Unifying discrete structural models and reduced-form models in credit risk using a jump-diffusion process

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SUMMARY

Merton (1974) pioneered the structural model using a diffusion process to model the firm value evolution. Since a sudden drop of firm value is impossible, Jones, Mason and Rosenfeld (1984) argue that the short term yield spread and the default probability are too small. Zhou (1997) uses a jump-diffusion process that is originally proposed by Merton (1976) to model the firm value process. However, a method for finding the jump distribution is not developed. In a reduced-form model, the default probability (or intensity of default) and the mean recovery rate are obtained from the market spread by using model-specific assumptions. However, the capital structure that triggers the default usually is not used. In this paper, we propose methods to remove the discrepancy of yield spreads between structural models and reduced-form models and unify these two models. We first show the equivalence of yield spreads between structural models and reduced-form models and then find the implied jump distribution based on the market spread. The mean recovery rate for multiple seniorities and the mean recovery rate are thus obtained.

Key words and phrases: Default risk; Jump-diffusion process; Structural model; Reduced-form model.
1 Introduction

The probability and severity of a possible future default event can be evaluated using either the realistic (or "real world") $P$-measure or the risk neutral $Q$-measure. Methods based on the $Q$-measure are usually more efficient than those based on the $P$-measure for evaluation of public firms while the later ones are widely used for private firms due to the lack of the credit spread data. Credit risk can be evaluated from either its cause or its effect in the $Q$-measure. From the view of point of the cause, a default event is triggered by a firm’s capital structure when the value of the firm falls below its financial obligation. The yield spreads are often different from the market spreads. This kind of model is referred as a structural model. From the view of point of the effect, the corporate bond price is lower than the government bond price by a spread which is a function of the default probability and the mean recovery rate. One key issue is that we know the product of the default probability and mean recovery rate but we are not able to identify these two from a credit spread. With model-specific assumptions, default probability and mean recovery rate may be identifiable. This kind of model is referred as a reduced-form model.

When default can occur any time before the debt’s maturity, the model is referred as a continuous model in this paper. When the default time is restricted to some specific points, we say that the model is discrete. We try to remove the discrepancy of yield spreads between discrete structural models and discrete reduced-form models and unify these two models. We show the equivalence of yield spreads between structural models and reduced-form models in section 4 and then find the implied jump distribution based on the market spread in sections 5.3 and 5.5. The yield spread and the mean recovery rate for a structural model based on a jump-diffusion process for multiple seniorities are obtained in sections 5.2 and 5.4.

2 Merton’s structural model

2.1 Merton’s structural model

Merton pioneered the structural credit risk model in 1974. He assumes that the market is perfect and frictionless thus that there are many investors who can sell, short, and buy as much as they want in continuous time. The term structure of interest rates is assumed to be flat and the interest is fixed for all time for both borrowing or lending. The Modigliani-Miller theorem holds so that the value of the firm is invariant to its capital structure. All random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$ for uncertainty under $Q$-measure probability. A firm issues equity and bonds of amount $D$ that mature at time $T$. Both are traded in the market. The value of a firm $V_t$ at time $t$ is assumed to be traded in the market and follow a diffusion process as

\[ dV_t = (\alpha V_t - c)dt + \sigma V_t dZ_t \]
where \( \sigma^2 \) is the instantaneous variance, \( \alpha \) is instantaneous return, \( c \) is total payout including dividends or coupons, and \( Z_t \) is a standard Brownian motion. A default event can only occur at its bond’s maturity and occurs only if the value of the firm \( V_T \) at bond maturity \( T \) falls below the debt obligation \( D \).

Merton (1974) argues that a derivative of the value of the firm \( C(V_t, t) \) must follow another diffusion process. Under the argument of no risk and no arbitrage of a representative agent’s trading strategy, the instantaneous return must equal riskless rate \( r \) and \( C(V_t, t) \) follows

\[
\frac{1}{2} \sigma^2 V_t^2 \frac{\partial^2 C(V_t, t)}{\partial V_t^2} + r V_t \frac{\partial C(V_t, t)}{\partial V_t} + \frac{\partial C(V_t, t)}{\partial t} - r C(V_t, t) = 0
\]

(1)

with boundary condition \( C(x, T) = g(x) \) for a payoff \( g(x) \) of a derivative at maturity time \( T \). The bonds and equity are considered as derivatives of the firm value so they satisfy equation (1). The total value of the bonds equals \( D \) times \( V(t, T) \), the price of the bond per dollar face amount. Then,

\[
V_t = DV(t, T) + e^{-r(T-t)}E_t^Q[(V_T - D)_+]
\]

Let the equity of the firm at time \( t \) or the stock price be denoted by \( S_t \). The price of the equity is considered as the price of an European call option on the value of the firm. Black and Scholes (1973) show that

\[
S_t = V_t \Phi[-h_1(t)] - D \exp^{-r(T-t)} \Phi[h_2(t)]
\]

where

\[
\begin{align*}
    h_1(t) &= -\ln\left(\frac{V_t}{D}\right) + \frac{(r + \frac{1}{2} \sigma^2)(T - t)}{\sigma(T - t)^{1/2}}, \\
    h_2(t) &= \ln\left(\frac{V_t}{D}\right) + \frac{(r - \frac{1}{2} \sigma^2)(T - t)}{\sigma(T - t)^{1/2}},
\end{align*}
\]

(2)

and \( \Phi[\cdot] \) is the cumulative distribution function for a standard normal random variable. The price of the bond per dollar face value is

\[
V(t, T) = \frac{V_t - S_t}{D} = e^{-r(T-t)} \left\{ \Phi[h_2(t)] + \frac{1}{d(t)} \Phi[h_1(t)] \right\},
\]

where \( d(t) = \frac{D e^{-r(T-t)}}{V_t} \).

Let \( R(t, T) \) represent the yield to maturity at time \( t \) of a risky debt that matures at time \( T \). The risk premium or the credit spread as given by Merton (1974) is

\[
R(t, T) - r = -\frac{1}{T-t} \ln \left\{ \Phi[h_2(t)] + \frac{1}{d(t)} \Phi[h_1(t)] \right\}.
\]

(3)

The value of the ratio \( \frac{V_T}{D} \) determines whether there is default or not. If \( \frac{V_T}{D} \geq 1 \), there is no default. If \( \frac{V_T}{D} < 1 \), then default happens. When a default event happens, the ratio \( \frac{V_T}{D} \) represents the recovery rate.
2.2 The default probability and the mean recovery rate in Merton’s model

In this section, we derive the default probability and the mean recovery in Merton’s model in the multiple seniorities case. They will be used to show the equivalence between structural models and reduced-form models in section 4. In the Merton (1974) model, \( \ln V_T \) follows a normal distribution

\[
\ln V_T \sim N \left( \ln V_0 + \left( r - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right)
\]

under \( Q \)-measure. Assuming that a company has survived \( t \) years and its random failure time is \( \tau_t \). At current time \( t \), the \( Q \)-measure probability that the firm will default at time \( T \) is

\[
\Pr_t^Q \{ \tau_t = T \} = \Pr_t^Q \{ V_T < D \} = \Phi(-h_2(t))
\]

where \( h_2(t) \) is defined by equation (2). At time \( t \), the probability that the firm will not default at time \( T \) is \( \Pr_t^Q (\tau_t > T) = 1 - \Phi(-h_2(t)) = \Phi(h_2(t)) \). The probability that \( \tau_t < T \) is zero since default before maturity is not allowed in Merton’s model. The value \( \frac{V_T}{D} \) determines whether a default event happens or not at time \( T \).

The recovery rate \( \delta(T) \) is the proportion \( V_T \) of \( D \) if the firm defaults at time \( T \). The formal definition of recovery rate is given by its distribution function

\[
F_{\delta(T)}^Q(w) = \begin{cases} 
0 & w \leq 0 \\
\frac{\Pr_t^Q \{ \frac{V_T}{D} < w \}}{\Pr_t^Q \{ \frac{V_T}{D} < 1 \}} & 0 < w < 1 \\
1 & w \geq 1.
\end{cases}
\]

The probability density function thus follows

\[
f_{\delta(T)}^Q(w) = \begin{cases} 
D \frac{f_{V_T}^Q(wD)}{\Pr_t^Q \{ V_T < D \}} & 0 \leq w < 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Without considering the seniority class of the bond, the recovery rate can be expressed as

\[
E_t^Q[\delta(T)] = \int_0^1 w \frac{D f_{V_T}^Q(wD)}{\Pr_t^Q \{ V_T < D \}} \, dw
\]

\[
= \int_0^D \frac{w}{D} \times \frac{f_{V_T}^Q(v)}{\Pr_t^Q \{ V_T < D \}} \, dv
\]

\[
= E_t^Q \left[ \frac{V_T}{D} \bigg| V_T < D \right]
\]

\[
= E_t^Q \left[ \frac{D - (D - V_T)_+}{D} \bigg| V_T < D \right]
\]

where \( \frac{D - (D - V_T)_+}{D} \) is the proportion that the bond holder receives at the maturity time \( T \). The expected value at time \( t \) of the bond holder’s position is
\[ E^Q_t \left[ \frac{D - (D - V_T)_+}{D} \right] = \frac{D - E^Q_t[D - V_T]}{D} \]
\[ = \frac{D - p_t e^{r(T-t)}}{D} \]
\[ = \Phi(h_1(t)) + \Phi(h_2(t)) \tag{4} \]

where \( p_t \) in equation (4) represents the time-\( t \) price European put option on the value of the firm.

