INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 5, Number 4, Pages 374–399

MULTISCALE FEM-FVM HYBRID METHOD FOR CONVECTION-DIFFUSION EQUATIONS WITH PERIODIC DISCONTINUOUS COEFFICIENTS IN GENERAL CONVEX DOMAINS

LIRUI SHEN, LIQUN CAO, AND YAU SHU WONG

Abstract. This paper presents the multiscale analysis and numerical algorithms for the convection-diffusion equations with rapidly oscillating periodic discontinuous coefficients. The multiscale asymptotic expansions are developed and an explicit rate of convergence is derived for the convex domains. An efficient multiscale hybrid FEM-FVM algorithm is constructed, and numerical experiments are reported to validate the predicted convergence results.

Key words. Convection-diffusion equation, convex domain, periodic discontinuous coefficients, composite materials, porous media, multiscale asymptotic expansion, finite element method, finite volume element method.

1. Introduction

In this paper, we consider the convection-diffusion equations with rapidly oscillating periodic discontinuous coefficients given as follows:

$$
(1)\qquad \begin{cases}
\mathcal{L}_\varepsilon(u^\varepsilon) \equiv \dfrac{\partial u^\varepsilon(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a^\varepsilon_{ij}(x,t)\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\Big) + \dfrac{\partial}{\partial x_i}\Big(b^\varepsilon_i(x,t)u^\varepsilon(x,t)\Big) + \\
\qquad\qquad a^\varepsilon_0(x,t)u^\varepsilon(x,t) = f(x,t), \quad (x,t)\in\Omega\times(0,T), \\
u^\varepsilon(x,t) = g_0(x,t), \qquad (x,t)\in\Gamma_0\times(0,T), \\
\sigma_\varepsilon(u^\varepsilon) \equiv \nu_i\Big(a^\varepsilon_{ij}(x,t)\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j} - b^\varepsilon_i(x,t)u^\varepsilon(x,t)\Big) = g_1(x,t), \\
\qquad\qquad (x,t)\in\Gamma_1\times(0,T), \\
u^\varepsilon(x,0) = \bar{u}_0(x), \qquad x\in\Omega,
\end{cases}
$$

where $\Omega\subset\mathbb{R}^d$ ($d\geq 1$) is a bounded convex Lipschitz domain with the boundary $\partial\Omega = \overline{\Gamma}_0\cup\overline{\Gamma}_1$, $\Gamma_0\cap\Gamma_1=\emptyset$, $\mathrm{meas}(\Gamma_0)>0$, where $\mathrm{meas}(\Gamma_0)$ denotes the Lebesgue's measure of Γ_0; $u^\varepsilon(x,t)$ is the unknown function, $f(x,t)$, $g_0(x,t)$, $g_1(x,t)$ and $\bar{u}_0(x)$ are known functions. Here, we focus on the following specified cases: $a^\varepsilon_{ij}(x,t) = a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^k})$, $b^\varepsilon_i(x,t) = b_i(\frac{x}{\varepsilon},\frac{t}{\varepsilon^k})$ and $a^\varepsilon_0(x,t) = a_0(\frac{x}{\varepsilon},\frac{t}{\varepsilon^k})$, $k=0,1$; $\nu=(\nu_1,\cdots,\nu_d)$ is the outward unit normal to Γ_1. Throughout the paper, the Einstein summation convention on repeated indices is adopted. By C we denote a positive constant independent of ε.

Let $\xi=\varepsilon^{-1}x$, $\tau=\varepsilon^{-k}t$, $k=0,1$. We make the following assumptions:

($\mathbf{A_1}$) For $k=0$, $a_{ij}(\xi,t)$, $b_i(\xi,t)$ and $a_0(\xi,t)$ are 1-periodic in ξ; For $k=1$, $a_{ij}(\xi,\tau)$ $b_i(\xi,\tau)$ and $a_0(\xi,\tau)$ are 1-periodic in ξ and τ_0-periodic in τ.

Received by the editors May 22, 2014.
2000 *Mathematics Subject Classification.* 65F10, 65W05.
This work is supported by the National Natural Science Foundation of China (grant # 91330202), National Basic Research Program of China (973 Program) (grant # 2010CB832702), Natural Sciences and Engineering Research Council of Canada and project supported by the Funds for Creative Research Groups of China (grant # 11321061).

($\mathbf{A_2}$) a_{ij}, b_i, $a_0 \in L^\infty(\mathbb{R}^d \times (0, +\infty))$; $a_{ij} = a_{ji}$; there are two positive constants γ_0 and γ_1 such that $\gamma_0\eta_i\eta_i \leq a_{ij}(\xi,\tau)\eta_i\eta_j \leq \gamma_1\eta_i\eta_i$, $\forall\ \eta = (\eta_1, \cdots, \eta_d) \in \mathbb{R}^d$ for *a.e.* $\xi \in \mathbb{R}^d$, $\tau \in (0, +\infty)$.

($\mathbf{A_3}$) $f \in L^2(0,T; L^2(\Omega))$, $g_0 \in L^2(0,T; H^{1/2}(\Gamma_0))$, $g_1 \in L^2(0,T; H^{-1/2}(\Gamma_1))$ and $\bar{u}_0 \in H^1(\Omega)$.

($\mathbf{A_4}$) Let $Q = (0,1)^d$ be the reference cell of the composite material. Assume that $Q \subset\subset Q'$ with $Q' = \bigcup_{m=1}^{L} \overline{D}_m$. Suppose that the boundaries of D_m belong to $C^{1,\gamma}$ for some $0 < \gamma < 1$ and that $a_{ij}(\xi,\tau) \in C^\mu(\overline{D}_m \times (0,+\infty))$, $b_i(\xi,\tau) \in C^\mu(\overline{D}_m \times (0,+\infty))$, $i,j = 1,\cdots,d$, for some constants $0 < \mu < 1$.

Problem (1) has a wide range of applications in fluid mechanics especially in highly heterogeneous media; and in the heat and mass transfer problems for composite materials or porous media (see [4, 22, 34]). It is noted that the problem often involves materials with a large number of heterogeneities (inclusions or holes). For homogenization results concerning linear parabolic equations with rapidly oscillating coefficients which depend on the spatial and time variables, we refer to Bensoussan, Lions and Papanicolaou [4] for the periodic cases and to Colombini and Spagnolo [17] for the general non-periodic case. For a type of nonlinear parabolic partial differential operators, Pankov [33] and Svanstedt [36] derived the G-convergence and the homogenization results. Zhikov, Kozlov and Oleinik [41] investigated the parabolic operators with almost periodic coefficients and presented convergence results for the asymptotic homogenization.

By introducing a cutoff function, Bensoussan, Lions and Papanicolaou [4] obtained the strong convergence result without an explicit rate for the first-order corrector of the linear parabolic equations in $L^2(0,T;H^1(\Omega))$. Brahim-Otsmane, Francfort and Murat [5] extended this result to $L^2(0,T;W^{1,1}(\Omega))$. Ming and Zhang [30] derived the convergence result with an explicit rate $\varepsilon^{1/2}$ for the case $k = 0$ under the assumption $u^0 \in H^{3,1}(\Omega \times (0,T))$, where $u^0(x,t)$ is the solution of the linear homogenized parabolic equation. Allegretto, Cao and Lin [2] investigated the higher-order multiscale method for the linear parabolic equations in the four specific cases $k = 0,1,2,3$, and obtained the convergence results with an explicit rate $\varepsilon^{1/2}$ under the assumption $u^0 \in H^{s+2,1}(\Omega \times (0,T))$, $s = 1,2$. It is well known that, for a bounded convex polygonal Lipschitz domain Ω, the assumptions $u^0 \in H^{s+2,1}(\Omega \times (0,T))$, $s = 1,2$ may be invalid. Thus the error estimates presented in [2] are not valid. On the other hand, many results are now available for the finite volume element methods (FVEM) for the elliptic and parabolic equations. For example, we refer to [3, 7, 27, 11, 14, 15, 18, 19, 20] for the early important results. Chou and Ye [13] presented the unified variational form of the conforming and nonconforming elements, and the DG for 2-D elliptic equations with the error estimates of triangular elements. Li et al. [25] and Luo et al. [26] developed FVEM and the stabilized FVEM for the Navier-Stokes equations in 2-D, and obtained the error estimates for triangular elements under the assumption $\Delta t = O(h)$. However, the error estimates for 3-D are not available. Under the assumption that the term of right side $f(x,t) \equiv 0$ is indispensable, Sinha and Geiser [35] investigated the FVEM for the 2-D and 3-D convection-diffusion equations, and derived the optimal error estimates for the triangular and tetrahedral elements. For other recent results in this topic, we refer to [31, 21, 29, 40].

In this paper, we present the following new results:

(i) The interior error estimates for the multiscale asymptotic solutions of the original problem (1) are obtained under the weaker assumptions (see Theorem 3.1).

By defining the boundary layer solutions, the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale approximate solutions of (1) in a bounded convex domain are derived (see Theorem 3.2).

(ii) The optimal error estimates of FVEM for solving the modified homogenized convection-diffusion equation (44) in a bounded convex Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ are given (see Theorem 4.2).

(iii) A multiscale hybrid FEM-FVM algorithm is developed and numerical examples modeling 2-D and 3-D problems in composite materials are reported.

The remainder of this paper is organized as follows. In section 2, we present the formal multiscale asymptotic expansions for the original problem (1). In section 3, we derive the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale approximate solutions in a bounded convex domain based on the boundary layer solutions. In section 4, we discuss numerical algorithms and the error estimates for the related problems. Finally, a multiscale hybrid FEM-FVM algorithm is presented, and numerical simulations for problems in 2-D and 3-D composite materials are reported.

2. Multiscale asymptotic expansions

In this section, we present the formal multiscale asymptotic expansions for the original problem (1). First, we introduce two sets of variables: x, $\xi = \varepsilon^{-1}x$ and t, $\tau = \varepsilon^{-k}t$, where $k = 0, 1$. We define the multiscale asymptotic expansions of (1) in the following forms:

$$
\begin{aligned}
u_1^\varepsilon(x,t) &= u^0(x,t) + \varepsilon N_m(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_m} + \varepsilon R_0(\xi,\tau)u^0(x,t),\\
u_2^\varepsilon(x,t) &= u_1^\varepsilon(x,t) + \varepsilon^2 N_{ml}(\xi,\tau)\tfrac{\partial^2 u^0(x,t)}{\partial x_m \partial x_l}\\
&\quad +\varepsilon^2 M_l(\xi,\tau)\tfrac{\partial u^0(x,t)}{\partial x_l} + \varepsilon^2 R_1(\xi,\tau)u^0(x,t).
\end{aligned} \tag{2}
$$

In the case of $k = 0$, the cell functions are defined by

$$
\left\{
\begin{array}{l}
\dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,t)\dfrac{\partial N_m(\xi,t)}{\partial \xi_j}\big) = -\dfrac{\partial}{\partial \xi_i}(a_{im}(\xi,t)), \quad \xi \in Q\\
\quad \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,t)\dfrac{\partial R_0(\xi,t)}{\partial \xi_j}\big) = \dfrac{\partial}{\partial \xi_i}(b_i(\xi,t)), \quad \xi \in Q,\\
N_m(\xi,t),\ R_0(\xi,t) \ \text{ are 1-periodic in } \ \xi,\\
\displaystyle\int_Q N_m(\xi,t)\mathrm{d}\xi = 0, \quad \int_Q R_0(\xi,t)\mathrm{d}\xi = 0,
\end{array}
\right. \tag{3}
$$

and

$$
\left\{
\begin{array}{l}
\dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,t)\dfrac{\partial N_{ml}(\xi,t)}{\partial \xi_j}\big) = -\dfrac{\partial}{\partial \xi_i}(a_{im}(\xi,t)N_l(\xi,t))\\
\quad -a_{mj}(\xi,t)\dfrac{\partial N_l(\xi,t)}{\partial \xi_j} - a_{ml}(\xi,t) + \hat{a}_{ml}(t), \quad \xi \in Q,\\
N_{ml}(\xi,t) \ \text{ is 1-periodic in } \ \xi, \ \displaystyle\int_Q N_{ml}(\xi,t)\mathrm{d}\xi = 0,
\end{array}
\right. \tag{4}
$$

where the elements of the homogenized matrix $(\hat{a}_{ml}(t))$ are calculated by

$$
\hat{a}_{ml}(t) = \int_Q \big[a_{ml}(\xi,t) + a_{mq}(\xi,t)\frac{\partial N_l(\xi,t)}{\partial \xi_q}\big]\mathrm{d}\xi.
$$

$$(5)\quad \begin{cases} \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,t)\dfrac{\partial M_l(\xi,t)}{\partial \xi_j}\big) = -\dfrac{\partial}{\partial \xi_i}\big[a_{il}(\xi,t)R_0(\xi,t) - b_i(\xi,t)M_l(\xi,t)\big] \\ \quad -a_{lj}(\xi,t)\dfrac{\partial R_0(\xi,t)}{\partial \xi_j} + b_l(\xi,t) - \hat{b}_l(t), \quad \xi \in Q, \\ M_l(\xi,t) \text{ is 1-periodic in } \xi, \quad \displaystyle\int_Q M_l(\xi,t)\mathrm{d}\xi = 0, \end{cases}$$

where the elements of the homogenized vector $(\hat{b}_l(t))$ are calculated by

$$\hat{b}_l(t) = \int_Q \big[b_l(\xi,t) - a_{lj}(\xi,t)\frac{\partial R_0(\xi,t)}{\partial \xi_j}\big]\mathrm{d}\xi.$$

$$(6)\quad \begin{cases} \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,t)\dfrac{\partial R_1(\xi,t)}{\partial \xi_j}\big) = \dfrac{\partial}{\partial \xi_i}\big(b_i(\xi,t)R_0(\xi,t)\big) + a_0(\xi,t) - \langle a_0\rangle(t), \ \xi \in Q, \\ R_1(\xi,t) \text{ is 1-periodic in } \xi, \quad \displaystyle\int_Q R_1(\xi,t)\mathrm{d}\xi = 0, \\ \langle a_0\rangle(t) = \displaystyle\int_Q a_0(\xi,t)\mathrm{d}\xi, \end{cases}$$

where $t \in (0,T)$ plays the role of a parameter.

