



# Nilpotency of group ring units symmetric with respect to an involution<sup>☆</sup>

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## ABSTRACT

Let  $F$  be an infinite field of characteristic different from 2. Let  $G$  be a torsion group having an involution  $*$ , and consider the units of the group ring  $FG$  that are symmetric with respect to the induced involution. We classify the groups  $G$  such that these symmetric units satisfy a nilpotency identity  $(x_1, \dots, x_n) = 1$ .

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## 1. Introduction

Let  $FG$  be the group ring of a group  $G$  over a field  $F$  of characteristic different from 2. If  $G$  is endowed with an involution  $*$ , then we can extend it  $F$ -linearly to an involution of  $FG$ , also denoted by  $*$ . An element  $\alpha \in FG$  is said to be symmetric with respect to  $*$  if  $\alpha^* = \alpha$ . We write  $(FG)^+$  for the set of symmetric elements, which are easily seen to be the linear combinations of the terms  $g + g^*$ , for all  $g \in G$ . Let  $\mathcal{U}^+(FG)$  denote the set of symmetric units.

The symmetric units have been the subject of a good deal of attention; indeed, it is interesting to know the extent to which these units determine the structure of the unit group of the group ring. Prior to the last couple of years, the attention had largely been devoted to the classical involution induced from the map  $g \mapsto g^{-1}$  on  $G$ . For example, in Giambruno–Sehgal–Valenti [4] and Sehgal–Valenti [15], the groups  $G$  were determined such that the units symmetric with respect to the classical involution satisfy a group identity. (We recall that a subset  $S$  of a group  $G$  is said to satisfy a group identity if there exists a nontrivial word  $w(x_1, \dots, x_n)$  in the free group  $\langle x_1, x_2, \dots \rangle$  such that  $w(s_1, \dots, s_n) = 1$  for all  $s_i \in S$ .) But recently, in [2], Giambruno–Polcino Milies–Sehgal established the corresponding result for an involution induced from an arbitrary involution on  $G$ , when  $G$  is a torsion group.

Particular group identities are also of interest. On any group, let

$$(x_1, x_2) = x_1^{-1}x_2^{-1}x_1x_2$$

and

$$(x_1, \dots, x_n, x_{n+1}) = ((x_1, \dots, x_n), x_{n+1}).$$

We say that  $\mathcal{U}^+(FG)$  is nilpotent if it satisfies the group identity  $(x_1, \dots, x_n)$ , for some  $n$ . When the involution on  $FG$  is the classical one, the groups  $G$  such that  $\mathcal{U}^+(FG)$  is nilpotent were determined in Lee [7] and Lee–Polcino Milies–Sehgal [8]. Our goal in this paper is to extend the result to arbitrary involutions, when  $F$  is an infinite field and  $G$  a torsion group.

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Recall that a nonabelian group  $G$  is said to be an LC-group (for “lack of commutativity”) if  $G$  is not abelian, and whenever  $g, h \in G$  and  $gh = hg$ , at least one of  $\{g, h, gh\}$  is central. These groups were introduced by Goodaire. A group  $G$  is an LC-group with a unique nonidentity commutator if and only if  $G/\zeta(G) \simeq C_2 \times C_2$ , where  $\zeta(G)$  is the centre of  $G$  (see [5, Proposition III.3.6]). An LC-group, endowed with an involution  $*$ , is said to be a special LC-group, or SLC-group, if it has a unique nonidentity commutator  $z$ , and for all  $g \in G$ , we have  $g^* = g$  if  $g$  is central, and otherwise,  $g^* = gz$ . Our main result is the following.

**Theorem.** *Let  $F$  be an infinite field of characteristic  $p > 2$  and  $G$  a torsion group having an involution  $*$ , and let  $FG$  have the induced involution. Suppose that  $\mathcal{U}(FG)$  is not nilpotent. Then  $\mathcal{U}^+(FG)$  is nilpotent if and only if  $G$  is nilpotent and  $G$  has a finite normal  $*$ -invariant  $p$ -subgroup  $N$  such that  $G/N$  is an SLC-group.*

## 2. Preliminaries

Here we present some necessary lemmas. We assume throughout that  $G$  is a group with involution  $*$ , and  $F$  is a field of characteristic  $p \neq 2$ . Our starting point is the main result of Giambruno–Polcino Milies–Sehgal [2].

