Group algebras of torsion groups and Lie nilpotence

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Abstract. Let $*$ be an involution of a group algebra $FG$ induced by an involution of the group $G$. For $\text{char } F \neq 2$, we classify the torsion groups $G$ with no elements of order 2 whose Lie algebra of $*$-skew elements is nilpotent.

1 Introduction

Let $F$ be a field and $FG$ the group algebra of a group $G$ over $F$. If $*$ is an involution of $FG$ then the set of skew elements $FG^*=\{x \in FG \mid x^* = -x\}$ is a Lie algebra. Here we are interested in classifying the groups $G$ for which such an algebra is nilpotent. We shall assume throughout that $\text{char } F \neq 2$. A natural involution of $FG$ to consider is the so-called classical involution, obtained by linearly extending the group involution $g \rightarrow g^{-1}$ to $FG$. For this involution the problem has been completely settled in [4] for groups with no elements of order 2 and in [5] for arbitrary groups.

We should mention that if we regard $FG$ as a Lie algebra under the usual Lie bracket, then from results in [9] the algebra $FG$ is nilpotent if and only if either $\text{char } F = 0$ and $G$ is abelian or $\text{char } F = p > 0$ and $G$ is a nilpotent group whose derived group is a finite $p$-group. The same classification holds if $G$ has no elements of order 2 and we only impose that $FG^*$ is Lie nilpotent under the classical involution; see [4].

Here we try to extend this result to an involution of $FG$ obtained as a linear extension of a group involution of $G$. We shall classify the groups $G$ for which $FG^*$ is Lie nilpotent when $G$ is a torsion group and has no elements of order 2. It turns out that the conclusion is much more involved than for the classical involution. Our main result is the following.

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Theorem 1.1. Let $F$ be a field of characteristic $p \neq 2$ and $G$ a torsion group with no elements of order 2. Let $\ast$ be an involution on $FG$ induced by an involution of $G$. Then the Lie algebra $FG^\ast$ is nilpotent if and only if $FG$ is Lie nilpotent or $\text{char } F = p > 2$ and the following conditions hold:

(i) the set $P$ of $p$-elements in $G$ is a subgroup;

(ii) $\ast$ is trivial on $G/P$;

(iii) there exist normal $\ast$-invariant subgroups $A$ and $B$ with $B \leq A$ such that $B$ is a finite central $p$-subgroup of $G$ and $A/B$ is central in $G/B$ with both $G/A$ and $\{a \in A \mid aa^\ast \in B\}$ finite.

Remark 1.2. It turns out that if the conditions of the above theorem are satisfied then $G$ is a $p$-abelian group as is pointed out in the proof.

2 Preliminaries

Throughout this paper $F$ will be a field of characteristic different from 2 and $\ast$ will denote an involution of $FG$ obtained as a linear extension of an involution of $G$.

Recall that for a prime $p$, an element $x \in G$ is called a $p$-element if its order is a power of $p$. We write

$$P = \{x \in G \mid x \text{ is a } p\text{-element}\} \quad \text{and} \quad G^+ = \{g \in G \mid g^\ast = g\}.$$

Also, a group $G$ is said to be $p$-abelian if $G^\prime$, the commutator group of $G$, is a finite $p$-group. We make the convention that a 0-abelian group is abelian.

We record some results that we shall use throughout the paper. The first result is due to Passi, Passman and Sehgal [9].

Theorem 2.1. Suppose that $\text{char } F = p \geq 0$. The group algebra $FG$ is nilpotent as a Lie algebra if and only if $G$ is a nilpotent and $p$-abelian group.

If $R$ is any ring with involution $\ast$, we denote by

$$R^\ast = \{x \in R \mid x^\ast = -x\}$$

its set of skew elements. Also, we write the Lie bracket $[x, y] = xy - yx$. Recall that if $x_1, \ldots, x_4$ are non-commuting variables,

$$\text{St}_4(x_1, \ldots, x_4) = \sum_{\sigma \in S_4} (\text{sgn } \sigma)x_{\sigma(1)} \cdots x_{\sigma(4)}$$

is the standard identity of degree 4.
Lemma 2.2 ([4]). Let $R$ be a semiprime ring with involution such that $2R = R$. If the Lie algebra $R^-$ is nilpotent, then $R^-$ is abelian, i.e., $[R^-, R^-] = 0$ and $R$ satisfies $\text{St}_4$, the standard identity in four variables.

