, but the list gets ugly to write. Of course, you might be tempted to conclude that jeez, there must be more Guys that Girls, since every Girl is dancing, and in fact, the Girls even have more than one partner, although each Guys are only dancing with a single Girl!

There are people who worry about these things. Some of them come up with ideas. Good ideas. So good, in fact, we call these ideas "Theorems." Here is a Theorem due to two Guys called **Schröder** and **Bernstein**. They should not be confused with the piano-playing kid in the Peanuts comic strip and the former orchestra conductor, although it is tempting to do so. Here is one way of interpreting their Theorem—probably a way they never thought of.

Schröder-Bernstein *Suppose you have a bunch of Guys and a bunch of Girls together in this HUGE room. We won't even pretend to say what we mean by a "bunch." Ok, suppose you are able to find* some *way of getting every Girl to dance with one, or even more than one Guy. Maybe some Guys aren't even dancing at this point. Maybe they're eating potato chips and drinking carrot juice, nectar of the gods of rabbits.*

Suppose that in the NEXT song, you find a NEW way of getting every Guy to dance with one, or even more than one Girl. This time, maybe some Girls are hitting the taco bar.

Then we, Mssrs. Schröder and Bernstein, GUARANTEE (and trust us, we're doctors) you that there is some THIRD way of pairing them up in the third dance so that each Guy has one Girl, each Girl has one Guy, and EVERYONE is dancing.[3]

This is truly marvelous. Why? Think about it. When we had the natural numbers dancing with the rationals, each Girl had (at least one) partner. By Schröder–Bernstein, we're half way there. If, in the second song, we pair up Guy 1 with Girl 1, Guy 2 with Girl 2, etc., then every Guy is dancing. That's the other half. Although these gentlemen refuse to tell us how to do it, they nevertheless GUARANTEED that there is some way of pairing them up so that EVERYONE is dancing with exactly one partner. That means that $\mathbb{Q}$ is countable. Cool.

Let's review a bit. The infinite sets $\mathbb{N}$, $\mathbb{E}$, $\mathbb{Z}$ and $\mathbb{Q}$ are all countable. They are all "the same size." This isn't helping. Your love life is at stake, the OOYA has just told you they love you infinitely many times more than you love them, and it seems as though infinity, while it can

[3] For the technically inclined, I agree that replacing injections by surjections may involve using the Axiom of Choice. Like you've never used the Axiom of Choice in your lives! Like, maybe every vector space has a basis, and you didn't use the Axiom of Choice to prove it. Sorry for the digression. In fact, I've just been informed by a Zermelo-Fraenkel-wise friend of mine (thanks, Ross) that in 1982, it was still an open question whether or not this version of the Schröder–Bernstein Theorem was equivalent to the Axiom of Choice. Anyhow, since it is common practice to use the Axiom of Choice (or one of its equivalent formulations, I don't even know why I brought it up).

wear lots of different disguises, can only be found in the "one size fits all" bin at your local department store.

Here's the advice you've been waiting for. Don't call Sue Johansen. Just tell the OOYA: "But I $\mathbb{R}$eally love you." This is going to melt the OOYA's heart. The reason?

Remember that the *real numbers* are just the numbers we can write as infinite decimals. Yeah, there are lots of these. Every natural number can be written as a decimal, for instance $4 = 4.000\ldots$. You may already know that every rational number (i.e. every fraction) can be written as a decimal with a *repeating* term at the end: for example, $1/7 = .142857142857142857\ldots$. But there are others, like $\pi = 3.141592653\ldots$, that are still real numbers, but can't be written as a fraction no matter how hard you try. They are real numbers, but they are not rational. We call 'em, are you ready? ...*irrational numbers*. (Well, what on earth would you have called them?) Let us write $\mathbb{R}$ for real numbers. Nifty notation, eh? How big is $\mathbb{R}$? Infinite? Sure, it contains $\mathbb{N}$, and $\mathbb{N}$ is infinite. Same size as $\mathbb{N}$? Let's think. Hmmm, not so obvious...not obvious that it's not, either ...hmmm ...er ...uh ...hmmmm ...

Fear not. We can do this. The idea is cool. $\mathbb{R}$eal cool. I shouldn't even be showing you something so cool at your age, but I can't help myself. It's weird though, at least the first time you see something like this. $\mathbb{R}$eal weird. That's what makes it so cool. Watch.