**Lemma 1.** *Let  $G$  be torsion and  $F$  infinite. Then the symmetric units of  $FG$  satisfy a group identity if and only if*

1.  $FG$  is semiprime and  $G$  is abelian or an SLC-group, or
2.  $FG$  is not semiprime, the  $p$ -elements of  $G$  form a (normal) subgroup  $P$ ,  $G$  has a  $p$ -abelian normal subgroup of finite index, and either
  - (a)  $G'$  is a  $p$ -group of bounded exponent, or
  - (b)  $G/P$  is an SLC-group and  $G$  contains a normal  $*$ -invariant  $p$ -subgroup  $B$  of bounded exponent, such that  $P/B$  is central in  $G/B$  and the induced involution acts as the identity on  $P/B$ .

Here, we recall (see [11, Theorems 4.2.12 and 4.2.13]) that  $FG$  is semiprime if and only if  $G$  has no finite normal subgroups with order divisible by  $\text{char } F$ . In particular, in the characteristic zero case,  $G$  can only be abelian or an SLC-group. But Jespers–Ruiz Marín proved the following in [6].

**Lemma 2.** *The following are equivalent:*

1.  $(FG)^+$  is commutative,
2.  $(FG)^+$  is central in  $FG$ , and
3.  $G$  is abelian or an SLC-group.

Thus, the characteristic zero case is completely resolved, and we need only concern ourselves with the  $p > 2$  case. We also have need of the following reduction, which is a simple consequence of [12, Lemma 2.1]. For convenience, we include the proof. If  $N$  is a normal subgroup of  $G$ , let  $\epsilon_N : FG \rightarrow F(G/N)$  be the obvious homomorphism, and let  $\Delta(G, N)$  be its kernel. If  $N$  is finite, let  $\hat{N} = \sum_{g \in N} g$ .

**Lemma 3.** *Let  $G$  be torsion and  $F$  infinite. Fix a group identity  $w(x_1, \dots, x_n)$ . If  $N$  is a  $*$ -invariant normal subgroup of  $G$ , and  $\mathcal{U}^+(FG)$  satisfies  $w$ , then so does  $\mathcal{U}^+(F(G/N))$ .*

**Proof.** Suppose that  $N$  is finite and its order is not divisible by  $\text{char } F$ . Then letting  $e = \frac{1}{|N|}\hat{N}$ , we see that  $e$  is a symmetric central idempotent. Furthermore,  $FGe \cong FG/\Delta(G, N) \cong F(G/N)$ , and these isomorphisms respect the involution. Thus, as  $FG = FGe \oplus FG(1 - e)$ , we see that  $\mathcal{U}(F(G/N))$  is isomorphic to a subgroup of  $\mathcal{U}(FG)$ , and this isomorphism respects the involution. Hence,  $\mathcal{U}^+(F(G/N))$  satisfies  $w$ .

Now let  $p > 2$  and suppose that  $N$  is a finite  $p$ -group. Letting  $F\bar{G} = F(G/N)$ , take  $\bar{\alpha} \in \mathcal{U}^+(F\bar{G})$ . Of course, we can lift  $\bar{\alpha}$  to  $\alpha \in (FG)^+$ . Similarly, if  $\bar{\beta} = \bar{\alpha}^{-1}$ , then we lift  $\bar{\beta}$  to  $\beta \in (FG)^+$ . But then  $\alpha\beta - 1 \in \Delta(G, N)$  which, by [14, Lemma 1.2.21], is a nilpotent ideal. Therefore, for some  $k \geq 0$ ,  $(\alpha\beta)^{p^k} = 1$  and, similarly,  $(\beta\alpha)^{p^k} = 1$ , hence  $\alpha$  is a symmetric unit. That is, every symmetric unit in  $F\bar{G}$  is the homomorphic image of a symmetric unit in  $FG$ . Thus,  $\mathcal{U}^+(F(G/N))$  satisfies  $w$ .

