Lie properties of symmetric elements in group rings II

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\textbf{A R T I C L E  I N F O}

\textbf{Article history:}
Received 16 August 2008
Available online 30 December 2008
Communicated by A.V. Geramita

\textbf{MSC:}
16S34
16R50

\textbf{A B S T R A C T}

Let $F$ be a field of characteristic different from 2, and $G$ a group with involution $\ast$. Write $(FG)^{\ast}$ for the set of elements in the group ring $FG$ that are symmetric with respect to the induced involution. Recently, Giambruno, Polcino Milles and Sehgal showed that if $G$ has no 2-elements, and $(FG)^{\ast}$ is Lie nilpotent (resp. Lie $n$-Engel), then $FG$ is Lie nilpotent (resp. Lie $m$-Engel, for some $m$). Here, we classify the groups containing 2-elements such that $(FG)^{\ast}$ is Lie nilpotent or Lie $n$-Engel.

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1. Introduction

Let $FG$ be the group ring of a group $G$ over a field $F$ of characteristic $p \neq 2$. If $\ast: G \rightarrow G$ is an involution, then it can be extended linearly to an involution on $FG$, also denoted by $\ast$. An element $\alpha \in FG$ is said to be symmetric if $\alpha^{\ast} = \alpha$, and we write $(FG)^{\ast}$ for the set of symmetric elements. These can easily be seen to be the linear combinations of $g + g^{\ast}$, for all $g \in G$.

For any group, we have the classical involution, given by $g^{\ast} = g^{-1}$ for all $g \in G$. There have been a number of papers (see, for instance [1,5,10–12]) devoted to determining the extent to which the Lie properties of the symmetric elements under the classical involution determine the Lie properties of the whole group ring. On any ring $R$, we let $[x_1, x_2] = x_1x_2 - x_2x_1$, and

$$[x_1, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}].$$

We say that a subset $\Lambda$ of $R$ is Lie nilpotent if there exists an $n$ such that $[r_1, \ldots, r_n] = 0$ for all $r_i \in \Lambda$, and $\Lambda$ is Lie $n$-Engel if $[r_1, r_2, \ldots, r_2] = 0$

for all $r_1, r_2 \in \Lambda$. In [5], Giambruno and Sehgal showed that if $G$ contains no 2-elements and $(FG)^{\ast}$ is Lie nilpotent, then so is $FG$. In [10], Lee classified the groups $G$, containing 2-elements, such that $(FG)^{\ast}$ is Lie nilpotent. He proved similar results for the Lie $n$-Engel property in [11].

Recently, there has been a considerable amount of work on involutions of $FG$ other than the classical one (see, for instance [2–4,6,9]). In particular, in [4], Giambruno, Polcino Milles and Sehgal showed that if $G$ has no 2-elements, and $(FG)^{\ast}$ is Lie nilpotent (resp. Lie $n$-Engel), then $FG$ is Lie nilpotent (resp. Lie $m$-Engel, for some $m$), for any involution $\ast$ on $G$.

Naturally, this cannot be expected to hold if the group contains 2-elements. Using the results in [4], we will establish the conditions under which $(FG)^{\ast}$ is Lie nilpotent or Lie $n$-Engel, when $G$ is a group containing 2-elements, with an arbitrary involution.

\textsuperscript{∗} This research was supported by NSERC of Canada.

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doi:10.1016/j.jpaa.2008.11.027
In order to state our main results, a definition is required. We recall that a group is said to be an LC-group (that is, it has the “lack of commutativity” property) if it is not abelian, but if $g, h \in G$, and $gh = hg$, then at least one of $g, h$ and $gh$ must be central. These groups were introduced by E. Goodaire. By [7, Proposition III.3.6], a group is an LC-group with a unique nonidentity commutator (which must, obviously, have order 2) if and only if $G/\zeta(G) \cong C_2 \times C_2$. Here, $\zeta(G)$ denotes the centre of $G$.

**Definition.** A group $G$ endowed with an involution $\ast$ is said to be a special LC-group, or SLC-group, if it is an LC-group, it has a unique nonidentity commutator $z$, and for all $g \in G$, we have $g^\ast = g$ if $g \in \zeta(G)$, and otherwise, $g^\ast = zg$.

We now present our main results of this paper.

