Central Units of Integral Group Rings

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Abstract

In this note we give a description of the central units of an integral group ring \( \mathbb{Z}G \) for an arbitrary group \( G \). We also give a set of generators of a subgroup of finite index in the centre of the unit group when \( G \) is any group whose FC-centre is finitely generated. Jespers, Parmenter and Sehgal did the same for finitely generated nilpotent groups.

1 Introduction

There is a classical result of G. Higman [1] that if \( A \) is a finite abelian group then any torsion unit of \( \mathbb{Z}A \), the integral group ring of \( A \), is trivial (i.e., of the form \( \pm a, a \in A \)). This was extended by Sehgal [4, Theorem I.3.5] to prove that if \( A \) is arbitrary abelian, then any unit \( \mu \) of \( \mathbb{Z}A \) can be written as a product \( \alpha a \), with \( \alpha \in \mathbb{Z}T \), where \( T \) denotes the torsion subgroup of

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A, and a ∈ A. Later it was proved by Sehgal [4, Theorem VI.3.22] that, under some stringent conditions, the last result extends to nilpotent groups. In another direction, one could ask if a similar description exists for central units in $\mathbb{Z}G$. Indeed, this was done by Jespers, Parmenter and Sehgal in [2] for finitely generated nilpotent groups. Using similar methods we extend this result to arbitrary groups. Recall that in a group $G$, the set

$$\Phi(G) = \{ g \in G \mid |G : C_G(g)| < \infty \}$$

is a characteristic subgroup of $G$, called the FC-centre (see [3, p. 115]).

**Theorem 1** Let $G$ be any group. Let $\Phi(G)$ denote the FC-centre of $G$ and let $T = T\Phi(G)$ be the torsion subgroup of $\Phi(G)$. Then, every central unit $\mu$ of $\mathbb{Z}G$ can be written in the form $\mu = \omega g$, with $\omega \in \mathbb{Z}T$ and $g \in \Phi(G)$; moreover, $\omega$ and $g$ commute.

Further, we are able to produce a finite set of generators for a subgroup of finite index in $Z(U(\mathbb{Z}G))$ when the FC-centre of $G$ is finitely generated. We shall always use $Z(H)$ to denote the centre of a group $H$.

**Theorem 2** Let $G$ be a group such that its FC-centre $\Phi = \Phi(G)$ is finitely generated. Let $\{z_1, \ldots, z_d\}$ be a set of generators of the centre of $G$. Then $\langle z_1, \ldots, z_d, b_1, \ldots, b_r \rangle$ is a subgroup of finite index in $Z(U(\mathbb{Z}G))$.

The elements $\{\tilde{b}_1, \ldots, \tilde{b}_r\}$ will be described in section 2. They are related to a set $\{b_1, \ldots, b_r\}$ of generators of a subgroup of finite index in $Z(U(\mathbb{Z}T))$ which could be, for example, the one obtained by Ritter and Sehgal [5, section 29].

Since any subgroup of a finitely generated nilpotent group is finitely generated Theorem 2 is also an extension of the results in [2].

## 2 Central units

The form of a central unit in $\mathbb{Z}G$ has been described in [2, Proposition 3], in the case when $G$ is a finitely generated nilpotent group. In this section, we shall prove that a similar result holds for arbitrary groups. To do so, we first consider the case where $G$ is an FC group and then show how to reduce the general case to this one.

**Proposition 2.1** Let $G$ be an FC group. Then, every central unit $\mu$ of $\mathbb{Z}G$ can be written in the form $\mu = \omega g$, with $\omega \in \mathbb{Z}T$ and $g \in G$. 
Proof. Given a central unit $\mu \in \mathbb{Z}G$, we can work in the integral group ring of the group generated by the elements in its support so, without loss of generality, we may assume that $G$ is finitely generated and hence that $T$ is finite. We can write $QT$ as a direct sum of simple components:

$$QT = \bigoplus_{j=1}^{r} A_j.$$ 

The central idempotents of $QT$ are not necessarily central in $\mathbb{Q}G$, so we let $G$ act by conjugation on these idempotents. Adding the components corresponding to all the idempotents in each orbit under this action, we can write

$$QT = \bigoplus_{i=1}^{n} R_i,$$

where each $R_i$ is a direct sum of simple rings and is invariant under conjugation by elements of $G$.

Now, set $F = G/T$. We can write $\mathbb{Q}G$ as a cross product:

$$\mathbb{Q}G = (QT) \ast F \cong \bigoplus_{i=1}^{n} (R_i \ast F).$$

Hence, the unit $\mu$ can be viewed as a tuple $\mu = (u_1, \ldots, u_n)$ where $u_i \in R_i \ast F$, $1 \leq i \leq n$, is of the form $u_i = \sum_h x_h f_h$, with $x_h \in R_h$, $f_h \in F$. Notice that we may assume that the elements in $F$ have been chosen in such a way that $f_l T \neq f_k T$ if $l \neq k$.

