INTEGRAL GROUP RINGS WITH TRIVIAL CENTRAL UNITS II*

Misha Dockuchaev and César Polcino Milies
Instituto de Matemática e Estatística, Universidade de São Paulo,
São Paulo, Brazil

Sudarshan K. Sehgal
Department of Mathematical & Statistical Sciences, University of Alberta,
Edmonton, Alberta, Canada

All groups G with trivial central units in ZG are classified. This was done for finite groups by Ritter and Sehgal [Ritter, J., Sehgal, S. K. (1990). Integral group rings with trivial central units. Proc. Amer. Math. Soc. 108:327–329].

Key Words: Central units; Group Rings.

Mathematics Subject Classification: Primary 20C05, 20C07, 16U60; Secondary 16S34, 20C10, 20C12.

1. INTRODUCTION

Let $\mathcal{U} = \mathcal{U}(ZG)$ be the unit group of the integral group ring $ZG$. It is a classical result of G. Higman (Sehgal, 1978, p. 57) that if $G$ is torsion then $\mathcal{U}$ is trivial if and only if $G$ is abelian of exponent 2, 3, 4 or 6 or $G = K_8 \times E$, the product of the quaternion group $K_8$ of order 8 and an elementary abelian 2-group $E$. Since triviality of $\mathcal{U}$ for torsion free groups $G$ is an open problem it is quite difficult to extend this result to arbitrary groups. However, if a mild condition is imposed on $G$ a classification can be seen from the results in Sehgal (1978) as we show in Sec. 5.

In 1990, Ritter and Sehgal classified finite groups $G$ so that the central units of $ZG$ are trivial, namely of the form $\pm g$, where $g$ is an element of the centre of $G$. A classification of arbitrary groups $G$ with trivial central units in $ZG$ was asked for in Sehgal (1993, Problem 26, p. 301). Recently, Parmenter (1999) renewed this call. We give below this classification in terms of the condition of Ritter and Sehgal (1990).

Received June 2003; Revised October 2003
*Communicated by E. Puczylowski.
Address correspondence to Misha Dokuchaev, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05315-970 São Paulo, Brazil; E-mail: dokucha@ime.usp.br
Theorem. The central units of $\mathcal{Z}G$ are trivial if and only if every finite normal subgroup $A$ of $G$ satisfies (RS) in $G$:

For all $a \in A$ and $j \in \mathbb{Z}$ with $(j, |a|) = 1$ we have $a^j \sim a^e$, $e = \pm 1$.

(RS)

2. PRELIMINARIES

Our notation is standard as in Sehgal (1993). For any ring $R$ (group $B$) we shall denote by $\mathfrak{Z}(R)(\mathfrak{Z}(B))$ its centre. In particular, we shall simply write $\mathfrak{Z}$ for $\mathfrak{Z}(\mathcal{U}ZG)$. A subgroup of $\mathcal{U}$ is called trivial if it consists of trivial units only. We recall

Theorem 1 (Ritter and Sehgal). If $G$ is a finite group then $\mathfrak{Z} = \mathfrak{Z}(\mathcal{U}ZG)$ consists of trivial units if and only if $G$ satisfies

For every integer $j$ with $(j, |G|) = 1$ and every $g \in G$ we have $g^j \sim g^e$, $e = \pm 1$.

(RS)

Lemma 2. For a finite group $G$ the following two conditions are equivalent:

1. $(j, |G|) = 1$ and every $g \in G$ we have $g^j \sim g^e$, $e = \pm 1$.

2. $g^j \sim g^j$, $e = \pm 1$.

(RS)

Proof. Since there are more integers $j$ such that $(j, |g|) = 1$ than those such that $(j, |G|) = 1$ it follows that (RS) is stronger. Thus (RS) $\Rightarrow$ (RS)$_1$, we have only to prove (RS)$_1$ $\Rightarrow$ (RS). Then we are given $g \in G$, $|g| = m$, $(j, m) = 1$ We have to show that $g^j \sim g^j$. Let us write $|G| = n = m_1s$ with $m_1$ an $m$-number and $(s, m) = 1$. Suppose that $j = j_1l$, $s_1$ an $s$-number, $(l, n) = 1$. We write $s = s's''$ where $s'$ an $s_1$-number an $s''$ an $s'$-number and set $j' = j + m_1s'$. We have

then $(j', s) = 1$ and $(j', m) = 1$, consequently, $(j', n) = 1$. Here, $g^{j'} = g^{j + m_1s''} = g^j$.