We denote by $\langle S \rangle$ the subgroup generated by a subset $S$ of a group. The following result was proved by Broche, Jespers, Polcino Milies and Ruiz [2].

Theorem 2.3. Let $\text{char} F \neq 2$ and let $\ast$ be an involution of $G$ linearly extended to $FG$. Then $FG^-$ is abelian if and only if either $G$ is abelian or one of the following conditions holds:

1. $K = \langle g \in G \mid g \notin G^+ \rangle$ is abelian;
2. $G$ contains an abelian subgroup of index 2 that is contained in $G^+$;
3. $\text{char} F = 3$, $|G'| = 3$, $G/G' = (G/G')^+$ and $g^3 \in G^+$, for all $g \in G$.

Suppose that $G$ is a torsion group with no elements of order 2 and $FG$ is a semiprime algebra. If $FG^-$ is Lie nilpotent, then $FG^-$ is commutative by Lemma 2.2, and so, by Theorem 2.3 either $G$ is abelian or one of the above three conditions holds. If (1) is satisfied, then $[G : K] \leq 2$ (see [7, Lemma 2.3]), hence the absence of elements of order 2 rules out (1) and (2). Also (3) is not possible as $FG$ is semiprime. In conclusion we have the following.

Corollary 2.4. Assume that $FG$ is semiprime and $G$ is a torsion group with no elements of order 2. Then $FG^-$ is Lie nilpotent if and only if $G$ is abelian.

Another important fact that we shall use is that the $p$-elements form a subgroup. This is the content of the following.

Lemma 2.5. Let $\text{char} F > 2$ and suppose that $G$ is a locally finite group. If $FG^-$ is Lie nilpotent then $P$ is a subgroup and, in case $G$ has no elements of order 2, $G/P$ is abelian.

Proof. Let $g, h \in P$ and $H = \langle g, h, g^*, h^* \rangle$. If $J$ is the Jacobson radical of $FH$, then $R = FH/J$ is a semisimple algebra with induced involution and the Lie algebra $R^-$ is nilpotent. Being a finite-dimensional semisimple algebra, $R$ is a finite direct sum of simple algebras $A_i$. By Lemma 2.2, $R$ satisfies $\text{St}_4$, hence each $A_i$ satisfies $\text{St}_4$. Now, it is well known that a simple algebra of dimension $m^2$ over its center satisfies no polynomial identity of degree less than $2m$. Hence we deduce that $R$ is isomorphic to a direct sum of simple algebras of dimension at most 4 over their center. But then by [3, Lemma 2.6] or [8], the $p$-elements of $H$ form a subgroup. In particular $gh \in P$ and $P$ is a subgroup.

Now, since $F(G/P)$ is a semiprime algebra with $F(G/P)^-$ Lie nilpotent, if $G$ has no elements of order 2 then $G/P$ is abelian by Corollary 2.4. □

Let $Z$ denote the center of the group $G$. In [4, Corollary] it was proved that if $\ast$ is the classical involution and $Z^2$ is infinite, and if $FG^-$ is Lie nilpotent of index $n$, then
also $FG$ is Lie nilpotent of index $n$. The proof of that result can be adapted, with the due changes, to our situation and we get the following result that we state without proof.

**Lemma 2.6.** Let $Z$ be the center of $G$ and suppose that $\tilde{Z} = \{z^{-1}z^* \mid z \in Z\}$ is infinite. If $FG^-$ is Lie nilpotent, then so is $FG$.

Another tool we shall need is the following lemma proved in [3, Lemma 2.9].

**Lemma 2.7.** Assume that $A$ is an abelian group with no elements of order 2 and let $*: A \to A$ be an automorphism of order 2. Then

$$A^2 \subseteq A_1 \times A_2,$$

where

$$A_1 = \{a \in A \mid a^* = a\} \quad \text{and} \quad A_2 = \{a \in A \mid a^* = a^{-1}\}.$$ 

Moreover if $A$ is a torsion group, then $A = A_1 \times A_2$.