If  $N$  is finite and  $p > 2$ , then we see from Lemma 1 that the  $p$ -elements of  $N$  form a (normal  $*$ -invariant) subgroup  $H$ . Thus, by our observations above, we can factor out  $H$  and then  $N/H$  to obtain our result.

Finally, let  $N$  be arbitrary. We see from Lemma 1 that  $G$  is locally finite. Thus, taking any  $\bar{\alpha}_1, \dots, \bar{\alpha}_n \in \mathcal{U}^+(F\bar{G})$ , and lifting the  $\bar{\alpha}_i$  and their inverses to symmetric elements of  $FG$ , we can find a finite  $*$ -invariant subgroup  $K$  of  $G$  containing the supports of all of these elements. Replacing  $G$  with  $K$  and  $N$  with  $K \cap N$ , we now apply the finite case to obtain the result.  $\square$

The classification of the groups  $G$  such that  $\mathcal{U}(FG)$  is nilpotent is found in [14, Section VI.3]. In particular, the following is proved.

**Lemma 4.** *If  $G$  is torsion, then  $\mathcal{U}(FG)$  is nilpotent if and only if  $G$  is nilpotent and  $p$ -abelian.*

We need several group-theoretic lemmas as well.

**Lemma 5.** *Let  $G$  be abelian and torsion. If  $G$  has no 2-elements, then  $G = G^+ \times H$ , where  $H = \{g \in G : g^* = g^{-1}\}$ .*

**Proof.** See [1, Corollary 2.10].  $\square$

**Lemma 6.** Let  $G$  be finite, and suppose that  $G = A \rtimes H$ , where  $A$  and  $H$  have relatively prime orders. Further suppose that  $A$  is abelian of odd order and  $*$ -invariant. If  $h \in H$  satisfies  $h^* \in hA$ , then there exists  $a \in A^+$  such that  $(ha)^* = ha$ .

**Proof.** See [3, Lemma 2.8 and Remark 3.1].  $\square$

**Lemma 7.** Suppose that  $G$  has an abelian normal torsion subgroup  $A$  without 2-elements. Further suppose that  $g \in G \setminus A$  satisfies  $g^* \in g^{-1}A$ . Then there exists  $a \in A^+$  such that  $(ga)^* = (ga)^{-1}$ .

**Proof.** See [1, Lemma 2.11].  $\square$

Finally, in any ring  $R$ , write  $[a_1, a_2] = a_1a_2 - a_2a_1$ , and  $[a_1, \dots, a_{n+1}] = [[a_1, \dots, a_n], a_{n+1}]$ . We say that a subset  $S$  of  $R$  is Lie nilpotent if there exists an  $n$  such that  $[s_1, \dots, s_n] = 0$  for all  $s_i \in S$ . The groups  $G$  such that  $FG$  is Lie nilpotent were classified in Passi–Passman–Sehgal [10].

**Lemma 8.**  $FG$  is Lie nilpotent if and only if  $G$  is nilpotent and  $p$ -abelian.

The groups  $G$  such that  $(FG)^+$  is Lie nilpotent were classified in Giambruno–Polcino Milies–Sehgal [1] and Lee–Sehgal–Spinelli [9].

**Lemma 9.** Suppose that  $p > 2$  and  $FG$  is not Lie nilpotent. Then  $(FG)^+$  is Lie nilpotent if and only if  $G$  is nilpotent and  $G$  has a finite normal  $*$ -invariant  $p$ -subgroup  $N$  such that  $G/N$  is an SLC-group.

**Proof.** See [9, Theorem 1].  $\square$

### 3. Proof of the main result

We assume throughout that  $F$  is an infinite field of characteristic  $p > 2$ , and  $G$  is a torsion group endowed with an involution  $*$ . Write  $P$  for the set of  $p$ -elements of  $G$ . Let us begin with

**Lemma 10.** Let  $G$  be a finite group of odd order. If  $\mathcal{U}^+(FG)$  is nilpotent, then  $G$  is nilpotent and  $p$ -abelian.