By (RS)$_1$ $g^j = g^{j'} \sim g^e$ as required.

Example 3. $m = |g| = 2^35^4$, $n = |G| = 2^53^45^67^211$, $m_1 = 2^55^6$, $j' = 3^2 \cdot 11 \cdot 13$, $s = 3^47^211$, $s' = 3^4 \cdot 11$, $j' = 3^2 \cdot 11 \cdot 13 + 2^5 \cdot 5^6 \cdot 7^2$, $s'' = 7^2$.

We need the next two results which are crucial for our purposes. The first result was proved for nilpotent groups in Jespers et al. (1996) and extended to arbitrary groups in Polcino Milies and Sehgal (1999).

Lemma 4. Let $G$ be any group. Then any central unit $\mu \in \mathcal{U}(\mathcal{Z}G)$ can be written as $\mu = gw$, $g \in G$, $w \in \mathcal{Z}T$, $gw = wg$ where $T$ is the torsion subgroup of $\Phi(G)$ the FC-subgroup of $G$. 
Proof. This is Theorem 1 of Polcino Milies and Sehgal (1999).

Lemma 5. Let $\mu$ be a central unit of $\mathbb{Z}G$ such that $\mu^n$ is trivial for some $n \in \mathbb{Z}$. Then $\mu$ is trivial.

Proof. This is Lemma 2.5 of Polcino Milies and Sehgal (1999).

3. OBSERVATIONS FOR FINITE GROUPS $G$

Suppose that $G$ is a finite group. Let $L$ be a splitting field of $\mathbb{Q}G$. Express $\mathbb{Q}G$ as a direct sum

$$\mathbb{Q}G = \bigoplus S$$

of simple rings $S$. Let $\chi$ be an absolutely irreducible character of $G$ and $D$ the corresponding representation. Then there exists a unique $S$ so that $D(S) \neq 0$. There is an embedding (see Huppert, 1967, p. 544):

$$D: S \rightarrow L_{n \times n}$$

For a class sum $C_g$ of $g \in G$, we have

$$D(C_g) = \frac{h}{n} \chi(g) I_n$$

with $h = |C_g|$, $n = \deg D$. This induces an isomorphism $\mathfrak{B}(S) \simeq \mathbb{Q}(\chi) = \mathbb{Q}\{\chi(g) : g \in G\}$. We have

$$\mathfrak{B}(\mathbb{Q}(G)) \simeq \bigoplus \mathbb{Q}(\chi)$$

with the map $C_g \rightarrow \sum \chi(g)$, $h = h_x$, $n = n_x$. For a normal subgroup $A$ of $G$ restricting to the subring $\mathfrak{B}(\mathbb{Q}G) \cap \mathbb{Q}A$, we have an injection

$$\mathfrak{B}(\mathbb{Q}G) \cap \mathbb{Q}A \hookrightarrow \bigoplus \mathbb{Q}(\chi(A))$$

where $\mathbb{Q}(\chi(A)) = \mathbb{Q}\{\chi(a) : a \in A\}$. The typical projection

$$\pi: \mathfrak{B}(\mathbb{Q}G) \cap \mathbb{Q}A \rightarrow \mathbb{Q}(\chi(A))$$

is onto. We have

Lemma 6. If $\mathbb{Z}A \cap \mathfrak{B}$ is trivial then $\mathbb{Q}(\chi(A))$ is $\mathbb{Q}$ or imaginary quadratic.

Proof. Let us write $K = \mathbb{Q}(\chi(A))$ and $O_K$ for its ring of integers. Then $\pi(\mathbb{Z}A \cap \mathfrak{B}(\mathbb{Q}G))$ is an order in $O_K$. If $O_K$ has an element of infinite order then
so does $\pi(\mathbb{Z}A \cap \mathfrak{B}(\mathbb{Q}G))$ and therefore also $\mathbb{Z}A \cap \mathfrak{B}(\mathbb{Q}G)$. This is a contradiction proving the lemma.