The expected value at time \( t \) of the bond holder’s position can also be written as
\[ E^Q_t \left[ \frac{D - (D - V_T)_+}{D} \right] = E^Q_t \left[ \frac{D - (D - V_T)_+}{D} | V_T < D \right] Pr^Q_t(V_T < D) \]
\[ + E^Q_t \left[ \frac{D - (D - V_T)_+}{D} | V_T \geq D \right] Pr^Q_t(V_T \geq D) \]
\[ = E^Q_t[\delta(T)]Pr^Q_t(V_T < D) + Pr^Q_t(V_T \geq D). \]

The mean recovery rate can be expressed as a function of the position of the bond holder as
\[ E^Q_t[\delta(T)] = \frac{E^Q_t \left[ \frac{D - (D - V_T)_+}{D} \right] - Pr^Q_t(V_T \geq D)}{Pr^Q_t(V_T < D)} \]
\[ = \frac{\Phi(h_1(t)) + \Phi(h_2(t)) - \Phi(h_2(t))}{\Phi(-h_2(t))} \]
\[ = \frac{\Phi(h_1(t))}{d(t)} \Phi(-h_2(t)). \tag{5} \]

We now show how to express the mean recovery rate in terms of the stock prices. Since the price of the European call option is regarded as the price of the equity, based on put-call parity, we can find the mean recovery rate based on the stock price \( S_t \).

**Theorem 2.1** In Merton’s (1974) model, the mean recovery can be written as the functions of the stock prices \( S_t \) as
\[ E^Q_t[\delta(T)] = 1 - \frac{d(t) \Phi(h_2(t))}{d(t) \Phi(-h_2(t))} - \frac{1}{d(t) \Phi(-h_2(t))} S_t \tag{6} \]

Proof: The relationship between the mean recovery rate and the European put price \( p_t \) of the value of the firm at time \( t \) is given by
\[ E^Q_t[\delta(T)] = \frac{E^Q_t \left[ \frac{D - (D - V_T)_+}{D} \right] - Pr^Q_t(V_T \geq D)}{Pr^Q_t(V_T < D)} \]
\[ = \frac{D - p_t e^{r(T-t)}}{D} - \Phi(h_2(t)) \]
\[ = \frac{\Phi(-h_2(t))}{e^{r(T-t)}} \frac{e^{r(T-t)}}{D}\Phi(-h_2(t)) p_t. \tag{7} \]
From equation (7), we have
\[
E^Q_t[\delta(T)] = 1 - \frac{e^{r(T-t)}}{D\Phi(-h_2(t))}p_t
\]
\[
= 1 - \frac{e^{r(T-t)}}{D\Phi(-h_2(t))}(S_t + De^{-r(T-t)} - V_t)
\]
\[
= 1 - \frac{d(t)\Phi(-h_2(t))}{d(t)\Phi(-h_2(t))} - \frac{S_t}{V_t}.
\]

Giving that a firm survives time \( t \), the probability density function for the recovery rate is
\[
f^Q_{\delta(T)}(w) = \frac{1}{\sqrt{2\pi(T-t)\sigma w\Phi(-h_2(t))}}e^{-\frac{(\ln(wD/V_t) - (r-\frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}}
\]
where \( 0 < w < 1 \).

We now consider the case of several bonds classes issued by the firm. The mean recovery rate is usually a function of the seniority of the bond. Assume that a firm issues bonds of five seniority classes: senior secured, senior unsecured, senior subordinated, subordinated, and junior subordinated (Carty and Lieberman, 1996). Each of the five seniority classes represents proportions \( p_1, p_2, \ldots, p_5 \) of the debt \( D \) where \( \sum_i p_i = 1 \). Similar to Madan and Unal (1995) and Lando (1997), we assume that the strict priority rule is applied so that senior claimants are fully paid before any money is distributed to junior claimants. The recovery rate for these five seniority classes are denoted by \( \delta_1(T), \delta_2(T), \ldots, \delta_5(T) \). Thus recovery rates for each seniority classes are
\[
\delta_1(T) = \min \left( \frac{\delta(T)}{p_1}, 1 \right),
\]
\[
\delta_2(T) = \min \left( \frac{\delta(T) - p_1}{p_2}, 1 \right),
\]
\[
\delta_3(T) = \min \left( \frac{\delta(T) - p_1 - p_2}{p_3}, 1 \right),
\]
\[
\delta_4(T) = \min \left( \frac{\delta(T) - p_1 - p_2 - p_3}{p_4}, 1 \right),
\]
\[
\delta_5(T) = \frac{\delta(T) - p_1 - p_2 - p_3 - p_4}{p_5}.
\]

The probability density functions of the recovery rates are:
\[
f^Q_{\delta_i(T)}(w) = \begin{cases} 
\frac{1}{\sqrt{2\pi(T-t)\sigma w\Phi(-h_2(t))}}e^{-\frac{(\ln(wD/V_t) - (r-\frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}}, & 0 < w < 1 \\
\frac{1}{p_1} \int_{p_1}^{1} \frac{1}{\sqrt{2\pi(T-t)\sigma w\Phi(-h_2(t))}}e^{-\frac{(\ln(uD/V_t) - (r-\frac{\sigma^2}{2})(T-t))^2}{2\sigma^2(T-t)}}, & w = 1 \\
0, & \text{otherwise}
\end{cases}
\]
default (a stopping time) be denoted as \( \tau \). All random variables are defined on a filtered probability space \((\Omega, F, (F_t), Q)\). We assume the existence of a default. In a discrete reduced-form model, the default time is restricted to some specific points for each class.

### 3 Reduced-form credit risk models

In a continuous reduced-form model, default can occur any time before the debt’s maturity. The credit risk can be represented by an intensity process and the recovery rate (or the loss given default). In a discrete reduced-form model, the default time is restricted to some specific points so that the intensity of default does not exist. The credit risk can be represented by the default probability and the recovery rate. We start with the continuous model in the next section.

#### 3.1 The first passage time and intensity process for a continuous model

We assume the existence of a Q-measure probability. As a result, there is no arbitrage opportunity. All random variables are defined on a filtered probability space \((\Omega, F, (F_t), Q)\). Let the time of default (a stopping time) be denoted as \( \tau \). A right-continuous one-jump process is defined as \( N(t) = 1_{\{\tau \leq t\}} \). Let \( F_\tau(t), S_\tau(t), \) and \( f_\tau(t) \) denote distribution function, survival function and the probability density function of \( \tau \) respectively. An intensity function is defined as

\[
\mu(t) = \lim_{h \to 0} \frac{1}{h} \Pr\{N(t + h) - N(t) = 1 \mid N(t) = 0\} = \frac{f_\tau(t)}{1 - F_\tau(t)} = -\frac{S_\tau'(t)}{S_\tau(t)}.
\]

An adapted process \( \{1 - N(t-)\mu(t) : t > 0\} \) is called an intensity process which is assumed to be non-negative and predictable with \( \int_0^T \mu(s) ds < \infty \) for all \( T > 0 \) almost surely. Artzner and
Delbaen (1995) show that the existence of an intensity process under $P$-measure guarantees the existence of an intensity process under any equivalent measure such as a $Q$-measure probability. The intensity of default at time $t$ in $Q$-measure is denoted by $\tilde{\mu}(t)$. The distribution function and survival function of time of default can be expressed as a function of the intensity function as $F_\tau(t) = 1 - S_\tau(t) = 1 - e^{-\int_0^t \mu(s) ds}$. Assuming that a company has survived $t$ years and its random failure time is $\tau_t$, we use two actuarial symbols to denote conditional probabilities by $s_p = \Pr\{\tau_t > t + s \mid \tau_t > t\} = e^{-\int_s^{t+s} \mu(s) ds}$ and $s_q = \Pr\{\tau_t \leq t + s \mid \tau_t > t\} = 1 - e^{-\int_s^{t+s} \mu(s) ds}$.

If $N(t)$ follows an inhomogeneous Poisson process with stochastic intensity $\mu(t, \omega)$ which depends on the realization of $\omega \in \Omega$, we have a Cox Process and $s_p = \mathbb{E}_t \left[ e^{-\int_s^{t+s} \mu(s, \omega) ds} \right]$. The amount recovered once default occurs and three major recovery schemes are introduced in the next section.

### 3.2 Recovery scheme

The recovery rate\(^1\) is defined as the ratio of debt recovered once a default event happens. It is denoted by $\delta(\tau)$ where $\tau$ is the time of default. There are three major recovery schemes modelling credit risk: recovery of par value (RPV), recovery of treasury value (RTV), and recovery of market value (RMV). Let $P(t, T)$ represent the time-$t$ price of a default-free zero-coupon bond paying one dollar at maturity time $T$ where $0 \leq t \leq T < \infty$, and let $V(t, T)$ represent the time-$t$ price of a defaultable corporate zero-coupon bond paying one dollar at maturity time $T$. The yields to maturity for the default-free bond and defaultable bond are denoted as $r(t, T)$ and $R(t, T)$ respectively. The bond prices can be written as

$$P(t, T) = \exp \left[ -r(t, T)(T - t) \right]$$

and

$$V(t, T) = \exp \left[ -R(t, T)(T - t) \right].$$

A time-$t$ value of a bank account process or the value of one dollar accumulated from time 0 to time $t$ is defined as

$$B(t) = \exp \left[ \int_0^t r(s) ds \right]$$

where $r(s)$ is the spot rate at time $s$.

When a default event happens and the firm is immediately liquidated at the time of default to pay the bond holders a fractional payment of face amount, the scheme is called recovery of par value (RPV). By assuming the existence of $Q$-measure probability, a risky bond price per dollar face amount under a RPV scheme can be written as

$$V(t, T) = \mathbb{E}_t^Q \left[ \frac{B(t)}{B(\tau_t)} \delta(\tau_t) 1_{(\tau_t \leq T)} + \frac{B(t)}{B(T)} 1_{(\tau_t > T)} \right].$$

\(^1\)For a structural model, the recovery rate is the ratio $V_{\tau_D}^T$ if default occurs. Throughout this paper, we use the notation $\mathbb{E}_t^Q[\delta(\tau)]$ for the conditional expectation $\mathbb{E}_t^Q \left[ \frac{V_{\tau_D}^T}{B(T)} \mid \tau_t \leq T \right]$. 