**Proof.** If  $FG$  is semiprime, then by Lemma 1,  $G$  is abelian or an SLC-group. Of course, an SLC-group cannot have odd order, so this case is done. Suppose that  $FG$  is not semiprime. In view of Lemma 1,  $P$  is a subgroup of  $G$ , and  $G/P$  is abelian or an SLC-group. Again, it cannot be an SLC-group, so  $G'$  is a  $p$ -group. It therefore suffices to show that  $G$  is nilpotent. By [13, 5.2.10], it is enough to show that  $G/P'$  is nilpotent. Noting that  $P'$  is  $*$ -invariant, we see from Lemma 3 that  $\mathcal{U}^+(F(G/P'))$  is nilpotent. Thus, we replace  $G$  with  $G/P'$  and assume that  $P$  is abelian.

Letting  $Q = G/P$ , we see that  $Q$  has an induced involution, since  $P$  is  $*$ -invariant. Thus, we have  $Q = Q^+ \times H$  as in Lemma 5. Take any  $xP \in Q^+$  (resp.  $H$ ). We claim that  $K = \langle P, x \rangle$  is abelian. It is clear that  $K$  is  $*$ -invariant, and the induced involution acts as the identity (resp. the classical involution) on  $K/P$ . Applying the Schur–Zassenhaus Theorem, we have  $K = P \rtimes L$ , for some  $p'$ -group  $L$ . Take any  $w \in L$ . We will show that  $\langle w, P \rangle = 1$ . By Lemmas 6 and 7, there exists  $c \in P$  such that  $(wc)^* = wc$  (resp.  $(wc)^{-1}$ ). Suppose that  $o(wc) = p^m n$ , where  $p$  does not divide  $n$ . Then  $o(wP) = o(wcP) = o((wc)^{p^m}P) = n$ . Thus,  $\langle P, w \rangle = \langle P, (wc)^{p^m} \rangle$ , and it suffices to replace  $w$  with  $(wc)^{p^m}$ . That is, we assume that  $w^* = w$  (resp.  $w^{-1}$ ) and  $w$  has  $p'$ -order. Take any  $y \in P$ , say  $o(y) = p^k$ . As  $P$  is abelian,  $(y + y^*)^{p^k} = 2$ , hence  $\frac{y+y^*}{2}$  is a symmetric unit of  $p$ -power order. Now, by [7, Lemma 3.1],  $\langle \mathcal{U}^+(FK) \rangle$  is a nilpotent group. Therefore, elements of relatively prime order commute.

We consider the two cases separately. Suppose that  $w^* = w$ . Then  $w$  is a symmetric unit of  $p'$ -order. Thus,  $[w, y + y^*] = 0$ . In other words,

$$wy + wy^* = yw + y^*w.$$

If  $wy \neq yw$ , then we must have  $wy = y^*w$  and  $wy^* = yw$ . But then  $y^* = w^{-1}yw$ , hence  $wy = w^{-1}yw^2$ , and therefore  $w^2$  commutes with  $y$ . As  $w$  has odd order,  $w$  and  $y$  commute, hence  $\langle w, P \rangle = 1$ .

Suppose, on the other hand, that  $w^* = w^{-1}$ . As  $w$  is a  $p'$ -element, choose a positive integer  $r$  such that  $w^{p^r} = w$ . Then  $(w + w^{-1})^{p^r} = w + w^{-1}$ . But if  $o(w) = 2e + 1$ , then

$$(1 + w^2)(1 - w^2 + w^4 - \dots + w^{4e}) = 1 + (w^{2e+1})^2 = 2,$$

hence  $1 + w^2$ , and therefore  $w + w^{-1} = w^{-1}(1 + w^2)$ , is a unit. That is,  $(w + w^{-1})^{p^r-1} = 1$ , and  $w + w^{-1}$  is a symmetric unit of  $p'$ -order. Thus, we have  $[w + w^{-1}, y + y^*] = 0$ . Now,  $w$  and  $w^{-1}$  lie in different cosets modulo  $P$ , hence  $[w, y + y^*] = 0$ . As before,  $wy = yw$ , and again,  $\langle w, P \rangle = 1$ . Therefore,  $P$  is central in  $K$ . As  $K = \langle P, x \rangle$ ,  $K$  is abelian, as claimed.

