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Group algebras of torsion groups and Lie nilpotence

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Abstract. Let $*$ be an involution of a group algebra FG induced by an involution of the group G. For $\operatorname{char} F \neq 2$, we classify the torsion groups G with no elements of order 2 whose Lie algebra of $*$-skew elements is nilpotent.

1 Introduction

Let F be a field and FG the group algebra of a group G over F. If $*$ is an involution of FG then the set of skew elements $FG^- = \{x \in FG \mid x^* = -x\}$ is a Lie algebra. Here we are interested in classifying the groups G for which such an algebra is nilpotent. We shall assume throughout that $\operatorname{char} F \neq 2$. A natural involution of FG to consider is the so-called classical involution, obtained by linearly extending the group involution $g \to g^{-1}$ to FG. For this involution the problem has been completed settled in [4] for groups with no elements of order 2 and in [5] for arbitrary groups.

We should mention that if we regard FG as a Lie algebra under the usual Lie bracket, then from results in [9] the algebra FG is nilpotent if and only if either $\operatorname{char} F = 0$ and G is abelian or $\operatorname{char} F = p > 0$ and G is a nilpotent group whose derived group is a finite p-group. The same classification holds if G has no elements of order 2 and we only impose that FG^- is Lie nilpotent under the classical involution; see [4].

Here we try to extend this result to an involution of FG obtained as a linear extension of a group involution of G. We shall classify the groups G for which FG^- is Lie nilpotent when G is a torsion group and has no elements of order 2. It turns out that the conclusion is much more involved than for the classical involution. Our main result is the following.

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Theorem 1.1. *Let F be a field of characteristic $p \neq 2$ and G a torsion group with no elements of order 2. Let $*$ be an involution on FG induced by an involution of G. Then the Lie algebra FG^- is nilpotent if and only if FG is Lie nilpotent or* $\operatorname{char} F = p > 2$ *and the following conditions hold*:

(i) *the set P of p-elements in G is a subgroup*;

(ii) *$*$ is trivial on G/P*;

(iii) *there exist normal $*$-invariant subgroups A and B with $B \leqslant A$ such that B is a finite central p-subgroup of G and A/B is central in G/B with both G/A and $\{a \in A \mid aa^* \in B\}$ finite.*

Remark 1.2. It turns out that if the conditions of the above theorem are satisfied then G is a p-abelian group as is pointed out in the proof.

2 Preliminaries

Throughout this paper F will be a field of characteristic different from 2 and $*$ will denote an involution of FG obtained as a linear extension of an involution of G.

Recall that for a prime p, an element $x \in G$ is called a p-element if its order is a power of p. We write

$$P = \{x \in G \mid x \text{ is a } p\text{-element}\} \quad \text{and} \quad G^+ = \{g \in G \mid g^* = g\}.$$

Also, a group G is said to be p-abelian if G', the commutator group of G, is a finite p-group. We make the convention that a 0-abelian group is abelian.

We record some results that we shall use throughout the paper. The first result is due to Passi, Passman and Sehgal [9].

Theorem 2.1. *Suppose that* $\operatorname{char} F = p \geqslant 0$. *The group algebra FG is nilpotent as a Lie algebra if and only if G is a nilpotent and p-abelian group.*

If R is any ring with involution $*$, we denote by

$$R^- = \{x \in R \mid x^* = -x\}$$

its set of skew elements. Also, we write the Lie bracket $[x, y] = xy - yx$. Recall that if $x_1, \dots, x_4$ are non-commuting variables,

$$\mathrm{St}_4(x_1, \dots, x_4) = \sum_{\sigma \in S_4} (\operatorname{sgn} \sigma) x_{\sigma(1)} \dots x_{\sigma(4)}$$

is the standard identity of degree 4.