For the case $k = 1$, we define the cell functions by

$$(7)\quad \begin{cases} \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,\tau)\dfrac{\partial N_m(\xi,\tau)}{\partial \xi_j}\big) = -\dfrac{\partial}{\partial \xi_i}(a_{im}(\xi,\tau)), \quad (\xi,\tau) \in Q \times (0,\tau_0), \\ \quad \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,\tau)\dfrac{\partial R_0(\xi,\tau)}{\partial \xi_j}\big) = \dfrac{\partial}{\partial \xi_i}(b_i(\xi,\tau)), \quad (\xi,\tau) \in Q \times (0,\tau_0), \\ N_m(\xi,\tau),\ R_0(\xi,\tau) \text{ are 1-periodic in } \xi \text{ and } \tau_0\text{-periodic in } \tau, \\ \displaystyle\int_Q N_m(\xi,\tau)\mathrm{d}\xi = 0, \quad \int_Q R_0(\xi,\tau)\mathrm{d}\xi = 0, \end{cases}$$

and

$$(8)\quad \begin{cases} \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,\tau)\dfrac{\partial N_{ml}(\xi,\tau)}{\partial \xi_j}\big) = -\dfrac{\partial}{\partial \xi_i}(a_{im}(\xi,\tau)N_l(\xi,\tau)) \\ \quad -a_{mj}(\xi,\tau)\dfrac{\partial N_l(\xi,\tau)}{\partial \xi_j} - a_{ml}(\xi,\tau) + \hat{a}_{ml}, \quad (\xi,\tau) \in Q \times (0,\tau_0), \\ N_{ml}(\xi,\tau) = 0, \quad (\xi,\tau) \in \partial Q \times (0,\tau_0), \end{cases}$$

where the elements of the homogenized matrix $(\hat{a}_{ml})$ are calculated by

$$\hat{a}_{ml} = \frac{1}{\tau_0}\int_0^{\tau_0}\int_Q \big[a_{ml}(\xi,\tau) + a_{mq}(\xi,\tau)\frac{\partial N_l(\xi,\tau)}{\partial \xi_q}\big]\mathrm{d}\xi\mathrm{d}\tau.$$

$$(9)\quad \begin{cases} \dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,\tau)\dfrac{\partial M_l(\xi,\tau)}{\partial \xi_j}\big) = -\dfrac{\partial}{\partial \xi_i}\big[a_{il}(\xi,\tau)R_0(\xi,\tau) - b_i(\xi,\tau)M_l(\xi,\tau)\big] \\ \quad -a_{lj}(\xi,\tau)\dfrac{\partial R_0(\xi,\tau)}{\partial \xi_j} + b_l(\xi,\tau) - \hat{b}_l, \quad (\xi,\tau) \in Q \times (0,\tau_0), \\ M_l(\xi,\tau) = 0, \quad (\xi,\tau) \in \partial Q \times (0,\tau_0), \end{cases}$$

where the elements of the homogenized vector $(\hat{b}_l)$ are calculated by

$$\hat{b}_l = \frac{1}{\tau_0}\int_0^{\tau_0}\int_Q \big[b_l(\xi,\tau) - a_{lj}(\xi,\tau)\frac{\partial R_0(\xi,\tau)}{\partial \xi_j}\big]\mathrm{d}\xi\mathrm{d}\tau.$$

$$
(10)\quad \begin{cases}
\dfrac{\partial}{\partial \xi_i}\big(a_{ij}(\xi,\tau)\dfrac{\partial R_1(\xi,\tau)}{\partial \xi_j}\big) = \dfrac{\partial}{\partial \xi_i}(b_i(\xi,\tau)R_0(\xi,\tau)) + a_0(\xi,\tau) - \langle a_0\rangle, \\
\qquad\qquad (\xi,\tau) \in Q \times (0,\tau_0), \\
R_1(\xi,\tau) = 0, \quad (\xi,\tau) \in \partial Q \times (0,\tau_0), \\
\langle a_0 \rangle = \dfrac{1}{\tau_0}\displaystyle\int_0^{\tau_0}\int_Q a_0(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau.
\end{cases}
$$

Remark 2.1. Under the assumptions $(\mathbf{A}_1) - (\mathbf{A}_2)$, the existence and uniqueness of the cell problems (3)-(10) can be established based on the Lax-Milgram lemma, see also [2].

The homogenized equation associated with problem (1) is written in the following unified form:

$$
(11)\quad \begin{cases}
\dfrac{\partial u^0(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\big(\hat{a}_{ij}\dfrac{\partial u^0(x,t)}{\partial x_j}\big) + \hat{b}_i\dfrac{\partial u^0(x,t)}{\partial x_i} + \langle a_0\rangle u^0(x,t) = f(x,t), \\
\qquad\qquad (x,t) \in \Omega \times (0,T), \\
u^0(x,t) = g_0(x,t), \quad (x,t) \in \Gamma_0 \times (0,T), \\
\hat{\sigma}(u^0) \equiv \nu_i\big(\hat{a}_{ij}\dfrac{\partial u^0(x,t)}{\partial x_j} - \hat{b}_i u^0(x,t)\big) = g_1(x,t), \quad (x,t) \in \Gamma_1 \times (0,T), \\
u^0(x,0) = \bar{u}_0(x), \quad x \in \Omega.
\end{cases}
$$

Remark 2.2. It can be proved that the homogenized coefficients matrix $(\hat{a}_{ij})$ is symmetric and positive-definite (see [2, 4, 23, 32]). Existence and uniqueness of the solution for the homogenized parabolic equation (11) can be established.

3. Convergence for the multiscale asymptotic method

In the previous section, we present the formal multiscale asymptotic expansions (2) for the original problem (1). In this section, we derive the convergence results for the multiscale asymptotic method in a bounded convex Lipschitz domain. As already discussed in the Introduction, for a bounded convex Lipschitz domain Ω, the assumptions $u^0 \in H^{s+2,1}(\Omega \times (0,T))$, $s = 1,2$ may be invalid. Thus the error estimates reported in [2] cannot be accepted. In this study, following the idea of [8], we first define a boundary layer solution. Then we derive the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale asymptotic solutions in a bounded convex Lipschitz domain Ω. Let $\overline{\Omega}_0 = \bigcup_{\mathbf{z}\in I_\varepsilon} \varepsilon(\mathbf{z} + \overline{Q})$, where $I_\varepsilon = \{\mathbf{z} = (z_1, \cdots, z_d) \in \mathbb{Z}^d : \varepsilon(\mathbf{z} + Q) \subset \Omega\}$. The interior subdomain Ω_0 and the boundary layer $\Omega_1 = \Omega \setminus \overline{\Omega}_0$ are shown in Figs.3.1 and 3.2. Let $\mathrm{dist}(\partial\Omega_0, \partial\Omega) > 2\varepsilon$. Since Ω is a bounded convex domain, this condition can be satisfied.

The boundary layer solutions $u_s^{\varepsilon,b}(x,t)$ are defined as follows:

$$
(12)\quad \begin{cases}
\mathcal{L}_\varepsilon(u_s^{\varepsilon,b}) = f(x,t), \quad (x,t) \in \Omega_1 \times (0,T), \\
u_s^{\varepsilon,b}(x,t) = g_0(x,t), \quad (x,t) \in \Gamma_0 \times (0,T), \\
\sigma_\varepsilon(u_s^{\varepsilon,b}) = g_1(x,t), \quad (x,t) \in \Gamma_1 \times (0,T), \\
u_s^{\varepsilon,b}(x,t) = u_s^{\varepsilon}(x,t), \quad (x,t) \in \Gamma^* \times (0,T), \\
u_s^{\varepsilon,b}(x,0) = \bar{u}_0(x), \quad x \in \Omega,
\end{cases}
$$

where $\mathcal{L}_\varepsilon$ and σ_ε have been given in (1), $u_s^\varepsilon(x,t)$, $s = 1,2$ are defined in (2), $\Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$, $s = 1,2$. The multiscale asymptotic solutions for problem (1) are

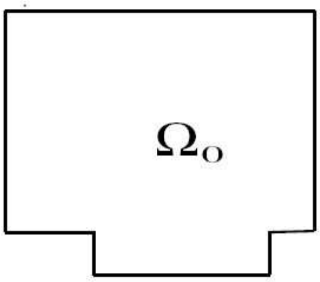

FIGURE 1. Interior sub-domain Ω_0.

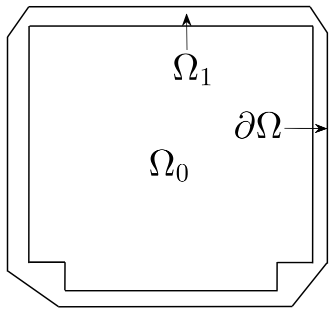

FIGURE 2. Boundary layer Ω_1.

defined as

$$U_s^\varepsilon(x,t) = \begin{cases} u_s^\varepsilon(x,t), & (x,t) \in \overline{\Omega}_0 \times (0,T), \\ u_s^{\varepsilon,b}(x,t), & (x,t) \in \Omega_1 \times (0,T), \end{cases} \tag{13}$$

where $u_s^{\varepsilon,b}(x,t)$, $s = 1,2$ have been given in (12), $u_s^\varepsilon(x,t)$, $s = 1,2$ denote the restrictions in a sub-domain $\Omega_0 \subset\subset \Omega$.

Theorem 3.1. Suppose that $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded convex domain with a Dirichlet's boundary $\partial\Omega$. Let $u^\varepsilon(x,t)$ be the unique weak solution of problem (1), and let $u^0(x,t)$ be the solution of the homogenized convection-diffusion equation (11), and $u_1^\varepsilon(x,t)$, $u_2^\varepsilon(x,t)$ be the first-order and the second-order multiscale asymptotic solutions defined in (2), respectively. Assume that $f \in L^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^s(\Omega''))$, $g_0 \in L^2(0,T;H^{1/2}(\partial\Omega))$, $\bar{u}_0 \in H^1(\Omega) \cap H^{s+1}(\Omega'')$, $s = 1,2$, where $\Omega_0 \subset\subset \Omega'' \subset\subset \Omega$. For the specified case $k = 0$, we assume that $a_{ij}(\frac{x}{\varepsilon},\cdot) \in C^1(0,T)$ for any fixed $x \in \Omega$. Under the assumptions $(\mathbf{A}_1) - (\mathbf{A}_4)$, then it holds

$$\sup_{0\leq t\leq T} \int_{\Omega_0} (u^\varepsilon(x,t) - u_s^\varepsilon(x,t))^2 \mathrm{d}x + \int_0^T \|u^\varepsilon(\cdot,t) - u_s^\varepsilon(\cdot,t)\|^2_{H^1(\Omega_0)} \mathrm{d}t \leq C\varepsilon, \tag{14}$$

for $k = 0$ and $s = 1,2$, or $k = 1$ and $s = 2$.

Proof. Let us introduce the following sub-domains:

$$\begin{aligned} \Omega' &= \{x \in \Omega : dist(x,\partial\Omega) \geq \varepsilon/2\}, \\ K_\varepsilon &= \{x \in \Omega : dist(x,\partial\Omega) \leq \varepsilon\}, \\ K'_\varepsilon &= \{x \in \Omega : \varepsilon \leq dist(x,\partial\Omega) \leq 2\varepsilon\}. \end{aligned}$$

Observe that the homogenized equation (11) is the initial-boundary value problem with smooth coefficients. Under the assumptions of this theorem, using the interior regularity estimates for the parabolic equations (see, e.g., [24, 37]), we infer that $u^0 \in H^1(0,T;H^{s+1}(\Omega'))$, $s = 1,2$. Now, introduce the cutoff function $m_\varepsilon(x)$ given by

$$\begin{aligned} & m_\varepsilon \in \mathcal{D}(\Omega) \\ & m_\varepsilon = 0, \quad \text{if} \quad \mathrm{dist}(x,\partial\Omega) \leq \varepsilon \\ & m_\varepsilon = 1, \quad \text{if} \quad \mathrm{dist}(x,\partial\Omega) \geq 2\varepsilon \\ & \varepsilon|\frac{\partial m_\varepsilon}{\partial x_i}| \leq C, \quad i = 1,2,\cdots,d. \end{aligned} \tag{15}$$

Define

$$
\begin{aligned}
(16) \qquad & \theta_1^\varepsilon(x,t) = u^0(x,t) + \varepsilon m_\varepsilon(x)\big\{N_m(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_m} + R_0(\xi,\tau)u^0(x,t)\big\},\\
& \theta_2^\varepsilon(x,t) = \theta_1^\varepsilon(x,t) + \varepsilon^2 m_\varepsilon(x)\{N_{ml}(\xi,\tau)\tfrac{\partial^2 u^0(x,t)}{\partial x_i \partial x_j} + \\
& \qquad M_l(\xi,\tau)\tfrac{\partial u^0(x,t)}{\partial x_l} + R_1(\xi,\tau)u^0(x,t)\big\}.
\end{aligned}
$$

Next we prove Theorem 3.1 for the case $k = 1$. Other cases are similar. For simplicity, we assume that $g_0(x,t) \equiv 0$. Let $\Omega_t = \Omega \times (0,t)$ and $(u,v)_{\Omega_T} = \int_0^t \int_\Omega uv \mathrm{d}x\mathrm{d}t$. For simplicity, let $a_{ij}^\varepsilon = a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon})$, $b_i^\varepsilon = b_i(\frac{x}{\varepsilon},\frac{t}{\varepsilon})$, $a_0^\varepsilon = a_0(\frac{x}{\varepsilon},\frac{t}{\varepsilon})$. For all $v \in L^2(0,T;H_0^1(\Omega))$, from (2) and (16), we obtain the following equation which holds in the sense of distribution:

$$
(17) \qquad (\mathcal{L}_\varepsilon(u^\varepsilon - \theta_2^\varepsilon), v)_{\Omega_T} = J^\varepsilon(v),
$$

where

$$
J^\varepsilon(v) = (K_1 + K_2 + K_3 + K_4, v)_{\Omega_T}
$$

and

$$
\begin{aligned}
K_1 = & -m_\varepsilon \frac{\partial N_m}{\partial \tau}\frac{\partial u^0}{\partial x_m} - m_\varepsilon \frac{\partial R_0}{\partial \tau}u^0 - \varepsilon m_\varepsilon N_m \frac{\partial^2 u^0}{\partial t \partial x_m} - \varepsilon m_\varepsilon R_0 \frac{\partial u^0}{\partial t}\\
& -\varepsilon m_\varepsilon \frac{\partial N_{ml}}{\partial \tau}\frac{\partial^2 u^0}{\partial x_m \partial x_l} - \varepsilon m_\varepsilon \frac{\partial M_l}{\partial \tau}\frac{\partial u^0}{\partial x_l} - \varepsilon m_\varepsilon \frac{\partial R_1}{\partial \tau}u^0\\
& -\varepsilon^2 m_\varepsilon N_{ml}\frac{\partial^3 u^0}{\partial t \partial x_m \partial x_l} - \varepsilon^2 m_\varepsilon M_l \frac{\partial^2 u^0}{\partial t \partial x_l} - \varepsilon^2 m_\varepsilon R_1 \frac{\partial u^0}{\partial t},
\end{aligned}
$$

$$
K_2 = (m_\varepsilon - 1)\big[(\hat{a}_{ij} - a_{ij}^\varepsilon)\frac{\partial^2 u^0}{\partial x_i \partial x_j} + (b_i^\varepsilon - \hat{b}_i)\frac{\partial u^0}{\partial x_i} + (a_0^\varepsilon - \langle a_0\rangle)u^0\big],
$$

