Group identities on symmetric units

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Introduction

Let $F$ be an infinite field of characteristic different from 2, $G$ a group and $\ast$ an involution of $G$ extended by linearity to an involution of the group algebra $FG$. Here we completely characterize the torsion groups $G$ for which the $\ast$-symmetric units of $FG$ satisfy a group identity. When $\ast$ is the classical involution induced from $g \mapsto g^{-1}$, $g \in G$, this result was obtained in [A. Giambruno, S.K. Sehgal, A. Valenti, Symmetric units and group identities, Manuscripta Math. 96 (1998) 443–461].

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is which properties of $R^+$ or $R^-$ can be lifted to $R$ (see [13]). For example, a classical result of Amitsur [1] states that if $R^+$ (or $R^-$) satisfies a polynomial identity, then so does $R$.

Group rings are naturally endowed with an involution; the one obtained as a linear extension of the involution of $G$ given by $g \mapsto g^{-1}$. We shall refer to this as the classical involution.

For this particular involution Giambruno, Sehgal and Valenti [6] classified groups so that $U^+(FG)$, the symmetric units, satisfy a group identity.

Recently, there has been a surge of activity in studying more general involutions of $FG$; namely, the maps obtained from arbitrary involutions of $G$, extended linearly to $FG$. Properties of $(FG)^+$ and $(FG)^-$ were considered in [2,14] and, recently, Gonçalves and Passman considered the existence of bicyclic units $u$ in the integral group rings such that the group $\langle u, u^* \rangle$ is free [9]. Marciniak and Sehgal had proved that, with respect to the classical involution, $\langle u, u^* \rangle$ is always free if $u \neq 1$ [16]. Also, Gonçalves and Passman constructed free pairs of unitary units in group algebras [10].

We classify group algebras $FG$ so that $U^+$ is GI. Our main result is stated below. Partial results were obtained by Dooms and Ruiz [3].

First we need some notation. Recall that as introduced by Edgar Goodaire a group $G$ is LC if for any pair of elements $g, h \in G$, it is the case that $gh = hg$ if and only if $g \in Z(G)$ or $h \in Z(G)$ or $gh \in Z(G)$ where $Z(G)$ is the centre of $G$. It turns out (see [12, page 98]) that the LC groups with a unique nonidentity commutator are precisely the noncommutative groups with $G/Z(G) \cong C_2 \times C_2$, where $C_2$ denotes the cyclic group of order 2. Such groups we shall call SLC groups when they are endowed with the involution

$$g^* = \begin{cases} g & \text{if } g \text{ is central}, \\ gz & \text{otherwise}, \end{cases}$$

where $z$ is the unique commutator.

**Main Theorem.** Let $G$ be a torsion group with an involution $*$ and $F$ an infinite field of characteristic $p \geq 0$, $p \neq 2$. Then we have the following:

(a) If $FG$ is semiprime then $U^+(FG) \in GI$ if and only if $G$ is abelian or an SLC group.
(b) If $FG$ is not semiprime then $U^+(FG) \in GI$ if and only if $P$, the set of $p$-elements, is a subgroup, $FG$ is PI and one of the following holds:
   (i) $G/P$ is abelian and $G'$ is of bounded $p$-power exponent.
   (ii) $G/P$ is SLC and $G$ contains a normal $*$-invariant $p$-subgroup $B$ of bounded exponent such that $P/B$ is central in $G/B$ and $*$ is trivial on $P/B$.

Throughout the paper by an involution of $FG$ we understand an involution of $G$ extended linearly to $FG$.

### 1. Prerequisites

In this section $R$ is a unitary algebra with an involution $*$ over an infinite commutative domain $A$ and the elements of $A$ are not zero-divisors in $R$. Also we assume that $\text{char } A \neq 2$.

**Sublemma 1.1.** (See [6, Lemma 2].) If $U^+(R)$ is GI then there exists a positive integer $N \geq 1$ such that

1. if $a \in R$ and $a^2 = 0$, then $(aa^*)^N = 0$;
2. if $s, t \in R^+$, $s^2 = t^2 = 0$, then $(sts^d)^N = 0$ for all $d \in R^+$.

**Sublemma 1.2.** (See [6, Lemma 3].) Let $R$ be a semiprime ring and suppose that $U^+(R)$ is GI. Let $s \in R^+$ be such that $s^2 = 0$.

1. If $t \in R^+$ is nilpotent then $sts = 0$.
2. If $x, y \in R$ are such that $xy = 0$ then $xsy = 0$. 
Sublemma 1.3. (See [6, Theorem 2.1].) Let $R$ be a semiprime ring such that $U^+(R)$ is GI. Then every symmetric idempotent of $R$ and $A^{-1}R$ is central.

Sublemma 1.4. Suppose $U^+(FG)$ satisfies a group identity and that $N = N^*$ is a nil but not nilpotent ideal of $FG$. Then $N$ satisfies a polynomial identity, $FG$ satisfies a multilinear generalized polynomial identity and $G$ contains a normal subgroup $A$ of finite index with $A^*$ nil.

Proof. For the first assertion see Remark 2 in [6, page 450] and for the second assertion see [7, page 633]. The existence of the group $A$ then is a theorem of Passman [17, page 202].

The next result we need is due to Passman [18, page 661].

Sublemma 1.5. Suppose that $A$ is an abelian normal subgroup of finite index $n$ in $G$. Suppose $I$ is a $G$-stable ideal of $FA$ which is nil of bounded degree $\leq p^k$. Then $I \cdot FG$ is an ideal of $FG$ which is nil of bounded degree $np^k$.

Sublemma 1.6. (See Jespers and Ruiz [14, page 730.]) Let $F$ be a field of characteristic $\neq 2$. Then $(FG)^+$ is commutative if and only if $G$ is abelian or an SLC group.

We record an easy observation which will be used without mention.

Sublemma 1.7. If $A$ is an abelian group with an involution $*$, then $A^2 = A_1 \times A_2$ where $A_1 = \{a \in A \mid a^* = a\}$, $A_2 = \{a \in A \mid a^* = a^{-1}\}$. If $A$ has no elements of order 2 then $A = A_1 \times A_2$.

Proof. See Lemma 2.9 of [5].

2. General results

Throughout this section we assume that $F$ is an infinite field, $\text{char} F \neq 2$, and $FG$ is endowed with an involution $*$ induced from an involution of $G$. We start with the following general result.

Lemma 2.1. Let $R$ be a finite dimensional semisimple algebra with involution over an infinite field $F$, $\text{char} F \neq 2$, and suppose that $U^+(R)$ satisfies a group identity. Then $R$ is a direct sum of simple algebras of dimension at most over their centres and the symmetric elements $R^+$ are central in $R$.

Proof. Let $R = e_1R \oplus \cdots \oplus e_kR$ be the decomposition of $R$ into simple components where the $e_i$’s are central idempotents.