Lemma 2.2 ([4]). *Let R be a semiprime ring with involution such that $2R = R$. If the Lie algebra R^- is nilpotent, then R^- is abelian, i.e., $[R^-, R^-] = 0$ and R satisfies* St_4, *the standard identity in four variables.*

We denote by $\langle S \rangle$ the subgroup generated by a subset S of a group. The following result was proved by Broche, Jespers, Polcino Milies and Ruiz [2].

Theorem 2.3. *Let* $\operatorname{char} F \neq 2$ *and let $*$ be an involution of G linearly extended to FG. Then FG^- is abelian if and only if either G is abelian or one of the following conditions holds*:

(1) $K = \langle g \in G \mid g \notin G^+ \rangle$ *is abelian*;

(2) *G contains an abelian subgroup of index 2 that is contained in G^+*;

(3) $\operatorname{char} F = 3$, $|G'| = 3$, $G/G' = (G/G')^+$ *and* $g^3 \in G^+$, *for all* $g \in G$.

Suppose that G is a torsion group with no elements of order 2 and FG is a semiprime algebra. If FG^- is Lie nilpotent, then FG^- is commutative by Lemma 2.2, and so, by Theorem 2.3 either G is abelian or one of the above three conditions holds. If (1) is satisfied, then $[G : K] \leqslant 2$ (see [7, Lemma 2.3]), hence the absence of elements of order 2 rules out (1) and (2). Also (3) is not possible as FG is semiprime. In conclusion we have the following.

Corollary 2.4. *Assume that FG is semiprime and G is a torsion group with no elements of order 2. Then FG^- is Lie nilpotent if and only if G is abelian.*

Another important fact that we shall use is that the p-elements form a subgroup. This is the content of the following.

Lemma 2.5. *Let* $\operatorname{char} F > 2$ *and suppose that G is a locally finite group. If FG^- is Lie nilpotent then P is a subgroup and, in case G has no elements of order 2, G/P is abelian.*

Proof. Let $g, h \in P$ and $H = \langle g, h, g^*, h^* \rangle$. If J is the Jacobson radical of FH, then $R = FH/J$ is a semisimple algebra with induced involution and the Lie algebra R^- is nilpotent. Being a finite-dimensional semisimple algebra, R is a finite direct sum of simple algebras A_i. By Lemma 2.2, R satisfies St_4, hence each A_i satisfies St_4. Now, it is well known that a simple algebra of dimension m^2 over its center satisfies no polynomial identity of degree less than $2m$. Hence we deduce that R is isomorphic to a direct sum of simple algebras of dimension at most 4 over their center. But then by [3, Lemma 2.6] or [8], the p-elements of H form a subgroup. In particular $gh \in P$ and P is a subgroup.

Now, since $F(G/P)$ is a semiprime algebra with $F(G/P)^-$ Lie nilpotent, if G has no elements of order 2 then G/P is abelian by Corollary 2.4. □

Let Z denote the center of the group G. In [4, Corollary] it was proved that if $*$ is the classical involution and Z^2 is infinite, and if FG^- is Lie nilpotent of index n, then also FG is Lie nilpotent of index n. The proof of that result can be adapted, with the due changes, to our situation and we get the following result that we state without proof.

Lemma 2.6. *Let Z be the center of G and suppose that $\tilde{Z} = \{z^{-1}z^* \mid z \in Z\}$ is infinite. If FG^- is Lie nilpotent, then so is FG.*

Another tool we shall need is the following lemma proved in [3, Lemma 2.9].

Lemma 2.7. *Assume that A is an abelian group with no elements of order 2 and let $* : A \to A$ be an automorphism of order 2. Then*

$$A^2 \subseteq A_1 \times A_2,$$

where

$$A_1 = \{a \in A \mid a^* = a\} \quad \text{and} \quad A_2 = \{a \in A \mid a^* = a^{-1}\}.$$

Moreover if A is a torsion group, then $A = A_1 \times A_2$.