$$
\begin{aligned}
K_3 = & \varepsilon^{-1}\frac{\partial}{\partial \xi_i}(a_{ij}^\varepsilon \frac{\partial N_m}{\partial \xi_j})(m_\varepsilon - 1)\frac{\partial u^0}{\partial x_m} + \varepsilon^{-1}\frac{\partial}{\partial \xi_i}(a_{ij}^\varepsilon \frac{\partial R_0}{\partial \xi_j})(m_\varepsilon - 1)u^0\\
& +\varepsilon \frac{\partial}{\partial \xi_i}(a_{ij}^\varepsilon N_{ml})m_\varepsilon \frac{\partial^3 u^0}{\partial x_j \partial x_m \partial x_l} + \varepsilon \frac{\partial}{\partial \xi_i}(a_{ij}^\varepsilon M_l)m_\varepsilon \frac{\partial^2 u^0}{\partial x_j \partial x_l}\\
& +\varepsilon \frac{\partial}{\partial \xi_i}(a_{ij}^\varepsilon R_1)m_\varepsilon \frac{\partial u^0}{\partial x_j} + a_{ij}^\varepsilon \frac{\partial N_m}{\partial \xi_j}\frac{\partial m_\varepsilon}{\partial x_i}\frac{\partial u^0}{\partial x_m} + a_{ij}^\varepsilon \frac{\partial R_0}{\partial \xi_j}\frac{\partial m_\varepsilon}{\partial x_i}u^0\\
& -\varepsilon \frac{\partial}{\partial \xi_i}(b_i^\varepsilon N_{ml})m_\varepsilon \frac{\partial^2 u^0}{\partial x_m \partial x_l} - \varepsilon \frac{\partial}{\partial \xi_i}(b_i^\varepsilon M_l)m_\varepsilon \frac{\partial u^0}{\partial x_l} - \varepsilon \frac{\partial}{\partial \xi_i}(b_i^\varepsilon R_1)m_\varepsilon u^0,
\end{aligned}
$$

$$
\begin{aligned}
K_4 = & \varepsilon a_{ij}^\varepsilon \frac{\partial N_{ml}}{\partial \xi_j}\frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial^2 u^0}{\partial x_m \partial x_l}) + \varepsilon a_{ij}^\varepsilon \frac{\partial M_l}{\partial \xi_j}\frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial u^0}{\partial x_l})\\
& +\varepsilon a_{ij}^\varepsilon \frac{\partial R_0}{\partial \xi_j}\frac{\partial}{\partial x_i}(m_\varepsilon u^0) + \varepsilon a_{ij}^\varepsilon N_m \frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial^2 u^0}{\partial x_j \partial x_m})\\
& +\varepsilon a_{ij}^\varepsilon R_0 \frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial u^0}{\partial x_j}) + \varepsilon^2 a_{ij}^\varepsilon N_{ml}\frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial^3 u^0}{\partial x_j \partial x_m \partial x_l})\\
& +\varepsilon^2 a_{ij}^\varepsilon M_l \frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial^2 u^0}{\partial x_j \partial x_l}) + \varepsilon^2 a_{ij}^\varepsilon R_1 \frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial u^0}{\partial x_j})\\
& -\varepsilon b_i^\varepsilon N_m \frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial u^0}{\partial x_m}) - \varepsilon b_i^\varepsilon R_0 \frac{\partial}{\partial x_i}(m_\varepsilon u^0) - \varepsilon^2 b_i^\varepsilon R_1 \frac{\partial}{\partial x_i}(m_\varepsilon u^0)\\
& -\varepsilon^2 b_i^\varepsilon N_{ml}\frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial^2 u^0}{\partial x_m \partial x_l}) - \varepsilon^2 b_i^\varepsilon M_l \frac{\partial}{\partial x_i}(m_\varepsilon \frac{\partial u^0}{\partial x_l}).
\end{aligned}
$$

Under the assumptions $(\mathbf{A}_1), (\mathbf{A}_2)$ and $(\mathbf{A}_4)$, and using Theorem 1.1 of [28], we can prove that

$$
\begin{aligned}
&N_m,\ R_0,\ N_{ml},\ M_l,\ R_1 \in L^2(0,\tau_0;W^{1,\infty}(Q)),\\
&\frac{\partial N_m}{\partial \tau},\ \frac{\partial R_0}{\partial \tau},\ \frac{\partial N_{ml}}{\partial \tau},\ \frac{\partial M_l}{\partial \tau},\ \frac{\partial R_1}{\partial \tau} \in L^2(0,\tau_0;L^\infty(Q)).
\end{aligned} \tag{18}
$$

Since $\int_Q \frac{\partial N_m}{\partial \tau}\mathrm{d}\xi = 0$, applying Lemma 1.6 and Lemma 1.5 of [32, p. 7-8], we get

$$
\begin{aligned}
\Big|(m_\varepsilon \frac{\partial N_m}{\partial \tau}\frac{\partial u^0}{\partial x_m}, v)_{\Omega_T}\Big| &\le C\varepsilon \int_0^T \|m_\varepsilon \nabla u^0\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}\mathrm{d}t\\
&\le C\varepsilon^{1/2}\|u^0\|_{L^2(0,T;H^2(\Omega'))}\|v\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned} \tag{19}
$$

Similarly, we have

$$
\Big|(m_\varepsilon \frac{\partial R_0}{\partial \tau}u^0, v)_{\Omega_T}\Big| \le C\varepsilon^{1/2}\|u^0\|_{L^2(0,T;H^1(\Omega'))}\|v\|_{L^2(0,T;H^1(\Omega))}. \tag{20}
$$

Combining (18), (19) and (20) gives

$$
|(K_1, v)_{\Omega_T}| \le C\big\{\varepsilon^{1/2}\|u^0\|_{L^2(0,T;H^2(\Omega'))} + \varepsilon\|u^0_t\|_{L^2(0,T;H^2(\Omega'))}\big\}\|v\|_{L^2(0,T;H^1(\Omega))}. \tag{21}
$$

Applying Lemma 1.5 of [32, p. 7] again, we get

$$
\begin{aligned}
\Big|((m_\varepsilon - 1)(\hat{a}_{ij} - a^\varepsilon_{ij})\frac{\partial^2 u^0}{\partial x_i \partial x_j}, v)_{\Omega_T}\Big| &\le C\int_0^T \|u^0\|_{H^2(K_\varepsilon)}\|v\|_{L^2(K_\varepsilon)}\mathrm{d}t\\
&\le C\varepsilon^{1/2}\|u^0\|_{L^2(0,T;H^2(\Omega))}\|v\|_{L^2(0,T;H^1(\Omega))},
\end{aligned} \tag{22}
$$

and consequently

$$
|(K_2, v)_{\Omega_T}| \le C\varepsilon^{1/2}\|u^0\|_{L^2(0,T;H^2(\Omega))}\|v\|_{L^2(0,T;H^1(\Omega))}. \tag{23}
$$

We observe that

$$
\begin{aligned}
&\varepsilon^{-1}\frac{\partial}{\partial \xi_i}(a^\varepsilon_{ij}\frac{\partial N_m}{\partial \xi_j})(m_\varepsilon - 1)\frac{\partial u^0}{\partial x_m} + a^\varepsilon_{ij}\frac{\partial N_m}{\partial \xi_j}\frac{\partial m_\varepsilon}{\partial x_i}\frac{\partial u^0}{\partial x_m}\\
&= \frac{\partial}{\partial x_i}(a^\varepsilon_{ij}\frac{\partial N_m}{\partial \xi_j}(m_\varepsilon - 1)\frac{\partial u^0}{\partial x_m}) - a^\varepsilon_{ij}\frac{\partial N_m}{\partial \xi_j}(m_\varepsilon - 1)\frac{\partial^2 u^0}{\partial x_i \partial x_m},
\end{aligned}
$$

and

$$
\begin{aligned}
&\varepsilon m_\varepsilon \frac{\partial}{\partial \xi_i}(a^\varepsilon_{ij}N_{ml})\frac{\partial^3 u^0}{\partial x_j \partial x_m \partial x_l} = \varepsilon^2 \frac{\partial}{\partial x_i}(m_\varepsilon a^\varepsilon_{ij}N_{ml}\frac{\partial^3 u^0}{\partial x_j \partial x_m \partial x_l})\\
&-\varepsilon^2 a^\varepsilon_{ij}N_{ml}\frac{\partial m_\varepsilon}{\partial x_i}\frac{\partial^3 u^0}{\partial x_j \partial x_m \partial x_l} - \varepsilon^2 m_\varepsilon a^\varepsilon_{ij}N_{ml}\frac{\partial^4 u^0}{\partial x_i \partial x_j \partial x_m \partial x_l}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big|(\frac{\partial}{\partial x_i}(a^\varepsilon_{ij}\frac{\partial N_m}{\partial \xi_j}(m_\varepsilon - 1)\frac{\partial u^0}{\partial x_m}), v)_{\Omega_T}\Big| = \Big|(a^\varepsilon_{ij}\frac{\partial N_m}{\partial \xi_j}(m_\varepsilon - 1)\frac{\partial u^0}{\partial x_m}, \frac{\partial v}{\partial x_i})_{\Omega_T}\Big|\\
&\le C\int_0^T \|u^0\|_{H^1(K_\varepsilon)}\|v\|_{H^1(K_\varepsilon)}\mathrm{d}t \le C\varepsilon^{1/2}\|u^0\|_{L^2(0,T;H^2(\Omega))}\|v\|_{L^2(0,T;H^1(\Omega))},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\Big|(a^\varepsilon_{ij}\frac{\partial N_m}{\partial \xi_j}(m_\varepsilon - 1)\frac{\partial^2 u^0}{\partial x_i \partial x_m}, v)_{\Omega_T}\Big| &\le C\int_0^T \|u^0\|_{H^2(K_\varepsilon)}\|v\|_{L^2(K_\varepsilon)}\mathrm{d}t\\
&\le C\varepsilon^{1/2}\|u^0\|_{L^2(0,T;H^2(\Omega))}\|v\|_{L^2(0,T;H^1(\Omega))},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&\Big|(\varepsilon^2 m_\varepsilon \frac{\partial}{\partial x_i}(a_{ij}^\varepsilon N_{ml}\frac{\partial^3 u^0}{\partial x_j\partial x_m\partial x_l}),v)_{\Omega_T}\Big| \\
&= \Big|(\varepsilon^2 m_\varepsilon a_{ij}^\varepsilon N_{ml}\frac{\partial^3 u^0}{\partial x_j\partial x_m\partial x_l},\frac{\partial v}{\partial x_i})_{\Omega_T}\Big| \\
&\le C\varepsilon^2\|u^0\|_{L^2(0,T;H^3(\Omega'))}\|v\|_{L^2(0,T;H^1(\Omega))},
\end{aligned}
\tag{26}
$$

$$
\Big|(\varepsilon^2 a_{ij}^\varepsilon N_{ml}\frac{\partial m_\varepsilon}{\partial x_i}\frac{\partial^3 u^0}{\partial x_j\partial x_m\partial x_l},v)_{\Omega_T}\Big| \le C\varepsilon^{3/2}\|u^0\|_{L^2(0,T;H^3(\Omega')}\|v\|_{L^2(0,T;H^1\Omega)},
\tag{27}
$$

$$
\Big|(\varepsilon^2 m_\varepsilon a_{ij}^\varepsilon N_{ml}\frac{\partial^4 u^0}{\partial x_i\partial x_j\partial x_m\partial x_l},v)_{\Omega_T}\Big| \le C\varepsilon^2\|u^0\|_{L^2(0,T;H^4(\Omega'))}\|v\|_{L^2(0,T;H^1(\Omega))}.
\tag{28}
$$

We can similarly estimate other terms in K_3, and obtain

$$
|(K_3,v)_{\Omega_T}| \le C\big\{\varepsilon^{1/2}\|u^0\|_{L^2(0,T;H^2(\Omega))}+\varepsilon^{3/2}\|u^0\|_{L^2(0,T;H^4(\Omega'))}\big\}\|v\|_{L^2(0,T;H^1(\Omega))}.
\tag{29}
$$

Using (18) and Lemma 1.5 of [32, p. 7], we can prove that

$$
|(K_4,v)_{\Omega_T}| \le C\varepsilon\|u^0\|_{L^2(0,T;H^4(\Omega'))}\|v\|_{L^2(0,T;H^1(\Omega))}.
\tag{30}
$$

Combining (21),(23),(29) and (30), it gives

$$
|J^\varepsilon(v)| \le C\varepsilon^{1/2}\|v\|_{L^2(0,T;H^1(\Omega))}.
$$

For the initial condition, we have

$$
u^\varepsilon(x,0)-\theta_2^\varepsilon(x,0)=\varepsilon\Psi_\varepsilon(x),
\tag{31}
$$

where

$$
\begin{aligned}
\Psi_\varepsilon(x) = m_\varepsilon(x)\big\{&N_m(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_m}+R_0(\xi,\tau)u^0(x,t)+\varepsilon N_{ml}(\xi,\tau)\frac{\partial^2 u^0(x,t)}{\partial x_m\partial x_l} \\
&+\varepsilon M_l(\xi,\tau)\frac{\partial u^0(x,t)}{\partial x_l}+\varepsilon R_1(\xi,\tau)u^0(x,t)\big\}|_{t=0}.
\end{aligned}
$$

By using (18) and the regularity of the solution $u^0(x,t)$, i.e. $u^0\in L^2(0,T;H^2(\Omega))$ (see, e.g., [24, 37]), we can show that $\|\Psi_\varepsilon\|_{L^2(\Omega)}\le C$, where C is a constant independent of ε. In a standard way, applying the Gronwall's inequality, it completes the proof of Theorem 3.1. □

Remark 3.1. It should be emphasized that, in order to derive the convergence results with an explicit rate $\varepsilon^{1/2}$ for the first-order and the second-order multiscale asymptotic solutions of problem (1) in general convex domains, Theorem 3.1 plays a key role. Allaire and Amar[1] obtained a similar convergence result in an interior subdomain $\Omega_0\subset\subset\Omega$ for elliptic equations. However, their method fails for parabolic equations.

Theorem 3.2. Suppose that $\Omega\subset\mathbb{R}^d$, $d\ge 1$ is a bounded convex domain with a pure Dirichlet's boundary $\partial\Omega$. Let $u^\varepsilon(x,t)$ be the unique weak solution of problem (1), and let $u^0(x,t)$ be the solution of the homogenized convection-diffusion equation (11), and $U_1^\varepsilon(x,t)$, $U_2^\varepsilon(x,t)$ be the first-order and the second-order multiscale approximate solutions defined in (13), respectively. Assume that $f\in L^2(0,T;L^2(\Omega))\cap H^1(0,T;H^s(\Omega''))$, $g_0\in L^2(0,T;H^{1/2}(\partial\Omega))$, $\bar{u}_0\in H^1(\Omega)\cap H^{s+1}(\Omega'')$, $s=1,2$. For the specified case $k=0$, we assume that $a_{ij}(\frac{x}{\varepsilon},\cdot)\in C^1(0,T)$ for any fixed $x\in\Omega$. Under the assumptions $(\mathbf{A}_1)-(\mathbf{A}_4)$, then it holds

$$
\sup_{0\le t\le T}\int_\Omega (u^\varepsilon(x,t)-U_s^\varepsilon(x,t))^2\mathrm{d}x+\int_0^T\|u^\varepsilon(\cdot,t)-U_s^\varepsilon(\cdot,t)\|_{H^1(\Omega)}^2\mathrm{d}t\le C\varepsilon,
\tag{32}
$$

for $k = 0$ and $s = 1, 2$, or $k = 1$ and $s = 2$.