**Proof.** If $b = a^2 \in A^2$, we can write

$$b = (aa^*)(a(a^*)^{-1})$$

with $aa^* \in A_1$, $a(a^*)^{-1} \in A_2$. This gives the required decomposition. \(\square\)

In the sequel we shall use the notation for $A_1$ and $A_2$ without mention. A first application of the decomposition given in the previous lemma is given in the following.

**Lemma 2.8.** Let $A$ be a $*$-invariant torsion abelian normal subgroup of $G$, with no elements of order 2.

1. If $x \in G \setminus A$ is such that $x^* = x^{-1}c$ with $c \in A$, then there exists a symmetric element $b \in A$ such that $(xb)^* = (xb)^{-1}$.

2. If $x \in G \setminus A$ is such that $x^* = xc$ with $c \in A$ and $x^* = y^{-1}xy$, for some $y \in A$, then there exists a symmetric element $b \in A$ such that $(xb)^* = xb$.

**Proof.** Write $A = A_1 \times A_2$ as in the previous lemma, and let $x \in G \setminus A$ be such that $x^* = x^{-1}c$ with $c \in A$. Notice that $xx^* = c$ is in $A$ and is symmetric, so $c \in A_1$. Also $x^{-1}cx \in A_1$. As $A_1$ has no elements of order 2, we can find $b \in A_1$ such that $b^2 = x^{-1}c^{-1}x$. This means that $b^{-1}x^{-1} = bx^{-1}c$ and thus $(xb)^{-1} = bx^{-1}c = (xb)^*$, as desired. This proves (1).

Now suppose that $x \in G \setminus A$ is such that $x^* = xc$ with $c \in A$ and $x^* = y^{-1}xy$, for some $y \in A$. Write $y = y_1y_2$ where $y_1^* = y_1$, $y_2^* = y_2^{-1}$. Since

$$x = x^{**} = (y^{-1}xy)^* = y^*y^{-1}xy(y^{-1})^*,$$
it follows that \((y^*y^{-1}, x) = 1\). Since \(y^*y^{-1} = y_2^{-1}y_1y_2^{-1}y_1^{-1} = (y_2^{-1})^2\), we conclude that \((y_2, x) = 1\). Thus we can write \(x^* = y_1^{-1}xy_1 = xc\) and \((xy_1)^* = y_1xc = xy_1\) follows.

\[\square\]

3 Finite groups

In this section we obtain a characterization of a finite group \(G\) of odd order such that the Lie algebra \(FG^-\) is nilpotent. We start with the following useful remark related to Lemma 2.8.

**Remark 3.1.** Let \(G = A \rtimes X\) be a finite group with involution \(*\) such that \((|A|, |X|) = 1\) and \(A^* = A\). If \(x \in X\) is such that \(x^* = xc\) with \(c \in A\), then \(x^* = y^{-1}xy\), for some \(y \in A\).

**Proof.** Let \(H = A \rtimes \langle x \rangle\). Since \(A^* = A\) we have \(H = A \rtimes \langle x^* \rangle\) and by the Schur–Zassenhaus theorem there exists \(y \in A\) such that \(\langle x^* \rangle = y^{-1}\langle x \rangle y\). So there exists \(i \geq 1\) such that \(x^* = y^{-1}x^iy\). Since \(x^* = xc\), \(x^i = x\) follows. \(\square\)

Next we prove the main result of this section.

**Theorem 3.2.** Let \(G\) be a finite group of odd order. Then \(FG^-\) is Lie nilpotent if and only if either \(FG\) is Lie nilpotent or char \(F = p > 2\), \(P\) is a subgroup, \(G/P\) is abelian and \(*\) is trivial on \(G/P\).

**Proof.** Suppose that \(FG^-\) is Lie nilpotent. If char \(F = 0\), \(FG\) is semiprime and \(G\) is abelian by Corollary 2.4. Hence we may assume that char \(F = p > 2\) and by Lemma 2.5, \(P\) is a subgroup of \(G\). Since \((|G/P|, |P|) = 1\), by the Schur–Zassenhaus theorem we can write \(G = P \rtimes X\) with \(X\) a \(p^2\)-group. Since \(FX\) is semiprime with \(FX^-\) Lie nilpotent, \(X\) must be abelian by Corollary 2.4.