**Theorem 1.** Let $F$ be a field of characteristic $p > 2$, and let $G$ be a group with involution $\ast$. Suppose that $FG$ is not Lie nilpotent. Then $(FG)^\ast$ is Lie nilpotent if and only if $G$ is nilpotent, and $G$ has a finite normal $\ast$-invariant $p$-subgroup $N$ such that $G/N$ is an SLC-group.

**Theorem 2.** Let $F$ be a field of characteristic $p > 2$, and let $G$ be a group with involution $\ast$. Suppose that $FG$ is not Lie $m$-Engel, for any $m$. Then $(FG)^\ast$ is Lie $n$-Engel if and only if $G$ is nilpotent, $G$ has a $p$-abelian $\ast$-invariant normal subgroup $A$ of finite index, and $G$ has a normal $\ast$-invariant $p$-subgroup $N$ of bounded exponent, such that $G/N$ is an SLC-group.

Here, we recall that $G$ is said to be $p$-abelian if $G'$ is a finite $p$-group, and 0-abelian means abelian.

2. Preliminaries and assumed results

Let us begin by explaining the need for the SLC-group. In [10,11], the exceptional cases turned out to involve Hamiltonian 2-groups, because they are nonabelian groups such that the symmetric elements in the group rings commute. Jespers and Ruiz Marín proved the following result for an arbitrary involution on $G$.

**Lemma 1.** Let $R$ be a commutative ring of characteristic different from 2, and $G$ a nonabelian group. Then the following are equivalent:
1. $(RG)^\ast$ is commutative,
2. $(RG)^\ast$ is the centre of $RG$, and
3. $G$ is an SLC-group.

**Proof.** See [9, Theorem 2.4]. $\Box$

This is especially helpful in combination with the following two results due to Giambruno, Polcino Milies and Sehgal.

**Lemma 2.** Let $R$ be a semiprime ring with involution, such that $2R = R$. If $R^\ast$ is Lie $n$-Engel, then $R^\ast$ is commutative.

**Proof.** See [4, Lemma 2.4]. $\Box$

**Lemma 3.** Suppose that char $F = p > 2$, and $(FG)^\ast$ is Lie $n$-Engel. Then the $p$-elements of $G$ form a (normal) subgroup of $G$.

**Proof.** See [4, Proposition 3.2]. (Although the groups there were implicitly assumed not to have 2-elements, that fact was not used in proving that the $p$-elements form a subgroup.) $\Box$

By a result of Passman (see [14, Theorem 4.2.12]), every group ring of characteristic zero is semiprime. Thus, the first two lemmas completely resolve this case, and we will assume for the rest of the paper that char $F = p > 2$. We will write $P$ for the set of $p$-elements of $G$, which we now know to be a normal subgroup. Furthermore, $P$ is invariant under $\ast$. Thus, if $(FG)^\ast$ is Lie $n$-Engel, then so is $(F(G/P))^\ast$. By [14, Theorem 4.2.13], since $G/P$ has no $p$-elements, $F(G/P)$ is semiprime. Thus, by Lemmas 1 and 2, we have the following.

**Lemma 4.** If $(FG)^\ast$ is Lie $n$-Engel, then $G/P$ is abelian or an SLC-group.

We will also need the classical results that classify the groups $G$ such that $FG$ is Lie nilpotent, or Lie $n$-Engel. The first is due to Passi, Passman and Sehgal [13], and the second due to Sehgal [16, Theorem V.6.1].

**Lemma 5.** If char $F = p \geq 0$, then $FG$ is Lie nilpotent if and only if $G$ is nilpotent and $p$-abelian.

**Lemma 6.** If char $F = 0$, then $FG$ is Lie $n$-Engel if and only if $G$ is abelian. If char $F = p > 0$, then $FG$ is Lie $n$-Engel if and only if $G$ is nilpotent and $G$ has a $p$-abelian normal subgroup $A$ with $(G : A) = p^k$ for some $k$.

The main results of Giambruno, Polcino Milies and Sehgal [4] are the following.