Proof. We prove this theorem only for the case $k = 1$, $s = 2$, and other cases are similar. Recalling the boundary layer solution $u_2^{\varepsilon,b}(x,t)$ defined in (12), and using (14) and the trace theorem, we get

$$\begin{aligned} \|u^\varepsilon - u_2^{\varepsilon,b}\|_{L^2(0,T;H^1(\Omega_1))} &\leq C\|u^\varepsilon - u_2^\varepsilon\|_{L^2(0,T;H^{1/2}(\Gamma^*))} \\ &\leq C\|u^\varepsilon - u_2^\varepsilon\|_{L^2(0,T;H^1(\Omega_0))} \leq C\varepsilon^{1/2}. \end{aligned} \tag{33}$$

We thus have

$$\begin{aligned} \|u^\varepsilon - U_2^\varepsilon\|_{L^2(0,T;H^1(\Omega))} &\leq \|u^\varepsilon - u_2^\varepsilon\|_{L^2(0,T;H^1(\Omega_0))} \\ &+\|u^\varepsilon - u_2^{\varepsilon,b}\|_{L^2(0,T;H^1(\Omega_1))} \leq C\varepsilon^{1/2}. \end{aligned} \tag{34}$$

□

Remark 3.2. Assume that $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ is the mixed boundary. Under the assumptions of Theorem 3.1 or Theorem 3.2, if the condition $u^0 \in L^2(0,T;H^3(\Omega))$ is imposed, then we can derive a similar convergence results to Theorem 3.1 or Theorem 3.2 for the case $k = 0$.

Remark 3.3. It should be emphasized that, in order to obtain the convergence results with an explicit rate $\varepsilon^{1/2}$ for the case $k = 1$, we need to apply the second-order multiscale asymptotic expansion defined in $(2)_2$ and (13). The numerical results presented in Section 5 show that the second-order correctors are necessary regardless of $k = 0$ or $k = 1$.

4. Numerical algorithms for related problems and error estimates

By adopting the standard notation $W^{m,q}(\Omega)$ for the Sobolev spaces in Ω with norm $\|\cdot\|_{m,q,\Omega}$ and semi-norm $|\cdot|_{m,q,\Omega}$, We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$ and denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$) with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$.

4.1. Adaptive finite element method (AFEM) for cell problems. For simplicity, we only consider the cell problems (7) for the case $k = 1$. As for cell problems (3)-(6), (8)-(10) and other cases, they are similar. Since the elements $a_{ij}(\xi,\tau)$ of the coefficients matrix of (7) are discontinuous, we employ an adaptive finite element method (see, e.g., [12, 39]). For convenience, we present *a posteriori* error estimate for solving the cell problems (7). Let $\mathcal{K}_\mu$ be a sequence of tetrahedrons for the reference cell Q, and let $\mathcal{F}_\mu$ be the set of faces not lying on ∂Q , $\mu \geq 0$, i.e. $\mathcal{F}_\mu \cap \partial Q = \emptyset$. Note that the tetrahedrons must be aligned with the boundary of Q to employ the periodic boundary conditions on the boundary ∂Q. The linear finite element space $\mathcal{W}_\mu$ in Q is defined by

$$\begin{aligned} \mathcal{W}_\mu(Q) = \{v \in C^0(\overline{Q})| \quad v|_K \in P_1(K), \quad v \quad \text{takes the same value} \\ \text{on the opposite faces of} \quad Q, \quad \forall K \in \mathcal{K}_\mu\}, \end{aligned} \tag{35}$$

where $P_1(K)$ is the space of all polynomials of degree, not exceeding 1, on each element K. For any $K \in \mathcal{K}_\mu$ and $F \in \mathcal{F}_\mu$, we denote the diameters of K and F by h_K and h_F, respectively.

Let $N_{m,\mu}(\xi,\tau)$ denote the approximate solution of $N_m(\xi,\tau)$, $m = 1, 2, \cdots, d$ in the linear finite element space $\mathcal{W}_\mu(Q)$, $\mu \geq 0$, respectively. By means of Theorem 5.2.1 of [39], a *posteriori* error estimate for $N_m(\xi,\tau)$ is given by

$$\|N_m - N_{m,\mu}\|_{H^1(Q)}^2 \leq C\Big(\sum_{K\in\mathcal{K}_\mu} \eta_K^2 + \sum_{F\in\mathcal{F}_\mu} \eta_F^2\Big), \tag{36}$$

where

$$\eta_K^2 = h_K^2 \int\limits_K \Big(\frac{\partial a_{im}(\xi,\tau)}{\partial \xi_i} + \frac{\partial}{\partial \xi_i}(a_{ij}(\xi,\tau)\frac{\partial N_{m,\mu}(\xi,\tau)}{\partial \xi_j})\Big)^2 \mathrm{d}\xi,$$

$$\eta_F^2 = h_F \int\limits_F \Big(\nu_i^F(a_{im}(\xi,\tau) + a_{ij}(\xi,\tau)\frac{\partial N_{m,\mu}(\xi,\tau)}{\partial \xi_j})\Big)^2 \mathrm{d}s,$$

where $\nu^F = (\nu_1^F, \cdots, \nu_d^F)$ is the outward unit normal to the element face $F \in \mathcal{F}_\mu$. Here, with a careful design of the mesh, $a_{im}(\cdot,\tau)$ is smooth on K for any fixed τ.

Under the assumptions $(\mathbf{A}_1)-(\mathbf{A}_2)$, for any fixed $\tau \in (0,\tau_0)$, it can be proved that $N_m(\cdot,\tau) \in H^1(Q)$, $m = 1,2,\cdots,d$ (see, e.g., [4, p. 13]). Furthermore, if the assumption (A_4) is imposed, it follows from Theorem 1.1 of [28] that $N_m(\cdot,\tau) \in W^{1,\infty}(Q)$ for any fixed $\tau \in (0,\tau_0)$, $m = 1,2,\cdots,d$. However, until now one cannot show that $N_m(\cdot,\tau) \in H^{1+\sigma}(Q)$, for some $\sigma > 0$. This is still an unsolved problem. In order to obtain the error estimates of $\|N_m(\cdot,\tau) - N_m^{h_0}(\cdot,\tau)\|_{H^1(Q)}$, where $N_m^{h_0}(\xi,\tau)$ is the finite element solution of $N_m(\xi,\tau)$ and h_0 is the final mesh parameter of the adaptive finite element method, following the idea of [9], we introduce the approximate technique of smooth coefficients for a_{ij}^ε. It is well known that the smooth functions are not dense in $L^\infty(Q)$. To this end, we choose a number $q > 2$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ (see Theorem 4.2 of [4, p. 38]). We can find a sequence of smooth functions $a_{ij}^{(\beta)} \in C^\infty(Q)$ such that

$$\|a_{ij} - a_{ij}^{(\beta)}\|_{L^q(Q)} \to 0, \quad \text{as} \quad \beta \to \infty. \tag{37}$$

Similarly to (7), let $N_m^{(\beta)}(\xi,\tau)$, $m = 1,2,\cdots,d$, be the solutions of the following equations:

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\big(a_{ij}^{(\beta)}(\xi,\tau)\dfrac{\partial N_m^{(\beta)}(\xi,\tau)}{\partial \xi_j}\big) = -\dfrac{\partial a_{im}^{(\beta)}(\xi,\tau)}{\partial \xi_i}, \quad (\xi,\tau) \in Q \times (0,\tau_0), \\ N_m^{(\beta)}(\xi,\tau) \ \text{ is 1-periodic in } \ \xi \ \text{ and } \ \tau_0-\text{periodic in } \ \tau, \\ \displaystyle\int_Q N_m^{(\beta)}(\xi,\tau)\mathrm{d}\xi = 0, \end{cases} \tag{38}$$

where $\tau \in (0,\tau_0) \subset (0,+\infty)$ plays the role of a parameter.

Subtracting (3) from (38), for $v(\xi,\tau) = N_m(\xi,\tau) - N_m^{(\beta)}(\xi,\tau)$, we get

$$\begin{aligned} \|N_m - N_m^{(\beta)}\|_{H^1(Q)}^2 &\le C\int_Q a_{ij}(\xi,\tau)\frac{\partial(N_m(\xi,\tau) - N_m^{(\beta)}(\xi,\tau))}{\partial \xi_j}\frac{\partial v(\xi,\tau)}{\partial \xi_i}\mathrm{d}\xi \\ &\le C\big\{\|a_{ij} - a_{ij}^{(\beta)}\|_{L^q(Q)}\|N_m^{(\beta)}\|_{W^{1,r}(Q)}\|v\|_{H^1(Q)} + \|a_{ij} - a_{ij}^{(\beta)}\|_{L^2(Q)}\|v\|_{H^1(Q)}\big\}, \end{aligned} \tag{39}$$

where C is a constant independent of β, ε.

Since $a_{ij}^{(\beta)} \in L^\infty(0,\tau_0; C^\infty(Q))$, we have $N_m^{(\beta)} \in L^\infty(0,\tau_0; W^{1,\infty}(Q))$. From (38), for any fixed $\tau \in (0,\tau_0)$, we get

$$\|N_m - N_m^{(\beta)}\|_{H^1(Q)} \le C\|a_{ij} - a_{ij}^{(\beta)}\|_{L^q(Q)} \to 0 \quad \text{as} \quad \beta \to +\infty. \tag{40}$$

For any fixed $\tau \in (0,\tau_0)$, Eq.(38) is an elliptic equation with smooth coefficients in a bounded convex domain Q, we deduce that $N_m^{(\beta)} \in H^2(Q)$. Let $N_{m,h_0}^{(\beta)}$ be the adaptive finite element solution of $N_m^{(\beta)}$ in $\mathcal{W}_\mu$, then we have (see, e.g., [16])

$$\|N_m^{(\beta)} - N_{m,h_0}^{(\beta)}\|_{H^1(Q)} \le Ch_0\|N_m^{(\beta)}\|_{H^2(Q)}, \quad \text{for any fixed } \ \tau \in (0,\tau_0). \tag{41}$$

Let $N_{m,h_0}(\xi,\tau)$ be the adaptive finite element solution of $N_m(\xi,\tau)$ in $\mathcal{W}_\mu$ for any fixed $\tau\in(0,\tau_0)$. Similarly to (40), we have

$$\|N_{m,h_0}-N^{(\beta)}_{m,h_0}\|_{H^1(Q)}\le C\|a_{ij}-a^{(\beta)}_{ij}\|_{L^q(Q)}\to 0 \quad \text{as} \quad \beta\to+\infty. \tag{42}$$

From (40), (41) and (42), using the triangle inequality and choosing a sufficiently large $\beta>0$, we get

$$\begin{aligned}\|N_m-N_{m,h_0}\|_{H^1(Q)} &\le \|N^{(\beta)}_m-N^{(\beta)}_{m,h_0}\|_{H^1(Q)}+\|N_m-N^{(\beta)}_m\|_{H^1(Q)}\\ &+ \|N_{m,h_0}-N^{(\beta)}_{m,h_0}\|_{H^1(Q)}\le Ch_0\|N^{(\beta)}_m\|_{H^2(Q)},\end{aligned}\tag{43}$$

where C is a constant independent of h_0, ε. Therefore, we obtain the error estimates of the adaptive finite element method for solving the cell problems.

Theorem 4.1. Let $N_m(\xi,\tau)$, $m=1,2,\cdots,d$, be the weak solutions of the cell problems (7), and let $N_{m,h_0}(\xi,\tau)$ be the associated adaptive finite element solutions. Under the assumptions $(\mathbf{A}_1)-(\mathbf{A}_2)$, it holds

$$\|N_m-N_{m,h_0}\|_{H^1(Q)}\le Ch_0,\quad m=1,2,\cdots,d,$$

where C is a constant independent of h_0, ε; h_0 is the final mesh parameter of the adaptive finite element method.

Remark 4.1. The error estimates of the adaptive finite element method for computing other cell functions can also be obtained similarly.

4.2. Finite volume element method (FVEM) for the homogenized convection -diffusion equation. In this section, we employ FVEM to solve the homogenized convection-diffusion equation (11) over a whole domain Ω using a coarse mesh and at a larger time step. The associated error estimates for FVEM are derived. In real simulations, we need to solve the modified homogenized convection-diffusion equation, namely we replace $\hat{a}_{ij}$,$\hat{b}_i$ in (11) with $\hat{a}^{h_0}_{ij}$, $\hat{b}^{h_0}_i$, respectively. For example, in the case of $k=1$, we have

$$\begin{aligned}\hat{a}^{h_0}_{ij}&=\frac{1}{\tau_0}\int_0^{\tau_0}\int_Q\big(a_{ij}(\xi,\tau)+a_{il}(\xi,\tau)\frac{\partial N^{h_0}_j(\xi,\tau)}{\partial\xi_l}\big)\mathrm{d}\xi\mathrm{d}\tau,\\ \hat{b}^{h_0}_i&=\frac{1}{\tau_0}\int_0^{\tau_0}\int_Q\big(b_i(\xi,\tau)-a_{ij}(\xi,\tau)\frac{\partial R^{h_0}_0(\xi,\tau)}{\partial\xi_j}\big)\mathrm{d}\xi\mathrm{d}\tau,\end{aligned}$$

where $N^{h_0}_j(\xi,\tau)$, $R^{h_0}_0(\xi,\tau)$ are the finite element solutions of $N_j(\xi,\tau)$, $R_0(\xi,\tau)$, respectively; h_0 is the final mesh parameter of the adaptive finite elements for computing $N_j(\xi,\tau)$ and $R_0(\xi,\tau)$.

Remark 4.2. If the mesh parameter $h_0>0$ is sufficiently small, then it holds

$$\begin{aligned}&\hat{a}^{h_0}_{ij}=\hat{a}^{h_0}_{ji},\ \ \forall i,j=1,2\cdots,d,\ \ \max_{i,j}|\hat{a}_{ij}-\hat{a}^{h_0}_{ij}|\le Ch_0^2,\\ &\max_i|\hat{b}_i-\hat{b}^{h_0}_i|\le Ch_0^2,\ \ \bar{\gamma}_0|\eta|^2\le\hat{a}^{h_0}_{ij}\eta_i\eta_j\le\bar{\gamma}_1|\eta|^2,\quad|\hat{b}^{h_0}_i|\le\bar{\beta}_1,\\ &\forall\eta=(\eta_1,\cdots,\eta_d)\in\mathbb{R}^d,\quad|\eta|^2=\eta_i\eta_i,\end{aligned}$$

where $C,\bar{\gamma}_0,\bar{\gamma}_1,\bar{\beta}_1$ are constants independent of h_0, ε. Furthermore, we can show that $\|u^0-u^{0,h_0}\|_{L^2(0,T;H^1(\Omega))}\le Ch_0^2$, where $u^0(x,t)$ and $u^{0,h_0}(x,t)$ are respectively the solutions of the homogenized convection-diffusion equation (11) and the associated modified equation, C is a constant independent of h_0, ε. Here, we omit the detailed proof and we refer the interested reader to [8, 9, 10].

For the sake of convenience, we define $u(x,t)$ is the solution of the modified homogenized convection-diffusion equation as follows:

$$
\begin{cases}
u_t - \nabla \cdot (A\nabla u - bu) + ru = f, \quad (x,t) \in \Omega \times (0,T), \\
u = g_0, \quad (x,t) \in \Gamma_0 \times (0,T), \\
\sigma(u) \equiv \nu_i \big(\hat{a}_{ij}^{h_0} \dfrac{\partial u}{\partial x_j} - \hat{b}_i^{h_0} u\big) = g_1, \quad (x,t) \in \Gamma_1 \times (0,T), \\
u(x,0) = \bar{u}_0(x), \quad x \in \Omega,
\end{cases} \tag{44}
$$

where $A = (\hat{a}_{ij}^{h_0})$, $b = (\hat{b}_1^{h_0}, \cdots, \hat{b}_d^{h_0})^T$, $r = \langle a_0 \rangle$, $\nabla\cdot$ and ∇ are the divergence and the gradient operators, respectively.