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When a default event happens and the firm is liquidated later at the time of the maturity of the bond to pay the bond holders a fractional payment of face amount, the scheme is called recovery of treasury value (RTV). The remaining value of the defaulted firm does not earn any interest during the period from default time to maturity. If it does earn interest, the scheme is equivalent to a recovery of par value scheme. A risky bond price for a RTV scheme under risk-neutral valuation can be written as

$$V(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)} \left[ \delta(\tau_t)1_{(\tau_t \leq T)} + 1_{(\tau_t > T)} \right] \right].$$

It is almost impossible that a firm can be liquidated immediately at the default time. The time of liquidation is usually at a later time and can be a random variable. In this paper, we adopt RTV scheme where the liquidation time is fixed as the time of maturity of the bond.

If a default event happens and the bond holders receive a fraction of the pre-default market value of bond from the liquidation of the firm at time of default, we call this recovery scheme the recovery of market value (RMV). The price of a risky bond can be written as

$$V(t, T) = E_t^Q \left[ \frac{B(t)}{B(\tau_t)} (1 - L_{\tau_t})V(\tau_t - , T)1_{(\tau_t \leq T)} + \frac{B(t)}{B(T)}1_{(\tau_t > T)} \right],$$

where $L_{\tau_t}$ represents the expected fractional loss in market value at time of default. Duffie and Singleton (1995) showed that the price of a risky bond can be generally written as

$$V(t, T) = E_t^Q \left[ \exp \left( - \int_t^T R(s) ds \right) X \right],$$

where $R(s)$ represents the default-adjusted short rate at time $s$ and $X$ is the promised payment at maturity in the event of no default and is 1 for a zero-coupon bond. An instantaneous credit spread or an instantaneous short-rate spread can be expressed as the product of the mean loss ratio and the intensity of default as

$$R(s) - r(s) = (1 - L_s)\tilde{\mu}(s)$$

When a firm issues bonds of several seniorities, we say that the strict priority rule is applied if the senior claimants are fully paid before any money distributed to the junior claimants.

In credit risk literature such as Jarrow, Lando, and Turnbull (1997), the recovery rate is usually assumed to be independent of the bankruptcy process in order to identify the mean from the default probability in a credit spread. However, based on the following lemma, the assumption is not necessary.

**Lemma 3.1**

$$E_t^Q[I_{(\tau_t \leq T)}\delta(\tau_t)] = Pr_t^Q{\tau_t \leq T}E_t^Q[\delta(\tau_t)]$$

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Proof:

\[ E_t^Q[I_{\{\tau_t \leq T\}} \delta(\tau_t)] = E_t^Q\{E_t^Q[I_{\{\tau_t \leq T\}} \delta(\tau_t) \mid I_{\{\tau_t \leq T\}}]\} \]
\[ = E_t^Q[I_{\{\tau_t \leq T\}} \delta(\tau_t) \mid I_{\{\tau_t \leq T\}} = 1] \Pr_t^Q\{\tau_t \leq T\} \]
\[ + E_t^Q[I_{\{\tau_t \leq T\}} \delta(\tau_t) \mid I_{\{\tau_t \leq T\}} = 0] \Pr_t^Q\{\tau_t > T\} \]
\[ = E_t^Q[\delta(\tau_t)] \Pr_t^Q\{\tau_t \leq T\} + 0 \times \Pr_t^Q\{\tau_t > T\} \]
\[ = E_t^Q[\delta(\tau_t)] \Pr_t^Q\{\tau_t \leq T\} \]

This result is used in the next section.

3.3 Risk neutral valuation and the yield spread for a reduced-form model

Based on the models of Jarrow and Turnbull (1995) and Jarrow, Lando, and Turnbull (1997), we assume the financial market is frictionless with a finite time horizon. By assuming the existence of an \(Q\)-measure probability, arbitrage opportunities are excluded. The price of a riskless bond can be written as

\[ P(t, T) = E_t^Q\left[ \frac{B(t)}{B(T)} \right] = E_t^Q\left[ \exp\left( -\int_t^T r(s) ds \right) \right]. \] (12)

Based on a RTV scheme and the assumption of the existence of a \(Q\)-measure probability, the price of a risky bond can be written as

\[ V(t, T) = E_t^Q\left[ B(t) \delta(\tau_t) 1_{\{\tau_t \leq T\}} + 1_{\{\tau_t > T\}} \right]. \] (12)

If the default-free spot rates and the default process are independent, the price of a risky bond is

\[ V(t, T) = P(t, T) \left\{ 1 - E_t^Q\left[ \{1 - \delta(\tau_t)\} 1_{\{\tau_t \leq T\}} \right] \right\} \] (13)

From Lemma 3.1, the yield spread is

\[ R(t, T) - r(t, T) = -\frac{1}{T - t} \ln \left\{ 1 - \left(1 - E_t^Q[\delta(\tau_t)]\right) \Pr_t^Q\{\tau_t \leq T\} \right\} \] (14)

where \(T - \tilde{q}_t\) represents the \(Q\)-measure probability that a firm will default before time \(T\) given it has not defaulted at time \(t\).

The product of the loss given default (LGD) and the default probability is given by

\[ \left\{ 1 - E_t^Q[\delta(\tau_t)] \right\} \Pr_t^Q\{\tau_t \leq T\} = \frac{P(t, T) - V(t, T)}{P(t, T)}. \] (15)

The ratio of risky bond price to riskless bond price can be obtained by

\[ 1 - \left\{ 1 - E_t^Q[\delta(\tau_t)] \right\} \Pr_t^Q\{\tau_t \leq T\} = \frac{V(t, T)}{P(t, T)}. \] (16)
There are two unknowns; \( \Pr_t \{ \tau_t \leq T \} \) and \( E_t^Q[\delta(\tau_t)] \), in one equation. As long as one of them is known, the term structure of default for the future can be derived. At least one more assumption is necessary to identify the mean recovery rate and default probability. Different choices of this assumption lead to different models.

### 3.4 Discrete time term structure and the forward spread

Li (1998) and Jarrow, Lando and Turnbull (1997) show the term structure of default in a discrete case when the recovery is assumed to be constant. We show in this section the general case where the recovery rate is a random variable and is not necessary independent of the occurrence of default. In a discrete time model, the 1-year forward rate for a riskless zero-coupon bond is defined as

\[
f(t; T, T + 1) \equiv -\ln \left\{ \frac{P(t, T + 1)}{P(t, T)} \right\},
\]

and the 1-year forward rate for a risky zero-coupon bond is defined as

\[
F(t; T, T + 1) \equiv -\ln \left\{ \frac{V(t, T + 1)}{V(t, T)} \right\}.
\]

If the default-free spot rates and the default process are independent in a RTV scheme, from equation (13), the forward rate for a risky zero-coupon bond can be written as

\[
F(t; T, T + 1) = f(t; T, T + 1) - \ln \left\{ \frac{1 - \left( 1 - E_t^Q[\delta(\tau_t; T)] \right)_{T-t\tilde{q}_t}}{1 - \left( 1 - E_t^Q[\delta(\tau_t; T + 1)] \right)_{T+1-t\tilde{q}_t}} \right\},
\]

From Lemma 3.1, the credit spread or the forward spread is given by

\[
F(t; T, T + 1) - f(t; T, T + 1) = \ln \left\{ \frac{1 - \left( 1 - E_t^Q[\delta(\tau_t; T)] \right)_{T-t\tilde{q}_t}}{1 - \left( 1 - E_t^Q[\delta(\tau_t; T + 1)] \right)_{T+1-t\tilde{q}_t}} \right\},
\]

where \( E_t^Q[\delta(\tau_t; T)] \) represents the \( Q \)-measure mean recovery rate if the maturity is time \( T \).

By letting \( T = t \), we have the spread for spot rate

\[
R(t, t + 1) - r(t, t + 1) = \ln \left\{ \frac{1}{1 - \left( 1 - E_t^Q[\delta(\tau_t; t + 1)] \right)_{1\tilde{q}_t}} \right\}.
\]

If the \( Q \)-measure mean recovery rate \( E_t^Q\{\delta(T)\} \) is known, the conditional \( Q \)-measure default
probability for the first period is:

\[ \tilde{p}_t = \frac{1 - e^{-[R(t,t+1) - r(t,t+1)]}}{1 - E_t^Q[\delta(\tau_t, t+1)]}. \]

The rest of the default probabilities for the future can be derived directly from equation (16)

\[ T-t \tilde{p}_t = \frac{1 - V(t,T)}{P(t,T)} \frac{1 - E_t^Q[\delta(\tau_t, T)]}{1 - E_t^Q[\delta(\tau_t, t+1)]}. \]

### 4 Unifying Merton’s structural model with a reduced-form model

In this section, we unify the structural model with the reduced-form model by showing that the yield spread of the Merton (1974) model is equivalent to the yield spread of a reduced-form model. After rearranging the yield spread in the Merton (1974) structural model, the price of risky bond for a structural model can be rewritten as the price of a contingent claim under risk-neutral valuation paying full obligation if there is no default and paying the recovery rate if default happens at maturity like a reduced-form model.

In the other direction, we start with a reduced-form model under risk-neutral valuation. The corresponding value of the firm process defines the default probability and mean recovery rate. From the yield spread formula equation (14) for a reduced-form model, we can derive Merton’s (1974) yield spread.