Proof. If $b = a^2 \in A^2$, we can write

$$b = (aa^*)(a(a^*)^{-1})$$

with $aa^* \in A_1$, $a(a^*)^{-1} \in A_2$. This gives the required decomposition. □

In the sequel we shall use the notation for A_1 and A_2 without mention. A first application of the decomposition given in the previous lemma is given in the following.

Lemma 2.8. *Let A be a $*$-invariant torsion abelian normal subgroup of G, with no elements of order 2.*

(1) *If $x \in G \setminus A$ is such that $x^* = x^{-1}c$ with $c \in A$, then there exists a symmetric element $b \in A$ such that $(xb)^* = (xb)^{-1}$.*

(2) *If $x \in G \setminus A$ is such that $x^* = xc$ with $c \in A$ and $x^* = y^{-1}xy$, for some $y \in A$, then there exists a symmetric element $b \in A$ such that $(xb)^* = xb$.*

Proof. Write $A = A_1 \times A_2$ as in the previous lemma, and let $x \in G \setminus A$ be such that $x^* = x^{-1}c$ with $c \in A$. Notice that $xx^* = c$ is in A and is symmetric, so $c \in A_1$. Also $x^{-1}cx \in A_1$. As A_1 has no elements of order 2, we can find $b \in A_1$ such that $b^2 = x^{-1}c^{-1}x$. This means that $b^{-1}x^{-1} = bx^{-1}c$ and thus $(xb)^{-1} = bx^{-1}c = (xb)^*$, as desired. This proves (1).

Now suppose that $x \in G \setminus A$ is such that $x^* = xc$ with $c \in A$ and $x^* = y^{-1}xy$, for some $y \in A$. Write $y = y_1y_2$ where $y_1^* = y_1$, $y_2^* = y_2^{-1}$. Since

$$x = x^{**} = (y^{-1}xy)^* = y^*y^{-1}xy(y^{-1})^*,$$

it follows that $(y^*y^{-1}, x) = 1$. Since $y^*y^{-1} = y_2^{-1}y_1y_2^{-1}y_1^{-1} = (y_2^{-1})^2$, we conclude that $(y_2, x) = 1$. Thus we can write $x^* = y_1^{-1}xy_1 = xc$ and $(xy_1)^* = y_1xc = xy_1$ follows. □

3 Finite groups

In this section we obtain a characterization of a finite group G of odd order such that the Lie algebra FG^- is nilpotent. We start with the following useful remark related to Lemma 2.8.

Remark 3.1. Let $G = A \rtimes X$ be a finite group with involution $*$ such that $(|A|, |X|) = 1$ and $A^* = A$. If $x \in X$ is such that $x^* = xc$ with $c \in A$, then $x^* = y^{-1}xy$, for some $y \in A$.

Proof. Let $H = A \rtimes \langle x \rangle$. Since $A^* = A$ we have $H = A \rtimes \langle x^* \rangle$ and by the Schur–Zassenhaus theorem there exists $y \in A$ such that $\langle x^* \rangle = y^{-1}\langle x \rangle y$. So there exists $i \geqslant 1$ such that $x^* = y^{-1}x^iy$. Since $x^* = xc$, $x^i = x$ follows. □

Next we prove the main result of this section.

Theorem 3.2. *Let G be a finite group of odd order. Then FG^- is Lie nilpotent if and only if either FG is Lie nilpotent or* $\operatorname{char} F = p > 2$, *$P$ is a subgroup, G/P is abelian and $*$ is trivial on G/P.*

Proof. Suppose that FG^- is Lie nilpotent. If $\operatorname{char} F = 0$, FG is semiprime and G is abelian by Corollary 2.4. Hence we may assume that $\operatorname{char} F = p > 2$ and by Lemma 2.5, P is a subgroup of G. Since $(|G/P|, |P|) = 1$, by the Schur–Zassenhaus theorem we can write $G = P \rtimes X$ with X a p'-group. Since FX is semiprime with FX^- Lie nilpotent, X must be abelian by Corollary 2.4.