Thus, if  $xP \in Q^+$  or  $xP \in H$ , then  $x$  centralizes  $P$ . As  $Q = Q^+ \times H$ ,  $P$  is central, and  $G/P$  is abelian, so  $G$  is nilpotent. We are done.  $\square$

The next step is to consider groups of even order.

**Lemma 11.** Let  $G$  be a finite group. If  $\mathcal{U}^+(FG)$  is nilpotent, then  $G$  is nilpotent, and either  $G$  is  $p$ -abelian or  $G/P$  is an SLC-group.

**Proof.** In view of Lemma 1, it remains only to show that  $G$  is nilpotent. Suppose this is not the case, and let  $G$  be a counterexample of smallest order. If  $\zeta(G) \neq 1$ , then as  $\mathcal{U}^+(G/\zeta(G))$  is nilpotent, we conclude that  $G/\zeta(G)$  is nilpotent, hence  $G$  is nilpotent, giving a contradiction. Thus, we assume that  $\zeta(G) = 1$ . If  $P' \neq 1$ , then again,  $\mathcal{U}^+(F(G/P'))$  is nilpotent, hence  $G/P'$  is nilpotent. By [13, 5.2.10],  $G$  is nilpotent. Once again, we have a contradiction, so we assume that  $P$  is abelian.

In view of the preceding lemma, we know that  $G$  has even order. Let  $Q = G/P$ . If  $Q$  is an SLC-group, then of course it has a central symmetric element of order 2. Suppose that  $Q$  is abelian. If not every element of order 2 in  $Q$  is symmetric, then take any nonsymmetric  $gP \in Q$  of order 2. Then  $gg^*P$  is a symmetric element of order 2.

In general, then, let  $zP \in (G/P)^+$  be central of order 2. Replacing  $z$  with a suitable  $p$ -power, assume that  $z$  has order 2. We claim that  $z$  centralizes  $P$ . If not, then  $\langle P, z \rangle$  is not nilpotent. As  $\langle P, z \rangle$  is  $*$ -invariant, we must have  $G = \langle P, z \rangle$ , by minimality of  $|G|$ . That is,  $G = P \rtimes \langle z \rangle$ .

Now, conjugation by  $z$  gives an involution of  $P$ . In view of Lemma 5, write  $P = P_1 \times P_2$ , where  $z$  centralizes  $P_1$  and inverts elements of  $P_2$ . Now,  $P_1 \leq \zeta(G) = 1$ . Therefore, we may assume that  $P = P_2$ , and  $z$  acts by inversion on  $P$ . (Hence, in particular, every subgroup of  $P$  is normal in  $G$ .) Write  $P = P^+ \times H$  as in Lemma 5. Now, if  $H \neq 1$ , then  $H \rtimes \langle z \rangle$  is a  $*$ -invariant dihedral group. Furthermore,  $*$  acts as the classical involution upon  $H$  and  $z$ , hence upon  $H \rtimes \langle z \rangle$ . But by the main results of [7], this is impossible. Therefore,  $H = 1$  and every element of  $P$  is symmetric. Take  $1 \neq g \in P$ . Then, as in the proof of the preceding lemma,  $g$  and  $z$ , being symmetric elements of relatively prime order, commute. But this is a contradiction, and the claim is proved.

By the Schur–Zassenhaus Theorem,  $G = P \rtimes X$  for some  $p'$ -subgroup  $X$ , and by [13, 9.1.3],  $X$  can be chosen in such a way that it contains  $z$ . Now,  $zP$  is central in  $G/P$ , hence for any  $x \in X$ ,  $(x, z) \in P \cap X = 1$ . Thus,  $z$  centralizes both  $X$  and  $P$  and therefore,  $z \in \zeta(G) = 1$ , giving us a final contradiction and completing the proof.  $\square$

Let us now consider the infinite case. We will be able to reduce to the case in which  $(FG)^+$  is Lie nilpotent. To this end, we have the following two lemmas, which are true for nontorsion groups and finite fields as well.

