ON THE ISOMORPHISM OF INTEGRAL GROUP RINGS. II

SUDARSHAN K. SEHgal

Reprinted from
The Canadian Journal of Mathematics
ON THE ISOMORPHISM OF INTEGRAL GROUP RINGS. II

SUDARSHAN K. SEHGAL*

1. Introduction. Let \( Z(G) \) denote the integral group ring of a group \( G \). Let \( C \) be the class of groups \( G \) with the property that for any isomorphism \( \theta: Z(G) \rightarrow Z(H) \), we have \( \theta(g) = \pm h, h \in H \), for all \( g \in G \). We study this class in § 2 and establish that it contains classes of torsion-free abelian groups, torsion abelian groups, and ordered groups.

In § 4, we prove the following result.

THEOREM. Let \( G \) be a group which contains a normal abelian subgroup \( A \) such that \( G/A \in C \). Suppose that \( \theta: Z(G) \rightarrow Z(H) \) is an isomorphism such that \( \theta(\Delta(G, A)) = \Delta(H, B) \) for a suitable normal subgroup \( B \) of \( H \). Then \( G \simeq H \).

(Jackson (3) and Whitcomb (6) proved the special case of this theorem when \( G \) is supposed to be finite metabelian. The lemmas needed are given in § 3. In § 5, we extend (to arbitrary finite groups) a result of Passman (4), proved by him for finite nilpotent groups. As a corollary, it follows that if two finite groups have isomorphic integral group rings, then they have isomorphic derived series.

In the last section we give an easy proof of a theorem of Banaschewski.

2. The class \( C \). The following lemma was proved in (5).

LEMMA 1. Let \( G \) be an abelian group. Suppose that \( \gamma \in Z(G) \) is such that \( \gamma^n = 1 \) for some natural number \( n \). Then \( \gamma = \pm t \), where \( t \) is a torsion element of \( G \).

LEMMA 2. Let \( G \) be a torsion-free abelian group. Then \( U_{Z(G)} \), the group of units of \( Z(G) \), is given by

\[ U_{Z(G)} = \pm G = \{ \pm g \mid g \in G \}. \]

Proof. We can suppose, without loss of generality, that \( G \) is finitely generated and hence free. Suppose that \( G = \langle x_1, \ldots, x_n \rangle \). For \( g = x_1^{a_1} \cdots x_n^{a_n} \), set \( \deg(g) = \sum \alpha_i \). Let \( \gamma, \mu \in Z(G) \) be such that \( \gamma \mu = 1 \). Let \( m \) and \( t \) be the maximum degrees of terms appearing in \( \gamma \) and \( \mu \), respectively. Due to the fact that \( Z(G) \) is free of zero divisors, it follows that \( \gamma \mu \) has a term of degree

Received April 23, 1968.

*I wish to thank the National Research Council of Canada for partial support during the time this work was done.

1182
m + t. Therefore m + t = 0. Similarly, supposing s and u to be the minimum degrees of terms appearing in γ and μ, respectively, we conclude that s + u = 0. Thus s ≤ m yields −s ≥ −m, i.e. u ≥ t, and hence u = t and m = s. Therefore, all the terms appearing in γ have degree m and those in μ are of degree −m.

Let us use induction on n, the number of generators of G. If n = 1, then clearly γ = ±x1α1. If n > 1, we use the same argument as above on the degree in x1, namely d1(g) = α1, and conclude that γ = xαγ1 and μ = x−αμ1, where γ1, μ1 ∈ Z(G1), G1 = ⟨x2, . . . , xn⟩.

Now by induction, γ1 = ±g1, g1 ∈ G, and γ = ±g, g ∈ G.

Remark. The last lemma follows from the next but we have included the direct proof because of its independent interest.

Lemma 3. Let G be an ordered group. Then

\[ U_{\mathbb{Z}(O)} = \pm G = \{ \pm g \mid g \in G \} \]

Proof. Let \( \gamma = \sum a_i g_i, \mu = \sum b_j h_j \) be such that \( \gamma \mu = 1 \). Suppose that

\[ g_s = \max \{ g_i \}, \quad g_1 = \min \{ g_i \}, \]

\[ h_t = \max \{ h_i \}, \quad h_1 = \min \{ h_i \}. \]

Then \( g_s h_t = 1 = g_s h_1 \). Since \( h_1 ≤ h_s \), it follows that \( h_1^{-1} ≥ h_s^{-1} \), i.e. \( g_1 ≥ g_s \), and therefore \( g_1 = g_s \). We conclude that \( γ = ±g \).