**Lemma 7.** If char $F \neq 2$, $G$ has no 2-elements and $(FG)^\ast$ is Lie nilpotent, then $FG$ is Lie nilpotent.

**Lemma 8.** If char $F \neq 2$, $G$ has no 2-elements and $(FG)^\ast$ is Lie $n$-Engel for some $n$, then $FG$ is Lie $m$-Engel for some $m$. 
Finally, we need to discuss polynomial identities. We say that an $F$-algebra $R$ satisfies a polynomial identity if there exists a nonzero $f(x_1, \ldots, x_n)$ in the free algebra on noncommuting indeterminates $F\{x_1, x_2, \ldots\}$ such that $f(r_1, \ldots, r_n) = 0$ for all $r_i \in R$.

If $R$ has an involution, then we say that $R$ satisfies a $*$-polynomial identity if there exists a nonzero $f(x_1, x_1^*, \ldots, x_i, x_i^*) \in F\{x_1, x_1^*, \ldots, x_n, x_n^*\}$ such that $f(r_1, r_1^*, \ldots, r_i, r_i^*) = 0$ for all $r_i \in R$. For instance, if $R^*$ is Lie $n$-Engel, then $R$ satisfies

\[
[x_1 + x_1^*, x_2 + x_2^*, \ldots, x_n + x_n^*] = 0.
\]

By Amitsur’s result (see [8, p. 196]), if $R$ satisfies a $*$-polynomial identity, then $R$ satisfies a polynomial identity. We have the following result due to Isaacs and Passman [14, Corollaries 5.3.8 and 5.3.10].

**Lemma 9.** If $\text{char } F = p \geq 0$, then $FG$ satisfies a polynomial identity if and only if $G$ has a $p$-abelian subgroup of finite index.

In particular, if $G$ is torsion and $(FG)^+$ is Lie $m$-Engel, then $G$ is locally finite.

### 3. Some reductions

Throughout, we let $G$ be a group and $F$ a field of characteristic $p > 2$. An element $\alpha \in FG$ is said to be skew if $\alpha^* = -\alpha$. We write $(FG)^-$ for the set of skew elements. Also, if $N$ is a normal subgroup of $G$, then we write $\Delta(G, N)$ for the kernel of the natural homomorphism $FG \to F(G/N)$, and we let $\Delta(G) = \Delta(G, G)$.

Let us begin with

**Lemma 10.** Suppose that $z$ is a central element of order $p$ in $G$, and $z^p = z$ or $z^{-1}$. Then $(FG)^+$ is Lie nilpotent (resp. Lie $n$-Engel, for some $n$) if and only if $(FG/(z))^+$ is Lie nilpotent (resp. Lie $m$-Engel, for some $m$).

**Proof.** Since $(z)$ is $*$-invariant, necessity is obvious. Assume that $(FG/(z))^+$ satisfies $[x_1, \ldots, x_n] = 0$. If $z^p = z$, assume that $k$ is odd. Take any $\alpha \in FG$. Then $[\alpha, \ldots, \alpha] \in \Delta(G, z)$ is $\alpha - (z^{-1})FG$, so let us say that $[\alpha, \ldots, \alpha] = (z^{-1})\alpha, \alpha \in FG$. Noting that $[(FG)^+, (FG)^+] \subseteq (FG)^-$, and $[(FG)^-, (FG)^+] \subseteq (FG)^+$, we observe that $(z^{-1})\alpha$ lies in $(FG)^+$ if $z^p = z$, and in $(FG)^-$ if $z^p = z^{-1}$.

Assume that $z^p = z$. Then

\[
(z^{-1})\alpha = ((z^{-1})\alpha)^* = (z^{-1})\alpha^* = (z^{-1})\left(\frac{\alpha + \alpha^*}{2}\right).
\]

Letting $\beta_1 = \frac{\alpha + \alpha^*}{2}$, we see that $\beta_1$ is symmetric, and

\[
[\alpha_1, \ldots, \alpha_k] = (z^{-1})\beta_1.
\]

Similarly,

\[
[\alpha_1, \ldots, \alpha_{2k-1}] = (z^{-1})[\beta_1, \alpha_{k+1}, \ldots, \alpha_{2k-1}],
\]

and by the same argument, this is $(z^{-1})^2\beta_2$ for some $\beta_2 \in (FG)^+$. Repeating this, we obtain

\[
[\alpha_1, \ldots, \alpha_r] = (z^{-1})^p\beta_p = 0,
\]

for some $r$, as required.