For simplicity, we assume that $\Gamma_1 = \emptyset$, $g_0(x,t) \equiv 0$ without loss of generality. Let $\Omega \subset \mathbb{R}^3$ be a bounded convex polygonal domain. Denote by $\mathcal{T}_h = \{e\}$ a family of some quasi-uniform tetrahedrons of Ω, by $\mathcal{T}_h^* = \{e^*\}$ the dual partition shown as in Fig. 3(see also [13],[31]). We introduce two finite dimensional spaces as follows:

$$
\mathcal{V}_h = \{v_h \in C^0(\overline{\Omega}) : \quad v_h|_e \in P_1(e), \ \forall e \in \mathcal{T}_h, \ v_h|_{\partial\Omega} = 0\},
$$
$$
\mathcal{Q}_h^* = \{w_h \in L^2(\Omega) : \quad w_h|_{e^*} \in P_0(e^*), \ \forall e^* \in \mathcal{T}_h^*, \ w_h|_{\partial\Omega} = 0\}.
$$

The functions spaces $C^0(\overline{\Omega})$, P_1 and P_0 can be found in [16].

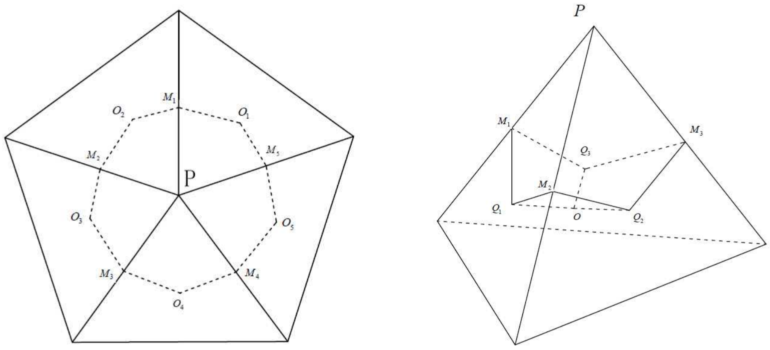

FIGURE 3. Dual partition on the triangular and tetrahedral mesh.

Define an interpolation operator $\mathcal{I}_h^* : \ H^2(\Omega) \to \mathcal{Q}_h^*$ by

$$
\mathcal{I}_h^* v = \sum_{x^{(i)} \in \mathcal{N}} v(x^{(i)}) \chi_i(x), \quad \forall v \in H^2(\Omega), \tag{45}
$$

where $\chi_i(x) = 1$, if $x \in e_i^*$; $\chi_i(x) = 0$, if $x \notin e_i^*$, $\mathcal{N}$ is the set of all nodes in $\mathcal{T}_h$, and e_i^* is the control volume which contains node $x^{(i)}$.

For any $v \in H^2(\Omega) \cap H_0^1(\Omega)$, multiplying Eq. $(44)_1$ by $\mathcal{I}_h^* v$, integrating and then summing over $e^* \in \mathcal{T}_h^*$, we get

$$
\begin{aligned}
&\sum_{e^* \in \mathcal{T}_h^*} \int_{e^*} u_t \mathcal{I}_h^* v \mathrm{d}x - \sum_{e^* \in \mathcal{T}_h^*} \int_{\partial e^*} (A\nabla u - bu) \cdot \nu_e^* \mathcal{I}_h^* v \mathrm{d}s \\
&\qquad + \sum_{e^* \in \mathcal{T}_h^*} \int_{e^*} r u \mathcal{I}_h^* v \mathrm{d}x = \sum_{e^* \in \mathcal{T}_h^*} \int_{e^*} f \mathcal{I}_h^* v \mathrm{d}x,
\end{aligned} \tag{46}
$$

where ν_e^* is the outward unit normal to ∂e^*. Let $M \geq 1$ be an integer and $\Delta t = T/M$ be the time step. Assume that u_h^n is the approximation in $\mathcal{V}_h$ of $u_h(x,t)$ at

$t = t_n = n\Delta t$. The fully finite volume element scheme is given by

$$(\bar{\partial} u_h^n, I_h^* v_h) + a_h^n(u_h^n, v_h) = (f^n, I_h^* v_h), \quad \forall v_h \in \mathcal{V}_h, \ \ n \geq 1, \tag{47}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$, $\bar{\partial} u_h^n = (u_h^n - u_h^{n-1})/\Delta t$ is a backward Euler difference quotient (see, e.g., [12, 37]), and the bilinear form $a_h^n(\cdot,\cdot)$ is given by

$$a_h^n(u,v) = -\sum_{e^* \in \mathcal{T}_h^*} \int_{\partial e^*} (A^n \nabla u - b^n u) \cdot \nu_e^* \mathcal{I}_h^* v \mathrm{d}s + \int_\Omega r^n u \mathcal{I}_h^* v \mathrm{d}x, \tag{48}$$

where $A^n = A(x, t_n)$,$b^n = b(x, t_n)$ and $r^n = r(x, t_n)$. Eq.(47) is rewritten as

$$(u_h^n, \mathcal{I}_h^* v_h) + \Delta t a_h^n(u_h^n, v_h) = (\Delta t f^n + u_h^{n-1}, \mathcal{I}_h^* v_h), \quad \forall v_h \in \mathcal{V}_h.$$

For the interpolation operator $\mathcal{I}_h^*$, we have the following lemma:

Lemma 4.1. (see [3, 7, 13, 18]) For $\forall e \in \mathcal{T}_h$, $\forall F_e \in \partial e$, $\forall v_h \in \mathcal{V}_h$, it holds

$$\begin{aligned} &\int_e (v_h - \mathcal{I}_h^* v_h)\mathrm{d}x = 0, \quad \int_{\partial e} (v_h - \mathcal{I}_h^* v_h)\mathrm{d}s = 0, \quad \|\mathcal{I}_h^* v_h\|_{0,e} \leq C\|v_h\|_{0,e}, \\ &\|v_h - \mathcal{I}_h^* v_h\|_{0,e} \leq C h_e |v_h|_{1,e}, \quad \|v_h - \mathcal{I}_h^* v_h\|_{0,F_e} \leq C h_e^{1/2} |v_h|_{1,e}, \end{aligned} \tag{49}$$

where h_e is the diameter of e, C is a constant independent of h_e.

If we define another bilinear form as follows

$$a^n(u,v) = \int_\Omega (A^n \nabla u - b^n u) \cdot \nabla v \mathrm{d}x + \int_\Omega r^n u v \mathrm{d}x, \tag{50}$$

and make the assumption:

$$a^n(v,v) \geq \gamma' \|v\|_{1,\Omega}, \quad \forall v \in H^1(\Omega), \tag{51}$$

where $\gamma' > 0$ is a constant, then we obtain the following proposition:

Proposition 4.1. We can prove that

$$a_h^n(u,v) = a^n(u,v) + a_r^n(u,v), \quad \forall u, v \in H^2(\mathcal{T}_h) \cap H^1(\Omega), \tag{52}$$

where $H^2(\mathcal{T}_h) = \{v \in \mathrm{L}^2(\Omega) : v|_e \in H^2(e), \forall e \in \mathcal{T}_h\}$, and that

$$\begin{aligned} a_r^n(u,v) = &-\int_{\partial\Omega} (A^n \nabla u - b^n u) \cdot \nu \mathcal{I}_h^* v \mathrm{d}s - \sum_{e \in \mathcal{T}_h} \int_e \nabla \cdot (A^n \nabla u - b^n u)(\mathcal{I}_h^* v - v)\mathrm{d}x \\ &+ \sum_{e \in \mathcal{T}_h} \int_{\partial e} (A^n \nabla u - b^n u) \cdot \nu_e (\mathcal{I}_h^* v - v)\mathrm{d}s + \sum_{e \in \mathcal{T}_h} \int_e r^n u (\mathcal{I}_h^* v - v)\mathrm{d}x. \end{aligned}$$

where ν and ν_e denote the outward unit normals to Ω and e, respectively. Furthermore, if $A^n \in (W^{1,\infty}(\Omega))^{3\times 3}$, $b^n \in (W^{1,\infty}(\Omega))^3$, $r^n \in L^\infty(\Omega)$, then

$$|a_r^n(u_h, v_h)| \leq Ch|u_h|_{1,\Omega}|v_h|_{1,\Omega}, \quad \forall u_h, \ v_h \in \mathcal{V}_h. \tag{53}$$

If $A^n \in \mathcal{Q}_h^{3\times 3}$, $b^n \in (W^{2,\infty}(\Omega))^3$, $r^n \in W^{1,\infty}(\Omega)$, then

$$|a_r^n(u_h, v_h)| \leq Ch^2|u_h|_{1,\Omega}|v_h|_{1,\Omega}, \quad \forall u_h, \ v_h \in \mathcal{V}_h, \tag{54}$$

where $\mathcal{Q}_h = \{v_h \in L^2(\Omega) : v_h|_e \in P_0(e), \forall e \in \mathcal{T}_h\}$.

Proof. For any vertex P of a given $e \in \mathcal{T}_h$, denote by K_P the control volume containing the node P and by $K_P|_e = D_P$ the part of K_P restricted onto e(see Fig. 3). Then the tetrahedron e can be divided into four parts(denoted by D_i, $i = 1, 2, 3, 4$). The boundary of D_i denoted by ∂D_i consists of two parts: Γ_i

and Γ_i^*. $\Gamma_i = \partial D_i \cap \partial e$ and $\Gamma_i^* = \partial D_i \setminus \Gamma_i$. Let $G := A^n \nabla u - b^n u$. For any $u, v \in H^2(\mathcal{T}_h) \cap H^1(\Omega)$, we have

$$\sum_{e^* \in \mathcal{T}_h^*} \int_{\partial e^*} G \cdot \nu_e^* \mathcal{I}_h^* v \mathrm{d}s = \sum_{e \in \mathcal{T}_h} \sum_{i=1}^{4} \int_{\Gamma_i^*} G \cdot \nu_e^* \mathcal{I}_h^* v \mathrm{d}s + \int_{\partial\Omega} G \cdot \nu \mathcal{I}_h^* v \mathrm{d}s, \tag{55}$$

and

$$\begin{aligned} \sum_{e \in \mathcal{T}_h} \sum_{i=1}^{4} \int_{\Gamma_i^*} G \cdot \nu_e^* \mathcal{I}_h^* v \mathrm{d}s &= \sum_{e \in \mathcal{T}_h} \sum_{i=1}^{4} \Big\{ \int_{D_i} \nabla \cdot G \mathcal{I}_h^* v \mathrm{d}x - \int_{\Gamma_i} G \cdot \nu_e \mathcal{I}_h^* v \mathrm{d}s \Big\} \\ &= \sum_{e \in \mathcal{T}_h} \int_e \nabla \cdot G \mathcal{I}_h^* v \mathrm{d}x - \sum_{e \in \mathcal{T}_h} \int_{\partial e} G \cdot \nu_e \mathcal{I}_h^* v \mathrm{d}s = \sum_{e \in \mathcal{T}_h} \int_e \nabla \cdot G(\mathcal{I}_h^* v - v) \mathrm{d}x \\ &- \sum_{e \in \mathcal{T}_h} \int_{\partial e} G \cdot \nu_e (\mathcal{I}_h^* v - v) \mathrm{d}s + \sum_{e \in \mathcal{T}_h} \int_e \nabla \cdot G v \mathrm{d}x - \sum_{e \in \mathcal{T}_h} \int_{\partial e} G \cdot \nu_e v \mathrm{d}s \\ &= \sum_{e \in \mathcal{T}_h} \int_e \nabla \cdot G(\mathcal{I}_h^* v - v) \mathrm{d}x - \sum_{e \in \mathcal{T}_h} \int_{\partial e} G \cdot \nu_e (\mathcal{I}_h^* v - v) \mathrm{d}s - \int_\Omega G \cdot \nabla v \mathrm{d}x. \end{aligned} \tag{56}$$

From (48), (50), (55) and (56), we get $a_h^n(u,v) = a_r^n(u,v) + a^n(u,v)$. For any $u_h, v_h \in \mathcal{V}_h$, it is obvious that

$$\begin{aligned} \Big| \sum_{e \in \mathcal{T}_h} \int_e \nabla \cdot (A^n \nabla u_h)(\mathcal{I}_h^* v_h - v_h) \mathrm{d}x \Big| &= \Big| \sum_{e \in \mathcal{T}_h} \int_e (\nabla \cdot A^n) \cdot \nabla u_h (\mathcal{I}_h^* v_h - v_h) \mathrm{d}x \Big| \\ \leq \sum_{e \in \mathcal{T}_h} C h_e \|A^n\|_{1,\infty,e} \|\nabla u_h\|_{0,e} |v_h|_{1,e} &\leq Ch \|A^n\|_{1,\infty} |u_h|_{1,\Omega} |v_h|_{1,\Omega}, \end{aligned} \tag{57}$$

$$\Big| \sum_{e \in \mathcal{T}_h} \int_e \nabla \cdot (b^n u_h)(\mathcal{I}_h^* v_h - v_h) \mathrm{d}x \Big| \leq Ch \|b^n\|_{1,\infty} |u_h|_{1,\Omega} |v_h|_{1,\Omega}. \tag{58}$$

For any given $e \in \mathcal{T}_h$, it follows from Lemma 4.1 that

$$\begin{aligned} \Big| \int_{\partial e} A^n \nabla u_h \cdot \nu_e (\mathcal{I}_h^* v_h - v_h) \mathrm{d}s \Big| &= \Big| \int_{\partial e} (A^n - A^n(P_e)) \nabla u_h \cdot \nu_e (\mathcal{I}_h^* v_h - v_h) \mathrm{d}s \Big| \\ \leq \int_{\partial e} C h_e |\nabla u_h| |\mathcal{I}_h^* v_h - v_h| \mathrm{d}s &\leq C h_e \|\nabla u_h\|_{0,\partial e} \|\mathcal{I}_h^* v_h - v_h\|_{0,\partial e} \\ \leq C h_e h_e^{-1/2} \|\nabla u_h\|_{0,e} h_e^{1/2} |v_h|_{1,e} &\leq C h_e |u_h|_{1,e} |v_h|_{1,e}, \end{aligned}$$

where P_e denotes the barycenter of ∂e, and consequently

$$\Big| \sum_{e \in \mathcal{T}_h} \int_{\partial e} A^n \nabla u_h \cdot \nu_e (\mathcal{I}_h^* v_h - v_h) \mathrm{d}s \Big| \leq Ch |u_h|_{1,\Omega} |v_h|_{1,\Omega}. \tag{59}$$

Define $Lip(\Omega) := \Big\{ v \in L^\infty(\Omega) : \ \sup \Big\{ \frac{|v(x) - v(y)|}{|x-y|} : x, y \in \Omega; x \neq y \Big\} < \infty \Big\}$. Since Ω is a bounded convex domain, one can deduce that $W^{1,\infty}(\Omega) \subset Lip(\Omega)$, see [6, p. 30]. We thus have

$$\sum_{e \in \mathcal{T}_h} \int_{\partial e} b^n u_h \cdot \nu_e (\mathcal{I}_h^* v_h - v_h) \mathrm{d}s = \sum_{F_e \in \mathcal{E}_h^0} \int_{F_e} [\![b^n u_h (\mathcal{I}_h^* v_h - v_h)]\!]_{F_e} \mathrm{d}s = 0, \tag{60}$$

where $\mathcal{E}_h^0 = \{F_e : \ F_e \in \partial e, F_e \notin \partial\Omega, \forall e \in \mathcal{T}_h\}$, $[\![w]\!]_{F_e} = w|_{\partial e_1} \cdot \nu_e^{(1)} + v|_{\partial e_2} \cdot \nu_e^{(2)}$, $F_e = e_1 \cap e_2$, $\nu_e^{(i)}$ is the unit normal vector on F_e outward to ∂e_i, $i = 1, 2$. For the last term in $a_r^n(u_h, v_h)$, we have

$$\Big|\sum_{e\in\mathcal{T}_h}\int_e r^n u_h(\mathcal{I}_h^* v_h - v_h)\mathrm{d}x\Big| \leqslant C\sum_{e\in\mathcal{T}_h}\|r^n\|_{0,\infty,e}h_e\|u_h\|_{0,e}|v_h|_{1,e} \\ \leqslant Ch\|r^n\|_{0,\infty}|u_h|_{1,\Omega}|v_h|_{1,\Omega}. \tag{61}$$

Estimate (53) follows from (57)-(61).