An immediate consequence of these lemmas is the following result.

Proposition 1. \( \mathcal{C} \) contains G if it is of any of the following types:

(i) torsion abelian,

(ii) torsion-free abelian,

(iii) ordered.

3. Some lemmas. For any normal subgroup H of G, let \( p_H : Z(G) → Z(G/H) \) be the linear extension of the natural homomorphism \( G → G/H \). Then \( \Delta(G, H) \), the kernel of \( p_H \), is the ideal generated by \( \{ (1 - h) \mid h \in H \} \). We shall write \( \Delta(H) \) instead of \( \Delta(G, H) \) if no confusion can arise. Furthermore, for \( \sum a_v g_v = γ \in Z(G) \), we denote by \( c(γ) \) the integer \( \sum_v a_v \). We need the following lemmas.

Lemma 4. Suppose that \( g \in G \) is such that \( g = 1 \mod(\Delta(G))^2 \). Then \( g \in G' \), the derived group of G.

Proof. Let us first suppose that G is abelian. We can assume that G is finitely generated, say, \( G = \langle g_1, g_2, . . . , g_r \rangle \). Let \( g = \prod g_i^{α_i} \). Since

\[ g - 1 = \sum a_i (g_i - 1) \mod(\Delta(G))^2, \]

we have \( \sum a_i (g_i - 1) \in (\Delta(G))^2 \). Therefore

\[ \sum a_i (g_i - 1) = \sum x(g_1^{α_1} . . . g_r^{α_r} - 1)(g_1^{β_1} . . . g_r^{β_r} - 1), \quad x \in Z(G). \]
Extend the endomorphism $\gamma_j: g_i \rightarrow g_i^{s_j}$ of $G$ to $Z(G)$. Applying this to both sides of (*) we obtain:

$$a_j(g_j - 1) = \sum x_{st}(g_j^s - 1)(g_j^t - 1), \quad x_{st} \in Z\langle g_j \rangle,$$

and therefore,

$$(a_j - y)(g_j - 1) = 0, \text{ where } y(g_j - 1) = \sum x_{st}(g_j^s - 1)(g_j^t - 1).$$

Suppose that $a_j \neq 0$, then since $c(y) \neq 0$, we can say that $(a_j - y) \neq 0$. It follows that $g_j$ has finite order $n_j$ and $a_j - y = m(1 + g_j + \ldots + g_j^{n_j-1})$, $m \in Z$. Therefore $a_j = mn_j$ and $g_j = 1$.

Now let $G$ be arbitrary and let $\beta: G \rightarrow G'G'$ be the natural map. Let $g^a \equiv 1 \mod (\Delta(G/G'))^2$, therefore $g^a = 1$ and $g \in G'$. This completes the proof of the lemma.

**Corollary 1.** $G/G' \cong \Delta(G)/(\Delta(G))^2$.

**Proof.** Let $\theta: G \rightarrow \Delta(G)/(\Delta(G))^2$ be given by $\theta(g) = (g - 1) \mod (\Delta(G))^2$. Then $\theta$ is a homomorphism as $gh - 1 = (g - 1) + (h - 1) + (g - 1)(h - 1)$. It is an epimorphism because of the same reason. By Lemma 4, the kernel of $\theta$ is $G'$, and the proof is complete.

The next corollary was proved by Higman (2) for finite abelian groups.

**Corollary 2.** Let $G$ be an abelian group such that $Z(G) \cong Z(H)$. Then $G \cong H$.

**Proof.** Let $\theta: Z(G) \rightarrow Z(H)$ be the given isomorphism. Then $c(\theta(g)) = \pm 1$. Normalize $\theta$ by defining $\mu: Z(G) \rightarrow Z(H)$ by $\mu(g) = c(\theta(g))\theta(g)$ for $g \in G$, and linear extension. It is easy to see that $\mu$ is an isomorphism and $c(\mu(g)) = 1$ for all $g \in G$. Now

$$\mu(\Delta(G)) = \Delta(H) \quad \text{and} \quad G \cong \Delta(G)/(\Delta(G))^2 \cong \Delta(H)/(\Delta(H))^2 \cong H.$$ 

Similarly, we have the following result.