It follows that \(G\) is a \(p\)-abelian group, and by Theorem 2.1, in order to complete the proof it is enough to show that if \(*\) is non-trivial on \(G/P\), then \(G\) is nilpotent. Now, since \(P\) is nilpotent, it is actually enough to prove that \(G/P'\) is nilpotent; see [11, p. 134].

If \(P' \neq 1\) we are done, by induction. Hence, we may assume that \(P' = 1\) and thus \(P\) is abelian. If we factor by a \(*\)-invariant subgroup of \(P\) contained in the center of \(G\), the induced involution is still non-trivial. Therefore, without loss of generality, we may assume that \(P\) contains no central elements in \(G\).

Write \(P = A = A_1 \times A_2\) and \(X = X_1 \times X_2\), where

\[X_1 = \{x \in X \mid x^* = x \mod A\} \quad \text{and} \quad X_2 = \{x \in X \mid x^* = x^{-1} \mod A\}.
\]

First we claim that \((A_2, X_2) = 1\). In fact, if \(x_2 \in X_2\), then \(x_2^* = x_2^{-1}c\), for some \(c \in A\), and by Lemma 2.8, there exists \(y \in A_1\) such that \((x_2y)^* = (x_2y)^{-1}\). Since \((x_2, A_2) = 1\) if and only if \((x_2y, A_2) = 1\), we may assume that \(x_2^* = x_2^{-1}\). But then \(H = \langle x_2, A_2 \rangle\), the subgroup generated by \(x_2\) and \(A_2\), is invariant under \(*\) and \(*\) is
Throughout this section we shall assume that $H$ is a nilpotent group, and so $(x_2, A_2) = 1$. This proves the claim.

Next we claim that $(A_1, X_2) = 1$. Let $x_2 \in X_2$ and $x_2^* = x_2^{-1}c$, for some $c \in A$. As above, by invoking Lemma 2.8 we may assume that $x_2^2 = x_2^{-1}$. For $a \in A_1$, $x_2 - x_2^{-1}, ax_2 - x_2^{-1}a \in FG^-$. Hence, for a suitable $n$, we have

$$[ax_2 - x_2^{-1}a, x_2^p - (x_2^{-1})^p] = 0.$$ 

Since $x_2$ is a $p'$-element, we get $[ax_2 - x_2^{-1}a, x_2 - x_2^{-1}] = 0$. Thus

$$ax_2^2 + x_2^{-1}ax_2^{-1} = x_2ax_2 + x_2^{-2}a$$

and so either $x_2ax_2 = ax_2^2$ or $x_2ax_2 = x_2^{-1}ax_2^{-1}$. In any case $ax_2^2 = x_2^2a$, and since $G$ has no elements of order 2, we get $ax_2 = x_2a$ and the claim is proved.

As an outcome of the previous claims we get that $G = X_2 \times (A \times X_1)$. Recall that $X_2 \neq 1$ by assumption.

We claim that $(A_2, X_1) = 1$. Let $a_2 \in A_2$, $x_1 \in X_1$ and pick $x_2 \in X_2$, $x_2 \neq 1$. By Lemma 2.8 and Remark 3.1, we may assume that $x_2^2 = x_2^{-1}$ and $x_1^* = x_1$. Thus

$$0 = [(x_1x_2 - x_2^{-1}x_1)^p, a_2 - a_2^{-1}] = [x_1^p (x_2^p - x_2^{-p}), a_2 - a_2^{-1}].$$

Since $G = X_2 \times (A \times X_1)$ and $x_2 \neq x_2^{-1}$, we conclude that $[x_1^p, a_2 - a_2^{-1}] = 0$, and so $[x_1, a_2 - a_2^{-1}] = 0$. It follows that $[x_1, a_2] = 0$, as desired.

In order to complete the proof it is enough to prove that $(A_1, X_1) = 1$. Let $x \in X_1$ and assume, as we may, that $x^* = x$. Then, for $a \in A_1$, we have $x^{-1}ax = (x^{-1}ax)^* = xax^{-1}$, and this says that $ax^2 = x^2a$. Since $G$ has no elements of order 2, we conclude that $ax = xa$. Thus $(A_1, X_1) = 1$ and $G$ is a nilpotent group.