We're going to see that there are so many real numbers even between 0 and 1 (and we'll let the Girls wear a T-shirt with a real number between 0 and 1 on the back) that no matter how we try to pair up the Guys (each wearing a T-shirt with a natural number on it) with the Girls, there'll always be some Girl who's not dancing. How? Here goes:

Suppose I'm wrong. That is, suppose one can find a Guy for each Girl. Let's check out the pairing (it's good to be nosy when you're in math). First, let's try an example to get a feeling for what's going to go wrong.

Guy No.	Girl No.
1	.231678...
2	.114632...
3	.048175...
4	.011364...
$\vdots$	$\vdots$

I claim that some Girl is not dancing. Which Girl? Here's how I'll find her.

Look at Guy 1. The first digit of his partner's number is 2. I'll pick 8.

Look at Guy 2. The second digit of his partner's number is 1. I'll pick 8.

Look at Guy 3. The third digit of his partner's number is 8. I'll pick 4.

Look at Guy 4. The fourth digit of his partner's number is 3. I'll pick 8. And so on.

Look at Guy k. If the k^{th} digit of his partner's number is 8, I'll pick 4. Otherwise, I'll always pick 8.

In this case, my choice is $.8848\ldots$.

You might say: "how do you know she's not further down the list?" Go ahead and say it. I'll answer:

She's not Guy 1's partner. Her first digit is wrong.
She's not Guy 2's partner, her second digit is wrong.
She's not Guy 3's partner, her third digit is wrong.

You might say: "Ok, I'll just add her to my list." Go ahead and say it. I'll use the same trick to find a NEW Girl who's not dancing. The point is not that there is a *single* Girl who never dances, but rather that no matter how you try to pair them up, in any given dance, at least one Girl is warming the bench.

Ok. That works great for this example, but we really want to know that it will work for *any* pairing, not just a rearrangement of this particular one. We'll start with an arbitrary list:

Guy No.	Girl No.
1	$.a_{11}\, a_{12}\, a_{13}\, a_{14}\, \ldots$
2	$.a_{21}\, a_{22}\, a_{23}\, a_{24}\, \ldots$
3	$.a_{31}\, a_{32}\, a_{33}\, a_{34}\, \ldots$
4	$.a_{41}\, a_{42}\, a_{43}\, a_{44}\, \ldots$
$\vdots$	$\vdots$

Here, a_{ij} refers to the j^{th} digit in the number on the back of the T-shirt for the Girl dancing with the i^{th} Guy. (I hope you got that. Think about it for a while if you have to. It's worth the effort.)

Like the great Victor von Frankenstein before me, I'll build my lonely Girl as follows:

If $a_{kk} = 8$, I'll choose a number $b_k = 4$.
If $a_{kk} \neq 8$, I'll choose a number $b_k = 8$.

The Girl who's not dancing is the Girl wearing the number $.b_1b_2b_3b_4\ldots$ on her T-shirt. The argument is the same as before. She's not Guy 1's partner. Her first digit is wrong.

She's not Guy 2's partner, her second digit is wrong.
She's not Guy 3's partner, her third digit is wrong, etc.

She's NOT DANCING!!! (Who would dance with a Girl with so many wrong digits?)

So, no matter how hard we try, we can never get all of the Girls to dance, because there are just *too many Girls*. There are more Girls than Guys. Lots more. Infinitely many more! The Guys are surrounded!

The moral of the story is (as if a story like this deserves a moral), the cardinality of $\mathbb{R}$ is *greater* than the cardinality of $\mathbb{N}$. We say that $\mathbb{R}$ is *uncountable*. What is the

cardinality of $\mathbb{R}$? Who knows? It is big. We sometimes call it c. What we do know is that there are as many real numbers as there are *subsets* of the natural numbers. This is a huge set. But we don't know if there is any infinite cardinal between the size of $\mathbb{N}$ and c. That would rule. But this is still just what we needed. We have now shown that there is more than one "infinity." (In fact, there are infinitely many, but we'll leave this to another time with another OOYA.)