If $e_i^* = e_i$, then $e_iR \cong M_{n_i}(D)$ for some finite dimensional central division algebra $D$ with induced involution. Since $U^+(e_iR)$ satisfies a group identity, as in the proof of [6, Lemma 4] we obtain that either $e_iR \cong D$ or $e_iR \cong M_2(K)$ where $K$ is a field and $*$ on $M_2(K)$ is symplectic. If the second case occurs, the symmetric elements are central. In case $e_iR \cong D$ is noncommutative, the centre of $D$ is a nonabsolute field. Hence, by invoking [11, Theorem 1.2], we get that $D$ is a 4-dimensional central division algebra with symplectic involution. Moreover in this case all the symmetric elements of $e_iR$ are central.

If $e_i \neq e_i^*$, then $e_iR$ and $e_i^*R$ are anti-isomorphic algebras. We get that $e_iR \oplus e_i^*R \cong M_{n_i}(D) \oplus M_{n_i}(D)$ with exchange involution given by $(a, b)^* = (b, a^*)$; hence $(e_iR \oplus e_i^*R)^+ \cong \{(a, a^*) \mid a \in M_{n_i}(D)\}$. Since $U^+(R)$ satisfies a group identity, it follows that also $U(M_{n_i}(D))$ must satisfy a group identity; since $F$ is an infinite field by [4, Corollary 3] and [8] we get a contradiction unless $M_{n_i}(D) = K$ is a field. Therefore the symmetric elements are central. The outcome is that $R^+$ is central in $R$.

Lemma 2.2. Suppose that $G$ is a torsion group and $FG$ is semiprime. If $U^+(FG)$ is GI, then $gh = hg$, for any $p^t$-elements $g, h \in G^+$. 
The next lemma is proved in [15] and [5].

**Lemma 2.3.** Let $F$ be a field of characteristic $p > 2$, $G$ a finite group and $J$ the Jacobson radical of $FG$. Suppose that $FG/J$ is isomorphic to a direct sum of simple algebras of dimension at most four over their centres. Then the $p$-elements of $G$ form a subgroup.

**Corollary 2.4.** Let $G$ be a locally finite group and $\text{char } F = p > 2$. If $U^+(FG)$ is GI, then the set $P$ of $p$-elements of $G$ is a subgroup.

**Proof.** Let $g, h \in P$ and let $H$ be the subgroup generated by $g, h, g^*, h^*$. Then $H$ is finite since $G$ is locally finite. Now, we combine the proof of [6, Lemma 4] together with [11, Theorem 1.1] as follows.

If $R = FH/J(FH)$ then $R$ is a semisimple algebra with induced involution and $U^+(R)$ still satisfies a group identity. By Lemma 2.1, we deduce that $R$ is isomorphic to a direct sum of simple algebras of dimension at most four over their centres. Hence the algebra $FH/J(FH)$ satisfies the hypotheses of the previous lemma and we obtain that $g h$ is a $p$-element. □

**Lemma 2.5.** Suppose that $U^+(FG)$ is GI. Let $H$ be an invariant subgroup of $G$ and $N$ a normal invariant subgroup of $G$. Then $U^+(FH)$ and $U^+(F(G/N))$ are both GI.

**Proof.** See [6, page 458]. □

Now we classify finite groups $G$ with involution so that $U^+(FG)$ is GI.

**Theorem 2.6.** Let $F$ be an infinite field of characteristic $p \geq 0$, $p \neq 2$, and $G$ a finite group. Then $U^+(FG)$ satisfies a group identity if and only if

1) $P$ is a normal subgroup of $G$,

2) $G/P$ is either abelian or an SLC group.

**Proof.** Suppose that $U^+(FG)$ is GI. Then by Corollary 2.4, $P$ is a subgroup. Also $U^+(F(G/P))$ is still GI. By Lemma 2.1 the symmetric elements of $F(G/P)$ commute. But then by Sublemma 1.6, $G/P$ is either abelian or an SLC group.

Conversely, if $G/P$ is either abelian or an SLC group, by Sublemma 1.6, the symmetric elements of $F(G/P)$ commute. Hence $(U^+(FG), U^+(FG)) \leq 1 + \Delta(G, P)$ where $\Delta(G, P)$ is the kernel of $F G \to F(G/P)$. This implies that $(x, y)^{p^2} \equiv 1$ is a GI for $U^+F(G)$, for some $n \geq 1$. □

3. Semiprime algebras

Throughout this section we assume that $G$ is a torsion group, $F$ is an infinite field of characteristic $p \geq 0$, but different from 2, and FG is a semiprime algebra with an involution induced from the group $G$ such that $U^+(FG)$ satisfies a group identity.
Definition 3.1.

\[ G_{p'} = \{ g \in G \mid g \text{ is a } p'-\text{element} \} . \]
\[ P = \{ g \in G \mid g \text{ is a } p-\text{element} \} . \]
\[ P_1 = \{ g \in G \mid g^p = 1 \} . \]

We agree that if \( p = 0 \) then \( P = 1, G = G_{p'} \).

Our notations are standard as in [19]. We write \( \Delta(G, N) \) for the kernel of \( FG \to F(G/N) \) and \( \Delta(G) \) for \( \Delta(G, G) \). Also we denote by \( \overline{g} \) the sum of all powers of \( g, \overline{g} = \sum_{i=1}^{n} g^i \).

Lemma 3.2. Let \( g \in G^\ast \cap G_{p'} \). Then \( gh = hg \), for all \( h \in G^\ast \cap P \).

Proof. Let \( H = (g, h) \) be the group generated by \( g \) and \( h \). Since \( FG \) is semiprime all symmetric idempotents are central, hence \( \hat{g} \) is central. This says that \( \hat{h}\hat{g}^{-1} = \hat{g} \) and, so, \( hg = g' h \), for some \( i \geq 1 \). Thus \( H \) is a finite group.

Let \( J = J(FH) \) be the Jacobson radical of \( FH \). Then \( FH/J \) is semisimple and still \( U^+(FH/J) \) satisfies a GI. Hence by Lemma 2.1, the symmetric elements of \( FH/J \) are central. Since by Theorem 2.6 the \( p \)-elements of \( H \) form a subgroup \( P \), then \( J = \Delta(H, P) \).

Since \( h \in H^+, hgh^{-1} - g \in \Delta(H, P) \). But by the above, \( hgh^{-1} = g^i \) and this says that \( hgh^{-1} \) and \( g \) commute. As a consequence, since \( (hgh^{-1} - g)^{p^i j} = 0 \), for some \( j \geq 1 \), we obtain that \( h^p g \hat{g}^{-1} = g^p \hat{g}^{-1} \). Since \( g \) is a \( p' \)-element, this implies that \( hgh^{-1} = g \), and we are done. \( \square \)

As a consequence of the above lemma and Lemma 2.2, we obtain the following.

Corollary 3.3. Let \( g \in G^\ast \cap G_{p'} \). Then \( gh = hg \), for all \( h \in G^\ast \).