Conversely, if $FG$ is Lie nilpotent, there is nothing to prove. Suppose that $P$ is a subgroup and $G/P$ is abelian with trivial involution. Then, for $g \in G$, $gP = g^*P$ implies $g^* = gb_g$ with $b_g \in P$. Thus

$$\sum_{g \in G} \zeta_g (g - g^*) = \sum_{g \in G} \zeta_g g(1 - b_g) \in \Delta(G, P),$$

the augmentation ideal of $P$ in $G$. This says that $FG^- \subseteq \Delta(G, P)$ and, since $\Delta(G, P)$ is nilpotent, $FG^-$ is Lie nilpotent and we are done. \qed

## 4 Torsion groups

Throughout this section we shall assume that $G$ is a torsion group with no elements of order 2 and $FG^-$ is Lie nilpotent. If char $F = 0$, then $FG$ is semiprime and, by Corollary 2.4, $G$ is abelian. Therefore throughout we shall assume that char $F = p > 2$.

Since $FG^-$ is Lie nilpotent, $FG$ satisfies a $*$-polynomial identity. Hence by a theorem of Amitsur [1], it also satisfies an ordinary polynomial identity. It then follows
from a theorem of Passman [10, p. 197] that $G$ has a normal $p$-abelian subgroup $A$ of finite index. We can assume that $A$ is $*$-invariant by replacing it by $A \cap A^*$. Since $G$ is torsion it also follows that $G$ is locally finite and by Lemma 2.5, $P$ is a subgroup and $G/P$ is abelian.

Therefore throughout we shall also assume that $G$ is a locally finite group with a normal subgroup $A$, which is $*$-invariant and such that $G/A$ is finite and $A'$ is a finite $p$-group. Moreover $P$ is a subgroup and $G/P$ is abelian.

Under the above hypotheses we start by proving the following result.

**Proposition 4.1.** If $G/A$ is cyclic of prime order, then $G'$ is a finite $p$-group.

**Proof.** From the hypotheses it follows that $G'$ is a $p$-group. Hence we only need to show that $G'$ is finite. To this end we may factor $G$ by any finite $*$-invariant normal subgroup. If $N$ is such a subgroup then $FG^-$ maps onto $F(G/N)^-$ under the natural map $FG \rightarrow F(G/N)$.

Since $A'$ is finite, by factoring by $A'$ we may assume that $A$ is abelian. As in Lemma 2.7 we write $A = A_1 \times A_2$.

Let $x \in G$ be such that $\langle xA \rangle = G/A$. Then, since $G/A$ has prime order, $x^* \equiv x^e \pmod {A}$, with $e = \pm 1$. If $x^* = x^e c$ for some $c \in A$, we factor by the normal and $*$-closure of $\langle c \rangle$ to assume that $x^* = x^e$, with $e = \pm 1$.

We assert that $A_p^m$ is central in $G$ for some $m$. If $x^* = x^{-1}$ then for some $m$ we have $0 = [x - x^{-1}, b^m - b^{-m}]$ for all $b \in A_2$. This implies that $[x, b^m - b^{-m}] = 0$ and so $[x, b^m] = 0$.

If $x^* = x$ then for all $a, b \in A_2$,

$$0 = [xa - a^{-1}x, b^m - b^{-m}] = [x, b^m - b^{-m}](1 - a^{-x}a^{-1})a. \tag{1}$$

Consider $H$, the normal and $*$-closure of the group $\langle a^{-x}a^{-1} | a \in A_2 \rangle$. If $H$ is infinite, from (1) we deduce that $[x, b^m - b^{-m}] = 0$, and so $[x, b^m] = 0$ for all $b \in A_2$. If $H$ is finite we can factor by $H$ to assume that $a^{-x}a^{-1} = 1$ for all $a \in A_2$. Now $a^{-x} = a$ implies $a^{-x^2} = a^{-1}$ and since there are no elements of order 2, we have $ax = xa$. In any case we have proved that $A_p^m$ is central in $G$, for some $m \geq 0$.