It follows that G is a p-abelian group, and by Theorem 2.1, in order to complete the proof it is enough to show that if $*$ is non-trivial on G/P, then G is nilpotent. Now, since P is nilpotent, it is actually enough to prove that G/P' is nilpotent; see [11, p. 134].

If $P' \neq 1$ we are done, by induction. Hence, we may assume that $P' = 1$ and thus that P is abelian. If we factor by a $*$-invariant subgroup of P contained in the center of G, the induced involution is still non-trivial. Therefore, without loss of generality, we may assume that P contains no central elements in G.

Write $P = A = A_1 \times A_2$ and $X = X_1 \times X_2$, where

$$X_1 = \{x \in X \mid x^* = x \bmod A\} \quad \text{and} \quad X_2 = \{x \in X \mid x^* = x^{-1} \bmod A\}.$$

First we claim that $(A_2, X_2) = 1$. In fact, if $x_2 \in X_2$, then $x_2^* = x_2^{-1}c$, for some $c \in A$, and by Lemma 2.8, there exists $y \in A_1$ such that $(x_2y)^* = (x_2y)^{-1}$. Since $(x_2, A_2) = 1$ if and only if $(x_2y, A_2) = 1$, we may assume that $x_2^* = x_2^{-1}$. But then $H = \langle x_2, A_2 \rangle$, the subgroup generated by x_2 and A_2, is invariant under $*$ and $*$ is the classical involution on H. By [4, Theorem], then H is a nilpotent group, and so $(x_2, A_2) = 1$. This proves the claim.

Next we claim that $(A_1, X_2) = 1$. Let $x_2 \in X_2$ and $x_2^* = x_2^{-1}c$, for some $c \in A$. As above, by invoking Lemma 2.8 we may assume that $x_2^* = x_2^{-1}$. For $a \in A_1$, $x_2 - x_2^{-1}, ax_2 - x_2^{-1}a \in FG^-$. Hence, for a suitable n, we have

$$[ax_2 - x_2^{-1}a, x_2^{p^n} - (x_2^{-1})^{p^n}] = 0.$$

Since x_2 is a p'-element, we get $[ax_2 - x_2^{-1}a, x_2 - x_2^{-1}] = 0$. Thus

$$ax_2^2 + x_2^{-1}ax_2^{-1} = x_2ax_2 + x_2^{-2}a$$

and so either $x_2ax_2 = ax_2^2$ or $x_2ax_2 = x_2^{-1}ax_2^{-1}$. In any case $ax_2^2 = x_2^2a$, and since G has no elements of order 2, we get $ax_2 = x_2a$ and the claim is proved.

As an outcome of the previous claims we get that $G = X_2 \times (A \rtimes X_1)$. Recall that $X_2 \neq 1$ by assumption.

We claim that $(A_2, X_1) = 1$. Let $a_2 \in A_2$, $x_1 \in X_1$ and pick $x_2 \in X_2$, $x_2 \neq 1$. By Lemma 2.8 and Remark 3.1, we may assume that $x_2^* = x_2^{-1}$ and $x_1^* = x_1$. Thus

$$0 = [(x_1x_2 - x_2^{-1}x_1)^{p^n}, a_2 - a_2^{-1}] = [x_1^{p^n}(x_2^{p^n} - x_2^{-p^n}), a_2 - a_2^{-1}].$$

Since $G = X_2 \times (A \rtimes X_1)$ and $x_2 \neq x_2^{-1}$, we conclude that $[x_1^{p^n}, a_2 - a_2^{-1}] = 0$, and so $[x_1, a_2 - a_2^{-1}] = 0$. It follows that $[x_1, a_2] = 0$, as desired.