If $A^n \in \mathcal{Q}_h^{3\times 3}$ and $b^n \in (W^{2,\infty}(\Omega))^3$, applying Lemma 4.1, we get

$$a_r^n(u_h, v_h) = \sum_{e\in\mathcal{T}_h}\int_e(\nabla\cdot(b^n u_h) + r^n u_h)(\mathcal{I}_h^* v_h - v_h)\mathrm{d}x, \quad \forall u_h,\, v_h \in \mathcal{V}_h. \tag{62}$$

For any $\varphi \in H^1(\Omega)$, $v_h \in \mathcal{V}_h$, it follows from Lemma 4.1 that

$$\begin{aligned}\Big|\sum_{e\in\mathcal{T}_h}\int_e \varphi(\mathcal{I}_h^* v_h - v_h)\mathrm{d}x\Big| &= \Big|\sum_{e\in\mathcal{T}_h}\int_e(\varphi - \mathcal{M}_e(\varphi))(\mathcal{I}_h^* v_h - v_h)\mathrm{d}x\Big| \\ &\le \sum_{e\in\mathcal{T}_h}\|\varphi - \mathcal{M}_e(\varphi)\|_{0,e}\|\mathcal{I}_h^* v_h - v_h\|_{0,e} \\ &\le Ch^2\sum_{e\in\mathcal{T}_h}|\varphi|_{1,e}|v_h|_{1,e} \le Ch^2|\varphi|_{1,\Omega}|v_h|_{1,\Omega},\end{aligned} \tag{63}$$

where $\mathcal{M}_e(\varphi) = \dfrac{1}{|e|}\displaystyle\int_e \varphi(x)\mathrm{d}x$. (54) follows from (62) and (63). Therefore, the proof of Proposition 4.1 is complete. □

We define a Ritz projection $R_h : H_0^1(\Omega) \to \mathcal{V}_h$ as follows:

$$a^n(R_h\varphi, v_h) = a^n(\varphi, v_h), \quad \forall v_h \in \mathcal{V}_h, \;\; \varphi \in H_0^1(\Omega).$$

One can directly check that (see also[12, p. 80])

$$\begin{aligned}\|\varphi - R_h\varphi\|_{0,\Omega} + h\|\varphi - R_h\varphi\|_{1,\Omega} &\le Ch^2|\varphi|_{2,\Omega}, \quad \forall\varphi \in H^2(\Omega)\cap H_0^1(\Omega), \\ \|\varphi - R_h\varphi\|_{0,\Omega} &\le Ch\|\varphi\|_{1,\Omega}, \quad \varphi \in H_0^1(\Omega).\end{aligned} \tag{64}$$

For $t = t_n$, we split the error into two parts: $u(\cdot, t_n) - u_h^n = \rho^n + \theta_h^n$, where $\rho^n = u(\cdot, t_n) - R_h u(\cdot, t_n)$, $\theta_h^n = R_h u(\cdot, t_n) - u_h^n$. It follows from (64) that

$$\|\rho^n\|_{0,\Omega} + h\|\rho^n\|_{1,\Omega} \le Ch^2|u(\cdot, t_n)|_{2,\Omega}. \tag{65}$$

Observe that

$$(\bar\partial\theta_h^n, v_h) + a^n(\theta_h^n, v_h) = F_h^n(v_h), \quad \forall v_h \in \mathcal{V}_h, \tag{66}$$

where

$$F_h^n(v_h) = (\bar\partial R_h u(\cdot, t_n) - u_t(\cdot, t_n), v_h) + X_h^n(v_h; u_h^n, f^n),$$
$$X_h^n(v_h; u_h^n, f^n) = (\bar\partial u_h^n, \mathcal{I}_h^* v_h - v_h) + a_r^n(u_h^n, v_h) - (f^n, \mathcal{I}_h^* v_h - v_h).$$

We first note that

$$\bar\partial R_h u(\cdot, t_n) - u_t(\cdot, t_n) = (R_h - I)\bar\partial u(\cdot, t_n) + (\bar\partial u(\cdot, t_n) - u_t(\cdot, t_n)) = \omega_1^n + \omega_2^n.$$

Following (7.40) of [12, p. 84], we obtain

$$\begin{aligned}\Delta t\|\omega_1^n\|_{0,\Omega} &\le Ch^2\int_{t_{n-1}}^{t_n}|u_t(\cdot, s)|_{2,\Omega}\mathrm{d}s, \\ \Delta t\|\omega_1^n\|_{0,\Omega} &\le Ch\int_{t_{n-1}}^{t_n}\|u_t(\cdot, s)\|_{1,\Omega}\mathrm{d}s, \\ \|\omega_2^n\|_{0,\Omega} &\le C\int_{t_{n-1}}^{t_n}\|u_{tt}(\cdot, s)\|_{0,\Omega}\mathrm{d}s,\end{aligned} \tag{67}$$

where C is a constant independent of h, Δt, ε; h and Δt are the mesh parameter and the time step for solving the modified homogenized convection-diffusion equation (44), respectively.

In order to estimate the term $X_h^n(v_h; u_h^n, f^n)$, we introduce the following hypotheses

$$
\begin{aligned}
(\mathbf{H}_1): \quad & A \text{ consists of elements } a_{ij} \in L^\infty(0,T;W^{1,\infty}(\Omega)),\\
& b \text{ has entries } b_i \in L^\infty(0,T;W^{1,\infty}(\Omega)),\ 0 \le r \in L^\infty(0,T;L^\infty(\Omega));\\
(\mathbf{H}_2): \quad & A \text{ consists of elements } a_{ij} \in W^{1,\infty}(0,T;Q_h),\\
& b \text{ has entries } b_i \in W^{1,\infty}(0,T;W^{2,\infty}(\Omega)),\ 0 \le r \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega)),
\end{aligned}
$$

and give the following proposition:

Proposition 4.2. For a given $u_h^n \in \mathcal{V}_h$ and $f^n = f(\cdot, t_n)$, $\forall v_h \in \mathcal{V}_h$, we can prove that
under the hypotheses $(\mathbf{H}_1)$:

$$
|X_h^n(v_h; u_h^n, f^n)| \le ChJ_0^n|v_h|_{1,\Omega}; \tag{68}
$$

under the hypotheses $(\mathbf{H}_2)$:

$$
|X_h^n(v_h; u_h^n, f^n)| \le \begin{cases} Ch^2J_1^n|v_h|_{1,\Omega}, \\ ChJ_1^n\|v_h\|_{0,\Omega}, \end{cases} \tag{69}
$$

where $J_0^n = |u_h^n|_{1,\Omega} + \|\bar{\partial}u_h^n\|_{0,\Omega} + \|f^n\|_{0,\Omega}$ and $J_1^n = |u_h^n|_{1,\Omega} + |\bar{\partial}u_h^n|_{1,\Omega} + |f^n|_{1,\Omega}$.

Proof. Applying Lemma 4.1, for $\varphi \in H^1(\Omega)$, we get

$$
\begin{aligned}
|(\varphi, \mathcal{I}_h^* v_h - v_h)| &= \Big| \sum_{e\in\mathcal{T}_h} \int_e \varphi(I_h^* v_h - v_h)\mathrm{d}x \Big| \\
&= \Big| \sum_{e\in\mathcal{T}_h} \int_e (\varphi - \mathcal{M}_e(\varphi))(\mathcal{I}_h^* v_h - v_h)\mathrm{d}x \Big| \\
&\le \sum_{e\in\mathcal{T}_h} \|\varphi - \mathcal{M}_e(\varphi)\|_{0,e}\|\mathcal{I}_h^* v_h - v_h\|_{0,e} \\
&\le \sum_{e\in\mathcal{T}_h} Ch_e|\varphi|_{1,e}h_e|v_h|_{1,e} \le Ch^2|\varphi|_{1,\Omega}|v_h|_{1,\Omega},
\end{aligned} \tag{70}
$$

where $\mathcal{M}_e(\varphi) = \dfrac{1}{|e|}\displaystyle\int_e \varphi(x)\mathrm{d}x$. On the other hand, we have

$$
\begin{aligned}
|(\varphi, \mathcal{I}_h^* v_h - v_h)| &= \Big| \sum_{e\in\mathcal{T}_h} \int_e \varphi(\mathcal{I}_h^* v_h - v_h)\mathrm{d}x \Big| \\
&\le \Big| \sum_{e\in\mathcal{T}_h} \int_e (\varphi - \mathcal{M}_e(\varphi))(\mathcal{I}_h^* v_h - v_h)\mathrm{d}x \Big| \\
&\le \sum_{e\in\mathcal{T}_h} \|\varphi - \mathcal{M}_e(\varphi)\|_{0,e}\Big(\|\mathcal{I}_h^* v_h\|_{0,e} + \|v_h\|_{0,e}\Big) \\
&\le C\sum_{e\in\mathcal{T}_h} h_e|\varphi|_{1,e}\|v_h\|_{0,e} \le Ch|\varphi|_{1,\Omega}\|v_h\|_{0,\Omega}.
\end{aligned} \tag{71}
$$

Similarly, we can show that

$$
|(\varphi, \mathcal{I}_h^* v_h - v_h)| \le Ch\|\varphi\|_{0,\Omega}|v_h|_{1,\Omega}. \tag{72}
$$

From (70)-(72), applying Proposition 4.1, we can complete the proofs of (68) and (69). □

Remark 4.3. If $h^2 \le \min(\frac{\gamma'}{4M_r C_1^* C^T}, \frac{\Delta t \gamma'}{2(C_1^*)^2})$ and $\Delta t \le \frac{1}{2}$ one can show that under the hypotheses $(\mathbf{H}_1)$

$$\Delta t \sum_{i=1}^{n} |J_0^i|^2 \le C\Big\{ \|u_h^0\|_{1,\Omega}^2 + \Delta t \sum_{i=1}^{n} \|f^i\|_{0,\Omega}^2 \Big\}$$

under the hypotheses $(\mathbf{H}_2)$

$$\Delta t \sum_{i=1}^{n} |J_1^i|^2 \le C\Big\{ \|u_h^0\|_{1,\Omega}^2 + \Delta t \sum_{i=1}^{n} \|f^i\|_{1,\Omega}^2 + \Delta t \sum_{i=1}^{n} \|\bar{\partial} f^i\|_{0,\Omega}^2 \Big\}.$$

Here, $M_r = \|r\|_{L^\infty(0,T;L^\infty(\Omega))}$, C_1^* and C^T are constants satisfying

$$\|v_h - \mathcal{I}_h^* v_h\|_{0,e} \le C_1^* h |v_h|_{1,e}, \quad \forall e \in \mathcal{T}_h, \forall v_h \in \mathcal{V}_h$$
$$\|v_h - \frac{1}{|e|}\int_e v_h(x)dx\|_{0,e} \le C^T h |v_h|_{1,e}, \quad \forall e \in \mathcal{T}_h, \forall v_h \in \mathcal{V}_h.$$

Next we give the error estimates of the finite volume element method for solving the modified homogenized convection-diffusion equation (44) in a whole domain Ω.

Theorem 4.2. Suppose $\Omega \subset \mathbb{R}^d$, $d \ge 1$ is a bounded convex Lipchitz domain. Let $u(x,t)$ be the unique weak solution of the modified homogenized convection-diffusion equation (44), and let $u_h^n = u_h(x,t_n)$ be the associated fully discrete finite volume element solution at time $t = t_n$ of $u(x,t)$ in a coarse mesh and at a larger time step. Let h be the mesh size and $\Delta t (< \frac{1}{2})$ be the time step. Set $E_n = u(x,t_n) - u_h^n$. The assumptions $(\mathbf{A}_1) - (\mathbf{A}_3)$ and the initial conditions

$$\|\bar{u}_h^0 - \bar{u}_0\|_{0,\Omega} + h\|\bar{u}_h^0 - \bar{u}_0\|_{1,\Omega} \le C h^2 |\bar{u}_0|_{2,\Omega} \tag{73}$$

are satisfied, where $\bar{u}_h^0(x)$ is the interpolation function of the initial function $\bar{u}^0(x)$ on $\mathcal{V}_h$. Under the hypothesis $(\mathbf{H}_1)$, it holds

(74)

$$\|E_n\|_{0,\Omega} + \Big(\sum_{i=1}^{n} \Delta t |E_i|_{1,\Omega}^2\Big)^{\frac{1}{2}} \le Ch\Big\{ h|u(\cdot,t_n)|_{2,\Omega} + h|\bar{u}_0|_{2,\Omega} + \Big(\sum_{i=1}^{n} \Delta t |u(\cdot,t_i)|_{2,\Omega}^2\Big)^{\frac{1}{2}}$$
$$+ h\Big(\int_0^{t_n} |u_t(\cdot,s)|_{2,\Omega}^2 \mathrm{d}s\Big)^{\frac{1}{2}} + \Big(\sum_{i=1}^{n} \Delta t |J_0^i|^2\Big)^{\frac{1}{2}}\Big\} + C\Delta t\Big(\int_0^{t_n} |u_{tt}(\cdot,s)|_{0,\Omega}^2 \mathrm{d}s\Big)^{\frac{1}{2}}.$$

Under the hypothesis $(\mathbf{H}_2)$, it holds

$$\begin{aligned} \|E_n\|_{0,\Omega} \le Ch^2\Big\{ &|u(\cdot,t_n)|_{2,\Omega} + |\bar{u}_0|_{2,\Omega} + \Big(\int_0^{t_n} |u_t(\cdot,s)|_{2,\Omega}^2 \mathrm{d}s\Big)^{\frac{1}{2}} \\ &+ \Big(\sum_{i=1}^{n} \Delta t |J_1^i|^2\Big)^{\frac{1}{2}}\Big\} + C\Delta t\Big(\int_0^{t_n} \|u_{tt}(\cdot,s)\|_{0,\Omega}^2 \mathrm{d}s\Big)^{\frac{1}{2}}, \end{aligned} \tag{75}$$

$$\begin{aligned} \|E_n\|_{1,\Omega} \le Ch\Big\{ &|u(\cdot,t_n)|_{2,\Omega} + |\bar{u}_0|_{2,\Omega} + h\Big(\int_0^{t_n} |u_t(\cdot,s)|_{2,\Omega}^2 \mathrm{d}s\Big)^{\frac{1}{2}} \\ &+ \Big(\sum_{i=1}^{n} \Delta t |J_1^i|^2\Big)^{\frac{1}{2}}\Big\} + C\Delta t\Big(\int_0^{t_n} \|u_{tt}(\cdot,s)\|_{0,\Omega}^2 \mathrm{d}s\Big)^{\frac{1}{2}}. \end{aligned} \tag{76}$$

where C is a constant independent of h and Δt, but dependent of T.