On the other hand, if $z^p = z^{-1}$, then

\[
-(z^{-1})\alpha = ((z^{-1})\alpha)^* = -(z^{-1})\alpha^*.
\]

That is,

\[
(z^{-1})\alpha = (z^{-1})\alpha^* = (z^{-1})\left(\frac{\alpha + \alpha^*}{2}\right),
\]

and we proceed as above.

The proof for the Lie $n$-Engel property is essentially identical, simply making $\alpha_i = \alpha_2$ for all $i \geq 3$. □

**Lemma 11.** Let $G$ be nilpotent, and let $N$ be a finite normal $*$-invariant $p$-subgroup of $G$. Then $(FG)^+$ is Lie nilpotent (resp. Lie $n$-Engel, for some $n$) if and only if $(FG/(z))^+$ is Lie nilpotent (resp. Lie $m$-Engel, for some $m$).

**Proof.** Necessity is clear. Let us prove sufficiency. We assume that $N \neq 1$. Since $G$ is nilpotent, we know that there exists $z \in \xi(G) \cap N$ with $o(z) = p$. If $z^p \neq z^{-1}$, then $zz^*$ is a symmetric element of order $p$ in $\xi(G) \cap N$. Replacing $z$ with $zz^*$ if necessary, we know that $z \in \xi(G) \cap N$, $o(z) = p$, and $z^p = z$. Thus, the previous lemma applies, and we know that $(FG)^+$ is Lie nilpotent (resp. Lie $n$-Engel) if and only if $(FG/(z))^+$ is Lie nilpotent (resp. Lie $m$-Engel). We can now apply the same process to $G/(z)$ and continue in this fashion until $N$ is exhausted. □

Suppose now that $G$ is a torsion group, and let $(FG)^+$ be Lie $n$-Engel. We will show that $G$ must be nilpotent, but to do so, we will need to borrow Lemmas 2.8 and 2.15 from [4].
Lemma 12. If $(FG)^+$ is Lie $n$-Engel, then for every symmetric $g \in G$, $g^p$ is central.

Lemma 13. Let $G$ be finite. If $G/P$ is abelian and $(FG)^+$ is Lie $n$-Engel, then $G$ is nilpotent.

We write $(g, h) = g^{-1}h^{-1}gh$ in any group.

Lemma 14. If $G$ is torsion and $(FG)^+$ is Lie $n$-Engel, then $G$ is nilpotent.

Proof. It will suffice to show that the $q$-elements form a group, for each prime $q$. If so, then $G = H \times K$, where every element of $H$ has odd order and $K$ is a $2$-group. By Lemmas 6 and 8, $H$ is nilpotent. Since $FK$ is semiprime, Lemmas 1 and 2 tell us that $K$ is abelian or an SLC-group. In the latter case, $K$ is abelian modulo its centre, so in any event, $K$, and therefore $G$, is nilpotent. Thus, since $G$ is locally finite, we may assume that $G$ is finite.

Our proof will be by induction on $|G|$. If $\zeta(G) \neq 1$, then since $\zeta(G)$ is $*$-invariant, we note that $(F(G/\zeta(G)))^+$ is still Lie $n$-Engel. By our inductive hypothesis, $G/\zeta(G)$ is nilpotent, hence $G$ is nilpotent. Therefore, we may assume that $G$ is centreless.

By Lemma 4, $G/P$ is abelian or an SLC-group. Also, by the Schur–Zassenhaus Theorem, $G = P \times Q$, for some $p'$-subgroup $Q$. Then $Q$ is abelian or an SLC-group, under the induced involution. In particular, $Q$ is nilpotent. Our proof will be complete once we have shown that $(P, x) = 1$ for all $x \in Q$. But for any such $x$, $xx^*$ is symmetric, and therefore by Lemma 12, $(xx^*)^p \in \zeta(G) = 1$. That is, $xx^* \in P$, so $x^* \equiv x^{-1}$ (mod $P$). Let $H = \langle P, x \rangle$. Then $H$ is a $*$-invariant subgroup of $G$, hence $(FH)^+$ is Lie $n$-Engel. Evidently $H/P$ is abelian. But then by Lemma 13, $H$ is nilpotent. Thus, $(P, x) = 1$, as required. \(\square\)

Now let $G$ be an arbitrary group, and let $(FG)^+$ be Lie $n$-Engel. By Lemma 4, $G/P$ is abelian or an SLC-group. For the case where $G/P$ is abelian, we have

Proposition 1. Suppose that $(FG)^+$ is Lie $n$-Engel and $G/P$ is abelian. Then $FG$ is Lie $m$-Engel, for some $m$.