**Lemma 7.** If symmetric elements of $\mathfrak{B} \cap \mathbb{Z}A$ are trivial then so are all elements of $\mathfrak{B} \cap \mathbb{Z}A$.

**Proof.** Let $u \in \mathbb{Z}A$, $u \in \mathfrak{B}$. Thus $u^* \in \mathbb{Z}A \cap \mathfrak{B}$. Further $(uu^*)^* = uu^*$ is a symmetric element of $\mathfrak{B} \cap \mathbb{Z}A$. Thus by assumption we have $uu^* = z \in A$. Writing $u = \sum u_g g$, we have

$$uu^* = \left( \sum u_g^2 \right) \cdot 1 + \cdots = z \in A.$$ It follows that $z = 1$ and $\sum u_g^2 = 1$. Thus $u = \pm g$, $g \in A$ as required.

**Theorem 8.** Let $A$ be a normal subgroup of a finite group $G$. Then $A$ satisfies (RS) in $G$ $\Rightarrow$ $\mathfrak{B} \cap \mathbb{Z}A$ consists of trivial units.

**Proof.** Suppose $u$ is a non trivial central unit in $\mathbb{Z}A$. We may assume $u$ is symmetric. Then

$$u = \sum q(a + a^{-1}), \quad q \in \mathbb{Q}$$

(we may have $a = a^{-1}$ and in this case $q = 1/2$). Let $\sigma$ be an automorphism of $\mathbb{Q}(\chi(A))$. Extend $\sigma$ to $\mathbb{Q}(\chi)$ and then to a cyclotomic field $\mathbb{Q}(\zeta)$ containing $\mathbb{Q}(\chi)$. If we express $\chi(a) = \sum \xi$, a sum of roots of unity, then

$$\chi^\sigma(a) = \sum \xi^j = \chi(a^j) \quad \text{with} \quad (j, |G|) = 1$$

$$= \chi(a^\sigma), \quad \varepsilon = \pm 1 \quad \text{by (RS)}$$

$$= \chi(a) \quad \text{or} \quad \overline{\chi}(a).$$

Then from $\chi(u) = \sum q(\chi(a) + \chi(a^{-1}))$, we conclude that $\chi^\sigma(u) = \chi(u)$ for all $\sigma$. It follows that $\chi(u) \in \mathbb{Q}$. Then it follows that the symmetric units in $\mathfrak{B} \cap \mathbb{Z}A$ are contained in $\sum^\oplus \mathbb{Q}$. Remember that $\mathfrak{B} \cap \mathbb{Z}A$ is an order in $\mathbb{Q}A \cap \mathfrak{B}(\mathbb{Q}G)$ which is a direct sum of fields $\sum^\oplus K$. This has a unique maximal order $\sum^\oplus \mathcal{O}_K$. Therefore, the symmetric units of $\mathfrak{B} \cap \mathbb{Z}A$ are contained in $\sum^\oplus \mathbb{Z}$. Thus $u^2 = 1$. This contradicts the assumption that $u$ is non trivial, proving the theorem.

**Theorem 9.** Suppose that $A$ is a normal subgroup of a finite group $G$. Then $\mathfrak{B} \cap \mathbb{Z}A$ trivial $\Rightarrow$ $A$ satisfied (RS) in $G$.

**Proof.** Let $a \in A$ be such that $(j, |a|) = 1$. Then we know by Lemma 6 that $\mathbb{Q}(\chi(A))$ is $\mathbb{Q}$ or imaginary quadratic. Observe that $j$ is an automorphism of $\mathbb{Q}(\chi(a))$ which extends to an automorphism $\sigma$ of $\mathbb{Q}(\chi(A))$. Then $\sigma$ is identity or $\sigma$. We have for all $\chi$:

$$\chi^\sigma(a) = \chi(a^j) = \chi(a) \quad \text{or} \quad \overline{\chi}(a)$$

$$\chi^\sigma(a^j) + \chi(a^{-j}) = \chi(a) + \chi(a^{-1}).$$
Define functions $T_g : \text{Irr}(G) \to \mathbb{C}by \chi \to \chi(g)$. These functions, one for each conjugacy class, are linearly independent. We have

$$T_{a^j} + T_{a^{-j}} = T_a + T_{a^{-1}}.$$

so that $T_{a^j} = T_a$ or $T_{a^{-j}}$. Thus $\chi(a^j) = \chi(a)$ or $\chi(a^{-j})$ for all $\chi$. Thus $a^j \sim a$ or $a^{-j} \sim a^{-1}$ as desired.