We now show that the credit spread of the Merton (1974) model is equivalent to the yield spread for a structural model. The yield spread of the Merton (1974) model is

\[ R(t, T) - r = -\frac{1}{T-t} \ln \left\{ \Phi(h_2(t)) + \frac{1}{d(t)} \Phi(h_1(t)) \right\} \]

\[ = -\frac{1}{T-t} \ln \left\{ 1 - \Phi(-h_2(t)) \left[ 1 - \frac{\Phi(h_1(t))}{d(t) \Phi(-h_2(t))} \right] \right\} \]

\[ = -\frac{1}{T-t} \ln \left\{ 1 - \Phi(-h_2(t)) \left[ 1 - E_t^Q[\delta(t)] \right] \right\} \]

\[ = -\frac{1}{T-t} \ln \left\{ 1 - E_t^Q\{V_T < D\} \left[ 1 - E_t^Q[\delta(T)] \right] \right\} \]

\[ = -\frac{1}{T-t} \ln \left\{ 1 - \{1 - E_t^Q[\delta(T)]\} \{1 - E_t^Q[\delta(T)]\} \right\} \]

which is the yield spread for a reduced-form model in equation (14). Starting from a reduced-form yield spread in equation (18), we can derive Merton’s (1974) risk premium by going from equation (18) to equation (17). After rearrangement, we have

\[ R(t, T) - r = -\frac{1}{T-t} \ln \left\{ 1 - E_t^Q[I(V_T < D)] \left[ E_t^Q[1 - \delta(T)] \right] \right\} \]
In the next section. Processes are too small to match market spreads. This leads to the use of the jump-diffusion process. However, Jones, Mason and Rosenfeld (1984) find that the yield spreads based on diffusion processes usually impose assumptions on the equation of the firm $V$. This shows that the Merton (1974) model equivalently defines a reduced-form model.

In general, the structural models and reduced-form models can be unified by

$$V_t = E_t^Q \left[ \frac{B(t)}{B(T)} V_T \right]$$

$$= E_t^Q \left[ \frac{B(t)}{B(T)} \left\{ (D - (D - V_T)_+) + (V_T - D)_+ \right\} \right]$$

$$= E_t^Q \left[ \frac{B(t)}{B(T)} \{ D - (D - V_T)_+ \} \right] + E_t^Q \left[ \frac{B(t)}{B(T)} (V_T - D)_+ \right]$$

$$= DE_t^Q \left[ \frac{B(t)}{B(T)} \left( D - (D - V_T)_+ \right) / D \right] + S_t$$

$$= DE_t^Q \left[ \frac{B(t)}{B(T)} \left\{ 1 - \left( 1 - \frac{V_T}{D} \right)_+ \right\} \right] + S_t$$

$$= DE_t^Q \left[ \frac{B(t)}{B(T)} \left\{ 1 I_{\{V_T \geq D\}} + \frac{V_T}{D} I_{\{V_T < D\}} \times I_{\{V_T < D\}} \right\} \right] + S_t$$

$$= DE_t^Q \left[ \frac{B(t)}{B(T)} \left\{ 1 I_{\{V_T \geq D\}} + \delta(T) I_{\{V_T < D\}} \right\} \right] + S_t$$

$$= DV(t, T) + S_t$$

(22)

Structural models usually impose assumptions on the value of the firm $V_t$ while reduced-form models usually impose assumptions on the components related to the equation

$$V(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)} \left\{ I_{\{V_T \geq D\}} + \delta(T) I_{\{V_T < D\}} \right\} \right].$$

However, Jones, Mason and Rosenfeld (1984) find that the yield spreads based on diffusion processes are too small to match market spreads. This leads to the use of the jump-diffusion process in the next section.
5 A jump-diffusion structural model

5.1 Introduction

In the Merton (1974) model, the firm value follows a diffusion process. A sudden jump in the firm value is impossible so that the short-term default probability is almost zero and is too small compared to the market spread. In order to capture default events which are caused either by a jump or a diffusion process, Zhou (1997) adopts Merton’s (1976) jump-diffusion process to model credit risk. The yield spread becomes more realistic compared to the market spread. Based on the equivalence of credit spreads shown in section 4, we use the jump-diffusion process proposed by Zhou (1997) modelling the firm value but we use the market spread to find the implied jump distribution. Thus, the discrepancy of credit spreads between structural models and reduced-form can be removed and the structural models and reduced-form models can be unified. We start with the following assumptions where assumptions 1 to 4 are based on Merton (1974) and Zhou (1997):

Assumptions 1

1. The market is perfect and frictionless. There are many investors who can sell, short and buy as much as they want in continuous time.

2. The term structure for interest rate is flat. The interest is fixed all time and is the same for borrowing and lending.

3. The Modigliani-Miller theorem holds, so that the value of the firm is invariant to its capital structure.

4. The firm value follows the jump-diffusion process proposed by Merton (1976). We assume that the capital asset pricing model (CAPM) holds for equilibrium returns. The diffusion process represents systematic risk and the jump components represent nonsystematic risk which has zero beta.

5. The firm issues equities and bonds. The bonds mature at the same time but differ only in the priority with which they are redeemed.

The value of a company consists of a diffusion process which represent the systematic risk and is correlated with the market and a jump process with its compensator which represents nonsystematic risk and is uncorrelated with the market:

\[ dV_t = \alpha V_t dt + \sigma V_t dZ_t + (J_{N_t} dN_t - \lambda \mu dt) V_t \]

where

\( V_t \): value of the firm at time \( t \),
\( \alpha \): instantaneous expected rate of return,
\( \sigma^2 \): instantaneous variance of return,
\(Z_t\): a standard Brownian motion,
\(N_t\): total number of jumps up to time \(t\),
\(J_{N_t}\): jump size as a proportion of \(V_t\),
and \(\{N_t; t \geq 0\}\) is a counting process which follows a Poisson process with parameter \(\lambda\) and is stochastically independent to \(Z_t\). The jump amplitudes \(J_1, J_2, \ldots\) are assumed to be independently and identically distributed whose moment generating function exists with mean \(\mu_J\) and variance \(\sigma^2_J\). The mean value of the instantaneous change of the value of a firm is
\[
E[dV_t] = \alpha V_t dt.
\]
The Poisson process does not play any role in deciding the expected return on the value of the firm but it allows the possibility of instantaneous jump of the value of the firm. The path of the value of the firm is different from a pure diffusion process and the default probability and recovery rate are also different. If we divide both sides by \(V_t\), we have,
\[
d\frac{V_t}{V_t} = (\alpha - \lambda \mu_J)dt + \sigma dZ_t + J_{N_t}dN_t.
\]
\(V_t\) is the value of a company and is assumed to be a nonnegative number. The percentage change \(\frac{dV_t}{V_t}\) is bounded below by \(-1\) and can go up to infinity. The range of the jump amplitude \(J\) can only take values in \([-1, \infty]\). Assume that \(Y\) is a nonnegative random variable and \(J_i = Y_i - 1\) for \(i = 1\) to \(N(t)\), then \(J\) can fall in \([-1, \infty]\). \(Y\) is an impulse function which takes value 1 when there is no jump and takes values from 0 to \(\infty\) other than 1 if there is a jump. Let
\[
Y(t) = \prod_{i=1}^{N(t)} Y_i = \prod_{i=1}^{N(t)} (J_i + 1).
\]
From Itô’s lemma, we have
\[
d \ln V_t = \left(\alpha - \frac{\sigma^2}{2} - \lambda \mu_J\right) dt + \sigma dZ_t + \ln Y_{N_t} dN_t.
\]
A derivative of the value of the firm \(C(V_t, t)\) must follow
\[
dC(V_t, t) = \left[(\alpha - \lambda \mu_J) \frac{\partial C(V_t, t)}{\partial V_t} + \frac{\partial C(V_t, t)}{\partial t} + \frac{1}{2}{\sigma^2 V_t}^2 \frac{\partial^2 C(V_t, t)}{\partial V_t^2}\right] dt
\]
\[+ \sigma V_t \frac{\partial C(V_t, t)}{\partial V_t} dZ_c + [C(V_t Y_{N_t}, t) - C(V_t, t)]dN_t.
\]
Merton (1976) argues that a representative agent can invest in the firm, the option and the riskless asset. If the jump components represent nonsystematic risk, the portfolio has zero beta. If the CAPM holds, the expected return on all zero-beta securities must equal riskless rate. We have \(\alpha = r\). The value of the firm thus follows
\[
\ln V_t = \ln V_0 + \left(r - \frac{\sigma^2}{2} - \lambda \mu_J\right) t + \sigma Z_t + \sum_{i=1}^{N_t} \ln Y_i.
\]
It can be shown that \( E[e^{-rt}V_t] = V_0 \) which is equivalent to saying that no arbitrage is allowed and showing thereby the existence of the \( Q \)-measure probability.

From the current value of the firm, we cannot find any information about the jumps. However, the prices of derivatives based on the value of the firm are functions of the jumps. The price of a bond can be regarded as a derivative of the value of the firm and thus carry the information about jumps. This is considered in the next section.

5.2 Unifying structural jump-diffusion model with reduced-form models: constant jumps

In this section, we obtain the mean recovery rate and the yield spread when the jump amplitude is a constant. In the next section, we will find a method to find the implied distributions of jump frequency and the constant amplitude so that the discrepancy of yield spreads between structural models and reduced-form models can be reduced and these two kinds of models can be unified.