In order to complete the proof it is enough to prove that $(A_1, X_1) = 1$. Let $x \in X_1$ and assume, as we may, that $x^* = x$. Then, for $a \in A_1$, we have $x^{-1}ax = (x^{-1}ax)^* = xax^{-1}$, and this says that $ax^2 = x^2a$. Since G has no elements of order 2, we conclude that $ax = xa$. Thus $(A_1, X_1) = 1$ and G is a nilpotent group.

Conversely, if FG is Lie nilpotent, there is nothing to prove. Suppose that P is a subgroup and G/P is abelian with trivial involution. Then, for $g \in G$, $gP = g^*P$ implies $g^* = gb_g$ with $b_g \in P$. Thus

$$\sum_{g \in G} \alpha_g(g - g^*) = \sum_{g \in G} \alpha_g g(1 - b_g) \in \Delta(G, P),$$

the augmentation ideal of P in G. This says that $FG^- \subseteq \Delta(G, P)$ and, since $\Delta(G, P)$ is nilpotent, FG^- is Lie nilpotent and we are done. □

4 Torsion groups

Throughout this section we shall assume that G is a torsion group with no elements of order 2 and FG^- is Lie nilpotent. If $\operatorname{char} F = 0$, then FG is semiprime and, by Corollary 2.4, G is abelian. Therefore throughout we shall assume that $\operatorname{char} F = p > 2$.

Since FG^- is Lie nilpotent, FG satisfies a $*$-polynomial identity. Hence by a theorem of Amitsur [1], it also satisfies an ordinary polynomial identity. It then follows from a theorem of Passman [10, p. 197] that G has a normal p-abelian subgroup A of finite index. We can assume that A is $*$-invariant by replacing it by $A \cap A^*$. Since G is torsion it also follows that G is locally finite and by Lemma 2.5, P is a subgroup and G/P is abelian.

Therefore throughout we shall also assume that G is a locally finite group with a normal subgroup A, which is $*$-invariant and such that G/A is finite and A' is a finite p-group. Moreover P is a subgroup and G/P is abelian.

Under the above hypotheses we start by proving the following result.

Proposition 4.1. *If G/A is cyclic of prime order, then G' is a finite p-group.*

Proof. From the hypotheses it follows that G' is a p-group. Hence we only need to show that G' is finite. To this end we may factor G by any finite $*$-invariant normal subgroup. If N is such a subgroup then FG^- maps onto $F(G/N)^-$ under the natural map $FG \to F(G/N)$.

Since A' is finite, by factoring by A' we may assume that A is abelian. As in Lemma 2.7 we write $A = A_1 \times A_2$.

Let $x \in G$ be such that $\langle xA \rangle = G/A$. Then, since G/A has prime order, $x^* \equiv x^\varepsilon \pmod{A}$, with $\varepsilon = \pm 1$. If $x^* = x^\varepsilon c$ for some $c \in A$, we factor by the normal and $*$-closure of $\langle c \rangle$ to assume that $x^* = x^\varepsilon$, with $\varepsilon = \pm 1$.

We assert that $A_2^{p^m}$ is central in G for some m. If $x^* = x^{-1}$ then for some m we have $0 = [x - x^{-1}, b^{p^m} - b^{-p^m}]$ for all $b \in A_2$. This implies that $[x, b^{p^m} - b^{-p^m}] = 0$ and so $[x, b^{p^m}] = 0$.

If $x^* = x$ then for all $a, b \in A_2$,

$$0 = [xa - a^{-1}x, b^{p^m} - b^{-p^m}] = [x, b^{p^m} - b^{-p^m}](1 - a^{-x}a^{-1})a. \tag{1}$$

Consider H, the normal and $*$-closure of the group $\langle a^{-x}a^{-1} \mid a \in A_2 \rangle$. If H is infinite, from (1) we deduce that $[x, b^{p^m} - b^{-p^m}] = 0$, and so $[x, b^{p^m}] = 0$ for all $b \in A_2$. If H is finite we can factor by H to assume that $a^{-x}a^{-1} = 1$ for all $a \in A_2$. Now $a^{-x} = a$ implies $a^{-x^2} = a^{-1}$ and since there are no elements of order 2, we have $ax = xa$. In any case we have proved that $A_2^{p^m}$ is central in G, for some $m \geqslant 0$.