**Corollary 3.** $Z(G) \cong Z(H) \Rightarrow G/G' \cong H/H'$.

**Lemma 5.** Let $N$ be a normal subgroup of $G$. Then

$$\gamma \in Z(N), \quad \gamma \equiv 0 \mod \Delta(G, G)\Delta(G, N) \Rightarrow \gamma \equiv 0 \mod (\Delta(N, N))^2.$$

**Proof.** Choose a set of coset representatives $\{g_i\}$ of $G$ mod $N$. Define for $g_i n \in G$, $\sigma(g_i n) = n$ and extend this linearly to $\sigma: Z(G) \rightarrow Z(N)$. Now $\gamma = \sum \gamma_i(n_i - 1)$, $\gamma_i \in \Delta(G, G)$, $n_i \in N$. Therefore

$$\gamma = \gamma \sigma = \sum \gamma_i(n_i - 1) \quad \text{and} \quad \gamma \in (\Delta(N, N))^2.$$

**Corollary 4.** Let $N$ be a normal subgroup of $G$. Then

$$g \in G, \quad g \equiv 1 \mod \Delta(G, G)\Delta(G, N) \Rightarrow g \in N'.$$

**Proof.** Since $g - 1 \equiv 0 \mod \Delta(G, G)\Delta(G, N)$, we have $gN = N$ and $g \in N$. 

1184  SUDARSHAN K. SEHGAL
Now by Lemma 5, $g - 1 \equiv 0 \mod \Delta(N, N)^2$, and hence $g \in N'$ by Lemma 4.

**Lemma 6.** Let $A$ be a normal subgroup of $G$. Then

$$A/A' \simeq \Delta(G, A)/\Delta(G)\Delta(G, A).$$

**Proof.** Define the map $\theta : A \to \Delta(G, A)/\Delta(G)\Delta(G, A)$ by $\theta(a) = 1 - a$. The kernel of $\theta$ is $A'$ by the last corollary. That $\theta$ is an epimorphism follows from the fact that

$$g(a - 1) = a - 1 \mod \Delta(G)\Delta(G, A).$$

**4. Proof of the main Theorem.**

**Proposition 2.** Suppose that $G$ has a normal abelian subgroup $A$ such that $G/A \in \mathcal{C}$. Suppose that there is a normalized isomorphism $\theta : Z(G) \to Z(H)$ such that $\theta(\Delta(G, A)) = \Delta(H, B)$ for a suitable normal subgroup $B$ of $H$. Then $B$ is abelian and $H/B \in \mathcal{C}$.

**Proof.** Since $\theta(\Delta(G, A)) = \Delta(H, B)$, we have:

$$Z(G/A) \simeq Z(G)/\Delta(G, A) \simeq Z(H)/\Delta(H, B) \simeq Z(H/B).$$

Let $\lambda : Z(G/A) \to Z(H/B)$ be the isomorphism. Then $\lambda(\bar{g}) = \bar{h} \in H/B$ for all $\bar{g} \in G/A$. Actually, $\lambda(G/A) = H/B$. Now, given an isomorphism $\mu : Z(H/B) \to Z(K)$, we need to prove that for $\bar{h} \in H/B$, $\mu(\bar{h}) = k \in K$. Consider $\mu \lambda : Z(G/A) \to Z(K)$. We know that $\bar{h} = \lambda(\bar{g})$ for some $\bar{g} \in G/A$. Since $G/A \in \mathcal{C}$, $(\mu \lambda)(\bar{g}) = k, k \in K$. It follows that $\mu(\bar{h}) = k$ and $H/B \in \mathcal{C}$. Furthermore, $B$ is abelian, since by Lemma 6:

$$A/A' \simeq \Delta(G, A)/\Delta(G)\Delta(G, A) \simeq \Delta(H, B)/\Delta(H)\Delta(H, B) \simeq B/B'.$$

**Remark.** It is easy to see that if in the above proposition $A = G'$, then $B = H'$ satisfies the condition $\theta(\Delta(G, A)) = \Delta(H, B)$.