Proof. By setting $\theta_h^0 = R_h\bar{u}_0 - \bar{u}_0 + \bar{u}_0 - \bar{u}_h^0$, and using (73), we have

$$\|\theta_h^0\|_{0,\Omega} + h\|\theta_h^0\|_{1,\Omega} \le Ch^2|u_0|_{2,\Omega}.$$

Next we only prove the estimates (75) and (76). The proof of (74) is similar. Setting $v_h = \theta_h^n$ in (66) and using (51), we thus get

$$(\bar{\partial}\theta_h^n, \theta_h^n) + \gamma'|\theta_h^n|_{1,\Omega}^2 \leqslant \|\omega_1^n + \omega_2^n\|_{0,\Omega}\|\theta_h^n\|_{0,\Omega} + |X_h^n(\theta_h^n; u_h^n, f^n)|.$$

It is obvious that

$$\Delta t(\bar{\partial}\theta_h^n, \theta_h^n) = \frac{1}{2}\|\theta_h^n\|_{0,\Omega}^2 + \frac{1}{2}\|\theta_h^n - \theta_h^{n-1}\|_{0,\Omega}^2 - \frac{1}{2}\|\theta_h^{n-1}\|_{0,\Omega}^2.$$

It follows from Proposition 4.2 that

$$|X_h^n(\theta_h^n; u_h^n, f^n)| \leqslant \frac{Ch^4}{\gamma'}|J_1^n|^2 + \gamma'|\theta_h^n|_{1,\Omega}^2.$$

Hence we have

$$\begin{aligned}\|\theta_h^n\|_{0,\Omega}^2 \leqslant \|\theta_h^{n-1}\|_{0,\Omega}^2 &+ Ch^4\int_{t_{n-1}}^{t_n}|u_t(\cdot,s)|_{2,\Omega}^2\mathrm{d}s + C(\Delta t)^2\int_{t_{n-1}}^{t_n}\|u_{tt}(\cdot,s)\|_{0,\Omega}^2\mathrm{d}s\\ &+ \frac{Ch^4}{\gamma'}\Delta t|J_1^n|^2 + \Delta t\|\theta_h^n\|_{0,\Omega}^2,\end{aligned}$$

and consequently

$$\begin{aligned}\|\theta_h^n\|_{0,\Omega}^2 \leqslant \|\theta_h^0\|_{0,\Omega}^2 &+ Ch^4\int_0^{t_n}|u_t(\cdot,s)|_{2,\Omega}^2\mathrm{d}s + C(\Delta t)^2\int_0^{t_n}\|u_{tt}(\cdot,s)\|_{0,\Omega}^2\mathrm{d}s\\ &+ \frac{Ch^4}{\gamma'}\sum_{i=1}^n\tau|J_1^i|^2 + \Delta t\sum_{i=1}^n\|\theta_h^i\|_{0,\Omega}^2.\end{aligned}$$

Since $\Delta t < \frac{1}{2}$, using the discrete Gronwall's inequality, we get

$$\begin{aligned}\|\theta_h^n\|_{0,\Omega}^2 \le C\Big\{&h^4|\bar{u}_0|_{2,\Omega}^2 + h^4\int_0^{t_n}|u_t(\cdot,s)|_{2,\Omega}^2\mathrm{d}s + h^4\sum_{i=1}^n\Delta t|J_1^i|^2\\ &+ (\Delta t)^2\int_0^{t_n}\|u_{tt}(\cdot,s)\|_{0,\Omega}^2\mathrm{d}s\Big\}.\end{aligned}\tag{77}$$

where C is a constant independent of h and Δt. Therefore, (75) follows from (65), (73) and (77).

It remains to prove (76). Taking $v_h = \bar{\partial}\theta_h^n$ in (66) and using Proposition 4.2, we obtain

$$\|\bar{\partial}\theta_h^n\|_{0,\Omega}^2 + a^n(\theta_h^n, \bar{\partial}\theta_h^n) \le \|\omega_1^n + \omega_2^n\|_{0,\Omega}\|\bar{\partial}\theta_h^n\|_{0,\Omega} + ChJ_1^n\|\bar{\partial}\theta_h^n\|_{0,\Omega}.\tag{78}$$

Define the following two bilinear forms

$$a_1^n(u,v) = \int_\Omega A^n\nabla u\nabla v\,dx + \int_\Omega r^n uv\mathrm{d}x, \quad a_2^n(u,v) = -\int_\Omega b^n u\nabla v\mathrm{d}x.$$

Hence we get

$$\begin{aligned}a^n(\theta_h^n, \bar{\partial}\theta_h^n) &= a_1^n(\theta_h^n, \bar{\partial}\theta_h^n) + a_2^n(\theta_h^n, \bar{\partial}\theta_h^n)\\ &= \frac{1}{\Delta t}a_1^n(\theta_h^n, \theta_h^n) - \frac{1}{\Delta t}a_1^n(\theta_h^n, \theta_h^{n-1}) + a_2^n(\theta_h^n, \bar{\partial}\theta_h^n).\end{aligned}\tag{79}$$

For the last two terms in (79), we have

$$\begin{aligned}&a_1^n(\theta_h^n, \theta_h^{n-1}) \le \frac{1}{2}a_1^n(\theta_h^n, \theta_h^n) + \frac{1}{2}a_1^n(\theta_h^{n-1}, \theta_h^{n-1}),\\ &|a_2^n(\theta_h^n, \bar{\partial}\theta_h^n)| \le C\|b^n\|_{1,\infty,\Omega}\|\bar{\partial}\theta_h^n\|_{0,\Omega}|\theta_h^n|_{1,\Omega}.\end{aligned}\tag{80}$$

Combining (78), (79) and (80) gives

$$
\begin{aligned}
a_1^n(\theta_h^n,\theta_h^n) \le{}& a_1^n(\theta_h^{n-1},\theta_h^{n-1}) + C\Delta t|\theta_h^n|_{1,\Omega}^2 + Ch^2\Delta t|J_1^n|^2\\
&+ Ch^4\int_{t_{n-1}}^{t_n}|u_t(\cdot,s)|_{2,\Omega}^2\mathrm{d}s + C(\Delta t)^2\int_{t_{n-1}}^{t_n}\|u_{tt}(\cdot,s)\|_{0,\Omega}^2\mathrm{d}s,
\end{aligned}
$$

Thanks to the assumption $(\mathbf{H}_2)$, we get

$$
|a_1^n(\theta_h^{n-1},\theta_h^{n-1}) - a_1^{n-1}(\theta_h^{n-1},\theta_h^{n-1})| \leqslant C\Delta t|\theta_h^{n-1}|_{1,\Omega}^2,
$$

and consequently

$$
\begin{aligned}
a_1^n(\theta_h^n,\theta_h^n) \le{}& a_1^{n-1}(\theta_h^{n-1},\theta_h^{n-1}) + C\Delta t(|\theta_h^{n-1}|_{1,\Omega}^2 + |\theta_h^n|_{1,\Omega}^2) + Ch^2\Delta t|J_1^n|^2\\
&+ Ch^4\int_{t_{n-1}}^{t_n}|u_t(\cdot,s)|_{2,\Omega}^2\mathrm{d}s + C(\Delta t)^2\int_{t_{n-1}}^{t_n}\|u_{tt}(\cdot,s)\|_{0,\Omega}^2\mathrm{d}s.
\end{aligned}
$$

Applying the discrete Gronwall's inequality again, we obtain

$$
\begin{aligned}
|\theta_h^n|_{1,\Omega}^2 \le{}& Ch^2\Big\{|\theta_h^0|_{1,\Omega}^2 + \sum_{i=1}^n\Delta t|J_1^i|^2 + h^2\int_0^{t_n}|u_t(\cdot,s)|_{2,\Omega}^2\mathrm{d}s\Big\}\\
&+ C(\Delta t)^2\int_0^{t_n}\|u_{tt}(\cdot,s)\|_{0,\Omega}^2\mathrm{d}s,
\end{aligned}
\tag{81}
$$

where C is a constant independent of h, Δt. Therefore (76) follows from (65), (73) and (81). □

Remark 4.4. As we already mentioned in Section 3, in a bounded convex domain, we need to define the boundary layer solutions. We employ the finite volume element method (FVEM) to solve the boundary layer equations (12). Due to space limitation, we omit the detailed process.

5. Multiscale hybrid FEM-FVM algorithm and numerical examples

5.1. Multiscale hybrid FEM-FVM algorithm. To summarize the above results, the multiscale hybrid FEM-FVM method for the convection-diffusion equation with rapidly oscillating coefficients can now be described as follows:

Step I: The adaptive finite element method (AFEM) is first applied to compute the cell functions $N_m(\xi,\tau)$, $N_{ml}(\xi,\tau)$, $M_l(\xi,\tau)$, $R_0(\xi,\tau)$ and $R_1(\xi,\tau)$ on $Q\times(0,T)$ (or $Q\times(0,\tau_0)$) defined in (3)-(10), where $m,l=1,2,\cdots,d$.

Step II: The finite volume element method (FVEM) is then used to solve the modified homogenized convection-diffusion equation (44) in a whole domain $\Omega\times(0,T)$ using a coarse mesh and a larger time step.

Step III: The finite volume element method (FVEM)is applied to solve the boundary layer equations over a subdomain $\Omega_1\times(0,T)$ using a fine mesh and a small time step, see (12).

Step IV: The derivatives $\dfrac{\partial u(x,t)}{\partial x_m}$ and $\dfrac{\partial^2 u(x,t)}{\partial x_m\partial x_l}$, $m,l=1,2,\cdots,d$ are computed by the finite difference method, where $u(x,t)$ is the unique solution of the modified homogenized convection-diffusion equation (44). The computational formulas can be found in [8, 9, 10].

Remark 5.1. From Theorems 3.2, 4.1 and 4.2, we can derive the convergence results of the multiscale hybrid FEM-FVM method for the original problem (1).

5.2. Numerical examples. To verify the numerical accuracy for the finite volume element method (FVEM) introduced in this paper, we first consider Example 5.1.

Example 5.1. Consider the convection-diffusion equation (44) with Dirichlet boundary condition with constant coefficients, in which

$$A = 0.1I, \quad b = (1,3,1)^T, \quad r = 1, \tag{82}$$

where I is an identity matrix, $\Omega = (0,1)^3$, $T = 1$, and the exact solution is given by

$$u(x,t) = e^{(1-0.5t)}\left(x - e^{\frac{x-1}{0.1}}\right)\left(y^2 - e^{\frac{3(y-1)}{0.1}}\right)\left(z - e^{\frac{z-1}{0.1}}\right). \tag{83}$$

The initial and boundary conditions for problem (82) and the term of right side $f(x,t)$ are determined by (83). As we already mentioned in (47), the finite volume element method in space and the backward Euler difference scheme in time are applied. We take the time step $\Delta t = 0.01$. Relative errors at $t = 1.0$ in norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ are displayed in Fig. 4, which support the error estimates obtained in Theorem 4.2. Here, h denotes the mesh size.

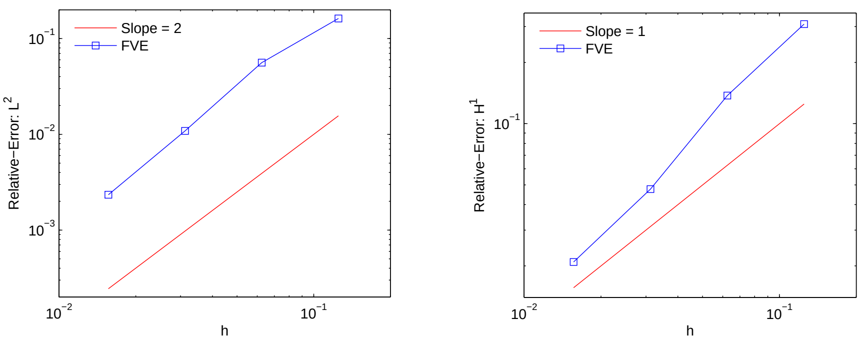

FIGURE 4. Relative errors at $t = 1.0$; Left: in $\|\cdot\|_{L^2(\Omega)}$; Right: in $\|\cdot\|_{H^1(\Omega)}$.

To validate the multiscale hybrid FEM-FVM algorithm and to confirm the theoretical analysis reported in this paper, numerical examples in 2-D and 3-D composite materials are presented.

Example 5.2. Consider the 2-D convection-diffusion equation with rapidly oscillating periodic discontinuous coefficients as follows:

$$\begin{cases} \dfrac{\partial u^\varepsilon(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\left(a_{ij}(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^k})\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\right) + \dfrac{\partial}{\partial x_i}\left(b_i(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^k})u^\varepsilon(x,t)\right) \\ \quad +a_0(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^k})u^\varepsilon(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ u^\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u^\varepsilon(x,0) = \bar{u}_0(x), \quad x \in \Omega, \end{cases} \tag{84}$$

For convenience, we choose $\Omega = (0,1)^2$ and the reference cell $Q = (0,1)^2$ which are shown in Fig. 5. We take $\varepsilon = \frac{1}{8}$, $f(x,t) = 100$, $\bar{u}_0(x) = 0$ and $T = 1$. Let $\xi = \varepsilon^{-1}x$ and $\tau = t/\varepsilon^k$, $k = 0, 1$.

Let S be a square inclusion of the reference cell Q, assume that the coefficients $a_{ij}(\xi,t)$, $b_i(\xi,t)$ and $a_0(\xi,t)$ are taken as follows:

$$a_{ij}(\xi,\tau) = \begin{cases} 10^{-2}a_{ij0}(\xi,\tau), & \xi \in S, \\ a_{ij0}(\xi,\tau), & \xi \in Q \setminus \overline{S}; \end{cases} \qquad b_i(\xi,\tau) = \begin{cases} 10^{-2}b_{i0}(\xi,\tau), & \xi \in S, \\ b_{i0}(\xi,\tau), & \xi \in Q \setminus \overline{S}; \end{cases}$$

$$a_0(\xi,\tau)=\begin{cases}10^{-2}a_{00}(\xi,\tau), & \xi\in S,\\ a_{00}(\xi,\tau), & \xi\in Q\setminus\overline{S},\end{cases}$$

where

$$a_{ij0}(\xi,\tau)=200(1+0.2\cos(2\pi\xi_1))(1+0.2\cos(2\pi\xi_2))(1+0.2\cos(2\pi\tau))\delta_{\mathrm{ij}},$$

$$b_{i0}(\xi,\tau)=100(1+0.4\cos(2\pi\xi_1)\cos(2\pi\xi_2))(1+0.3\cos(2\pi\tau)),$$

$$a_{00}(\xi,\tau)=100(1+0.2\cos(2\pi\tau)),\quad \delta_{\mathrm{ij}}\text{ is the Kronecker symbol.}$$

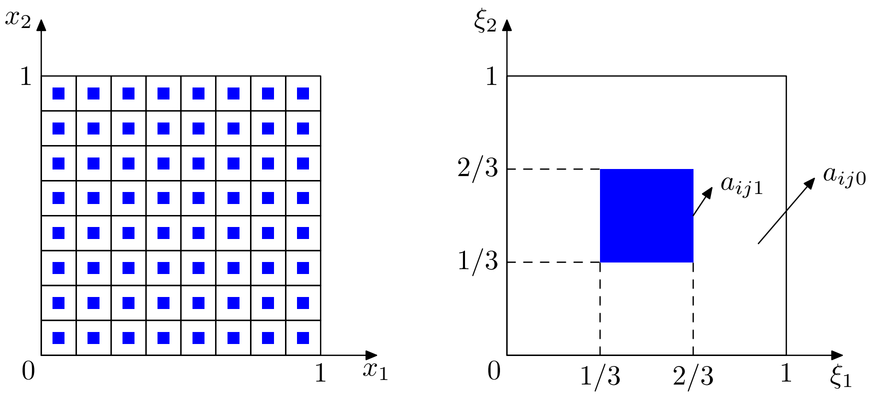

FIGURE 5. Left: A whole domain Ω; Right: the reference cell Q.