Lemma 3.4. Let \( g \in G \). If \( gg^\ast \in G_{p'} \), then \( gg^\ast = g^\ast g \).

Proof. Since \( gg^\ast \in G_{p'} \), then \( \hat{gg^\ast} \) is central in \( FG \). Thus \( g^{-1} \hat{gg^\ast} g = \hat{gg^\ast} \), which says that \( g^\ast g = (gg^\ast)^i \), for some \( i \geq 1 \). By exchanging the role of \( g \) and \( g^\ast \), we also get that \( gg^\ast = (g^\ast g)^j \), for some \( j \geq 1 \).

Now, \( g^\ast 2 g = g^\ast (gg^\ast)^j = (g^\ast g)^j g^\ast \), and by induction it is easily seen that

\[ g^k g = (g^\ast g)^u g^{k-1} \]

where \( u = i^{k-1} \). Similarly one can prove the formulas

\[ g^k g^l = \begin{cases} (g^\ast g)^v (g^\ast)^{k-l} & \text{if } k \geq l, \\ (g^\ast g)^v g^{l-k} & \text{if } l \leq k, \end{cases} \]

for some integer \( v \geq 1 \). Similar formulas hold for the products of the type \( g^k g^s l \). Since \( gg^\ast = (g^\ast g)^j \), then we get the formulas

\[ g^k g^s l = \begin{cases} (g^\ast g)^w g^{k-l} & \text{if } k \geq l, \\ (g^\ast g)^w g^{s-l-k} & \text{if } l \leq k, \end{cases} \]

for some \( w \geq 1 \). All these relations say that the group \( H = (g, g^\ast) \), generated by \( g \) and \( g^\ast \) is finite. By Theorem 2.6, \( H/P \) is either abelian or an SLC group. Thus \( gg^\ast P = g^\ast g P \). Since \( gg^\ast = (g^\ast g)^j \), then \( (gg^\ast, g^\ast g) = 1 \) and this implies that \( (gg^\ast)^n = (g^\ast g)^{pn} \), for some \( n \geq 0 \). But \( gg^\ast \) and \( g^\ast g \) are \( p' \)-elements which implies that \( gg^\ast = g^\ast g \). \( \square \)
Corollary 3.5. If \( g \in P_1 \) and \( gg^* \in G_{p'} \), then \( gg^* = 1 \).

Proof. \( gg^* \in G_{p'} \cap P = 1. \) □

Lemma 3.6. If \( g \in G^+ \cap G_{p'} \), then \( gh = hg \), for all \( h \in G_{p'} \).

Proof. Let \( h \in G_{p'} \) be such that \( gh \neq hg \). Let \( o(hh^*) = p^kt \) with \( (p, t) = 1 \) and \( k \geq 0, \ t \geq 1 \). Write \( x = (hh^*)t^j \) where \( 2r + 1 = p^k \) is such that \( xx^* = (hh^*)p^j \). Then \( (xx^*)^j = 1 \) and, since \( (p, t) = 1 \), by Lemma 3.4 we have \( xx^* = x^* \).

Let \( H \) be the subgroup generated by \( g, x \) and \( x^* \). Then, since \( \widehat{G}/o(g) \) is a symmetric idempotent, it is central. Thus \( xg^i = x^ig \), for some \( i \geq 1 \). Similarly one gets that \( x^ig = g^ix^* \), for some \( j \geq 1 \). These relations say that \( H \) is a finite group. Hence, by Theorem 2.6, \( H/P \) is commutative or SLC. In any case \( gP \), being symmetric, is central in \( H/P \). This says that \( (x, g) \in P \), hence \( g \in G_{p'} \), for some \( m \geq 0 \). But \( (x, g) = xgx^{-1}g^{-1} = g^igj^{-1} = g^j \) is a \( p' \)-element, forcing \( (x, g) = 1 \).

Now, by Corollary 3.3, \( gh^* = hh^*g \) hence, recalling that \( x = (hh^*)t^j \) and \( gx = xg \) we get that \( gh = hg \). □

Lemma 3.7. Let \( \text{char} \ F = p > 2 \) and let \( g, h \in G \). If \( g \in G_{p'} \), then \( \widehat{gh} \neq 0 \). In fact, each \( g^{i}h^{j} \in \text{Supp} \ \widehat{gh} \).

Proof. If \( \langle g \rangle \cap \langle h \rangle = 1 \) then all \( g^{i}h^{j} \) are distinct and there is nothing to prove. So let \( \langle g \rangle \cap \langle h \rangle = \langle g^{d} \rangle = \langle h^{d} \rangle, \ g^{d} = h^{d} \neq 1 \). Suppose \( g^{i}h^{j} = g^{k}h^{m} \). Then

\[
\begin{align*}
g^{i-t} = h^{m-j} \quad \text{so} \quad i & \equiv \ell \mod d, \quad j \equiv m \mod t. \\
\end{align*}
\]

Let us fix \( i \) and \( j \). Then

\[
\begin{align*}
g^{i}h^{j} = g^{i}g^{d}g^{-d}h^{j} = g^{i+d}h^{i-t} = \ldots.
\end{align*}
\]

There are \( o(g)/d \) equal values. Thus the coefficient of \( g^{i}h^{j} \) is \( o(g)/d \neq 0 \). □

Lemma 3.8. If \( g \) is an element of \( G \) of order \( p \) then \( gg^* = g^*g \).

Proof. We can assume by Corollary 3.5 that \( gg^* \notin G_{p'} \). Then by Sublemma 1.2 we have \( (g-1)gg^* = 0 \), and so

\[
\begin{align*}
\widehat{gg^*} = \widehat{gg^*}.
\end{align*}
\]

If the terms on the right are not distinct then \( (gg^*)^{i}g^{k} = (gg^*)^{j}g^{m} \) for some \( i, k, \ell, m \). As \( g \) is of prime order we conclude that \( g \in \langle gg^* \rangle \) and \( (g, g^*g) = 1 \). Thus \( gg^* = g^*g \) and we are done. We are left with the case that all the terms on the right are distinct. Moreover, the finite set \( S = \langle gg^* \rangle \langle g \rangle \) satisfies \( S \cdot S \subset S \). Thus \( S \) is a group \( \langle g, g^* \rangle \). Let us write, for convenience, \( S = G \) and \( gg^* = t \). Then \( U^+(FG) = Gt^1 \) and \( P \) is a subgroup. Therefore, \( G \) is a \( p \)-group with \( t \) a subgroup of index \( p \). So \( t \) is normal in \( G \). Suppose \( o(t) = p^k, \ k > 1 \). Then \( \text{Aut}(t) \) is cyclic of order \( p^{k-1}(p-1) \). The subgroup of order \( p \) is generated by \( t \rightarrow t^{1+p^{k-1}} \). We have that \( g^* = g^{-1}t \) has order \( p \) as \( g \in P_1 \). Therefore,

\[
\begin{align*}
1 = (g^{-1}t)^p &= t^g \cdot t^{g^2} \cdot \ldots \cdot t^{g^p} - t \cdot t^g \cdot \ldots \cdot t^{g^{p-1}} \\
&= t^{1+(1+cp^{k-1})+(1+cp^{k-1})^2+\ldots+(1+cp^{k-1})^{p-1}} \quad \text{for some} \ 1 \leq c \leq p-1
\end{align*}
\]

\[
\begin{align*}
= t^p
\end{align*}
\]
as the exponent is congruent to \( p \) mod \( p^k \). We have proved that \( \langle g, g^* \rangle \) has order \( p^2 \) and therefore is abelian as desired. \( \square \)

