

#102

**ZASSENHAUS CONJECTURE** – Just as the only roots of unity in a cyclotomic field  $\mathbf{Q}(\zeta)$  are of the form  $\pm\zeta^i$ , there is the classical *theorem* of G. Higman stating that the torsion units in the integral group ring  $\mathbf{Z}G$  of a finite **Abelian group** are of the form  $\pm g$ ,  $g \in G$ . Of course, if  $G$  is non-Abelian, then any conjugate of  $\pm g$  is also of finite order; however, these are not all the torsion units in  $\mathbf{Z}G$ . The famous *Zassenhaus conjecture* says that for a **finite group**  $G$  all torsion units of  $\mathbf{Z}G$  are rationally conjugate to  $\pm g$ ,  $g \in G$ :

ZC1) Let  $u \in \mathbf{Z}G$ ,  $u^n = 1$  for some  $n$ ; then  $u = \pm x^{-1}gx$  for some  $g \in G$  and some unit  $x \in \mathbf{Q}G$ .

This conjecture was proved to be true by A. Weiss, first for  $p$ -groups [16] and then for nilpotent groups [17] (cf. also **Nilpotent group**). In fact, Weiss proved the following stronger Zassenhaus conjecture for nilpotent groups:

ZC3) If  $H$  is a finite subgroup of units of augmentation one in  $\mathbf{Z}G$ , then there exists a unit  $x \in \mathbf{Q}G$  such that  $x^{-1}Hx \subseteq G$ .

A special case of this is the following conjecture:

ZC2) If  $H$  is a subgroup of  $ZG$  of augmentation one of order  $|G|$  such that  $ZG = ZH$ , then there exists a unit  $x \in QG$  with  $x^{-1}Hx = G$ .

This last conjecture was earlier proved by K. Roggenkamp and L.R. Scott [12] for nilpotent groups. Subsequently, they also gave a counterexample to ZC2) (unpublished), which appears in a modified form in [5].

Clearly, ZC3) implies ZC1) and ZC2). Also, ZC2) implies that if two group rings  $ZG$  and  $ZH$  are isomorphic, then the groups  $G$  and  $H$  are isomorphic. This *isomorphism problem* was proposed in [3]:

$$ZG \simeq ZH \Rightarrow G \simeq H. \quad (1)$$

Of course, then, (1) is true for nilpotent groups. Moreover, it was proved by A. Whitcomb [18] that (1) is true for metabelian groups. M. Hertweck [2] has given a counterexample to (1).

Conjecture ZC1) is open in general (as of 2000). Besides nilpotent groups, it is known to be true for certain split metacyclic groups [10]: If  $G = \langle a \rangle \rtimes \langle b \rangle$  is the semi-direct product of two cyclic groups  $\langle a \rangle$  and  $\langle b \rangle$  of relative prime orders, then ZC1) holds for  $G$ .

This result has been strengthened to ZC3) [15].

There are several useful and interesting extensions of the above conjectures. Suppose that  $A$  is a **normal subgroup** of index  $n$  in  $G$ . Then  $ZG$  can be represented by  $(n \times n)$ -matrices over  $ZA$ . Any torsion unit  $u$  of  $ZG$  that is mapped by the natural homomorphism  $G \rightarrow G/A$  to  $1 \in Z(G/A)$  gives rise to a torsion matrix  $U \in \text{SGL}_n(ZA)$ . Here,  $\text{SGL}_n(ZA)$  denotes the subgroup of the **general linear group**  $\text{GL}_n(ZA)$  consisting of the matrices  $U$  that are mapped by the augmentation homomorphism  $ZA \rightarrow Z$ , when applied to each entry, to the identity matrix. Thus, ZC1) translates to the question about *diagonalization* of  $U$  in  $\text{GL}_n(QA)$ : Is a torsion matrix  $U \in \text{SGL}_n(ZG)$ , where  $G$  is a finite group, conjugate in  $(QG)_{n \times n}$  to a matrix of the form  $\text{diag}(g_1, \dots, g_n)$ ,  $g_i \in G$ ?