We assume that the jump amplitude is a constant. Let

\[ Y_i = J_i + 1 = s \quad 0 < s < 1, \quad i \in \{0, 1, 2\ldots\}. \]

The means satisfy \( \mu_J = \mu_Y - 1 = s - 1 \). From equation (23), the value of the firm follows

\[ \ln V_t = \ln V_0 + \left[ r - \frac{\sigma^2}{2} - \lambda(s - 1) \right] t + \sigma Z_t + N_t \ln s. \]

Conditioned on the number of jump, the distribution of the value of the firm follows

\[ \ln V_t \mid N_t = n \sim N \left\{ \ln V_0 + \left[ r - \frac{\sigma^2}{2} - \lambda(s - 1) \right] t + n \ln s, \sqrt{\sigma^2 t} \right\}. \]

The probability of default at bond’s maturity can be obtained conditioned on the number of jumps

\[ \Pr_t^Q(\tau_t = T) = \sum_{n=0}^{\infty} \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!} \Phi \left( \frac{-\ln V_T + [r - \frac{\sigma^2}{2} - \lambda(s - 1)](T-t) + n \ln s}{\sqrt{\sigma^2(T-t)}} \right) \]

\[ = \sum_{n=0}^{\infty} g(n; t, T) \Phi \left( -h_{2,n}^c(t) \right) \]

where

\[ g(n; t, T) = \frac{[\lambda(T-t)]^n e^{-\lambda(T-t)}}{n!}, \]

\[ h_{2,n}^c(t) = \frac{\ln V_T + [r - \frac{\sigma^2}{2} - \lambda(s - 1)](T-t) + n \ln s}{\sqrt{\sigma^2(T-t)}}. \]
Theorem 5.1 If the value of a firm process follows a jump-diffusion process where the jump amplitude is constant and the assumptions of section 5.1 hold, then the mean recovery rate is given by

\[
E_t^Q[\delta(T)] = \frac{\sum_{n=0}^{\infty} g(n; t, T) \Phi(h_{1,n}^c(t))}{d(t) \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_{2,n}^c(t))}
\]

(24)

where \(d(t) = \frac{D e^{-(T-t)}}{V_T} \) and \(h_{i,n}^c(t) = -\frac{\ln \left( \frac{c}{d} \right) + [r + \frac{\sigma^2}{2} - \lambda(s-1)](T-t) + n \ln s}{\sqrt{\sigma^2(T-t)}} \).

Proof:

Given the number of jumps \(N_{[t,T]} = n \) equals \(n \) in the time period \([t, T]\), the European put price \(p_t \) at time \(t \) can be derived as

\[
p_t \mid N_{[t,T]} = n = De^{-r(T-t)} \Phi(-h_{2,n}^c(t)) - V_t \Phi(h_{1,n}^c(t)).
\]

The unconditional European call price \(c_t \) at time \(t \) can be written as

\[
p_t = \sum_{n=0}^{\infty} g(n; t, T) [De^{-r(T-t)} \Phi(-h_{2,n}^c(t)) - V_t \Phi(h_{1,n}^c(t))].
\]

The mean of the bond holder’s position at current time \(t \) is

\[
E_t^Q \left[ \frac{D - (D - V_T)_+}{D} \right] = D - E_t^Q [D - V_T]_+ = \frac{D - p_t e^{r(T-t)}}{D} = D - e^{r(T-t)} \sum_{n=0}^{\infty} g(n; t, T) [De^{-r(T-t)} \Phi(-h_{2,n}^c(t)) - V_t \Phi(h_{1,n}^c(t))]
\]

\[
= \sum_{n=0}^{\infty} g(n; t, T) \left\{ \frac{V_t e^{r(T-t)}}{D} \Phi(h_{1,n}^c(t)) + \Phi(h_{2,n}^c(t)) \right\}
\]

\[
= \sum_{n=0}^{\infty} g(n; t, T) \left\{ \frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi(h_{2,n}^c(t)) \right\}.
\]

The mean recovery rate at time \(t \) can be expressed as

\[
E_t^Q[\delta(T)] = \frac{E_t^Q \left[ \frac{D - (D - V_T)_+}{D} \right] - \Pr_t^Q \{V_T \geq D \}}{\Pr_t^Q \{V_T < D \}}
\]

\[
= \sum_{n=0}^{\infty} g(n; t, T) \left\{ \frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi(h_{2,n}^c(t)) \right\} - \sum_{n=0}^{\infty} g(n; t, T) \Phi(h_{2,n}^c(t))
\]

\[
= \sum_{n=0}^{\infty} g(n; t, T) \Phi(h_{1,n}^c(t)) \frac{d(t)}{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_{2,n}^c(t))}.
\]

\(\square\)
Theorem 5.2 If the value of a firm process follows a jump-diffusion process where the jump size is constant and the assumptions of section 5.1 hold, then the yield spread is given by

\[ R(t, T) - r = -\frac{1}{T - t} \ln \left\{ \sum_{n=0}^{\infty} g(n; t, T) \left[ \frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi(h_{2,n}^c(t)) \right] \right\}. \]  

(25)

Proof:

From a reduced-form model under risk-neutral valuation in a RTV scheme, the equation (12)

\[ V(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)}(\delta(T)1_{(\tau \leq T)} + 1_{(\tau > T)}) \right] \]

can be written as equation (14) as

\[ R(t, T) - r = -\frac{1}{T - t} \ln \left\{ \sum_{n=0}^{\infty} g(n; t, T) \Phi(h_{1,n}^c(t)) \right\} \]

\[ = -\frac{1}{T - t} \ln \left\{ \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_{2,n}^c(t)) \right\} \]

\[ = -\frac{1}{T - t} \ln \left\{ \sum_{n=0}^{\infty} g(n; t, T) \left[ \frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi(h_{2,n}^c(t)) \right] \right\}. \]

This equation shows the yield spread for the corresponding value of a firm process in a reduced-form model. When there is no jump or equivalently let \( s = 0 \) or \( \lambda = 0 \), the equation (25) reduces to Merton’s (1974) credit spread formula equation (3) that

\[ R(t, T) - r = -\frac{1}{T - t} \ln \left\{ \Phi[h_2(t)] + \frac{1}{d(t)}\Phi[h_1(t)] \right\}, \]

which is solved by using boundary conditions on the differential-integral equation. Alternatively, we can write equation (25) in terms of the bond price as

\[ \frac{V(t, T)}{P(t, T)} = \sum_{n=0}^{\infty} g(n; t, T) \left[ \frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi(h_{2,n}^c(t)) \right]. \]  

(26)

Although the probability of default and mean recovery rate both have explicit forms, we still have difficulty calculating them because we do not know the jump frequency and the jump size. We will address this in the next section.
If the value of a firm process follows a jump-diffusion process where the jump size rate for class 2, \(\delta_2(T)\), is constant and the assumptions of section 5.1 hold, then we have:

**Theorem 5.3**

The credit spread for the lower seniority can be expressed as

\[
R_2(t, T) - r = -\frac{1}{T-t} \ln \left\{ 1 - \left( 1 - E_t^Q[\delta_2(T)] \right) \Pr_t^Q \{ V_T < D \} \right\}.
\]

The bond price with higher seniority is

\[ V_1(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)} \{ \delta_1(T)1_{(V_T < D)} + 1_{(V_T \geq D)} \} \right] \]

\[ = E_t^Q \left[ \frac{B(t)}{B(T)} \{ \delta_1^*(T)1_{(V_T < D_{\text{p}1})} + 1_{(D_{\text{p}1} \leq V_T < D) + 1_{(V_T \geq D)}} \} \right] \]

\[ = E_t^Q \left[ \frac{B(t)}{B(T)} \{ \delta_1^*(T)1_{(V_T < D_{\text{p}1})} + 1_{(V_T \geq D_{\text{p}1})} \} \right]. \]

The bond price for the lower seniority class can be expressed as

\[ V_2(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)} \{ \delta_2(T)1_{(V_T < D)} + 1_{(V_T \geq D)} \} \right] \]

\[ = E_t^Q \left[ \frac{B(t)}{B(T)} \{ \delta_2^*(T)1_{(V_T < D_{\text{p}1})} + 1_{(V_T \geq D_{\text{p}1})} \} \right]. \]

From Lemma (3.1), the credit spread of the bond for the higher seniority class can be expressed as

\[ R_1(t, T) - r = -\frac{1}{T-t} \ln \left\{ 1 - \left( 1 - E_t^Q[\delta_1(T)] \right) \Pr_t^Q \{ V_T < D \} \right\}. \]

The credit spread for the lower seniority can be expressed as

\[ R_2(t, T) - r = -\frac{1}{T-t} \ln \left\{ 1 - \left( 1 - E_t^Q[\delta_2(T)] \right) \Pr_t^Q \{ V_T < D \} \right\}. \]

**Theorem 5.3** If the value of a firm process follows a jump-diffusion process where the jump size is constant and the assumptions of section 5.1 hold, then we have:

a)

\[
\frac{1 - E_t^Q[\delta_1(T)]}{1 - E_t^Q[\delta_2(T)]} = \frac{P(t, T) - V_1(t, T)}{P(t, T) - V_2(t, T)}.
\]

b)

\[
E_t^Q[\delta_1(T)] = 1 - \frac{\sum_{n=0}^{\infty} g(n; t, T) \left[ \Phi \left( -h_{2,n}^c(t) \right) - \frac{\Phi(h_{2,n}^c(t))}{d(t)} \right]}{\sum_{n=0}^{\infty} g(n; t, T) \Phi (-h_{2,n}(t))}. \]

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\( E_t^Q [\delta_2(T)] \)
\[
= 1 - \frac{\sum_{n=0}^{\infty} g(n; t, T) \left\{ \Phi (-h_{2,n}(t)) - p_1 \Phi (-h_{2,n}(t)) \right\}}{(1 - p_1) \sum_{n=0}^{\infty} g(n; t, T) \Phi (-h_{2,n}(t))}. \tag{29}
\]

Proof:
From equation (15), we have
\[
\left\{ 1 - E_t^Q [\delta_i(T)] \right\} \Pr_t^Q \{ V_T < D \} = \frac{P(t, T) - V_i(t, T)}{P(t, T)} \quad i = 1, 2. \tag{30}
\]