**Remark.** If in the above proposition $G/A$ is ordered (in particular, torsion-free abelian) we do not need to assume the existence of $B$, since in this case it can be proved that there always exists a $B$ with the property

$$\theta(\Delta(G, A)) = \Delta(H, B).$$

**Theorem 1.** Suppose that $G$ has a normal abelian subgroup $A$ such that $G/A \in \mathcal{C}$. Suppose that there is an isomorphism $\theta : Z(G) \to Z(H)$ such that $\theta(\Delta(G, A)) = \Delta(H, B)$ for a suitable normal subgroup $B$ of $H$. Then $G \simeq H$.

**Proof.** We can assume that $\theta$ is normalized; then due to Proposition 2, $B$ is abelian and $H/B \in \mathcal{C}$. For $g \in G$, let $\theta(g) = \gamma$. Then since $H/B \in \mathcal{C}$, we have:

$$\gamma = h_1 \mod (\Delta(H, B)), \quad h_1 \in H.$$
Now

\[ \gamma = h_1 + \sum_{b \in B} \alpha_b (1 - b), \quad \alpha_b \in Z(H), \]

\[ = h_1 + \sum_{b \in B} n_b (1 - b) \mod (\Delta(H) \cdot \Delta(B)), \text{ where } n_b = c(\alpha_b), \]

\[ = h_1 + 1 - \prod b^{n_b} \mod (\Delta(H) \cdot \Delta(B)), \]

\[ = h_1 \prod b^{-n_b} \mod (\Delta(H) \cdot \Delta(B)). \]

Thus for \( g \in G \), if \( \theta(g) = \gamma \), there exists \( h_\gamma \in H \) such that

\[ \gamma = h_\gamma \mod (\Delta(H) \cdot \Delta(B)). \]

This \( h_\gamma \) is unique since if \( h' = h \mod (\Delta(H) \cdot \Delta(B)) \), then \( h'h^{-1} \in B' = \{1\} \) by Corollary 4, and hence \( h' = h \). Define \( \lambda: G \to H \) by \( \lambda(g) = h_\gamma \in H \). Then \( \lambda \) is a well-defined homomorphism. Notice that we have:

\[ g \xrightarrow{\theta} \gamma \xrightarrow{\phi_1} h_\gamma \xrightarrow{\theta^{-1}} \mu \xrightarrow{\phi_2} g_\mu \quad (g, g_\mu \in G), \]

where \( \phi_2 \) is the map obtained (as was \( \phi_1 \)) by using \( \theta^{-1} \) instead of \( \theta \). We see that \( g_\mu = g \) since

\[ (\phi_2\theta^{-1}\phi_1\theta)(g) = \phi_2\theta^{-1}(\theta(g) + \delta), \text{ where } \delta \in \Delta(H)\Delta(B), \]

\[ = \phi_2(g + \beta), \quad \text{ where } \theta^{-1}(\delta) = \beta \in \Delta(G)\Delta(A), \]

\[ = g \quad (\text{due to uniqueness of } g_\mu). \]

Thus \( \phi_2\theta^{-1}\phi_1\theta = I_G \), and similarly \( \phi_1\theta\phi_2\theta^{-1} = I_H \). It follows that \( \lambda = \phi_1\theta \) is an isomorphism.

5. **Group rings of finite groups.** Suppose that \( \theta: Z(G) \to Z(H) \) is a normalized isomorphism, where \( G \) (and hence \( H \)) is a finite group. We proved in (5) that for any normal subgroup \( A \) of \( G \), \( \theta(\sum_{a \in A} a) = (\sum_{b \in B} b) \) and \( A \leftrightarrow B = \Phi(A) \) is a one-to-one correspondence between normal subgroups of \( G \) and those of \( H \). This correspondence preserves union, intersection, and order. This is a result of Passman (4) who also proved the next proposition for finite nilpotent groups.

**Proposition 3.** Suppose that \( \Phi(A) = B \). Then

(i) \( Z(G/A) \simeq Z(H/B) \),

(ii) \( A/A' \simeq B/B' \), and

(iii) \( \Phi(A') = B' \).