If $A_2^{p^m}$ is infinite, then FG is Lie nilpotent by Lemma 2.6 and we are done, by Theorem 2.1. Therefore we may assume that $A_2^{p^m}$ is finite and, by factoring with it, we may assume that $A_2^{p^m} = 1$.

We shall now reduce the proof to the case $A_2 = 1$ in a way similar to [3]. Define

$$B = (x, A_2) = \{(x, a_2) \mid a_2 \in A_2\}.$$

Notice that B is a subgroup since $(x, ab) = (x, a)(x, b)$, i.e., the product of commutators is a commutator.

We claim that B is finite. Suppose to the contrary. Then, since $A_2^{p^m} = 1$, B is of bounded exponent. Then by [11, Theorem 4.3.5], $B = \prod_i B_i$, an infinite direct product of cyclic groups.

For an arbitrary $s \geqslant 1$ we shall produce elements $a_1, \dots, a_s \in A_2$ such that, after a possible renumbering of the indices, $1 \neq (x, a_i) \in B_i$ and

$$e = [x, a_1 - a_1^{-1}, \dots, a_s - a_s^{-1}] \neq 0.$$

For $s = 1$, we pick $1 \neq (x, a_1) \in B_1$; then $[x, a_1 - a_1^{-1}] \neq 0$ as $a_i^2 \neq 1$. Suppose we have already picked $a_1, \dots, a_{s-1}$ as desired. Then the normal closure N of $\langle a_1, \dots, a_{s-1} \rangle$ is finite abelian, as each a_i has a finite number of conjugates in G. Thus there exists an index s so that $B_s \cap N = 1$. Since every element of B is a commutator, we may choose $a_s \in A_2$ such that $1 \neq (x, a_s) \in B_s$ and $(x, a_s) \notin N$, so $a_s \notin N$. Write

$$0 \neq [x, a_1 - a_1^{-1}, \dots, a_{s-1} - a_{s-1}^{-1}] = x\alpha,$$

with $\alpha \in FN$. Then

$$e = [x, a_1 - a_1^{-1}, \dots, a_s - a_s^{-1}] = [x\alpha, a_s - a_s^{-1}] = x(a_s - a_s^{-1} - a_s^x + a_s^{-x})\alpha.$$

We observe that since $a_s, (x, a_s) \notin N$, then $a_s N$ cannot equal $a_s^{-1} N$ or $a_s^x N$. Thus $x a_s \alpha \neq 0$ and $e \neq 0$, as desired.

If $x^* = x^{-1}$, we get that $[x - x^*, a_1 - a_1^{-1}, \dots, a_s - a_s^{-1}] \neq 0$ for all $s \geqslant 1$, and this is a contradiction. In case $x^* = x$ we take an element $b \in A_2$ and compute

$$e' = [xb - b^{-1}x, a_1 - a_1^{-1}, \dots, a_s - a_s^{-1}] = e(b - b^{-x}) = e(1 - b^{-x}b^{-1})b.$$

If $e' = 0$, then $e(1 - b^{-x}b^{-1}) = 0$ and we consider the normal and $*$-closure H of $\langle b^{-x}b^{-1} \mid b \in A_2 \rangle$. If H is infinite, then since $e(1 - b^{-x}b^{-1}) = 0$ we have $e = 0$ and this is a contradiction. Hence H must be finite and we can factor by H to assume that $b^{-x}b^{-1} = 1$ for all $b \in A_2$. Now $b^{-x} = b$ implies $b^{-x^2} = b^{-1}$. Since there are no elements of order 2, this gives that $bx = xb$.