**Lemma 3.9.**

(i) If \( g \in P_1 \) and \( h \in P \) then \( g \) normalizes \( (h) \);

(ii) \( P_1 \) is a normal abelian subgroup of \( G \);

(iii) If \( g \) is an element of order \( p \) such that \( g^* = g \) or \( g^{-1} \) and \( h \in G' \) then \( h \) normalizes \( (g) \).

**Proof.** (i) Let us first assume that \( g^* = g \) or \( g^{-1} \). Then by Sublemma 1.2, \( (1-h)\hat{g}h = 0 \) which implies

\[
\hat{g}h = h\hat{g}.
\]

If all the terms on the right are not distinct then \( g \in (h) \) and the assertion follows. Otherwise, \( S = \langle g \rangle (h) \) is a \( p \)-group having a subgroup \( (h) \) of index \( p \). Then \( (h) \triangleleft S \). It remains to consider the case that \( \langle g \rangle \) is not \( * \)-invariant. Then by the last lemma, \( \langle g, g^* \rangle = A \) is an abelian group of order \( p^2 \). We can write \( A = (a_1) \times (a_2) \) with \( a_1^2 = a_1, a_2^2 = a_2^{-1} \). Then by the above case, \( a_1 \) and \( a_2 \) normalize \( (h) \) and \( g \) normalizes \( (h) \). This proves (i).

(ii) Let \( x, y \in P_1 \). Then by (i) \( x \) normalizes \( (y) \) and \( y \) normalizes \( (x) \). Thus we have \( xy = yx \) and \( P_1 \) is abelian and of course normal.

(iii) We have, again by Sublemma 1.2, that

\[
(h-1)\hat{g}h = 0.
\]

Therefore, \( h\hat{g}h = \hat{g} \) which is not zero by Lemma 3.7. It follows that \( \langle g \rangle (h) \) is a group. Then

\[
P_1 \ni g^h = g_1h_1, \quad g_1 \in \langle g \rangle, \ h_1 \in \langle h \rangle.
\]

Thus \( h_1 \in g_1^{-1}P_1 = P_1 \) implies \( h_1 = 1 \). Hence \( \langle g \rangle \) is normalized by \( h \). \( \square \)

**Proposition 3.10.** \( \Delta(G, P_1) \) is nil.

**Proof.** Let \( 0 \neq \gamma \in \Delta(G, P_1) \). Decompose each element in the support of \( \gamma \) into a product of finite sets of \( p \)-elements \( \{x_i\} \) and \( p^e \)-elements \( \{y_j\} \). We may suppose \( G = \langle P_1, x_1, \ldots, y_1, \ldots \rangle \). Let \( H = \langle P_1, x_1, \ldots \rangle, \ G = \langle H, y_1, \ldots \rangle \). Write \( x_i = x \). Then \( P_1 \) acts as automorphisms of \( \langle x \rangle \). Thus \( C_{P_1}(x) \) is of finite index in \( P_1 \). Consequently, \( C = \bigcap C_{P_1}(x_i) \) is of finite index and central in \( H \). We can take \( C \) to be \( * \)-invariant by considering \( C \cap C^* \). Let us write \( C = \Pi (c_i) \cdot \Pi (d_i) \)

\[
c_i^x = c_i, \quad d_i^x = d_i^{-1}, \quad c_i^y \in \langle c_i \rangle, \quad d_i^y \in \langle d_i \rangle
\]

for all \( y = y_j \).

Each \( y \) normalizes \( \langle c_i \rangle, \langle d_i \rangle \) and so also \( C \). This insures that \( C \triangleleft G \). In \( FG(C) \) we have \( \gamma \in \Delta(G, P_1) \) which is nilpotent. So \( \gamma^p \in \Delta(G, C) \) and hence we may assume \( \gamma \in \Delta(G, C) \). Since \( \gamma = \sum \gamma_i(1 - c_i), \ c_i \in C \), we can say that \( \gamma \in \Delta(G, D) \) where \( D \) is finite. But every finite subgroup \( D \) of \( C \) is contained in a finite normal subgroup \( E = \langle c_1 \rangle \times \cdots \times \langle c_i \rangle \times \langle d_1 \rangle \times \cdots \times \langle d_m \rangle \). We have

\[
\gamma \in \Delta(G, E) = F \Delta(E)
\]

which is nilpotent as claimed. \( \square \)

**Proposition 3.11.** If \( FG \) is semiprime then \( P = 1 \).
in the centre of $G$ since $g\in H$. Our aim is to prove that $H$ is a finite group. To this end, since $g$ and $g^*$ commute, the element $\widetilde{gg^*}$ is a multiple of a symmetric idempotent, hence it lies in the centre of $H$. Notice that since $g\in G_p'$, by Lemma 3.7, $\widetilde{gg^*}\neq 0$. Thus from $h\widetilde{gg^*}h^{-1} = \widetilde{gg^*}$, we deduce that $hgh^{-1} = g^ig^{j^*}$, for some $i, j$. Recall that $g^ig^{j^*} \in Z(H)$, the centre of $H$, for all $u \geq 1$. We deduce that

$$hg = cg^kh \quad \text{or} \quad hg = cg^{k^*}h,$$

for some central element $c$. By using $h^*$ instead of $h$ we also get

$$h^*g = c_1g^lh^* \quad \text{or} \quad h^*g = c_1g^{l^*}h^*,$$

for some $c_1 \in Z(H)$. Similar expressions hold for $h^*g^*$ and $hg^*$. The outcome of this is that $H$ is a finite group. Hence by Theorem 2.6, $H$ is either abelian or SLC. But then by Sublemma 1.6, the symmetric elements of $FH$ commute. In particular $[g + g^*, h + h^*] = 0$. Since $g$ and $h$ are arbitrary in $G$ this implies that $[FG^+, FG^+] = 0$. Again, by Sublemma 1.6 we get that $G$ is either abelian or an SLC group.

Sufficiency follows from Sublemma 1.6. □

4. A crucial special case

In this section we assume that $\text{char} \ F = p > 2$ and $U^+(FG)$ is $GI$. We shall handle the case when $G = A \rtimes \langle t \rangle$ where $A$ is an abelian normal subgroup and $t$ is of prime order $q \neq 2$.