If $A_p^m$ is infinite, then $FG$ is Lie nilpotent by Lemma 2.6 and we are done, by Theorem 2.1. Therefore we may assume that $A_p^m$ is finite and, by factoring with it, we may assume that $A_p^m = 1$.

We shall now reduce the proof to the case $A_2 = 1$ in a way similar to [3]. Define

$$B = (x, A_2) = \{(x, a_2) | a_2 \in A_2\}.$$ 

Notice that $B$ is a subgroup since $(x, ab) = (x, a)(x, b)$, i.e., the product of commutators is a commutator.

We claim that $B$ is finite. Suppose to the contrary. Then, since $A_2^p = 1$, $B$ is of bounded exponent. Then by [11, Theorem 4.3.5], $B = \prod B_i$, an infinite direct product of cyclic groups.
For an arbitrary \( s \geq 1 \) we shall produce elements \( a_1, \ldots, a_s \in A_2 \) such that, after a possible renumbering of the indices, \( 1 \neq (x, a_i) \in B_i \) and

\[
eq \begin{bmatrix} x, a_1 - a_1^{-1}, \ldots, a_s - a_s^{-1} \end{bmatrix} \neq 0.
\]

For \( s = 1 \), we pick \( 1 \neq (x, a_1) \in B_1 \); then \( [x, a_1 - a_1^{-1}] \neq 0 \) as \( a_1^2 \neq 1 \). Suppose we have already picked \( a_1, \ldots, a_{s-1} \) as desired. Then the normal closure \( N \) of \( \langle a_1, \ldots, a_{s-1} \rangle \) is finite abelian, as each \( a_i \) has a finite number of conjugates in \( G \). Thus there exists an index \( s \) so that \( B_s \cap N = 1 \). Since every element of \( B \) is a commutator, we may choose \( a_s \in A_2 \) such that \( 1 \neq (x, a_s) \in B_s \) and \( (x, a_s) \notin N \), so \( a_s \notin N \). Write

\[
0 \neq \begin{bmatrix} x, a_1 - a_1^{-1}, \ldots, a_{s-1} - a_{s-1}^{-1} \end{bmatrix} = xa,
\]

with \( a \in FN \). Then

\[
eq \begin{bmatrix} x, a_1 - a_1^{-1}, \ldots, a_s - a_s^{-1} \end{bmatrix} = [xa, a_s - a_s^{-1}] = x(a_s - a_s^{-1} - a_s^x + a_s^{-x})a.
\]

We observe that since \( a_s, (x, a_s) \notin N \), then \( a_sN \) cannot equal \( a_s^{-1}N \) or \( a_s^2N \). Thus \( xax \neq 0 \) and \( e \neq 0 \), as desired.

If \( x^s = x^{-1} \), we get that \( [x - x^s, a_1 - a_1^{-1}, \ldots, a_s - a_s^{-1}] \neq 0 \) for all \( s \geq 1 \), and this is a contradiction. In case \( x^s = x \) we take an element \( b \in A_2 \) and compute

\[
eq \begin{bmatrix} xb - b^{-1}x, a_1 - a_1^{-1}, \ldots, a_s - a_s^{-1} \end{bmatrix} = e(b - b^{-x}) = e(1 - b^{-x}b^{-1})b.
\]

If \( e' = 0 \), then \( e(1 - b^{-x}b^{-1}) = 0 \) and we consider the normal and \( * \)-closure \( H \) of \( \langle b^{-x}b^{-1} | b \in A_2 \rangle \). If \( H \) is infinite, then since \( e(1 - b^{-x}b^{-1}) = 0 \) we have \( e = 0 \) and this is a contradiction. Hence \( H \) must be finite and we can factor by \( H \) to assume that \( b^{-x}b^{-1} = 1 \) for all \( b \in A_2 \). Now \( b^{-x} = b \) implies \( b^{-x^2} = b^{-1} \). Since there are no elements of order 2, this gives that \( bx = xb \).

Now \( e' = e(b - b^{-1}) \). So if \( e' = 0 \) then \( eb^2 = e \), which cannot hold for all \( b \) as \( e \neq 0 \) and \( A_2 \) is infinite. This is the final contradiction and we have proved that \( B \) is finite.