Now $e' = e(b - b^{-1})$. So if $e' = 0$ then $eb^2 = e$, which cannot hold for all b as $e \neq 0$ and A_2 is infinite. This is the final contradiction and we have proved that B is finite.

If we now factor G by the normal and $*$-closure of the finite group B, we may assume that A_2 is central. Consequently, by Lemma 2.6 we may assume that A_2 is finite. Hence in order to prove that G' is finite, by factoring with the normal and $*$-closure of A_2, we may assume that $A_2 = 1$. Thus $A = A_1$.

If $x^* = x$ for any $a \in A$ we have $x^{-1}ax = (x^{-1}ax)^* = xax^{-1}$, which implies that $x^2 a = ax^2$, and so $xa = ax$. This gives that $(x, A) = 1$. Thus $G' = 1$.

Suppose now that $x^* = x^{-1}$. Since the Lie algebra FG^- is nilpotent, it has non-zero center ζ. Let $0 \neq \alpha \in \zeta$ and write $\alpha = \sum_{i=0}^{r} \alpha_i x^i$ with $\alpha_i \in FA$. Since $A = A_1$, we have $\alpha_0 = 0$, so $\alpha_i \neq 0$ for some $i \neq 0$. Since every non-identity element of $\langle x \rangle$ is the square of a generator, we may assume $\alpha_2 \neq 0$.

We claim that α_2 commutes with x. In fact, $\alpha(x - x^{-1}) = (x - x^{-1})\alpha$, and we equate the coefficients of x. Since $\alpha_0 = 0$, αx and $x\alpha$ have no x components, we easily get that $\alpha_2 x = x\alpha_2$ and the claim is established.

Now $\alpha(ax - x^{-1}a) = (ax - x^{-1}a)\alpha$, for all $a \in A$, and we equate the coefficients of x. Since $\alpha_0 = 0$, $\alpha_2 x = x\alpha_2$ and αax, $ax\alpha$ have no x components, we get that $\alpha_2(a^{x^2} - a) = 0$ for all $a \in A$. Multiplying by a^{-1}, we see that $\alpha_2((a, x^2) - 1) = 0$ for all $a \in A$ and this says that $\alpha_2\Delta((A, x^2)) = 0$, where $\Delta((A, x^2))$ is the augmentation ideal of (A, x_2). Since α_2 is non-zero, this implies that (A, x^2) is a finite group. Furthermore, since x^2 generates $\langle x \rangle$, it follows that $(A, x^2) = G'$. So G' is finite. □

Proposition 4.1 can be easily improved as shown in the following result.

Corollary 4.2. *G' is a finite p-group.*

Proof. Recall that as in the previous proof, we are allowed to factor G by any finite $*$-invariant normal subgroup. Hence, by factoring by A' we may assume that A is abelian.

We shall prove the corollary by induction on $m = [G : A]$. Take $x \in G \backslash A$. Suppose first that $xx^* \equiv 1 \pmod{A}$. Then $x^* \equiv x^{-1} \pmod{A}$. Let y be a power of x such that yA in G/A is of prime order. If H is the subgroup generated by y and A, then H is $*$-invariant and by the last proposition, (A, y) is finite. Factoring by the normal $*$-closure of (A, y) we may assume that $(A, y) = 1$. Let S be the normal $*$-closure of $\langle y \rangle$. Since $[G : A] < \infty$, S is a finite subgroup and by factoring by S we may assume that $y = 1$. It follows that $[G : A] < m$ and by induction G' is finite.

If $xx^* \not\equiv 1 \pmod{A}$, we let z be a power of xx^* such that zA in G/A is of prime order and we proceed as in the above case using the element z instead of y. □

The next step is to deal with the case when $*$ is non-trivial on G/P.