Claim 1. With the notations above, each coefficient $x_j$ is a unit in $R_i$.

Proof. Notice first that, since $\mu$ is also central in $\mathbb{Q}G$, for each element $f \in F$ we have that $f \mu = \mu f$. Hence, in a component of the form $R_i \ast F$ we can compute:

$$fu_i = \sum f_j x_j f_j = \sum x_j^{-1} f_j f_j = \sum x_j^{-1} \tau f_j f_j,$$

with $\tau \in U(R_i)$

and, taking into account the fact that $F = G/T$ is abelian, we also have

$$u_i f = \sum x_j f_j f_j = \sum x_j f f_j = \sum x_j f f_j.$$

So, we see that

$$x_j^{-1} = x_j,$$ (1)

Notice that if we write $u_i$ as a tuple in the direct sum $A_1 \oplus \cdots \oplus A_n$, the definition of $R_i$ implies that any two components can be switched by a conjugation by an element of $G$, so equation (1) actually means that $\mu = (a, \ldots, a)$ is a diagonal element.
Also, for an element \( t \in T \), we have that:

\[
\begin{align*}
tu_i &= \sum t \cdot x_j \overline{f}_j, \\
u_i t &= \sum x_j \overline{f}_j, t = \sum x_j t \overline{f}_j.
\end{align*}
\]

Because of the choice of the elements \( f_j \), since \( tu = ut \) we have that

\[
x_j t \overline{f}_j^{-1} = tx_j.
\]

Equation (2) shows that:

\[
x_j R_i = R_i x_j.
\]

so

\[
R_i x_j R_i = R_i x_j.
\]

Since \( R_i \) is a direct sum of simple components, it follows that, for each simple component \( A \) of \( R_i \), we have that \( Ax_j A = Ax_j \). Now, if \( x_j \neq 0 \) since \( x_j \) is diagonal, its projection in each simple component is non-zero, so \( Ax_j A \) is a non trivial two-sided ideal. Then, it follows that \( Ax_j A = A \); consequently \( Ax_j = A \) showing that \( x_j \) is invertible in each component. Hence, \( x_j \) is invertible in \( R_i \).

\textbf{Claim 2.} Each component \( u_i \) is actually of the form \( u_i = x_j \overline{f}_j \)

\textbf{Proof.} In fact, each component \( u_i \) is not a zero divisor and the group \( F \) is abelian, torsion free and thus ordered so, using the fact that the coefficients \( x_j \) are not zero divisors, a standard argument shows that \( u_i \) must be trivial in \( R_i \ast F \), as desired.

Now we are ready to prove our main statement. We can write \( \mu \) in the form

\[
\mu = \sum \alpha_i \overline{f}_i \in \oplus_i R_i \ast F, \quad \alpha_i \in R_i, \quad f_i \in F.
\]

Collecting together coefficients whenever \( f_i T = f_j T \) and changing notation we can write \( \mu \) in the form

\[
\mu = \sum \alpha_i' \overline{f}_i \in \oplus_i R_i \ast F, \quad \alpha_i' \in R_i, \quad f_i \in F, \quad f_i T \neq f_j T \text{ if } i \neq j.
\]

Since \( G \) is a finitely generated FC group, it is central-by-finite, so there exists a positive integer \( k \) such that \( g^k \in Z(G) \), for all \( g \in G \).

We compute:
\[ \mu^k = \sum_i (\alpha_i \overline{f}_i)^k = \sum_i \beta_i \overline{f}_i^k \in \oplus_i R_i \ast F, \quad \beta_i \in R_i, \quad f_i \in F. \tag{3} \]

Notice that, since \( G/T \) is torsion-free abelian, we also have that \( f_i^{kT} \neq f_j^{kT} \) whenever \( i \neq j \).

Now, since each \( f_i^k \) is central, we have that:

\[
\begin{align*}
t_{\mu^k} &= \sum t\beta_i \overline{f}_i^k, \\
\mu^k t &= \sum \beta_i \overline{f}_i^k t = \sum \beta_i t \overline{f}_i^k
\end{align*}
\]

So \( \beta_t = t\beta_i \), for all \( t \in T \), showing that the ring \( R \) generated by all the coefficients \( \beta_i \) is commutative and, in fact, \( R \subset \mathbb{Z}(\mathbb{Z}^2T) \).