Proof. Since $G/P$ is abelian, the torsion elements of $G$ form a subgroup, $T$. By the previous lemma, $T$ is nilpotent. Thus, the 2-elements form a normal subgroup $N$ of $G$. Hence, if $g \in G$ and $a \in N$, we see that $(g, a) \in P \cap N = 1$, so $N$ is central. Of course, $(F(G/N))^+$ is Lie $n$-Engel and $G/N$ contains no 2-elements. Thus, by Lemmas 6 and 8, $G/N$ is nilpotent and contains a $p$-abelian normal subgroup $A/N$ of finite $p$-power index. Since $N$ is central, $G$ is nilpotent, $A$ is of finite $p$-power index, and $(A/N)^\prime = A^\prime/N \simeq A^\prime/(A^\prime \cap N)$. But $G^\prime$ is a $p$-group, hence $A^\prime \cap N = 1$. As $(A/N)^\prime$ is a finite $p$-group, so is $A^\prime$. Lemma 6 completes the proof. \(\square\)

Similarly, if $(FG)^+$ is Lie nilpotent, then by Lemma 7 we get that $(G/N)^\prime$ is a finite $p$-group. But $(G/N)^\prime \simeq G^\prime$ and by Lemma 5 we obtain

Proposition 2. Suppose that $(FG)^+$ is Lie nilpotent and $G/P$ is abelian. Then $FG$ is Lie nilpotent.

4. Proofs of the main results

Having dispensed with the groups $G$ such that $G/P$ is abelian, we can now consider the case in which $G/P$ is an SLC-group. We begin with

Lemma 15. Suppose that the torsion elements of $G$ form a nilpotent group and $G$ has a normal $*$-invariant $p$-subgroup $N$ such that $G/N$ is an SLC-group. Then $G$ has a central symmetric element $z$ of order 2 such that $(G/z)^\prime$ is a subgroup of $N(z)/(z)$, hence a $p$-group.

Proof. Suppose that $z^N$ is the unique nonidentity commutator in $G/N$. Evidently $o(z^N) = 2$. Thus, $o(z) = 2p^k$ for some $k$. It follows that $o(z^k) = 2$, and $z^N = z^pN$. Replacing $z$ with $z^p$, we may assume that $o(z) = 2$. By the definition of an SLC-group, $z$ is central modulo $N$. But also, we know that the 2-elements of $G$ form a normal subgroup, $H$, hence for any $g \in G$, $(g, z) \in N \cap H = 1$. Thus, $z \in \zeta(G)$. Again, by the definition of an SLC-group, $(zN)^\prime = zN$, under the induced involution. That is, $z^* = za$ for some $a \in N$. But $z$ and $a$ lie in the torsion part of $G$, which is nilpotent. Thus, $z$ and $a$ commute, and if $a \neq 1$, then $za$ is not a 2-element. Therefore, $z^* = z$. Now,

$$((G/z)/(N/(z/z'))) \simeq (G/(N(z))'(z)/(N(z))) = 1,$$

since $G^\prime \leq N \times \langle z \rangle$, and we are done. \(\square\)

We can now see that $G$ must be nilpotent.

Lemma 16. Suppose that $(FG)^+$ is Lie $n$-Engel and $G/P$ is an SLC-group. Then $G$ is nilpotent.

Proof. Since $G/P$ is an SLC-group, it is clear that $G/P$ is nilpotent. Thus, the torsion elements of $G$ form a group which, by Lemma 14, must be nilpotent. Choose $z$ as in the previous lemma, taking $N = P$. Then $G^\prime \leq P \times \langle z \rangle$. Thus, $G(z)$ is abelian modulo its group of $p$-elements. By Lemma 6 and Proposition 1, $G/z$ is nilpotent. Since $z$ is central, we are done. \(\square\)

We can take care of the first of our main results, with the help of one more lemma.