For the time being, when the OOYA says: "I love you infinitely many times more than that," just smile and say:

"Oh, you just love me a countable number of times more than that. My love is $\mathbb{R}$eal, c?"

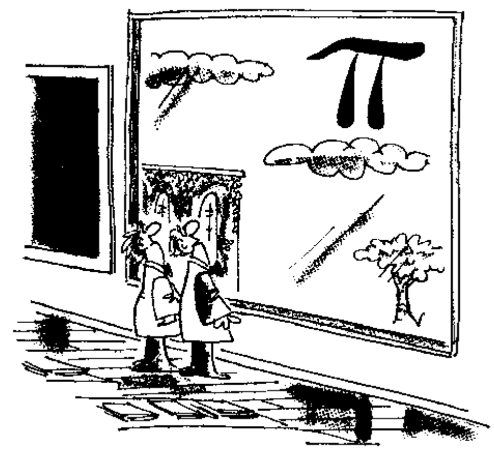
About the author. No one ever invites him to dances and he has no idea why. It's not like he talks about this stuff in public or anything. Ok, like, maybe a little. But only if someone asks him, like you did. I've got to face it. He just lives vicariously through cardinal numbers. You can send him an E-mail at: `L.Marcoux@ualberta.ca` . Check out his web page at: http://www.math.ualberta.ca/~lmarcoux/lmarcoux.html

Math Jokes

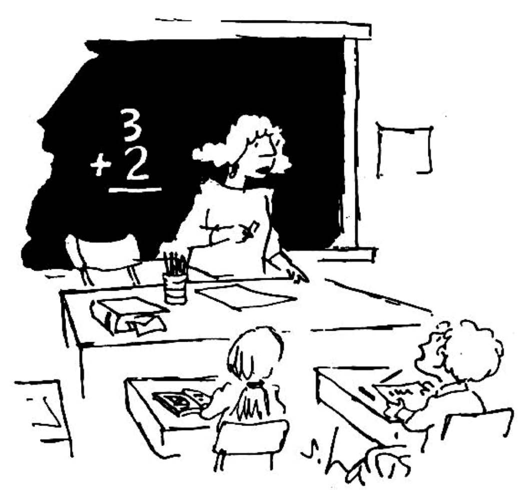
Weiner was in fact very absent minded. The following story is told about him: When they moved from Cambridge to Newton his wife, knowing that he would be absolutely useless on the move, packed him off to MIT while she directed the move. Since she was certain that he would forget that they had moved and where they had moved to, she wrote down the new address on a piece of paper, and gave it to him. Naturally, in the course of the day, an insight occurred to him. He reached in his pocket, found a piece of paper on which he furiously scribbled some notes, thought it over, decided there was a fallacy in his idea, and threw the piece of paper away. At the end of the day he went home (to the old address in Cambridge, of course). When he got there he realized that they had moved, that he had no idea where they had moved to, and that the piece of paper with the address was long gone. Fortunately inspiration struck. There was a young girl on the street and he conceived the idea of asking her where he had moved to, saying, "Excuse me, perhaps you know me. I'm Norbert Weiner and we've just moved. Would you know where we've moved to?" To which the young girl replied, "Yes daddy, mommy thought you would forget."

The capper to the story is that I asked his daughter (the girl in the story) about the truth of the story, many years later. She said that it wasn't quite true – that he never forgot who his children were! The rest of it, however, was pretty close to what actually happened....

I will never forget the day in statistics when, the Professor, who had all of the traditional looks of one (white hair, tweed jacket with leather elbow patches) was writing on the board $X_i \quad Y_j$; when one of the students asked, *"Don't you mean $X_j \quad Y_i$?"* The Prof looked at the board a bit, then erased the marks with his sleeve, and said; *"Yes, you are correct. Quite often I will say one thing, write another, and be thinking a third. What I am thinking is correct, and you will be tested on."* Every jaw in the classroom hit the floor! (Bert Tagge)



A researcher tried jalapenos on a stomach of an ulcer patient, and the ulcer went away. The researcher published an article "Jalapenos Cure Stomach Ulcers." The next patient subjected to the same treatment died. The researcher published a follow-up article "More Detailed Study Reveals That Jalapenos Cure 50% Of Stomach Ulcers." Zdislav V. Kovarik



"Do we need this even if we're not planning to go to college?"