If we now factor \( G \) by the normal and \( * \)-closure of the finite group \( B \), we may assume that \( A_2 \) is central. Consequently, by Lemma 2.6 we may assume that \( A_2 \) is finite. Hence in order to prove that \( G' \) is finite, by factoring with the normal and \( * \)-closure of \( A_2 \), we may assume that \( A_2 = 1 \). Thus \( A = A_1 \).

If \( x^s = x \) for any \( a \in A \) we have \( x^{-1}ax = (x^{-1}ax)^* = xax^{-1} \), which implies that \( x^2a = ax^2 \), and so \( xa = ax \). This gives that \( (x, A) = 1 \). Thus \( G' = 1 \).

Suppose now that \( x^s = x^{-1} \). Since the Lie algebra \( FG^- \) is nilpotent, it has non-zero center \( \zeta \). Let \( 0 \neq a \in \zeta \) and write \( x = \sum_{i=0}^{t'} a_i x_i' \) with \( x_i \in FA \). Since \( A = A_1 \), we have \( a_0 = 0 \), so \( x_i \neq 0 \) for some \( i \neq 0 \). Since every non-identity element of \( \langle x \rangle \) is the square of a generator, we may assume \( x_2 \neq 0 \).

We claim that \( x_2 \) commutes with \( x \). In fact, \( x(x - x^{-1}) = (x - x^{-1})x \), and we equate the coefficients of \( x \). Since \( a_0 = 0 \), \( xa \) and \( xz \) have no \( x \) components, we easily get that \( x_2x = x x_2 \) and the claim is established.
Now $\alpha(ax-x^{-1}a) = (ax-x^{-1}a)x$, for all $a \in A$, and we equate the coefficients of $x$. Since $x_0 = 0$, $x_2x = x_2x$, and $2ax$, $axx$ have no $x$ components, we get that $x_2(a^x - a) = 0$ for all $a \in A$. Multiplying by $a^{-1}$, we see that $x_2((a,x^2) - 1) = 0$ for all $a \in A$ and this says that $x_2\Delta((A,x^2)) = 0$, where $\Delta((A,x^2))$ is the augmentation ideal of $(A,x_2)$. Since $x_2$ is non-zero, this implies that $(A,x^2)$ is a finite group. Furthermore, since $x^2$ generates $\langle x \rangle$, it follows that $(A,x^2) = G'$. So $G'$ is finite.

Proposition 4.1 can be easily improved as shown in the following result.

**Corollary 4.2.** $G'$ is a finite $p$-group.

**Proof.** Recall that as in the previous proof, we are allowed to factor $G$ by any finite $*$-invariant normal subgroup. Hence, by factoring by $A'$ we may assume that $A$ is abelian.

We shall prove the corollary by induction on $m = [G : A]$. Take $x \in G \setminus A$. Suppose first that $xx^* = 1 \mod A$. Then $x^* = x^{-1} \mod A$. Let $y$ be a power of $x$ such that $yA$ in $G/A$ is of prime order. If $H$ is the subgroup generated by $y$ and $A$, then $H$ is $*$-invariant and by the last proposition, $(A,y)$ is finite. Factoring by the normal $*$-closure of $(A,y)$ we may assume that $(A,y) = 1$. Let $S$ be the normal $*$-closure of $\langle y \rangle$. Since $[G : A] < \infty$, $S$ is a finite subgroup and by factoring by $S$ we may assume that $y = 1$. It follows that $[G : A] < m$ and by induction $G'$ is finite.

If $xx^* \neq 1 \mod A$, we let $z$ be a power of $xx^*$ such that $zA$ in $G/A$ is of prime order and we proceed as in the above case using the element $z$ instead of $y$.

The next step is to deal with the case when $*$ is non-trivial on $G/P$.