Lemma 17. If $A$ is an abelian torsion group containing no 2-elements, with involution $*$, then $A = A_1 \times A_2$, where $A_1$ is the set of symmetric elements of $A$, and $A_2 = \{a \in A : a^* = a^{-1}\}$.

Proof. See [4, Corollary 2.10]. \(\square\)
Proof of Theorem 1. Suppose that \((FG)\) is Lie nilpotent but \(FG\) is not. We know that \(G/P\) is SLC. Also, \(G\) is nilpotent, and \(G\) has a central symmetric element \(z\) of order 2 such that \(G^z \leq P(z)\). Furthermore, \((F(G/z))\) is Lie nilpotent. Thus, since \(G/z\) is abelian modulo its group of \(p\)-elements, Proposition 2 and Lemma 5 tell us that \(G/z\) is a finite \(p\)-group. Hence, \(G\) has order \(2p^n\) for some \(n\). Let \(H = G' \cap P\). Then \(H\) is a finite normal \(*\)-invariant \(p\)-subgroup, so by Lemma 11, it suffices to consider \(G/H\). Now \((G/H)' = G'H/H = G' \cap P\). That is, we may assume that \(G\) has order 2, hence \(G^z = \{e\}\).

In particular, \((P, G) \leq P \cap \{z\} = 1\), and \(P\) is central. Hence, by Lemma 17, \(P = P_1 \times P_2\), where every element of \(P_1\) is symmetric and \(*\) acts as the classical involution on \(P_2\). By the proof of [3, Theorem 2] and its corollary, we see that if \(P_2\) is infinite, then \(FG\) must be Lie nilpotent. Since \(G^z\) is not a \(p\)-group, this is impossible. Therefore, \(P_2\) is a finite central \(*\)-invariant \(p\)-subgroup, and it suffices to consider \(G/P_2\). That is, we may assume that every element of \(P\) is symmetric. We claim that in this case, \(G\) is an SLC-group.

First notice that if \(g \in G\), and \(gP\) is central in \(G/P\), then for any \(h \in G\), \((g, h) \in P \cap \{z\} = 1\), hence \(g \in \zeta(G)\). Thus, by definition of an SLC-group, we see that \((gp)^* = gp\) if \(g\) is central, and otherwise, \((gp)^* = zgP\). That is, \(g^* = ga_1\) or \(zg\) for some \(a \in P\). If \(g^* = ga\), then \(g = (g^*)^* = (ga)^* = g^*a = g^*a^* = g^*,\) since \(a\) is central and symmetric. That is, \(a^2 = 1\). Since \(a\) is a \(p\)-element, \(g^* = g\) when \(g\) is central. If \(g^* = zg\), then \(g = (g^*)^* = (zg)g = zg^*,\) as \(z\) is central and has order 2. Again, \(a = 1\), hence \(g^* = zg\) when \(g\) is not central. This is the required action of the involution on an SLC-group. We already know that \(G\) has a unique nonidentity commutator. It only remains to check that \(G\) is an LC-group. Suppose \(g, h \in G\), \((g, h) = 1\), and neither \(g\) nor \(h\) is central. Then \(g^* = zg, h^* = zh\), hence \((gh)^* = h^*g^* = (zg)(zh) = gh\), since everything commutes and \(z^2 = 1\). That is, \(gh\) is central and \(G\) is an SLC-group.

Sufficiency follows immediately from Lemmas 1 and 11.

We need one last lemma before we prove Theorem 2.

Lemma 18. Suppose \(G\) is nilpotent and \(G\) contains an abelian normal subgroup \(A\) of finite index. If \(G^z\) is a \(p\)-group of bounded exponent, then \(FG\) is Lie \(n\)-Engel.

Proof. From Lemma 6, we see that we need only show that \(A\) may be chosen in such a way that \((G : A)\) is a \(p\)-power. Since \(G/A\) is a finite nilpotent group, we may choose a normal subgroup \(H\) of \(G\) containing \(A\) such that \(G/H\) is a \(p\)-group, and \(H/A\) is a \(p'\)-group. Then \(H = (H : A)\) is abelian, then we will be done. Let \(k = (H : A)\). Then \((H^k, A)\) is a \(p'\)-group. But \(G^z\) is a \(p\)-group. Thus, \(A \leq \zeta(H)\). Therefore, \(H/\zeta(H)\) is a finite \(p'\)-group. By [16, Theorem 1.4.2], \(H^*\) is a finite \(p'\)-group. But \(H^*\) is a \(p\)-group. Thus, \(H\) is abelian.