**Lemma 12.** *Let  $G$  be any group having an abelian normal  $*$ -invariant subgroup  $A$  of finite index, as well as an infinite normal  $*$ -invariant  $p$ -subgroup  $B$  of bounded exponent. Then  $A \cap B$  contains an infinite direct product of nontrivial finite  $p$ -groups, each of which is  $*$ -invariant and normal in  $G$ .*

**Proof.** By [13, 4.3.5],  $M = A \cap B$  is a direct product of cyclic groups, say  $M = \prod_{i \in I} A_i$ . As  $(B : A \cap B) \leq (G : A) < \infty$ , we see that  $M$ , and therefore  $I$ , is infinite. Let  $J$  be any finite subset of  $I$ . We claim that there is a nontrivial finite subgroup  $N$  of  $H = \prod_{i \in I \setminus J} A_i$  such that  $N$  is  $*$ -invariant and normal in  $G$ .

Let  $\{x_1, \dots, x_t\}$  be a transversal of  $A$  in  $G$ . Since  $(M : H) < \infty$ , we have  $(M : x_r H x_r^{-1}) < \infty$ ,  $1 \leq r \leq t$ . Similarly,  $(M : H^*) < \infty$ , hence  $(M : (x_r^{-1})^* H^* x_r^*) < \infty$ ,  $1 \leq r \leq t$ . Let  $K$  be the intersection of all of the subgroups  $x_r H x_r^{-1}$  and  $(x_r^{-1})^* H^* x_r^*$ . Then  $(M : K) < \infty$ , hence  $K$  is infinite. Take  $1 \neq a \in K$  and  $g \in G$ . Then  $g = b x_s$  for some  $b \in A$  and  $1 \leq s \leq t$ . But then  $g^{-1} a g = x_s^{-1} a x_s$  and  $g^{-1} a^* g = x_s^{-1} a^* x_s$ . However,  $a \in x_s H x_s^{-1}$ , hence  $g^{-1} a g \in H$ . Furthermore,  $a \in (x_s^*)^{-1} H^* x_s^*$ , hence  $a^* \in x_s H x_s^{-1}$ . By the same argument,  $g^{-1} a^* g \in H$ .

That is, the subgroup of  $G$  generated by all of the conjugates of  $a$  and all of the conjugates of  $a^*$  is contained in  $H$ . As  $A \cap B$  is abelian and torsion, and there are only finitely many such conjugates, we obtain a finite normal  $*$ -invariant subgroup of  $G$ , contained in  $H$ , and the claim is proved.

The lemma is now proved by induction. Let  $J_0$  be the empty set, and find a nontrivial finite  $*$ -invariant normal subgroup of  $G$  contained in  $M$ , say  $N_1 \leq \prod_{i \in J_1} A_i$ , where  $J_1$  is finite. Now construct a nontrivial finite  $*$ -invariant normal subgroup of  $G$  contained in  $\prod_{i \in I \setminus J_1} A_i$ , say  $N_2 \leq \prod_{i \in J_2} A_i$ , where  $J_2$  is finite and disjoint from  $J_1$ . Now work in  $I \setminus (J_1 \cup J_2)$ , and so forth. The proof is complete.  $\square$

**Lemma 13.** *Let  $G$  be any group having an abelian normal  $*$ -invariant subgroup of finite index, as well as an infinite normal  $*$ -invariant  $p$ -subgroup of bounded exponent. If  $\mathcal{U}^+(FG)$  satisfies  $(x_1, \dots, x_n) = 1$ , then  $(FG)^+$  satisfies  $[x_1, \dots, x_n] = 0$ .*

**Proof.** Suppose this is not the case, and take  $\alpha_1, \dots, \alpha_n \in (FG)^+$  such that  $[\alpha_1, \dots, \alpha_n] \neq 0$ . By the previous lemma, we have an infinite direct product  $N = A_1 \times A_2 \times \dots$  of nontrivial  $*$ -invariant finite  $p$ -subgroups, each of which is normal in  $G$ . Let  $X$  be a transversal to  $N$  in  $G$ , and write

$$[\alpha_1, \dots, \alpha_n] = \sum_j \beta_j x_j,$$

with  $\beta_j \in FN$  and  $x_j \in X$ . Choose an  $m$  so that the supports of all of the  $\beta_j$  are contained in  $F(A_1 \times \dots \times A_m)$ . For each  $i$ ,  $1 \leq i \leq n$ , let  $\eta_i = \hat{A}_{m+i}$ . Then each  $\eta_i$  is symmetric, central and square-zero, hence  $1 + \eta_i \alpha_i \in \mathcal{U}^+(FG)$  (as its inverse is  $1 - \eta_i \alpha_i$ ).