4. PROOF OF THEOREM

We try to reduce the problem to finite groups $G$. We begin with

Lemma 10. Let $A$ be a finite normal subgroup of $G$. Then there exists a finite group $H$ containing $A$ as a normal subgroup such that $\mathfrak{Z} \cap ZA = \mathfrak{Z}(\mathcal{U}ZH) \cap ZA$.

Proof. Let $C$ be the centralizer of $A$ in $G$. Then $C \triangleleft G$ and the index $(G : C)$ is finite. Let $H = A \times G/C$ where $a^g = a^c, \forall a \in A, \ g \in G/C$. Then $H$ is finite. Moreover, an element of $\mathfrak{Z}A$ is in $\mathfrak{Z}$ if and only if it is central in $ZH$.

Lemma 11. $\mathfrak{Z}$ trivial $\Rightarrow$ Every finite normal subgroup $A$ of $G$ satisfies (RS) in $G$.

Proof. Let $H$ be as in Lemma 10. Then $\mathfrak{Z}(\mathcal{U}ZH) \cap ZA$ is trivial. Then it follows by Theorem 9 that $A$ satisfies (RS) in $H$, hence also in $G$.

Lemma 12. Let $A$ be a finite normal subgroup of $G$. Then

A satisfies (RS) in $G \Rightarrow \mathfrak{Z} \cap ZA$ is trivial.

Proof. Let $H$ be as in Lemma 10. Then $A$ satisfies (RS) in $G$ and hence in $H$ which is finite. Also $\mathfrak{Z} \cap ZA = \mathfrak{Z}(\mathcal{U}ZH) \cap ZA$. By Theorem 8, we conclude that $\mathfrak{Z} \cap ZA$ is trivial.

Lemma 13. If every finite normal subgroup of $G$ satisfies (RS) in $G$ then $\mathfrak{Z}$ is trivial.

Proof. Let $\mu \in \mathfrak{Z}$. Then $\mu = \omega g$ with $\omega \in ZT, \ g \in G$ where $T$ is a finite normal subgroup contained in $\phi(G)$ and $\omega g = g\omega$ by Lemma 4. Consider the element

$$\mu^* = \omega g g^{-1} \omega^* = \omega \omega^* \in ZT.$$

This element is central in $ZG$. It follows by Lemma 12 that $\mu^*$ is trivial. Thus $\mu$ is trivial as well, as desired.


5. TRIVIAL UNITS

As mentioned in the introduction, the following result can be read off from Sehgal (1978). The condition of nilpotence of $G/T$ in the following can be relaxed.
Theorem 14. Let $G$ be an arbitrary group and $T$ the set of its torsion elements. Suppose $\mathcal{U}(\mathbb{Z}G) = \mathcal{U}$ is trivial. Then

1. Every subgroup $(t)$, $t \in T$, is normal in $G$ and hence $T$ is a subgroup.
2. Either $T$ is abelian of exponent 2, 3, 4 or 6 or $T = K_8 \times E$ where $K_8$ is the quaternion group of order 8 and $E^2 = 1$.

Conversely, if $G$ satisfies (1) and (2) and $G/T$ is nilpotent then $\mathcal{U}$ is trivial.

Proof. Assume $\mathcal{U}$ is trivial. As seen on page 51 of Sehgal (1978), $\mathbb{Z}G$ has no nilpotent elements and so every idempotent of $\mathbb{Q}G$ is central and for every $t \in T$, we have $(t) \trianglelefteq G$. Then $T$ is a subgroup and (2) is Higman's theorem. Conversely, it follows by Lemma 3.22, p. 194 of Sehgal (1978) that $\mathcal{U} = (\mathcal{U}ZT)G$. The result now follows from the one for $T$. \qed

ACKNOWLEDGMENT

This work was supported by CNPq and FAPESP, Brazil and NSERC, Canada.

REFERENCES