Finally, we have the

Proof of Theorem 2. Suppose that \((FG)\) is Lie \(n\)-Engel. We already know that \(G\) is nilpotent and \(FG\) satisfies a polynomial identity, hence by Lemma 9, \(G\) has a normal \(p\)-abelian subgroup \(A\) of finite index. Replacing \(A\) with \(A \cap A^z\), we can assume that \(A\) is \(*\)-invariant. Thus, to prove the necessity part of Theorem 2, it remains only to check that \(G\) has a \(*\)-invariant normal \(p\)-subgroup \(N\) of bounded exponent such that \(G/N\) is an SLC-group. Given \(z\) as in Lemma 15, we know that \((F(G/z))\) is Lie \(n\)-Engel hence, by Lemma 15 and Proposition 1, \((FG/z)\) is Lie \(m\)-Engel. In particular, \(0 = [g, h, \ldots, h] = [g, h^m]_{p^n\text{ times}}\) for all \(g, h \in G/z\). That is, \((G/z)\) is a \(p\)-group of bounded exponent modulo its centre. It follows from [16, Corollary 1.4.3] that \((G/z)'\) is a \(p\)-group of bounded exponent. Thus, \(G' \cap P\) is a normal \(*\)-invariant \(p\)-subgroup of \(G\) of bounded exponent. Factoring it out, we may assume that \(P\) is central, and \(G' = \langle z \rangle\). Thus, by Lemma 17, \(P = P_1 \times P_2\), where every element of \(P_1\) is symmetric and \(*\) acts as the classical involution on \(P_2\). Since \(P_2\) is central and \(*\)-invariant, we can consider \(G/P_2\). It follows exactly as in the proof of Theorem 1 that \(G/P_2\) is an SLC-group. Thus, if we can show that \(P_2\) has bounded exponent, the necessity part of the theorem will be complete.

Suppose \(P_2\) has unbounded exponent. Choose \(x, y \in G\) such that \((x, y) = z\), with \(z\) as above. Given the action of \(*\) on \(G/P_2\), since \(xP_2\) is not central in \(G/P_2\), we have \((xP_2)^* = xzP_2\). Thus, \(x^* = xza_1\) for some \(a_1 \in P_2\), and similarly \(y^* = yza_2\), for some \(a_2 \in P_2\). Since \(z\) and \(P_2\) are central, it follows that \(y \) and \(y^* \) commute. Thus, for any \(a \in P_2\), \(0 = [xa + (xa)^*, ya + (ya)^*, \ldots, ya + (ya)^*]_{p^n \text{ times}}\) \(= [xa + x^*a^*, (ya + y^*a^*)^{p^n}] = [xa + xza_1a^{-1}, y^{p^n}a^{p^n} + y^{p^n}za_2^{p^n}a^{-p^n}]\).

Expanding this last expression, we get sums and differences of eight group elements. To obtain zero, each group element must equal at least one other. In particular, \(xy^{p^n}a^{p^n+1}\) must equal one of the other seven group elements. Setting it equal to each of the other terms, and recalling that \(a\) is central, we will, in every case but one, be setting a positive power of \(a\) equal to a group element determined by \(x\) and \(y\). Since \(P_2\) has unbounded exponent, we can choose \(a\) in such a way that this does not happen. The one remaining case is \(xy^{p^n}a^{p^n+1} = y^{p^n}x^{p^n+1}\). Thus, \(xy^{p^n} = y^{p^n}x\). But \(xy = yzx\), and since \(z\) is central of order 2, \(xy^{p^n} = y^{p^n}xz\), and we have a contradiction. Thus, \(P_2\) has bounded exponent, and we have proved the necessity part.
Let us now discuss the sufficiency. Choosing $A$ as in the statement of the theorem, we know from Lemma 11 that it suffices to consider $G/A$. Thus, we will assume that $A$ is abelian. Choose $z$ as in Lemma 15. We know that $G/(z)$ is nilpotent, $(G/(z))'$ is a $p$-group of bounded exponent, and $A(z)/(z)$ is an abelian subgroup of finite index. Thus, by Lemma 18, $F(G/(z))$ is Lie $m$-Engel for some $m$. Hence, if $\alpha, \beta \in FG$, then

$$[\alpha, \beta, \ldots, \beta] \in \Delta(G, (z)) = (z - 1)FG.$$  

In particular, we may as well assume that $\alpha \in (z - 1)FG$. Therefore, if any term appears in $\beta$ that is a multiple of $1 + z$, we can drop that term, as it will become zero in our Lie product.