Let \( A \) be the central subgroup generated by all the elements of the form \( f_i^k \). Since \( R \) and \( A \) also commute, this shows that

\[ u^k \in RA, \]

the commutative group ring of a finitely generated, torsion-free abelian group \( A \).

Set \( N = \text{rad}(R) \). Since idempotents of \( R/N \) can be lifted to idempotents of \( R \) and \( R \subset \mathbb{Z}T \) contains no nontrivial idempotents, it follows that \( R/N \) contains no nontrivial idempotents. So, it follows from [4, Theorem I.3.5] that \((R/N)A\) has only trivial units. Hence:

\[ \mu^k = \beta f^k + \nu, \] where \( \nu \) is nilpotent.

Comparing with the expression of \( \mu^k \) given in equation (3) we see that \( \nu = 0 \). Hence, \( \mu^k = \beta f^k \) as required. \( \Box \)

We shall now show how to extend this result to the general case.

**Proof of Theorem 1.**

For each finite conjugacy class \( C \) of \( G \) consider the class sum \( \gamma = \sum_{x \in C} x \).
It is well-known that the set of all of these class sums forms a \( \mathbb{Z} \)-basis for the centre of the group ring [3, Lemma 4.1.1]. This means that \( \mu \) is also a central unit in \( \mathbb{Z}[\Phi(G)] \), so we can apply the proposition above and it follows immediately that \( \mu = \omega g \), with \( \omega \in \mathbb{Z}T \), where \( T = T(\Phi(G)) \) and \( g \in \Phi(G) \).

Since \( \mu \) is central, we have that \( \mu^g = \omega^g g = \omega g \) so \( \omega^g = g \) as stated. \( \Box \)

As an easy consequence, we can give an elementary proof of [4, I.1.7].
Corollary 2.2 Any central unit of finite order in \( \mathbb{Z}G \) is trivial.

Proof. Let \( \mu \) be a central unit of finite order \( n \). Writing \( \mu = \omega g \) as in Theorem 1, we have that

\[
\mu^n = \omega^ng^n.
\]

This means that \( g^n = \omega^{-n} \in \mathbb{Z}T \). This implies that \( g^n \in T \) and thus \( g \in T \). It follows that \( \mu \in \mathbb{Z}T \) where \( T \) is finite, so the result follows immediately from Higman's Theorem [5, I.1.7]. \( \square \)

We remark that, since finitely generated FC groups are residually finite, the same proof as in [2, Corollary 4] now gives the following.

Corollary 2.3 Let \( T \) be the torsion subgroup of the FC-centre of a group \( G \). If \( Z(U(\mathbb{Z}T)) \) is trivial then \( Z(U(\mathbb{Z}G)) \) is also trivial.

Lemma 2.4 Let \( G \) be a group and \( Z(G) \) its centre. If \( \gamma \in 1+\Delta(G)\Delta(Z(G)) \) is such that \( \gamma^n = 1 \) for some positive integer \( n \), then \( \gamma = 1 \).

Proof. Since \( \gamma \) can be written as finite sum of the form \( \gamma = 1 + \sum (1 - g)(1 - z) \), we can assume that \( \gamma \in 1 + \Delta(G)\Delta(A) \), where \( G \) is a finitely generated group and \( A \) is a finitely generated central subgroup.

If \( A = 1 \) there is nothing to prove. We use induction on the rank of \( A \) plus the order of its torsion subgroup to conclude that taking \( z \in A \) and putting \( \bar{G} = G/(z) \) we have that \( \bar{\gamma} = 1 \) in \( \bar{G} \).

It follows that \( \gamma = 1 + \delta, \delta \in \Delta(G', (z)) \). Thus, there is a central element in the support of \( \gamma \). Remembering that \( \gamma \) is a torsion element, it follows from [5, Proposition 47.3] that \( \gamma = z_0 \), \( z_0 \in Z(G) \). We conclude that

\[
z_0 - 1 \in \Delta(G)\Delta(Z(G))
\]

and thus \( z_0 \in 1 + \Delta(Z(G))^2 \) so \( z_0 \) belongs in the second dimension subgroup of \( Z(G) \). Hence \( z_0 = 1 \) and \( \gamma = 1 \). \( \square \)

Lemma 2.5 Let \( G \) be a group and let \( \mu \in \mathbb{Z}G \) be a central unit. If there exists a positive integer \( n \) such that \( \mu^n \) is trivial, then \( \mu \) itself is a trivial unit.