**Proof.** Since the annihilator of \( \sum_{a \in A} a \) in \( Z(G) \) is \( \Delta(G, A) \), it follows that \( \theta(\Delta(G, A)) = \Delta(H, B) \). Therefore

\[ Z(G/A) \simeq Z(G)/\Delta(G, A) \simeq Z(H)/\Delta(H, B) \simeq Z(H/B). \]


To prove (ii) we only have to notice that Lemma 6 yields:
\[ A/A' \cong \Delta(G, A)/\Delta(G)\Delta(G, A) \cong \Delta(H, B)/\Delta(H)\Delta(H, B) \cong B/B'. \]
We now prove (iii). Suppose that \( \Phi(A') = C \). Then we have a sequence of isomorphisms
\[ Z(G/A') \rightarrow Z(G)/\Delta(A') \rightarrow Z(H)/\Delta(C) \rightarrow Z(H/C), \]
where \( \bar{g} \rightarrow g \mod \Delta(A') \rightarrow \theta(g) \mod \Delta(C) \rightarrow \theta(g) \mod C \). Thus we have the isomorphism \( \lambda: Z(G/A') \rightarrow Z(H/C) \) given by \( \lambda(\bar{g}) = \theta(g) \mod C \). Now \( A/A' \) is a normal subgroup of \( G/A' \) and \( \Phi(A/A') \) is a normal subgroup of \( H/C \). From
\[ \lambda\left( \sum_{\bar{a} \in A} \bar{a} \right) = \left( \sum_{a \in A} \theta(a) \mod C \right) = \left( \sum_{b \in B} b \mod C \right), \]
we conclude that \( \Phi(A/A') = B/C \). Since \( A/A' \) is abelian, \( B/C \) is abelian and \( C \supseteq B' \). Therefore, due to (ii) and equality of orders of \( A/A' \) and \( B/C \), we have \( C = B' \). This completes the proof.

**Theorem 2.** Suppose that \( G \) is a finite group and \( Z(G) \cong Z(H) \). Then the derived series of \( G \) and \( H \) are isomorphic. In particular, if \( G \) is solvable, then so is \( H \).

**Proof.** Suppose that \( G_i \) and \( H_i \) are \( i \)th terms of the derived series of \( G \) and \( H \), respectively. Let \( \theta: Z(G) \rightarrow Z(H) \) be the normalized isomorphism and \( \Phi(G_i) = H_i \). Then by Proposition 3 (iii), \( \Phi(G_{i+1}) = H_{i+1} \). Also \( \theta(\Delta(G, G_i)) = \Delta(H, H_i) \), and by Lemma 6,
\[ G_i/G_{i+1} \cong \Delta(G, G_i)/\Delta(G) \cdot \Delta(G, G_i) \cong \Delta(H, H_i)/\Delta(H) \cdot \Delta(H, H_i) \cong H_i/H_{i+1}. \]
This completes the proof by induction.

6. A theorem of Banaschewski. For \( \gamma = \sum_{g \in G} a_g g \in Z(G) \), let \( \gamma^* = \sum_{g \in G} a_g g^{-1} \). Then
(i) \( (\gamma + \mu)^* = \gamma^* + \mu^* \),
(ii) \( (\gamma\mu)^* = \mu^*\gamma^* \), and
(iii) \( (\alpha\gamma)^* = \alpha\gamma^* \) for \( \alpha \in G \).
We say that a map \( \theta: Z(G) \rightarrow Z(H) \) is \(*\)-preserving if \( \theta(\gamma^*) = (\theta(\gamma))^* \) for all \( \gamma \in Z(G) \). We offer a simple proof of the following theorem proved by Banaschewski (1) for finite groups.

**Theorem 3.** Suppose that \( \theta: Z(G) \rightarrow Z(H) \) is a \(*\)-preserving isomorphism. Then \( G \cong H \).

**Proof.** Let \( \theta(g) = \gamma = \sum a_h h \); then \( 1 = \theta(gg^{-1}) = \theta(g)\theta(g^*) = \gamma\gamma^* \). Now
\[ \gamma\gamma^* = \left( \sum a_h^2 \right) \cdot 1 + \sum_{h \neq e} \beta_h h \]
implies that \( \sum a_h^2 = 1 \) and \( \gamma = \pm h, h \in H \). Thus \( G \cong H \).
REFERENCES


University of Alberta,
Edmonton, Alberta