**Theorem 4.1.** Suppose $G$ is a torsion group and $F$ is an infinite field of characteristic $p > 2$. Let $*$ be an involution of $G$ such that $U^+(FG)$ is $GI$. Then $FG$ is $PI$ and $P$ is a subgroup.

**Proof.** If $FG$ is semiprime then $G$ has an abelian subgroup of index at most four. Thus $FG \in PI$. Let $N$ be the sum of all nilpotent ideals of $FG$. Then $N^s = N$. Let $\phi$ be the $FC$-subgroup of $G$ and $\phi_p(G) = (\phi \cap P)$. If $N$ is nonzero, nilpotent then $\phi_p(G)$ is finite, $F(G/\phi_p(G))$ is semiprime and so $PI$. It follows that $FG$ is also $PI$. If $N$ is nonzero nil but not nilpotent then by Sublemma 1.4, $FG$ satisfies a generalized polynomial identity. It follows that $P$ is a subgroup and

$$(G : \phi) < \infty, \quad (\phi : 1) < \infty.$$

Either $G/P$ is abelian or $G$ has a subgroup $H$ of index four such that $H'$ is a $p$-group. Therefore either $\phi'$ is a finite $p$-group and $FG$ is $PI$ as asserted or $FH$ is $PI$ and thus again $FG$ is $PI$. Hence $G$ is locally finite and it follows from Corollary 2.4 that $P$ is a subgroup. □

The next result is the heart of the matter.

We are now able to classify semiprime group algebras $FG$ when $U^+(FG)$ is $GI$.

**Theorem 3.12.** Let $F$ be an infinite field of characteristic different from 2 and $G$ a torsion group such that $FG$ is semiprime. Then $U^+(FG)$ satisfies a group identity if and only if $G$ is either abelian or an SLC group.

**Proof.** Let $g, h \in G$ and let $H = (g, g^*, h, h^*)$ be the subgroup generated by $g$, $h$ and their *. By Lemma 3.6, $gg^*$ and $hh^*$ being symmetric commute with $g$ and $h$. Thus $gg^* = g^*g$ and $hh^* = h^*h$ lie in the centre of $H$. This follows that $H$ is a finite group. If $g$ and $g^*$ commute, the element $\widetilde{gg^*}$ is a multiple of a symmetric idempotent, hence it lies in the centre of $H$. Notice that since $g \in G_p'$, by Lemma 3.7, $\widetilde{gg^*} \neq 0$. Thus from $h\widetilde{gg^*}h^{-1} = \widetilde{gg^*}$, we deduce that $hgh^{-1} = g^ig^{j^*}$, for some $i, j$. Recall that $g^ig^{j^*} \in Z(H)$, the centre of $H$, for all $u \geq 1$. We deduce that

$$hg = cg^kh \quad \text{or} \quad hg = cg^{k^*}h,$$

for some central element $c$. By using $h^*$ instead of $h$ we also get

$$h^*g = c_1g^lh^* \quad \text{or} \quad h^*g = c_1g^{l^*}h^*,$$

for some $c_1 \in Z(H)$. Similar expressions hold for $h^*g^*$ and $hg^*$. The outcome of this is that $H$ is a finite group. Hence by Theorem 2.6, $H$ is either abelian or SLC. But then by Sublemma 1.6, the symmetric elements of $FH$ commute. In particular $[g + g^*, h + h^*] = 0$. Since $g$ and $h$ are arbitrary in $G$ this implies that $[FG^+, FG^+] = 0$. Again, by Sublemma 1.6 we get that $G$ is either abelian or an SLC group.

Sufficiency follows from Sublemma 1.6. □
Theorem 4.2. Suppose that $F$ is an infinite field of characteristic $p > 2$ and that $G$ is a torsion group, $G = A \rtimes (t)$ where $A$ is an abelian subgroup and $t$ is of prime order $q$. Further suppose that $G$ has an involution $*$ such that $t^* = t$ or $t^{-1}$ and $U^+(FG)$ is $G1$. If $(A, t)$ is a $p$-group then it has bounded exponent.

Proof. Since $(A^k, t) = (A, t)^k$ and $(A, t)$ is a $p$-group, we can assume that $A$ has no 2-elements. As usual we write $A = A_1 \times A_2$ where $A_1 = \{a \in A \mid a^* = a\}$ and $A_2 = \{a \in A \mid a^* = a^{-1}\}$. The subgroups $A_1$ and $A_2$ need not be normal in $G$ but it suffices to prove that $(A_1, t)$ and $(A_2, t)$ are of bounded exponent. We shall discuss separately two cases according as $q \neq p$ or $q = p$. Let us set up some notations. Write $\tau = \sum t^i$. Then $\tau^2 = q\tau$, $\tau a\tau = tr(a)\tau$, where $tr(a) = \sum d^i$ is central.

Case 1. $q \neq p$.

(i) Let us look at $(A_1, t)$, $t^* = t^{-1}$. We shall use Sublemma 1.1 which says that if $\alpha^2 = 0$ then $(\alpha\alpha^*)^{pN} = 0$ for a fixed $N$. Let $a \in A_1$. Write $\alpha = \tau a(t - 1)$. Then $\alpha^* = (t^{-1} - 1)a\tau$ and

\[
\alpha\alpha^* = \tau a(t - 1)(t^{-1} - 1)a\tau
= \tau a(2 - t - t^{-1})a\tau
= 2\tau a^2\tau - \tau ata\tau - \tau at^{-1}a\tau
= 2\tau a^2\tau - \tau a\tau^*a\tau - \tau a\tau^*\tau
= 2(\tau a^2\tau - \tau (aa^*)\tau)
= 2(tr(a^2) - tr(aa^*))\tau.
\]

Since $q \neq p$, $\tau p^k = q^{p^k-1}\tau \neq 0$ we can conclude that $(tr(a^2) - tr(aa^*))^{pN} = 0$ and consequently,

\[
(tr(a^2))^{pN} = (tr(aa^*))^{pN}.
\]

We get

\[
(a^2 + a^{2t} + \cdots + a^{2t^{p-1}})^{pN} = (aa^t + a^t a^{2t} + \cdots)^{pN},
\]

\[
b^2 + b^{2t} + \cdots + b^{2t^{p-1}} = bb^t + b^t b^{2t} + \cdots
\]

where $b = a^{pN}$. In case $q = 2$ we get $b = b^t$, so let $q \neq 2$. Either $b^2 = b^{2t^k}$ which means $(b^2, t^k) = 1$, $(b^2, t) = 1$ and so $(b, t) = 1$ as $q$ is prime and we have no 2-elements in $(A, t)$. This means $(a^{pN}, t) = 1$ and $(a, t)^{pN} = 1$ as desired. Or we have $b^2 = b^{2t^k} = b^{t^{k+1}}$ for some $k$. We deduce $b^{t^{k+1}+t^k-2} = 1$. Thus the operator $t^{k+1}+t^k-2$ vanishes on $B = A^{pN}$. Also, $t^q - 1 = 0$. We claim.