From equation (30), we have the relationship between the mean recovery rate of two different seniority classes in terms of bond’s price as
\[
\frac{1 - E_t^Q [\delta_1(T)]}{1 - E_t^Q [\delta_2(T)]} = \frac{P(t, T) - V_1(t, T)}{P(t, T) - V_2(t, T)}. \]

The mean recovery rate \( \delta_i^*(T) \) can be expressed as
\[
E_t^Q [\delta_i^*(T)] = E_t^Q \left[ \frac{\delta(T)}{P_1} \middle| V_T < Dp_1 \right]
= E_t^Q \left[ \frac{V_T}{Dp_1} \middle| V_T < Dp_1 \right].
\]

We can consider \( \delta_i^*(T) \) as the recovery of another bond with a liability of \( Dp_1 \). Thus the mean recovery rate \( \delta_i^*(T) \) in \( Q \)-measure can be calculated from equation (24) as
\[
E_t^Q [\delta_i^*(T)] = \frac{\sum_{n=0}^{\infty} g(n; t, T) \Phi (h_{i,n}^*(t))}{d^*(t) \sum_{n=0}^{\infty} g(n; t, T) \Phi (-h_{2,n}^*(t))}
\]

where
\[
d^*(t) = \frac{Dp_1}{V_t e^{r(T-t)}} = p_1 d(t),
\]
\[
h_{1,n}^*(t) = -\frac{\ln \frac{V_t}{Dp_1} + \left[ r + \frac{\sigma^2}{2} - \lambda (s - 1) \right](T-t) + n \ln s}{\sqrt{\sigma^2 (T-t)}},
\]
\[
h_{2,n}^*(t) = \frac{\ln \frac{V_t}{Dp_1} + \left[ r - \frac{\sigma^2}{2} - \lambda (s - 1) \right](T-t) - n \ln s}{\sqrt{\sigma^2 (T-t)}}.
\]

The probability that \( V_T < Dp_1 \) is
\[
Pr_t^Q \{ V_T < Dp_1 \} = \sum_{n=0}^{\infty} g(n; t, T) \Phi \left( -h_{2,n}^*(t) \right).
\]
The mean recovery rate for the higher seniority can be derived as

\[
E^Q_t[\delta_1(T)] = E^Q_t[\Phi(t)]\Pr_t^Q\{V_T < Dp_1\} + 1\Pr_t^Q\{Dp_1 \leq V_T < D\}
\]

\[
= \frac{\sum_{n=0}^{\infty} g(n; t, T)\Phi(h_{1_n}^*(t)) - \sum_{n=0}^{\infty} g(n; t, T)\Phi(-h_{2_n}^*(t))}{\sum_{n=0}^{\infty} g(n; t, T)\Phi(-h_{2_n}^*(t))}
\]

Now consider

\[
p_1\left[\frac{\delta(T)}{p_1} \wedge 1\right] + (1 - p_1)\frac{(\delta(T) - p_1)_+}{1 - p_1} = [\delta(T) \wedge p_1] + (\delta(T) - p_1)_+ = \delta(T).
\]

Taking the expected valued at time \(t\), we have

\[
p_1E^Q_t\left[\frac{\delta(T)}{p_1} \wedge 1\right] + (1 - p_1)E^Q_t\left[\frac{(\delta(T) - p_1)_+}{1 - p_1}\right] = E^Q_t[\delta(T)]
\]

and

\[
p_1E^Q_t[\delta_1(T)] + (1 - p_1)E^Q_t[\delta_2(T)] = E^Q_t[\delta(T)].
\]

Thus, the mean recovery rate for the lower seniority can be derived as

\[
E^Q_t[\delta_2(T)] = \frac{E^Q_t[\delta(T)] - p_1E^Q_t[\delta_1(T)]}{1 - p_1}
\]

\[
= \frac{\sum_{n=0}^{\infty} g(n; t, T)\Phi(h_{1_n}^*(t)) - \sum_{n=0}^{\infty} g(n; t, T)\Phi(-h_{2_n}^*(t))}{\sum_{n=0}^{\infty} g(n; t, T)\Phi(-h_{2_n}^*(t))}
\]

Substituting \(E^Q_t[\delta_1(T)]\) and \(E^Q_t[\delta_2(T)]\) into equation (27), we have

\[
\frac{\sum_{n=0}^{\infty} g(n; t, T)\Phi(-h_{2_n}^*(t)) - \phi(h_{1_n}^*(t))}{\sum_{n=0}^{\infty} g(n; t, T)\Phi(-h_{2_n}^*(t))}
\]

\[
= \frac{P(t) - V_1(t, T)}{P(t) - V_2(t, T)}.
\]
After rearrangement, we have

\[ \sum_{n=0}^{\infty} g(n; t, T) \left[ p_1 \Phi(-h_{1,n}^c(t)) - \frac{\Phi(h_{1,n}^c(t))}{d(t)} \right] \]

\[ \sum_{n=0}^{\infty} g(n; t, T) \left\{ [\Phi(-h_{2,n}(t)) - p_1 \Phi(-h_{2,n}^c(t))] - \frac{[\Phi(h_{1,n}(t)) - \Phi(h_{1,n}^c(t))]}{d(t)} \right\} \]

\[ = \frac{p_1 [P(t, T) - V_1(t, T)]}{(1 - p_1) [P(t, T) - V_2(t, T)]} \quad (31) \]

There are two unknown parameters \( s \) and \( \lambda \) in equation (31). With only one constraint, we are not able to obtain unique solutions for \( s \) and \( \lambda \). With one more constraint (26)

\[ \frac{V(t, T)}{P(t, T)} = \sum_{n=0}^{\infty} g(n; t, T) \left\{ \frac{\Phi(h_{1,n}^c(t))}{d(t)} + \Phi(h_{2,n}^c(t)) \right\} \]

where \( V(t, T) = p_1 V_1(t, T) + (1 - p_1) V_2(t, T) \), we can solve \( s \) and \( \lambda \) numerically. The two spreads of bonds issued by one firm with different seniorities reflect what the investors expect about the jump behavior in \( Q \)-measure.

If the jump size and jump frequency can be derived uniquely, then the default probability and mean recovery rate are unique. If there are infinite combinations of jump size and jump frequency, the \( Q \)-measure is not unique. In the case where a firm issue bonds of more than two seniorities, the jump size and jump frequency calculated using the first two seniorities should be consistent with those calculated using other seniorities. If not, the no arbitrage assumption is violated due to the investors insufficient knowledge about the jump size and jump frequency. The arbitrage opportunity will vanish if the arbitragers know this opportunity and trade on this opportunity.

5.4 Unifying structural jump-diffusion model with reduced-form models: lognormal jumps

In this section, we obtain the mean recovery rate and the yield spread when the jump amplitude follows a lognormal distribution. In the next section, we will find a method to find the implied distributions of jump frequency and amplitude so that the structural model and the reduced-form model can be unified. We assume that the impulse function \( Y \) or the jump amplitude plus one, \( J + 1 \), is lognormally distributed as

\[ \ln Y_i \sim N(\mu_1, \sigma_1) \quad i \in \{0, 1, 2,...\} \]

with probability density function as

\[ f(y) = \frac{1}{\sqrt{(2\pi)\sigma_1 y}} e^{-\frac{(\ln y - \mu_1)^2}{2\sigma_1^2}} \]
where $\mu_y = \mu_J + 1 = e^{\mu_1 + \sigma_1^2}$ and $\sigma_Y = \sigma_J = e^{2(\mu_1 + \sigma_1^2)} - e^{2\mu_1 + \sigma_1^2}$. From equation (23), the value of a firm follows

$$\ln V_t = \ln V_0 + \left[ r - \frac{\sigma^2}{2} - \lambda(\mu_Y - 1) \right] t + \sigma Z_t + \sum_{i=1}^{N_t} \ln Y_i.$$  

Conditioning on the number of jumps, the value of a firm follows

$$\ln V_t |_{N_t=n} \sim N \left\{ \ln V_0 + \left[ r - \frac{\sigma^2}{2} - \lambda(\mu_Y - 1) \right] t + n\mu_1, \sqrt{\sigma^2 t + n\sigma_1^2} \right\}.$$  

At the current time $t$, the probability of default at the bond’s maturity is derived by Zhou (1997) as

$$P_{t_t}^Q(\tau_t = T) = \sum_{n=0}^{\infty} g(n; t, T) \Phi \left( \frac{-\ln V_t + [r - \frac{\sigma^2}{2} - \lambda(\mu_Y - 1)](T - t) + n\mu_1}{\sqrt{\sigma^2(T - t) + n\sigma_1^2}} \right)$$

where

$$h_{2,n}(t) = \frac{-\ln V_t + [r - \frac{\sigma^2}{2} - \lambda(\mu_Y - 1)](T - t) + n\mu_1}{\sqrt{\sigma^2(T - t) + n\sigma_1^2}}.$$  

**Theorem 5.4** If the value of a firm process follows a jump-diffusion process where the jump size is lognormally distributed and the assumptions of section 5.1 hold, then the mean recovery rate is given by

$$E_t^Q[\delta(T)] = \frac{\sum_{n=0}^{\infty} g(n; t, T) \Phi(h_{1,n}(t))}{d(t) \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_{2,n}(t))}$$

where $h_{1,n}(t) = \frac{-\ln V_t + [r + \frac{\sigma^2}{2} - \lambda(\mu_Y - 1)](T - t) + n\mu_1}{\sqrt{\sigma^2(T - t) + n\sigma_1^2}}$.