Therefore,

$$(1 + \eta_1 \alpha_1, \dots, 1 + \eta_n \alpha_n) = 1$$

and it is easy to see by induction that

$$(1 + \eta_1 \alpha_1, \dots, 1 + \eta_n \alpha_n) = 1 + \eta_1 \cdots \eta_n [\alpha_1, \dots, \alpha_n].$$

That is,

$$0 = \eta_1 \cdots \eta_n [\alpha_1, \dots, \alpha_n] = \eta_1 \cdots \eta_n \sum_j \beta_j x_j.$$

Therefore,

$$\eta_1 \cdots \eta_n \beta_j = 0,$$

for all  $j$ . But each  $\beta_j \in F(A_1 \times \cdots \times A_m)$ , so by definition of the  $\eta_i$ , we have  $\beta_j = 0$ , and hence  $[\alpha_1, \dots, \alpha_n] = 0$ , giving us a contradiction.  $\square$

In particular, we can deal with one case of the main result.

**Lemma 14.** *Let  $G/P$  be abelian. If  $\mathcal{U}^+(FG)$  is nilpotent, then so is  $\mathcal{U}(FG)$ .*

**Proof.** By Lemma 1,  $G$  is locally finite. Thus, considering finite subgroups of  $G$ , we see from Lemma 11 that  $G = H \times P$ , where  $H$  is an abelian  $p'$ -group and  $P$  is a  $p$ -group. In view of Lemma 4, we need to show that  $G$  is nilpotent and  $p$ -abelian. Thus, it suffices to assume that  $G$  is a  $p$ -group. By [14, Lemma V.4.2], we will be done if we show that  $G'$  is finite. So suppose that  $G'$  is infinite. By Lemma 1,  $G'$  has bounded exponent, and it is  $*$ -invariant. Lemma 1 also tells us that  $G$  has a  $p$ -abelian normal subgroup  $A$  of finite index. Replacing  $A$  with  $A \cap A^*$ , we may assume that  $A$  is  $*$ -invariant. We know that  $\mathcal{U}^+(F(G/A'))$  is nilpotent, and  $G'$  is finite if and only if  $(G/A)'$  is finite. Thus, replacing  $G$  with  $G/A'$ , we assume that  $A$  is abelian. Then by the last lemma,  $(FG)^+$  is Lie nilpotent hence, by Lemmas 8 and 9,  $G$  is  $p$ -abelian. We are done.  $\square$

In order to handle the case in which  $G/P$  is an SLC-group, we need to strengthen the conclusion of [9] a bit. Let  $R$  be any ring and  $\Lambda$  a subset of  $R$ . We let  $\Lambda^{(1)} = R$ , and for each  $i \geq 1$ , let  $\Lambda^{(i+1)}$  be the (associative) ideal of  $R$  generated by the elements  $[a, b]$ , with  $a \in \Lambda^{(i)}$ ,  $b \in \Lambda$ . We say that  $\Lambda$  is strongly Lie nilpotent if there exists a positive integer  $n$  so that  $\Lambda^{(n)} = 0$ . The following lemma does not depend upon  $F$  being infinite or  $G$  being torsion.

**Lemma 15.** *If  $G$  is nilpotent, and  $G$  has a finite normal  $*$ -invariant  $p$ -subgroup  $N$  such that  $G/N$  is an SLC-group, then  $(FG)^+$  is strongly Lie nilpotent.*

**Proof.** Our proof is by induction on  $|N|$ . If  $N = 1$ , then  $G$  is an SLC-group, and by Lemma 2, the symmetric elements are central in  $FG$ , hence  $((FG)^+)^{(2)} = 0$ .