Proposition 4.3. *If $*$ is non-trivial on G/P, then G is nilpotent and FG is Lie nilpotent.*

Proof. Since $*$ is non-trivial on G/P, G is locally nilpotent. In fact, if H is a finite $*$-invariant subgroup of G, take $t \in G$ such that $t^* \not\equiv t \pmod{P}$ and let $K = \langle H, t, t^* \rangle$. Then the p-elements of K form a subgroup P_1 and $*$ is non-trivial on K/P_1. By Theorem 3.2 it follows that K is nilpotent. Since G is locally nilpotent, we may write $G = P \times Q$ with Q an abelian p'-group.

Notice that in order to prove that G is nilpotent, it suffices to prove that P is nilpotent. But this follows from [12, Lemma 4.2, p. 150], as P' is finite. □

We can now prove the main result of this paper.

Proof of Theorem 1.1. Suppose that FG is not Lie nilpotent but FG^- is Lie nilpotent. We shall prove the necessity of the conditions. By Corollary 4.2 it follows that G' is a finite p-group. Moreover by Proposition 4.3, $*$ is trivial on G/P. Since G' is finite we deduce by a theorem of Hall [6] that $Z^{(2)}$, the second center of G, is of finite index in G. Furthermore, $B = (Z^{(2)}, G) \leqslant Z \cap G'$ is a finite central p-group which is $*$-invariant. Thus G/B is not nilpotent as otherwise G would be nilpotent and FG Lie nilpotent.

Let $A = Z^{(2)}$. Then A is $*$-invariant, A/B is central in G/B and $F(G/B)^-$ is Lie nilpotent. Hence, if $(A/B)_2 = \{aB \in A/B \mid a^*B = a^{-1}B\}$ is infinite, so that there are infinitely many $aB \in A/B$ with $(aB)^{-1}(aB)^* = a^{-1}a^*B = a^{-2}B$, then, since squaring elements is a bijection on B/A, Lemma 2.6 shows that $F(G/B)$ is Lie nilpotent and G/B is nilpotent. This is a contradiction. Thus $(A/B)_2$ is finite and the proof of the necessity is complete.

It remains to prove the sufficiency of (i), (ii), (iii). Suppose that we are given $1 \leqslant B \leqslant A \leqslant G$ as in (iii). Since G/B is centre-by-finite, by Schur's theorem ([12, p. 39]) $(G/B)'$ is finite. Thus G' is finite and G is a BFC group. From (ii) it follows that G/P is abelian so that G' is a finite p-group.

Write $G/B = H$ and $A/B = C$. Then $H \geqslant C > 1$ where C is central of finite index in H. Let $x_1, \dots, x_l$ be a transversal of C in H and let K be the normal and $*$-closure of the group they generate. Then K is a finite group and by (i) and (ii) can be written as LM where L is a normal p-group and M is an abelian p'-group with $*$ trivial on LM/L.

If we write $H = CLM$, an arbitrary element $h \in H$ can be written as $h = z\pi\alpha$, where $z \in C$, $\pi \in L$, $\alpha \in M$. We decompose $C = C_1 \times C_2$, and C_2 is a p-group by (ii) and is finite by (iii). If we write $z = z_1 z_2$, with $z_1 \in C_1$, $z_2 \in C_2$, then $h = z_1 z_2 \pi\alpha$ and

$$h^* = \alpha^* \pi^* z_2^{-1} z_1 = z_1 z_2^{-1} \alpha \pi' \pi^*,$$

for some $\pi' \in L$ and $\pi^* \in L$. Thus

$$h - h^* = z_1(z_2\pi\alpha - z_2^{-1}\alpha\pi'\pi^*) = z_1\alpha(z_2\pi^\alpha - z_2^{-1}\pi'\pi^*) \in \Delta(H, S)$$

where $S = \langle L, C_2 \rangle$ is a normal finite p-subgroup of H. Looking at this relation in G we deduce that for all $g \in G$, $g - g^* \in \Delta(G, N)$ where N is a finite normal p-subgroup as B is a finite central p-group. Since $\Delta(N)$ is nilpotent it follows that FG^- is Lie nilpotent as desired. ☐

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