**Proposition 4.3.** If $*$ is non-trivial on $G/P$, then $G$ is nilpotent and $FG$ is Lie nilpotent.

**Proof.** Since $*$ is non-trivial on $G/P$, $G$ is locally nilpotent. In fact, if $H$ is a finite $*$-invariant subgroup of $G$, take $t \in G$ such that $t^* \neq t \mod P$ and let $K = \langle H, t, t^* \rangle$. Then the $p$-elements of $K$ form a subgroup $P_1$ and $*$ is non-trivial on $K/P_1$. By Theorem 3.2 it follows that $K$ is nilpotent. Since $G$ is locally nilpotent, we may write $G = P \times Q$ with $Q$ an abelian $p'$-group.

Notice that in order to prove that $G$ is nilpotent, it suffices to prove that $P$ is nilpotent. But this follows from [12, Lemma 4.2, p. 150], as $P'$ is finite.

We can now prove the main result of this paper.

**Proof of Theorem 1.1.** Suppose that $FG$ is not Lie nilpotent but $FG^-$ is Lie nilpotent. We shall prove the necessity of the conditions. By Corollary 4.2 it follows that $G'$ is a finite $p$-group. Moreover by Proposition 4.3, $*$ is trivial on $G/P$. Since $G'$ is finite we deduce by a theorem of Hall [6] that $Z^{(2)}$, the second center of $G$, is of finite index in $G$. Furthermore, $B = (Z^{(2)}, G) \leq Z \cap G'$ is a finite central $p$-group which is $*$-invariant. Thus $G/B$ is not nilpotent as otherwise $G$ would be nilpotent and $FG$ Lie nilpotent.
Let $A = Z^{(2)}$. Then $A$ is $\ast$-invariant, $A/B$ is central in $G/B$ and $F(G/B)^\ast$ is Lie nilpotent. Hence, if $(A/B)_2 = \{ aB \in A/B \mid a^\ast B = a^{-1}B \}$ is infinite, so that there are infinitely many $aB \in A/B$ with $(aB)^{-1}(aB)^\ast = a^{-1}a^\ast B = a^{-2}B$, then, since squaring elements is a bijection on $B/A$, Lemma 2.6 shows that $F(G/B)$ is Lie nilpotent and $G/B$ is nilpotent. This is a contradiction. Thus $(A/B)_2$ is finite and the proof of the necessity is complete.

It remains to prove the sufficiency of (i), (ii), (iii). Suppose that we are given $1 \leq B \leq A \leq G$ as in (iii). Since $G/B$ is centre-by-finite, by Schur’s theorem ([12, p. 39]) $(G/B)'$ is finite. Thus $G'$ is finite and $G$ is a BFC group. From (ii) it follows that $G/P$ is abelian so that $G'$ is a finite $p$-group.

Write $G/B = H$ and $A/B = C$. Then $H \supseteq C > 1$ where $C$ is central of finite index in $H$. Let $x_1, \ldots, x_l$ be a transversal of $C$ in $H$ and let $K$ be the normal and $\ast$-closure of the group they generate. Then $K$ is a finite group and by (i) and (ii) can be written as $LM$ where $L$ is a normal $p$-group and $M$ is an abelian $p'$-group with $\ast$ trivial on $LM/L$.

If we write $H = CLM$, an arbitrary element $h \in H$ can be written as $h = z\pi x$, where $z \in C$, $\pi \in L$, $x \in M$. We decompose $C = C_1 \times C_2$, and $C_2$ is a $p$-group by (ii) and is finite by (iii). If we write $z = z_1z_2$, with $z_1 \in C_1$, $z_2 \in C_2$, then $h = z_1z_2\pi x$ and

$$h^* = x^\ast \pi^* z_2^{-1}z_1 = z_1z_2^{-1}\pi x^\ast \pi^*,$$

for some $\pi' \in L$ and $\pi^* \in L$. Thus

$$h - h^* = z_1(z_2\pi x - z_2^{-1}\pi x^\ast \pi^*) = z_1x(z_2\pi x - z_2^{-1}\pi^* x^\ast) \in \Delta(H, S)$$

where $S = \langle L, C_2 \rangle$ is a normal finite $p$-subgroup of $H$. Looking at this relation in $G$ we deduce that for all $g \in G$, $g - g^* \in \Delta(G, N)$ where $N$ is a finite normal $p$-subgroup as $B$ is a finite central $p$-group. Since $\Delta(N)$ is nilpotent it follows that $FG^\ast$ is Lie nilpotent as desired. $\square$

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