So assume that  $N \neq 1$ . As  $G$  is nilpotent, there exists  $z \in N \cap \zeta(G)$  with  $o(z) = p$ . Then  $z^* \in N \cap \zeta(G)$  as well. If  $z^* \neq z^{-1}$ , then  $zz^*$  is a symmetric central element of order  $p$ . Thus, we may replace  $z$  with  $zz^*$  if necessary and assume that  $\langle z \rangle$  is  $*$ -invariant. Of course,  $|N/\langle z \rangle| < |N|$ , so by our inductive hypothesis, we may assume that  $(F\bar{G})^+$  is strongly Lie nilpotent, where  $\bar{G} = G/\langle z \rangle$ .

If  $((F\bar{G})^+)^{(n)} = 0$ , then  $((FG)^+)^{(n)} \subseteq \Delta(G, \langle z \rangle) = (z - 1)FG$ . By [7, Lemma 5.8], there exists an  $m$  so that  $((FG)^+)^{(n+m)} \subseteq (z - 1)^2FG$ , and so forth, so that there is a  $k$  such that  $((FG)^+)^{(k)} \subseteq (z - 1)^pFG = 0$ . We are done.  $\square$

**Proof of Theorem.** Let  $\mathcal{U}^+(FG)$  be nilpotent. By Lemma 1,  $G/P$  is abelian or an SLC-group. In the former case, Lemma 14 says that  $\mathcal{U}(FG)$  is nilpotent. Thus, let us assume that  $G/P$  is SLC. By Lemma 1,  $G$  is locally finite, so considering finite subgroups, Lemma 11 tells us that  $G = H \times P$ , where  $H$  is an SLC-group and  $P$  is a  $p$ -group. Now, an SLC-group is nilpotent, and by Lemmas 4 and 14,  $P$  is nilpotent. Thus,  $G$  is nilpotent, and it remains only to prove the existence of the subgroup  $N$ . By Lemma 1,  $G$  has a  $p$ -abelian normal subgroup  $A$  of finite index. Replacing  $A$  with  $A \cap A^*$ , we may assume that  $A$  is  $*$ -invariant. Also, we can factor out the finite  $*$ -invariant  $p$ -subgroup  $A'$  without harming our conclusion. Thus, we assume that  $A$  is abelian.

If  $P$  is finite, then letting  $N = P$  completes the proof, so assume that  $P$  is infinite. Let  $B$  be the subgroup described in Lemma 1. If  $B$  is finite, then let  $N = B$ . Then  $G/B = H \times (P/B)$ ,  $P/B$  is abelian, and  $*$  acts as the identity on  $P/B$ . It is easy to see, in this case, that  $G/B$  is also an SLC-group. Finally, suppose that  $B$  is infinite. Then by Lemma 13,  $(FG)^+$  is Lie nilpotent. Lemmas 8 and 9 now provide us with the necessary subgroup  $N$ .

Conversely, suppose that  $G$  is nilpotent and  $G$  has a finite normal  $*$ -invariant  $p$ -subgroup  $N$  such that  $G/N$  is an SLC-group. By Lemma 15,  $(FG)^+$  is strongly Lie nilpotent. Let us say that  $((FG)^+)^{(n)} = 0$ . It now follows as in the proof of [7, Theorem 2] that for all  $i \geq 2$  and all  $\alpha_1, \dots, \alpha_i \in \mathcal{U}^+(FG)$  we have

$$(\alpha_1, \dots, \alpha_i) \in 1 + ((FG)^+)^{(i)}.$$

In particular,  $(\alpha_1, \dots, \alpha_n) = 1$ . We are done.  $\square$

Noting, for torsion groups  $G$ , that  $\mathcal{U}(FG)$  is nilpotent if and only if  $FG$  is Lie nilpotent, and comparing our theorem with Lemma 9, we obtain the following consequence.

**Corollary.** *Let  $F$  be an infinite field of characteristic different from 2 and  $G$  a torsion group. Let  $*$  be an involution on  $FG$  induced from an involution on  $G$ . Then  $\mathcal{U}^+(FG)$  is nilpotent if and only if  $(FG)^+$  is Lie nilpotent.*

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