Sublemma 4.3. There exist polynomials $f(x), g(x) \in \mathbb{Z}[x]$ so that

\[
f(x)(x^q - 1) + g(x)(x^{k+1} + x^k - 2) = z_k(x - 1)
\]

for some $z_k \in \mathbb{Z}$.

Proof. In $\mathbb{Q}[x]$ the greatest common divisor

\[
(x^q - 1, x^{k+1} + x^k - 2) = x - 1.
\]
This is because if a polynomial other than \( x - 1 \) divides both these polynomials then it has a factor \( x - \xi \) in \( \mathbb{C}[x] \) for some root of unity \( \xi \in \mathbb{C} \). Then \( \xi^{k+1} + \xi^k = 2 \). The only roots of unity having this property are \( \xi^k = 1 = \xi^{k+1} \) and \( \xi = 1 \). Therefore in \( \mathbb{Q}[x] \) we can find polynomials \( f_1(x), g_1(x) \) so that

\[
f_1(x)(x^q - 1) + g_1(x)(x^{k+1} + x^k - 2) = (x - 1).
\]

Clearing the denominators we get \( f(x) \) and \( g(x) \in \mathbb{Z}[x] \)

\[
f(x)(x^q - 1) + g(x)(x^{k+1} + x^k - 2) = z_k(x - 1)
\]

for some \( z_k \in \mathbb{Z} \). This proves the sublemma. \( \square \)

So we find that if \( b^2 = b^{t^k}b^t \) then

\[
bf(t)(x^q - 1) \cdot bg(t)(x^{k+1} + x^k - 2) = b^{z_k(t-1)}
\]

\[
(b^{z_k})^{t^{-1}} = 1, \quad (t, b^{z_k}) = 1.
\]

Therefore for all \( b \in B \) we have

\[
(t, b^2) = 1 \quad \text{where} \quad z = \prod z_k.
\]

Since \( (A, t) \) is a \( p \)-group we may take \( z \) to be a prime power \( p^\ell \). We have proved

\[
(t, a^{p^{N+\ell}}) = (t, a)^{p^{N+\ell}} = 1.
\]

Thus \( (t, A_1) \) has bounded exponent if \( t^a = t^{-1} \).

(ii) Now we consider the case \( t^a = t \) which will be dealt with in a similar fashion. Take \( a \in A_1 \) and let \( \alpha = \tau a(t - 1) \). Then \( \alpha^* = (t - 1)a\tau \).

\[
\alpha\alpha^* = \tau a(t - 1)(t - 1)a\tau
\]

\[
= \tau a^2\tau + \tau a^2a\tau - 2\tau a\tau a\tau
\]

\[
= (tr(a^2) + tr(a^2a) - 2tr(aa^r))\tau.
\]

By Sublemma 1.1, \( (\alpha\alpha^*)^{p^N} = 0 \) for a fixed \( N \). This means

\[
(tr(a^2) + tr(a^2a) - 2tr(aa^r))^{p^N} = 0.
\]

Writing \( a^{p^N} = b \) we obtain

\[
tr(b^2) + tr(b^2b) - 2tr(bb^r) = 0.
\]

In case \( q = 2 \) we get \( b = b^t \). So let \( q \neq 2 \). There are three possible equalities for \( b^2 \) in this expression. If \( b^2 = (b^2)^t \) then \( b^2 = (b^2)^t \) and \( (b, t) = 1 \) as desired. If \( b^2 = (b^2b)^s \), a term from the second trace then \( b^2 = b^{tkq + tk}b^{t^k} \) which means the operator \( t^{k+2} + t^k - 2 \) vanishes on \( B = A^{p^N} \). This implies as in
the last case that \( t = 1 \) on \( B \) (remember that \( q \) is odd and \((-1)\) is not a root of \( x^q - 1 \)). The same conclusion follows in the last equality \( b^2 = (bb^s)^s \). We have proved \((A_1, t)\) is of bounded exponent.

(iii) We shall now prove that \((A_2, t)\) has bounded exponent if \( t^s = t^{-1} \). We pick an \( a \in A_2 \) and write \( \alpha = t\alpha^{-1}(1 - \alpha^{-1}) \). Then \( \alpha^s = (1 - \alpha)t\alpha^{-1} \). It follows by Sublemma 1.1 that \((\alpha\alpha^s)^pN = 0 \) for a fixed \( N \). As in [6, page 459],

\[
0 = (\alpha\alpha^s)^pN = (2q - tr((t, a)) - tr((t^{-1}, a))\tau)^pN.
\]

It follows as in [6] that \((t, a)^pN = 0 \). It remains to consider.

(iv) \((A_2, t)\) when \( t^s = t \). Let \( a \in A_2 \). Write \( \alpha = \tau a(1 - 1) \). Then \( \alpha^s = (t - 1)a^{-1}\tau \), \( \alpha\alpha^s = \tau a(1 + t^2 - 2t)a^{-1}\tau \). We have

\[
\alpha\alpha^s = \tau^2 + \tau at^2a^{-1}\tau - 2\tau ata^{-1}\tau, \\
= \tau^2 + \tau((t^2, a^{-1}))\tau - 2\tau((t, a^{-1}))\tau \\
= (q + \tau((t^2, a^{-1})) - 2\tau((t, a^{-1})))\tau.
\]

Therefore

\[
0 = (\alpha\alpha^s)^pN = (q + tr((t^2, a^{-1})) - 2 tr((t, a^{-1})))^pN(\tau)^pN.
\]

It follows that either \((t, a)^pN = 1 \) or \((t^2, a^{-1})^pN = 1 \). In any case we deduce that \((a, t)^pN = 1 \) as we wanted. This finishes the case when \( q \neq p \). We consider now

**Case 2.** \( q = p \).

(i) We first look at \((A_2, t)\), with \( t^s = t \) or \( t^{-1} \), we have \( t^p = 1 \). Let \( a \in A_2 \). Then \( \tau \) and \( a^{-1}\tau a \) are symmetric, square zero elements. Therefore, by Sublemma 1.1, we conclude that \( \alpha = a^{-1}\tau a \tau = tr(a)a^{-1}\tau \) satisfies \( \alpha^pN = 0 \). As in Passman [18, page 659] we conclude

\[
tr(b^{-1}) tr(b) = 0 \quad \text{where} \quad b = a^pN.
\]

It follows as in [18] that \((A_2, t)^pN+1 = 1 \).