Now, if $\beta \in (FG)^+$, then $\beta$ is a linear combination of terms of the form $g + g^*$. Let $W/N = \zeta(G/N)$. By definition of the involution on an SLC-group, we know that $g + g^* = g + ga$ for some $a \in N$, if $g \in W$, and otherwise, $g + g^* = g + gza$ for some $a \in N$. If $g \in W$, then $g + g^* \in FW$. If $g \notin W$, then $g + g^* = g + gza + ga - ga = (z + 1)ga + g(1 - a)$.

As we discussed above, we can ignore the $(z + 1)ga$ term, and what is left lies in $\Delta(G, N)$.

Let $I = \Delta(G, N)$. Then we have $\beta + I = \gamma + I$, where $\gamma = \sum \lambda_i w_i$, with $\lambda_i \in F$, $w_i \in W$. Modulo $N$, $W$ is abelian. Thus, for any $k$, $(\beta + I)^k = \sum \lambda_i^k w_i^k + I$. Now, $W$ is nilpotent and has an abelian normal subgroup, $W \cap A$, of finite index. Also, $W' \leq N$. Thus, by Lemma 18, $FW$ is Lie $p^k$-Engel, for some $k$. It follows that $W^{p^k} \subseteq \zeta(W)$. Thus, $\beta^{p^k} = \rho + \delta$, where $\rho \in F(\zeta(W), \delta \in \Delta(G, N)$.

We claim that $\zeta(W) \leq \zeta(G)$. Indeed, $\zeta(W)$ is an abelian normal subgroup in $G$. Furthermore, by definition of an SLC-group, $G^{\zeta} \subseteq W$. Thus, $(\gamma^{\zeta}, \zeta(W)) = 1$. It follows from [16, Lemma V.6.2] that $(G, \zeta(W))$ is a 2-group. But $(G, \zeta(W)) \subseteq (G, W) \subseteq N$, by definition of $W$. Since $N$ is a $p$-group, $\zeta(W)$ is central in $G$. That is, $\beta^{p^k}$ is the sum of a central element and an element of $\Delta(G, N)$. If we can show that $\Delta(G, N)$ is nil of bounded exponent, then we will have $\beta^{p^k/r} = \rho^{p^k} + \delta^{p^k} = \rho^{p^k}$, for some suitable $r$, and this is central. Thus,

$$[\alpha, \beta, \ldots, \beta] = 0,$$

as required.

Since $N/(N \cap A)$ is a finite $p$-group, we know from [16, Lemma I.2.21] that $\Delta(N/(N \cap A))$ is nilpotent. Say $(\Delta(N/(N \cap A)))^{p^k} = 0$. Then $(\Delta(N))^{p^k} \subseteq \Delta(N, N \cap A)$. Since $N$ and $N \cap A$ are normal subgroups of $G$, it follows immediately that $(FG\Delta(N))^{p^k} \subseteq FG\Delta(N, N \cap A)$. That is, $(\Delta(G, N))^{p^k} \subseteq FG\Delta(N, N \cap A)$. Thus, it suffices to show that $FG\Delta(N, N \cap A)$ is nil of bounded exponent. But this is $FG\Delta(N \cap A) = FG\Delta(A, N \cap A)$. By [15, Lemma 3.2], it suffices to show that $\Delta(A, N \cap A)$ is nil of bounded exponent. (Note that all fields in [15] were assumed to be infinite, but the field size was not used in the proof of that lemma.) However, $A$ is abelian and $N \cap A$ is a $p$-group of bounded exponent. It follows easily that $\Delta(A, N \cap A)$ is nil of bounded exponent, and this completes the proof. \(\square\)

References