Proof. Since \( \mu^n \in Z(G) \) we have that \( \mu^n = 1 \) in \( \mathbb{Z}[G/Z(G)] \). Thus \( \mu \) is a central unit of finite order in this group ring, so it is trivial by(2.2).
Hence, there exists an element $g \in G$ such that $\overline{\mu} = \overline{g}$ in the quotient, so we can write

$$\mu \equiv g \mod(\Delta(G, Z(G)))$$

Using the Whitcomb argument (see [5, Theorem 30.5]), it can be easily shown that there exists an element $g_1$ of the form $g_1 = gz$ with $z \in Z(G)$ such that

$$\mu \equiv g_1 \mod(\Delta(G)\Delta(Z(G))).$$

Thus

$$\mu^n \equiv g_1^n \mod(\Delta(G)\Delta(Z(G))),$$

hence

$$(\mu g_1^{-1})^n \equiv 1 \mod(\Delta(G)\Delta(Z(G))).$$

Since by Lemma 2.4 the group of invertible elements in $1 + \Delta(G)\Delta(Z(G))$ is torsion free it follows that $\mu g_1^{-1} = 1$ and thus $\mu = g_1 \in G$, as desired. \(\square\)

**Proposition 2.6** Let $G$ be a group such that its FC-centre $\Phi = \Phi(G)$ is finitely generated and let $S = Z(U(ZG)) \cap Z(U(ZT))$. Then $SZ(G)$ is a subgroup of finite index in $Z(U(ZG))$ and $Z(U(ZG))$ itself is finitely generated.

**Proof.** Since $T$ is finite, it follows that $Z(U(ZT))$ is finitely generated and thus, its subgroup $S$ is also finitely generated.

We have that $Z(G) \subset Z(\Phi)$ and $Z(\Phi)$ is of finite index in $\Phi$, which is finitely generated. Hence $Z(\Phi)$ is finitely generated and so is $Z(G)$.

We claim that $Z(U(ZG))$ is of bounded exponent over $SZ(G)$. In fact, given $\mu \in Z(U(ZG))$ we write it in the form $\mu = \omega g$ with $\omega \in U(ZT)$ and $g \in G$. Then $gT \in Z(G/T)$. If we set $k = |\text{Aut} T|$, we have that $(g^k, t) = 1$, $\forall t \in T$. Given any element $x \in G$, we have that:

$$(g^{k[T]}t)^x = (g^{kx}[T]) = (g^k t)[T], \text{ for some } t \in T.$$ 

Hence

$$(g^{k[T]}t)^x = g^{k[T]}t[T] = g^{k[T]}.$$ 

So, if we set $h = k[T]$ we have that $g^h \in Z(G)$.

Now $\mu^h = \omega^g g^h$ so $\omega^h = \mu^h g^{-h} \in Z(U(ZG)) \cap Z(ZT)$. Consequently, $\mu^h \in S Z(G)$.

Finally, we observe that $T(Z(U(ZG)))$ is trivial by (2.2), so it is finite. It follows that $Z(U(ZG))$ is finitely generated, as desired. \(\square\)
Let \( \{b_1, b_2, \ldots, b_r\} \) be any set of generators of a subgroup of finite index in \( Z(U(\mathbb{Z}T)) \). For example, this could be the set of generators explicitly constructed by Ritter-Sehgal (see [5, Theorem 29.2]). Let \( X \) be a transversal of the centralizer \( C_G(T) \). For each element \( b_i \) we define:

\[
\tilde{b}_i = \prod_{x \in X} b_i^x.
\]

Notice that this product is independent of the order of its factors since they belong to \( Z(U(\mathbb{Z}T)) \) which is commutative and is normalized by \( G \). Clearly \( \tilde{b}_i \in Z(U(\mathbb{Z}G)) \), \( 1 \leq i \leq r \).

Let \( \{\alpha_1, \ldots, \alpha_s\} \) be a set of generators of \( S \). Since \( \langle b_1, \ldots, b_r \rangle \) is of finite index in \( Z(U(\mathbb{Z}T)) \), there exists a positive integer \( m \) such that:

\[
\alpha_i^m \in \langle b_1, \ldots, b_r \rangle, \quad 1 \leq i \leq s.
\]

Hence, each element \( \alpha_i^m \) can be written as a product:

\[
\alpha_i^m = \prod b_j.
\]

So,

\[
\alpha_i^m \prod_{x \in X} = \prod_{x \in X} (\alpha_i^m)^x = \prod_{x \in X} \tilde{b}_j.
\]

This shows that \( \alpha_i^m \prod_{x \in X} \in \langle \tilde{b}_1, \ldots, \tilde{b}_r \rangle, \quad 1 \leq i \leq s \). It follows that \( \langle \tilde{b}_1, \ldots, \tilde{b}_r \rangle \) is a subgroup of finite index in \( S \).

Since we have shown that \( |Z(U(\mathbb{Z}G)) : SZ(G)| \) is finite, the proof of Theorem 2 follows from the considerations above.

References


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