(ii) Now we take up the case \((A_1, t)\). Let \( a \in A_1 \) and \( t^s = t \) or \( t^{-1} \). Then \( \alpha = a^{-1}\tau a \) is a square zero element. It follows by Sublemma 1.1 that \((\alpha\alpha^s)^pN = 0 \). We have

\[
0 = (a^{-1}\tau a \tau a^{-1})^pN = (a^{-1}\tau a^2 \tau a^{-1})^pN \\
= (a^{-1} tr(a^2)) \tau a^{-1})^pN = (tr(a^2))^pN (a^{-1}\tau a^{-1})^pN.
\]

Observe that

\[
(a^{-1}\tau a^{-1})(a^{-1}\tau a^{-1}) = a^{-1} tr(a^{-2}) \tau a^{-1} \\
= tr(a^{-2}) \tau a^{-1}
\]

and thus \((a^{-1}\tau a^{-1})^pN = (tr a^{-2})^pN-1(a^{-1}\tau a^{-1}) \). We conclude
Then again by Passman [18], \( (A_1, t)^{p^N} = 0 \). \( \square \)

5. Proof of main theorem

In this section we assume that \( G \) is a torsion group, \( F \) is an infinite field of characteristic \( p > 2 \) and \( U^+ = U^+(FG) \) is GI. It follows from Theorem 4.1 that \( P \) is a subgroup of \( G \).

Theorem 5.1. Suppose that \( A \) is an abelian normal invariant subgroup of finite index in \( G \). If \( G/P \) is abelian then \( (G, A) \) is of bounded exponent.

Proof. We shall prove the theorem by induction on \( (G : A) \). We know by the finite case that \( G/A \) is solvable. Suppose there exists a proper normal \( \ast \)-invariant subgroup \( A \subset N \subset G \). Then \( (N, A) \) is of bounded exponent. Also, \( U^+ F(N/(N, A)) \in GI \). So we can factor by \( (N, A) \) and assume that \( A \) is central in \( N \). It follows by Schur's Theorem [19, page 39] that \( N' \) is finite. Now, \( N \) being normal and \( \ast \)-invariant, we can factor by \( N' \) and assume that \( N \) is abelian. We have \( (G : N) < (G : A) \). It follows by induction that \( (G, N) \) is of bounded exponent. We are done, unless \( G = \langle A, t \rangle \), \( t^q \in A \) for a prime number \( q \) and \( t^n = t^{-1} \) (mod \( A \)). Suppose \( t^q = a \). Then the normal and \( \ast \)-closure of \( (a) \) is finite. Factoring we can assume that \( a = 1 \). So \( G = A \times (t) \). Also, \( t^n = c^e \) (mod \( A \)), \( e = \pm 1 \), say, \( t^n = t^e c \), \( c \in A \). Factoring by the normal and \( \ast \)-closure of \( (c) \) we can assume that \( t^n = t^e \). The result follows from Theorem 4.2. \( \square \)

Corollary 5.2. Suppose that \( G/P \) is abelian. Then \( G' \) is of bounded \( p \)-power exponent.

Proof. Since \( FG \) is \( PI \) we have

\[
G \triangleright A \triangleright A' \triangleright 1
\]

with \( (G : A) \) and \( (A' : 1) \) both finite. Since \( G' \) is a \( p \)-group we can factor by \( A' \) to assume that \( A \) is abelian. Then it follows by Theorem 5.1 that \( (G, A) \) is of bounded \( p \)-power exponent. Factoring by \( (G, A) \) we can assume that \( A \) is central in \( G \). We are finished now by applying Schur's Theorem [19, page 39]. \( \square \)

Corollary 5.3. If \( G/P \) is SLC, then \( G' \) is of exponent \( 2p^N \) for a fixed \( N \).

Proof. We have \( G \triangleright H \triangleright P \triangleright 1 \) where \( G/P \) is an SLC group and \( G = \langle H, x, y \rangle \). \( H/P \) is central in \( G/P \), \( x^2 \in H, y^2 \in H, xy = yx \), \( z \in H \), a unique commutator in \( G/P \). We have

\[
G = H_1 H_2 \quad \text{where} \quad H_1 = \langle H, x \rangle \text{ and } H_2 = \langle H, y \rangle
\]

are both \( \ast \)-invariant and abelian mod \( P \). Thus by the last corollary \( (H_1, P) \) and \( (H_2, P) \) are both of bounded \( p \)-power exponent. Factoring by the product of these two normal subgroups we get that \( P \) is central in \( G \) and that \( G = P \times X \) where \( X \) is SLC. Thus \( G' = X' \) is of order \( 2 \). This completes the proof. \( \square \)

Corollary 5.4. If \( G/P \) is SLC, then \( (G, P) \) is of finite exponent \( p^N \).

Let \( K_8 \) be the quaternion group of order 8, \( D_8 \) the dihedral group of order 8, and \( D_8^+ = \langle a, b \mid a^8 = 1 = b^2, a^b = a^5 \rangle \).
Lemma 5.5. Every SLC group has a $*$-invariant homomorphic image equal to $K_8$ or $D_8$ or $S_8^+$. 

Proof. Since $G/Z \cong C_2 \times C_2$ we can assume by taking a suitable homomorphic image that $G$ is finite. Then by [12, page 129] we can write $G$ as $D \times A$ where $D$ is indecomposable SLC, $A$ is central and

$$G = \langle x, y, Z(D) \rangle$$

$$Z(D) = \langle t_1 \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle, \quad o(t_i) = 2^{m_i}, \quad m_1 \geq 1, \quad m_2, m_3 \geq 0, \quad \text{and}$$

$$z = (x, y) = t_1^{2^{m_1-1}}.$$

If $m_1 = 1$ we set $N = \langle t_2 \rangle \times \langle t_3 \rangle \times A$ and $\overline{G} = G/N = \langle \bar{x}, \bar{y} \rangle$ is nonabelian of order 8 and so isomorphic to $K_8$ or $D_8$. Let us now suppose that $m_1 \geq 2$. Then set $N = \langle t_1^{2^{m_1-2}} \rangle \times \langle t_2 \rangle \times \langle t_3 \rangle \times A$. In this case, $\overline{G} = G/N \cong (x, y, t \mid t^4 = 1, \; t \text{ central}, \; (x)^2 \in \langle t \rangle, \; (y)^2 \in \langle t \rangle, \; (x, y) = t^2)$. If $x^2 = t^2$ set $x_1 = xt$ and $(x_1)^2 = x_1t^2 = t$. If $x^2 = t^2$ then $x_1^2 = (xt)^2 = 1$. Consequently, we may assume that $x^2, y^2 \in \{1, t\}$. We have three possibilities:

$$G_1 = \langle x, y, t \mid x^2 = 1 = y^2 = t^4, \; (x, y) = t^2 \rangle$$

$$G_2 = \langle x, y, t \mid x^2 = t, \; y^2 = 1, \; t^4 = 1, \; (x, y) = t^2 \rangle$$

$$G_3 = \langle x, y, t \mid x^2 = t, \; y^2 = t, \; t^4 = 1, \; (x, y) = t^2 \rangle.$$ 

In the last case $(xy)^2 = xxyx = x^2t^2y^2 = 1$ and so $G_3 \cong G_2$. The group $G_1$ is nonabelian of order 8 and $G_2$ is $D_8^+$ as claimed. 