Proof:

At the current time $t$, given the number of jumps $N_{t,T}$ in the time period $[t, T]$, the European put price can be derived as

$$p_t |_{N_t=n} = D e^{-r(T-t)} \Phi(-h_{2,n}(t)) - V_t \Phi(h_{1,n}(t)).$$

The unconditional European call price at time $T$ is

$$p_t = \sum_{n=0}^{\infty} g(n; t, T) [D e^{-r(T-t)} \Phi(-h_{2,n}(t)) - V_t \Phi(h_{1,n}(t))].$$
The mean of the bond holder’s position at current time \( t \) is
\[
E^Q_t \left[ \frac{D - (D - V_T)_+}{D} \right] = \frac{D - E^Q_t[D - V_T]}{D} = \frac{D - pe^{r(T-t)}}{D} = D - e^{r(T-t)} \sum_{n=0}^\infty g(n; t, T) \left[ De^{-r(T-t)} \Phi(-h_{2,n}(t)) - V_t \Phi(h_{1,n}(t)) \right] \]
\[
= \sum_{n=0}^\infty g(n; t, T) \left\{ \frac{V_t e^{rT}}{D} \Phi(h_{1,n}(t)) + \Phi(h_{2,n}(t)) \right\} \]
\[
= \sum_{n=0}^\infty g(n; t, T) \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi(h_{2,n}(t)) \right\} .
\]
The mean recovery rate can be expressed as
\[
E^Q_t[\delta(T)] = \frac{E^Q_t \left[ \frac{D-(D-V_T)_+}{D} \right] - Pr^Q_t \{ V_T \geq D \}}{Pr^Q_t \{ V_T < D \}} = \frac{\sum_{n=0}^\infty g(n; t, T) \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi(h_{2,n}(t)) \right\} - \sum_{n=0}^\infty g(n; t, T) \Phi(h_{2,n}(t))}{d(t) \sum_{n=0}^\infty g(n; t, T) \Phi(-h_{2,n}(t))}.
\]

**Theorem 5.5** If the value of a firm process follows a jump-diffusion process where the jump size is lognormal distributed and the assumption of section 5.1 hold, then the yield spread is given by
\[
R(t, T) - r = -\frac{1}{T-t} \ln \left[ \sum_{n=0}^\infty g(n; t, T) \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi(h_{2,n}(t)) \right\} \right].
\] (34)

Proof:

The yield spread from a reduced-form model based on the RTV scheme is
\[
R(t, T) - r = -\frac{1}{T-t} \ln \left[ 1 - \left( 1 - E^Q_t(\delta(T)) \right) Pr^Q_t(\tau_t \leq T) \right] \]
\[
= -\frac{1}{T-t} \ln \left[ 1 - \left( 1 - \sum_{n=0}^\infty g(n; t, T) \Phi(h_{1,n}(t)) \right) \right]
\[
\times \sum_{n=0}^\infty g(n; t, T) \Phi(-h_{2,n}(t)) \]
\[
= -\frac{1}{T-t} \ln \left[ 1 - \left( \sum_{n=0}^\infty g(n; t, T) \left[ \Phi(-h_{2,n}(t)) - \frac{\Phi(h_{1,n}(t))}{d(t)} \right] \right) \right] \]
\[
= -\frac{1}{T-t} \ln \left[ \sum_{n=0}^\infty g(n; t, T) \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi(h_{2,n}(t)) \right\} \right].
\]
An explicit form for credit spread is thus obtained. When there is no jump or equivalently letting \( \lambda = 0 \), the equation (34) reduces to a Merton’s (1974) credit spread formula equation (3)

\[
R(t, T) - r = -\frac{1}{T-t} \ln \left\{ \Phi[h_2(t)] + \frac{1}{d(t)} \Phi[h_1(t)] \right\}.
\]

Alternatively, we have

\[
\frac{V(t, T)}{P(t, T)} = \sum_{n=0}^{\infty} g(n; t, T) \left\{ \frac{\Phi(h_{1,n}(t))}{d(t)} + \Phi(h_{2,n}(t)) \right\}.
\]

(35)

In order to calculate the default probability and mean recovery rate, we need to know the mean and variance of the normal jumps. There are three unknown parameter, \( \lambda, \mu_Y, \) and \( \sigma_Y \), we at least need two more constraints besides equation (35) to derive these parameters. We will introduce it in the next section.

5.5 Finding the implied jump distribution: lognormal jumps

In this section, we try to find the implied jump distribution when a firm issue bonds of three or more seniorities. We start with a case where a firm issues bonds consisted of three seniority classes with maturity at time \( T \). The higher seniority class represents \( p_1 \) proportion of the total debt, the second lower one represents \( p_2 \) proportion of the debt, and the third seniority class represents \((1-p_1-p_2)\) proportion of the debt. The recovery rate for the class 1 given the bond default is \( \delta_1(T) = \min \left( \frac{\delta(T)}{p_1}, 1 \right) \). Let \( \delta_1^*(T) \) represent the recovery rate for class 1 only when the recovery is less than 1. The recovery rate for class 2 is \( \delta_2(T) = \min \left( \frac{(\delta(T)-p_1)}{p_2}, 1 \right) \). Let \( \delta_2^*(T) \) represent the recovery rate for the seniority class 2 only when the recovery is less than 1. The recovery rate for class 3 is \( \delta_3(T) = \frac{(\delta(T)-p_1-p_2)}{1-p_1-p_2} \). Following the results of section (5.3), the bond price of seniority class 1 is

\[
V_1(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)} \left\{ \delta_1(T)1_{(V_T<D)} + 1_{(V_T\geq D)} \right\} \right]
\]

\[
= E_t^Q \left[ \frac{B(t)}{B(T)} \left\{ \delta_1^*(T)1_{(V_T<D_{p_1})} + 1_{(V_T\geq D_{p_1})} \right\} \right].
\]

The bond price of seniority class 2 can be expressed as

\[
V_2(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)} \left\{ \delta_2(T)1_{(V_T<D)} + 1_{(V_T\geq D)} \right\} \right]
\]

\[
= E_t^Q \left[ \frac{B(t)}{B(T)} \left\{ \delta_2^*(T)1_{[V_T<D_{p_1+p_2}]} + 1_{[V_T\geq D_{p_1+p_2}]} \right\} \right].
\]

The bond price of seniority class 3 can be expressed as

\[
V_3(t, T) = E_t^Q \left[ \frac{B(t)}{B(T)} \left\{ \delta_3(T)1_{(V_T<D)} + 1_{(V_T\geq D)} \right\} \right]
\]

\[
= E_t^Q \left[ \frac{B(t)}{B(T)} \left\{ \delta_3^*(T)1_{(V_T<D)} + 1_{(V_T\geq D)} \right\} \right].
\]
Theorem 5.6 If the value of a firm process follows a jump-diffusion process where the jump size is lognormal distributed and the assumptions of section 5.1 hold, then we have:

a) 
\[
\frac{1 - E_Q^t [\delta_1(T)]}{1 - E_Q^t [\delta_2(T)]} = \frac{P(t, T) - V_1(t, T)}{P(t, T) - V_2(t, T)}
\]

and 
\[
\frac{1 - E_Q^t [\delta_1(T)]}{1 - E_Q^t [\delta_3(T)]} = \frac{P(t, T) - V_1(t, T)}{P(t, T) - V_3(t, T)}.
\]

b) 
\[
E_Q^t [\delta_1(T)] = 1 - \sum_{n=0}^{\infty} g(n; t, T) \left[ \Phi\left( -h_{2,n}^*(t) \right) - \Phi\left( h_{1,n}(t) \right) \right] \sum_{n=0}^{\infty} g(n; t, T) \phi(h_{2,n}(t)).
\]

c) 
\[
E_Q^t [\delta_2(T)] = 1 - \sum_{n=0}^{\infty} g(n; t, T) \left[ \frac{\Phi(h_{1,n}(t)) - \Phi(h_{2,n}(t))}{d(t)p_1} + (p_1 + p_2) \Phi(-h_{2,n}^*(t)) - p_1 \Phi(-h_{2,n}^*(t)) \right] \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_{2,n}(t)).
\]

d) 
\[
E_Q^t [\delta_3(T)] = 1 - \sum_{n=0}^{\infty} g(n; t, T) \left[ \frac{\Phi(h_{1,n}(t)) - \Phi(h_{2,n}(t))}{d(t)} + \Phi(-h_{2,n}^*(t)) - (p_1 + p_2) \Phi(-h_{2,n}^*(t)) \right] \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_{2,n}(t)).
\]

Proof:

From equation (15), the product of the loss given default and the default probability is given by
\[
\left\{ 1 - E_t^Q [\delta_i(T)] \right\} P_t^Q \{ V_T < D \} = \frac{P(t, T) - V_i(t, T)}{P(t, T)} \quad i = 1, 2, 3.
\]

From equation (41), the relationship between the mean recovery rate of two different seniorities in term of the bond’s prices is given by
\[
\frac{1 - E_t^Q [\delta_1(T)]}{1 - E_t^Q [\delta_2(T)]} = \frac{P(t, T) - V_1(t, T)}{P(t, T) - V_2(t, T)}
\]

and
\[
\frac{1 - E_t^Q [\delta_1(T)]}{1 - E_t^Q [\delta_3(T)]} = \frac{P(t, T) - V_1(t, T)}{P(t, T) - V_3(t, T)}.
\]
The mean recovery rate $E_t^Q[\delta^*_1(T)]$ can be expressed as

$$E_t^Q[\delta^*_1(T)] = E_t^Q \left[ \frac{\delta(T)}{p_1} \Big| V_T < D_{p_1} \right] = E_t^Q \left[ \frac{V_T}{D_{p_1}} \Big| V_T < D_{p_1} \right].$$