This was answered positively in [16] for  $p$ -groups (cf. also  **$p$ -group**). See [1] for an explicit example of a matrix  $U \in \text{SGL}_6(\mathbb{Z}(C_6 \times C_6))$  that cannot be diagonalized but for which  $U^6 = I$ . Such a matrix  $U$  exists for a finite nilpotent group  $G$  and some  $n$  if and only if  $G$  has at least two non-cyclic Sylow  $p$ -subgroups [1] (cf. also **Sylow subgroup**).

However, it was proved in [6] that if  $n = 2$  and  $G$  is finite Abelian, then  $U$  is conjugate in  $(QG)_{n \times n}$  to  $\text{diag}(g_1, g_2)$ . This has been extended to  $n \leq 5$  in [9], bridging the gap between 2 and 6.

The Zassenhaus conjectures and the isomorphism problem have also been studied for infinite groups  $\Gamma$ . The statements remain the same and the group  $\Gamma$  is arbitrary. A counterexample to ZC1) was provided in [8].

Conjecture ZC2) also does not hold for infinite groups, as shown by S.K. Sehgal and A.E. Zalesskiĭ (see [14, p. 279]).

However, one can ask if any torsion unit  $U \in \text{SGL}_n(\Gamma)$  can be stably diagonalized to  $\text{diag}(\gamma_1, \dots, \gamma_n)$ ,  $\gamma_i \in \Gamma$ . This has been proved [7] to be true for  $p$ -elements  $U$  when  $\Gamma$  is nilpotent. The isomorphism problem also has a positive answer for finitely-generated nilpotent groups of class 2, cf. [11]. In general for nilpotent groups the problem remains open (as of 2000).

## References

- [1] CLIFF, G., AND WEISS, A.: 'Finite groups of matrices over group rings', *Trans. Amer. Math. Soc.* **352** (2000), 457–475.
- [2] HERTWECK, M.: 'A solution of the isomorphism problem for integral group rings'.
- [3] HIGMAN, G.: 'Units in group rings': *D. Phil. Thesis Univ. Oxford*, 1940.
- [4] HIGMAN, G.: 'The units of group rings', *Proc. London Math. Soc.* **46** (1940), 231–248.
- [5] KLINGLER, L.: 'Construction of a counterexample to a conjecture of Zassenhaus', *Commun. Algebra* **19** (1991), 2303–2330.
- [6] LUTHAR, I.S., AND PASSI, I.B.S.: 'Torsion units in matrix group rings', *Commun. Algebra* **20** (1992), 1223–1228.
- [7] MARCINIAK, Z., AND SEHGAL, S.K.: 'Finite matrix groups over nilpotent group rings', *J. Algebra* **181** (1996), 565–583.
- [8] MARCINIAK, Z., AND SEHGAL, S.K.: 'Zassenhaus conjecture and infinite nilpotent groups', *J. Algebra* **184** (1996), 207–212.
- [9] MARCINIAK, Z., AND SEHGAL, S.K.: 'Torsion matrices over abelian group rings', *J. Group Th.* **3** (2000), 67–75.
- [10] MILIES, C. POLCINO, RITTER, J., AND SEHGAL, S.K.: 'On a conjecture of Zassenhaus on torsion units in integral group rings, II', *Proc. Amer. Math. Soc.* **97** (1986), 201–206.
- [11] RITTER, J., AND SEHGAL, S.K.: 'Isomorphism of group rings', *Archiv Math.* **40** (1983), 32–39.
- [12] ROGGENKAMP, K., AND SCOTT, L.: 'Isomorphisms for  $p$ -adic group rings', *Ann. Math.* **126** (1987), 593–647.
- [13] SEHGAL, S.K.: *Topics in group rings*, M. Dekker, 1978.
- [14] SEHGAL, S.K.: *Units in integral group rings*, Longman, 1993.
- [15] VALENTI, A.: 'Torsion units in integral group rings', *Proc. Amer. Math. Soc.* **120** (1994), 1–4.
- [16] WEISS, A.: 'Rigidity of  $p$ -adic  $p$ -torsion', *Ann. of Math.* **127** (1988), 317–332.
- [17] WEISS, A.: 'Torsion units in integral group rings', *J. Reine Angew. Math.* **415** (1991), 175–187.
- [18] WHITCOMB, A.: 'The group ring problem', *PhD Thesis Univ. Chicago* (1968).

S.K. Sehgal

MSC1991: 20Dxx, 20C05