Lemma 5.6. Suppose $FG$ is $PI$ and let $B$ be a $*$-invariant normal $p$-subgroup of $G$ of bounded exponent. Then $\Delta(G, B)$ is nil of bounded degree.

Proof. Since $FG$ is $PI$, $G$ has a normal $p$-abelian subgroup $A$ of finite index, $G \triangleleft A \triangleleft A' \triangleleft 1$. Since $A'$ is finite, $\Delta(G, A') = FG\Delta(A')$ is nilpotent. By factoring with $A'$ we assume that $A$ is abelian. Let $B_0 = A \cap B$ then $(B : B_0)$ is finite and

$$\Delta(G, B)^p = FG(\Delta B)^p \subseteq FG\Delta(B, B_0) = FG\Delta(B_0) = FGI$$

where $I = \Delta(A, B_0)$. Then $I$ is a $G$-stable ideal of $FA$. Moreover, $I$ is nil of bounded degree as $A$ is abelian. It follows by Sublemma 1.5 that $\Delta(G, B)$ is nil of bounded degree, as claimed. 

Proof of Main Theorem. (a) is Theorem 3.12. To prove (b) recall, by Theorem 4.1, that $FG$ is $PI$. Thus $G$ is locally finite and so $P$ is a subgroup. Therefore, $F(G/P)$ is semiprime and $G/P$ is abelian or SLC by (a). If $G/P$ is abelian then $G'$ is of bounded $p$-power exponent by Corollary 5.2, proving (i).

We shall now prove (ii). We know by Corollary 5.4 that $(G, P)$ is of bounded $p$-power exponent. Factoring by $(G, P)$ we can assume that $P$ is central in $G$. Thus $G = P \times L$ where $L$ is an SLC group. Moreover we know from Lemma 5.5 that $L$ has a homomorphic image the quaternion group $K_8$ or the dihedral group $D_8$ of order 8 or $D_8^+$ of order 16. Let us decompose $P = P_1 \times P_2$. As usual, $P_1 = \{x \in P \mid x^8 = x\}$ and $P_2 = \{x \in P \mid x^{x^{-1}} = x^{-1}\}$. Then we observe that $\ast$ on $K_8$ induced from the SLC group $L$ is the same as the classical involution. It was proved in [6] that in this case $P_2$ must be of bounded exponent.

Let us now assume that we have a homomorphic image $D_8 = \{a, b \mid a^4 = 1 = b^2, \; ab = a^{-1}\}$. Write $a^2 = x$ as the unique commutator of the SLC group. Let $x \in P_2$. Consider the element

$$\alpha = (x + x^{-1}z)(a + b)(1 - z).$$

Then $\alpha^z = (1 - z)(az + bx)(x^{-1} + xz)(a + b)z = (1 - z)(x + x^{-1}z)(a + b) = \alpha$ is symmetric. Also, $\beta = (x + x^{-1}z)(b - a)(1 - z)$ is symmetric. In the representation
where

\[ \alpha \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x - x^{-1} & 0 \\ 0 & x - x^{-1} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \].

Also,

\[ \beta \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x - x^{-1} & 0 \\ 0 & x - x^{-1} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \].

We know by Sublemma 1.1 that \((\alpha \beta)^N = 0\) for a fixed \(N\). Consequently, \((x - x^{-1})^N = x^{-1}(x^2 - 1)^N = 0\). It follows that \(x\) has bounded order \(p^N\).

It remains to consider the case when \(D_8^+\) is a factor of \(L\):

\[ D_8^+ = \langle a, b \mid a^8 = 1 = b^2, a^b = a^5 \rangle. \]

As before, we denote the unique commutator by \(z\). Let \(x \in P_2\). We need two square zero elements

\[ \alpha = (x + x^{-1})z(1 - b)\alpha(1 + b)(1 - z) \quad \text{and} \quad \beta = (x + x^{-1})z(1 + b)\alpha(1 - b)(1 - z). \]

Since \(((1 - b)\alpha(1 + b))^s = -(1 - b)\alpha(1 + b) + ((1 + b)\alpha(1 - b))^s = -(1 + b)\alpha(1 - b)\) it follows that \(\alpha^s = \alpha\) and \(\beta^s = \beta\). Let us consider the homomorphism

\[ F(\langle x \rangle \times D_8^+) \rightarrow eF(\langle x \rangle \times D_8^+), \]

with \(e\) a suitable central idempotent of \(F(D_8^+)\), given by

\[ a \rightarrow \begin{bmatrix} \zeta & 0 \\ 0 & -\zeta \end{bmatrix}, \quad b \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad x \rightarrow \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \]

where \(\zeta\) is a primitive 8th root of unity. Then \(z \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\). We see that

\[ (1 - b)\alpha(1 + b) \rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & -\zeta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2\zeta \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \]

\[ (1 + b)\alpha(1 - b) \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & -\zeta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2\zeta \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \]

Then

\[ \alpha \rightarrow 4\zeta (x - x^{-1}) \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad \beta \rightarrow 4\zeta (x - x^{-1}) \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \]

\[ \alpha \beta \rightarrow 16\zeta^2 (x - x^{-1})^2 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 32\zeta^2 (x - x^{-1})^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \]
\[(\alpha \beta)^2 \rightarrow 2(32\zeta^2)(x-x^{-1})^4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \]

\[(\alpha \beta)^N \rightarrow 2^r \xi^s (x-x^{-1})^{2N} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ for some } r, s. \]

But we know by Sublemma 1.2 that \((\alpha \beta)^N = 0\) for a fixed \(N\). It is easily seen that \((x-x^{-1})^{2N} = 0\) and so \((x-x^{-1})^{p^N} = 0\). It follows that \(x\) has a bounded order.

We remark that the formal elements \(\alpha, \beta\) picked for \(D_8^+\) work for \(D_8\) as well if \(\zeta\) is a 4th root of unity.

In all cases, we set \(B = P_2\). This satisfies the conditions (ii) of the statement.

**Sufficiency:** In case \(G/P\) is abelian and \(G'\) is of bounded \(p\)-power exponent then \(U\) is GI [18]. Let us now assume that \(G/P\) is SLC and that we have \(B\) as in the statement. We know by Lemma 5.6 that \(\Delta(G, B)\) is nil of bounded \(p\)-power degree. In order to prove that \(U^+\) is GI we may factor by \(B\). Thus we have that \(P\) is central in \(G\), \(x^a = x\) for all \(x \in P\); \(G/P\) is a \(p'\)-SLC group. Hence \(G = P \times H\) where \(H\) is a \(p'\)-SLC group. It follows that \(G\) is an SLC group and \([[FG]^+, (FG)^+]] = 0\). Therefore \(\langle \gamma, \mu \rangle = 1\) for all \(\gamma, \mu \in U^+\). This completes the proof of the theorem. \(\square\)

References