We can consider $\delta^*_1(T)$ as the recovery of another bond with a liability of $D_{p_1}$. The mean recovery rate $\delta^*_1(T)$ in $Q$-measure can be derived from equation (33)

$$E_t^Q[\delta^*_1(T)] = \frac{\sum_{n=0}^\infty g(n; t, T)\Phi(h^*_1, n(t))}{d^*(t)\sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t))}$$

where

$$d^*(t) = \frac{D_{p_1}}{V_t e^{r(T-t)}} = p_1d(t),$$
$$h^*_1, n(t) = -\ln \frac{V_t}{D_{p_1}} + \frac{(r + \sigma^2 - \lambda_1, r)(T-t) + n\mu_1}{\sqrt{\sigma_1^2(T-t) + n\sigma_1^2}},$$
$$h^*_2, n(t) = -\ln \frac{V_t}{D_{p_1}} + \frac{(r - \sigma^2 - \lambda_1, r)(T-t) + n\mu_1}{\sqrt{\sigma_1^2(T-t) + n\sigma_1^2}}.$$

The probability that $V_T < D_{p_1}$ is

$$\Pr_t^Q \{ V_T < D_{p_1} \} = \sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t)).$$

The mean recovery rate for the higher seniority can be given by

$$E_t^Q[\delta^*_1(T)] = E_t^Q[\delta^*_1(T)]\Pr_t^Q \{ V_T < D_{p_1} \} + \Pr_t^Q \{ D_{p_1} \leq V_T < D \}$$

$$= \frac{\sum_{n=0}^\infty g(n; t, T)\Phi(h^*_1, n(t))}{d^*(t)\sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t))} \sum_{n=0}^\infty g(n; t, T)\Phi(h^*_1, n(t))$$

$$+ \frac{\sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t)) - \sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t))}{\sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t))}$$

$$= \frac{\sum_{n=0}^\infty g(n; t, T)\Phi(h^*_1, n(t))}{d^*(t)\sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t))} + 1 - \frac{\sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t))}{\sum_{n=0}^\infty g(n; t, T)\Phi(-h^*_2, n(t))}.$$

Now consider

$$p_1 \left[ \frac{\delta(T)}{p_1} \right] + p_2 \left[ \frac{(\delta(T) - p_1)\delta(T) p_2 + 1}{-p_2} \right] + (1 - p_1 - p_2) \left[ \frac{\delta(T) - p_1 - p_2}{1 - p_1 - p_2} \right] = \delta(T).$$
Taking the expected valued at time \( t \), we have
\[
p_1E_t^Q \left[ \frac{\delta(T)}{p_1} \times 1 \right] + p_2E_t^Q \left[ \frac{(\delta(T) - p_1) + p_2}{p_1} \times 1 \right] + (1 - p_1 - p_2)E_t^Q \left[ \frac{(\delta(T) - p_1 - p_2) + p_2}{1 - p_1 - p_2} \right] = E_t^Q[\delta(T)],
\]
and
\[
p_1E_t^Q[\delta_1(T)] + p_2E_t^Q[\delta_2(T)] + (1 - p_1 - p_2)E_t^Q[\delta_3(T)] = E_t^Q[\delta(T)].
\]
(42)

The mean recovery rate for seniority class 3, \( E_t^Q[\delta_3(T)] = E_t^Q \left[ \frac{(\delta(T) - p_1 - p_2) + p_2}{1 - p_1 - p_2} \right] \), can be considered as that in a two seniority classes case but the first seniority represents the proportion \( p_1 + p_2 \) of the debt. Following equation (29), we have
\[
E_t^Q[\delta_3(T)]
= 1 - \sum_{n=0}^{\infty} g(n; t, T) \left[ \frac{\Phi(h_1^{*,n}(t)) - \Phi(h_1,n(t))}{d(t)} + \Phi(-h_2,n(t)) - (p_1 + p_2)\Phi(-h_2^{*,n}(t)) \right]
(1 - p_1 - p_2) \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t))
\]
where
\[
h_1^{*,n} = -\frac{V_i}{\sigma^2 + \lambda T} + [r + \frac{\sigma^2}{T} + \lambda(\mu_Y - 1)](T - t) - n\mu_Y
\]
\[
h_2^{*,n} = \frac{V_i}{\sigma^2 + \lambda T} + [r - \frac{\sigma^2}{T} + \lambda(\mu_Y - 1)](T - t) - n\mu_Y
\]
Substituting \( E_t^Q[\delta_1(T)] \) and \( E_t^Q[\delta_3(T)] \) into equation (42), we have
\[
E_t^Q[\delta_2(T)]
= E_t^Q[\delta(T)] - p_1E_t^Q[\delta_1(T)] - (1 - p_1 - p_2)E_t^Q[\delta_3(T)]

= \frac{p_2}{d(t)} \sum_{n=0}^{\infty} g(n; t, T) \Phi(h_1,n(t)) - \frac{p_2}{d(t)} \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t)) + \frac{p_2}{d(t)} \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2^{*,n}(t))

= 1 - \frac{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t)) - \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2^{*,n}(t))}{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t))}

= 1 - \frac{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t)) - \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2^{*,n}(t))}{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t))}

\]
Substituting \( E_t^Q[\delta_1(T)] \) and \( E_t^Q[\delta_2(T)] \) into equation (36), we have
\[
\frac{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t)) - \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t))}{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t))}
\]
\[
\frac{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t)) - \sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t))}{\sum_{n=0}^{\infty} g(n; t, T) \Phi(-h_2,n(t))}
\]
\[
P(t, T) - V_1(t, T)
\]
\[
P(t, T) - V_2(t, T)
\]
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After rearrangement, we have
\[
\frac{\sum_{n=0}^{\infty} g(n; t, T) \left[ p_1 \Phi \left( -h^*_{2,n}(t) \right) - \frac{\Phi(h^*_{1,n}(t))}{d(t)} \right]}{\sum_{n=0}^{\infty} g(n; t, T) \left[ \Phi(h^*_{1,n}(t)) - \Phi(h^*_{2,n}(t)) + (p_1 + p_2) \Phi \left( -h^*_{2,n}(t) \right) - p_1 \Phi \left( -h^*_{2,n}(t) \right) \right]}
= \frac{p_1[P(t, T) - V_1(t, T)]}{p_2[P(t, T) - V_2(t, T)].}
\] (43)

Substituting \( E_t^Q[\delta_1(T)] \) and \( E_t^Q[\delta_3(T)] \) into equation (37), we have
\[
\frac{\sum_{n=0}^{\infty} g(n; n, T) \left[ \Phi \left( -h^*_{2,n}(t) \right) - \frac{\Phi(h^*_{1,n}(t))}{d(t)p_V^*} \right]}{\sum_{n=0}^{\infty} g(n; n, T) \Phi \left( -h^*_{2,n}(t) \right)}
= \frac{P(t, T) - V_1(t, T)}{P(t, T) - V_3(t, T)}.
\]

After rearrangement, we have
\[
\frac{\sum_{n=0}^{\infty} g(n; n, T) \left[ p_1 \Phi \left( -h^*_{2,n}(t) \right) - \frac{\Phi(h^*_{1,n}(t))}{d(t)} \right]}{\sum_{n=0}^{\infty} g(n; n, T) \left[ \Phi(h^*_{1,n}(t)) - \Phi(h^*_{2,n}(t)) + (p_1 + p_2) \Phi \left( -h^*_{2,n}(t) \right) - (p_1 + p_2) \Phi \left( -h^*_{2,n}(t) \right) \right]}
= \frac{p_1[P(t, T) - V_1(t, T)]}{(1 - p_1 - p_2)[P(t, T) - V_3(t, T)].}
\] (44)

There are three unknown parameters \( \lambda, \mu_Y, \) and \( \sigma_Y \) in equation (43) and equation (44). With one more equation (35)
\[
V(t, T) = \frac{P(t, T)}{P(t, T)} = \sum_{n=0}^{\infty} \frac{g(n; n, T) \left[ \Phi(h^*_{1,n}(t)) + \Phi(h^*_{2,n}(t)) \right]}{d(t)p_V^*}.
\] (45)

where \( V(t, T) = p_1V_1(t, T) + p_2V_2(t, T) + (1 - p_1 - p_2)V_3(t, T) \), we can solve the parameters numerically. The \( Q \)-measure probability of default and recovery rate can be obtained.

In the case where a firm issues bonds with, say, five seniority classes, the jump frequency and mean and variance of the normal jump should be consistent using any three of the seniority classes otherwise arbitrage opportunities exist. When traders notice arbitrage opportunities, they will trade so that the opportunities will disappear.

An alternative method of calculating the parameters is through grouping. If we treat classes 2, 3, and 4 as one class, then there are in total three regrouped classes. Class 1 and 5 can stay the same. However, the new class 2 will represent \( (p_2 + p_3 + p_4) \) proportion of the total debt. The mean recovery rate for the new class 2 is thus given by
\[
E[\delta^*_{\text{new}}(T)] = \frac{p_2E[\delta_2(T)] + p_3E[\delta_3(T)] + p_4E[\delta_4(T)]}{p_2 + p_3 + p_4}.
\] (46)

and a mean bond price is given by
\[
V^\text{new}_2(t, T) = \frac{p_2V_2(t, T) + p_3V_3(t, T) + p_4V_4(t, T)}{p_2 + p_3 + p_4}.
\] (47)
6 Conclusion

The reduced-form model uses the market spread to find the mean recovery rate and the default probability. In a different direction, the structural model use the capital structure to find the default probability and the mean recovery rate. The results are usually different. Specifically, the yield spread is usually too small in the Merton (1974) diffusion model compared to the market spread while the yield spread in the Zhou (1997) jump-diffusion model can approach the market spread. In this paper, we use the market spread and the firm’s capital structure at the same time to find the default probability and the mean recovery rate. In the multiple seniorities case, we find the implied jump distribution from the credit spread. To achieve this goal, we show the equivalence of credit spreads between the structural model and the reduced-form model. The degree of freedom is increase by adding a jump process to the diffusion process but is reduced by the market spread. By using the market spread to find the implied the jump distribution, the discrepancy of credit spreads between the structural model and the reduced-form model is removed and the structural model and the reduced-form model can be unified when the default can only occur at the maturity.

References