In order to demonstrate the numerical accuracy of the present method, the exact solution $u^\varepsilon(x,t)$ of the original problem (84) must be available. Since the exact solution is not available. we replace $u^\varepsilon(x,t)$ by the numerical solution computed using a linear finite volume element method with a fine mesh and a small time step. This is a time-consuming task, and we only present the results for the case $\varepsilon=\frac{1}{8}$. The computational procedure is now briefly described. Firstly, we employ the linear triangle elements to solve cell problems (13)-(20). The linear finite volume elements is then applied to solve the modified homogenized convection-diffusion equation (44) in a coarse mesh using a larger time step. In particular, the computational cost are shown in Table 1.

TABLE 1. Computational costs for Example 5.2 with $k=1$ and $\varepsilon=1/8$.

DOF	the original problem	the homogenized one	cell ones
number of nodes	9,409	1,369	1,369
number of elements	18,432	2,592	2,592
the time step	1.25e-3	1.0e-2	-

Set $e_0=u^\varepsilon-u^0$, $e_1=u^\varepsilon-u_1^\varepsilon$, $e_2=u^\varepsilon-u_2^\varepsilon$. For simplicity, denote by $\|\cdot\|_{L^2(L^2)}$ the norm $\|\cdot\|_{L^2(0,T;L^2(\Omega))}$ and by $\|\cdot\|_{L^2(H^1)}$ the norm $\|\cdot\|_{L^2(0,T;H^1(\Omega))}$. The relative errors for the homogenization method ($s=0$), the first-order ($s=1$) and the second-order ($s=2$) multiscale methods in Example 5.2 are reported in Table 2.

The computational results for Example 5.2 with $k=0,1$ in line $x_1=x_2$ at time $t=0.6$ are displayed in Fig. 6. The figure shows that the exact solution exhibits oscillating behaviors at the interface between two materials and that the second-order multiscale solution makes a good agreement with the exact solution, while the

TABLE 2. Relative errors for Example 5.2 with $k = 0, 1$ and $\varepsilon = 1/8$.

s		0	1	2
$k=0$	$\frac{\|e_s\|_{L^2(L^2)}}{\|u^\varepsilon\|_{L^2(L^2)}}$	6.6931e-02	5.8982e-02	9.1313e-03
	$\frac{\|e_s\|_{L^2(H^1)}}{\|u^\varepsilon\|_{L^2(H^1)}}$	7.1000e-01	6.4740e-01	7.1110e-02
$k=1$	$\frac{\|e_s\|_{L^2(L^2)}}{\|u^\varepsilon\|_{L^2(L^2)}}$	1.6032e-01	1.5725e-01	1.4088e-01
	$\frac{\|e_s\|_{L^2(H^1)}}{\|u^\varepsilon\|_{L^2(H^1)}}$	7.1739e-01	6.5704e-01	1.2741e-01

homogenized and first-order multiscale methods do not even capable of capturing the oscillatory behaviors.

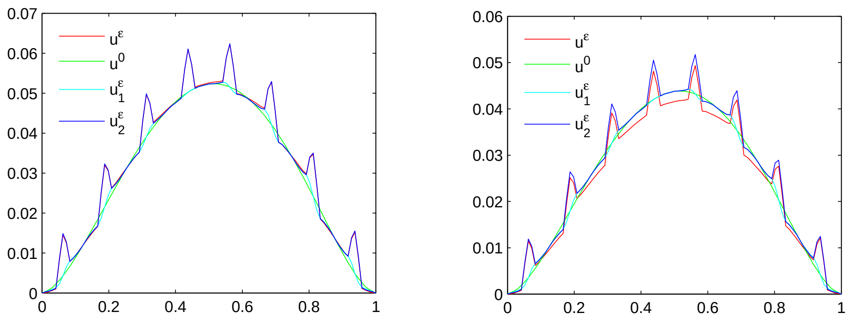

FIGURE 6. Solutions on line $x_1 = x_2$; Left: $k = 0$. Right: $k = 1$.

Example 5.3. Now, consider the same equation as Example 5.2 but in 3-D case, where, for simplicity, a whole domain $\Omega = (0, 1)^3$ and the reference cell $Q = (0, 1)^3$ are shown in Fig. 7. We take $\varepsilon = \frac{1}{8}$, $f(x, t) = 100$, $\bar{u}_0(x) = 0$ and $T = 1$. Let $\xi = \varepsilon^{-1}x$ and $\tau = t/\varepsilon^k$, $k = 0, 1$.

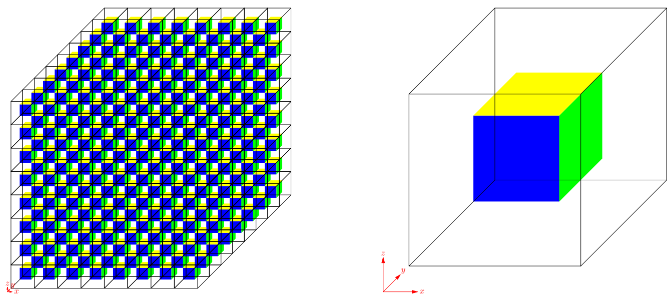

FIGURE 7. Left: A whole domain Ω; Right: the reference cell Q.

Let B be a cube inclusion of the reference cell Q, assume that the coefficients $a_{ij}(\xi, t)$, $b_i(\xi, t)$ and $a_0(\xi, t)$ are taken as follows:

$$a_{ij}(\xi,\tau) = \begin{cases} 10^{-2}a_{ij0}(\xi,\tau), & \xi \in B, \\ a_{ij0}(\xi,\tau), & \xi \in Q \setminus \overline{B}; \end{cases} \quad b_i(\xi,\tau) = \begin{cases} 10^{-2}b_{i0}(\xi,\tau), & \xi \in B, \\ b_{i0}(\xi,\tau), & \xi \in Q \setminus \overline{B}; \end{cases}$$

$$a_0(\xi,\tau) = \begin{cases} 10^{-2}a_{00}(\xi,\tau), & \xi \in B, \\ a_{00}(\xi,\tau), & \xi \in Q \setminus \overline{B}, \end{cases}$$

where

$$\begin{aligned} a_{ij0}(\xi,\tau) &= 200(1+0.2\mathrm{cos}(2\pi\xi_1))(1+0.2\mathrm{cos}(2\pi\xi_2))(1+0.2\mathrm{cos}(2\pi\xi_3)) \\ &\qquad (1+0.2\mathrm{cos}(2\pi\tau))\delta_{\mathrm{ij}}, \\ b_{i0}(\xi,\tau) &= 100(1+0.4\mathrm{cos}(2\pi\xi_1)\mathrm{cos}(2\pi\xi_2)\mathrm{cos}(2\pi\xi_3))(1+0.3\mathrm{cos}(2\pi\tau)), \\ a_{00}(\xi,\tau) &= 100(1+0.2\mathrm{cos}(2\pi\tau)). \end{aligned}$$

Similar to the Example 5.2, we use the linear finite volume element method to solve the problem (84) using a fine mesh and at a small time step. The numerical solution is regarded as the reference solution of the original problem (84). Comparison of the computational costs and the relative errors are reported in Table 3 and 4, respectively.

TABLE 3. Computational costs for Example 5.3 with $k = 1$ and $\varepsilon = 1/8$.

DOF	the original problem	the homogenized one	cell ones
number of nodes	887,753	87,598	88,127
number of elements	5,048,234	470,489	472,685
the time step	1.25e-3	1.0e-2	-

TABLE 4. Relative errors for Example 5.3 with $k = 0, 1$ and $\varepsilon = 1/8$.

s		0	1	2
$k=0$	$\frac{\|e_s\|_{L^2(L^2)}}{\|u^\varepsilon\|_{L^2(L^2)}}$	5.0974e-02	4.7331e-02	1.2145e-02
	$\frac{\|e_s\|_{L^2(H^1)}}{\|u^\varepsilon\|_{L^2(H^1)}}$	5.7061e-01	5.4708e-01	7.8330e-02
$k=1$	$\frac{\|e_s\|_{L^2(L^2)}}{\|u^\varepsilon\|_{L^2(L^2)}}$	1.5499e-01	1.5388e-01	1.4388e-01
	$\frac{\|e_s\|_{L^2(H^1)}}{\|u^\varepsilon\|_{L^2(H^1)}}$	5.8312e-01	5.6073e-01	1.4279e-01

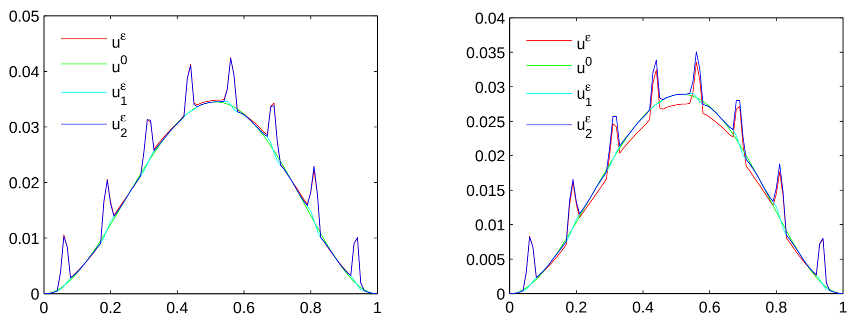

FIGURE 8. Solutions on line $x_1 = x_2 = x_3$; Left: $k = 0$; Right: $k = 1$.

Similar to the Example 5.2, the second-order multiscale solution is more accurate than those resulted by the homogenized and first-order multiscale methods (see

Table 4 and Fig. 8). Moreover, the multiscale methods provide a tremendous saving in computing resource, in particular, for the three-dimensional cases and when the value of ε is sufficiently small (see Table 3).

Remark 5.2. When solving the convection-diffusion equations with homogeneous Neumann boundary conditions on the surfaces of cavities in porous media, we usually fill these cavities with an almost degenerated phase, which is also called the hole-filling method, see [38]. It should be mentioned that the present method is suitable for solving this kind of problem.

References

[1] G. Allaire and M. Amar, Boundary layer tails in periodic homogenization, ESAIM: Control, Optim. Cal. Var., 4 (1999) 209–243.

[2] W. Allegretto, L.Q. Cao and Y.P. Lin, Multiscale asymptotic expansion for second order parabolic equations with rapidly oscillating coefficients, Discrete Contin. Dyn. Sys., 20 (2008) 543–576.

[3] R.E. Bank and D.J. Rose, Some error estimates for the box methods, SIAM J. Numer. Anal., 24 (1987) 777–787.

[4] A. Bensoussan, J. Lions and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.

[5] S. Brahim-Otsmane, G.A. Francfort and F. Murat, Correctors for the homogenization of the wave and heat equations, J. Math. Pures Appl., 71 (1992) 197–231.

[6] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods, Second Edition, Springer, 2007.

[7] Z. Cai and S. McCormick, On the accuracy of the finite volume element method for diffusion equations on composite grid, SIAM J. Numer. Anal., 27 (1990) 636–655.

[8] L.Q. Cao, L. Zhang, W. Allegretto and Y. P. Lin, Multiscale asymptotic method for steklov eigenvalue equations in composite media, SIAM J. Numer. Anal., 51 (2013) 273–296.

[9] L.Q. Cao, Y. Zhang, W. Allegretto and Y.P. Lin, Multiscale asymptotic method for Maxwell's equations in composite materials, SIAM J. Numer. Anal., 47 (2010) 4257–4298.

[10] L.Q. Cao and J.Z. Cui, Asymptotic expansions and numerical algorithms of eigenvalues and eigenfunctions of the Dirichlet problem for second order elliptic equations in perforated domains, Numer. Math., 96 (2004) 525–581.

[11] P. Chatzipantelidis, R. D. Lazarov and V. Thomee, Error estimates for the finite volume element method for parabolic equations in convex polygonal domains, Numer. Methods PDE, 20 (2004) 650–674.

[12] Z.M. Chen and H.J. Wu, Selected Topics in Finite Element Methods, Science Press, 2010.

[13] S.H. Chou and X. Ye, Unified analysis of finite volume methods for second order elliptic problems, SIAM J. Numer. Anal., 45 (2007) 1639–1653.

[14] S.H. Chou and P.S. Vassilevski, A general mixed covolume framework for constructing conservative schemes for elliptic problems, Math. Comp., 68 (1999) 991–1011.

[15] S.H. Chou and Q. Li, Error estimates in L^2, H^1 and L^∞ in covolume methods for elliptic and parabolic problems, a unified approach, Math. Comp., 69 (2000) 103–120.

[16] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[17] F. Colombini and S. Spagnolo, Sur la convergence des solutions d'equations paraboliques, J. Math.Pures Appl., 56 (1977) 263–306.

[18] R. E. Ewing, R. D. Lazarov and Y.P. Lin, Finite volume element approximations of nonlocal in time one-dimensional flows in porous media, Computing, 64 (2000) 157–182.

[19] R. E. Ewing, R. D. Lazarov and Y.P. Lin, Finite volume element approximations of nonlocal reactive flows in porous media, Numer. Methods PDE, 16 (2000) 285–311.

[20] R.E. Ewing, T. Lin and Y.P. Lin, On the accuracy of the fintie volume element method based on piecewise linear polynomials, SIAM J. Numer. Anal., 39 (2002) 1865–1888.

[21] R. Eymard, D. Hilhorst and M. Vohralík, A combined finite volume-finite element scheme for the discretization of strongly nonlinear convection- diffusion reaction problems on nonmatching grids, Numer. Methods PDE, 26 (2010) 612–646.

[22] U. Hornung, Homogenization and Porous Media, Springer-Verlag, New York, 1997.

[23] V.V. Jikov , S.M. Kozlov, and O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, Berlin, 1994.

[24] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uraltseva, Linear and Quasilinear Parabolic Equations, Nauka, Moskow, 1967.
[25] H. Li, Z.D. Luo, P. Sun and J. An, A finite volume element formulation and error analysis for the non-stationary conduction-convection problem, J. Math. Anal. Appl., 396 (2012) 864–879.
[26] Z.D. Luo, H. Li and P. Sun, A fully discrete stabilized mixed finite volume element formulation for the non-stationary conduction-convection problem, J. Math. Anal. Appl., 404 (2013) 71–85.
[27] R. H. Li, Z. Y. Chen and W. Wu, Generalized Difference Methods for Differential Equations, Marcel Dekker, New York, 2000.
[28] Y. Y. Li and L. Nirenberg, Estimates for elliptic systems from composite material, Comm. Pure Appl. Math., 56 (2003) 0892–0925.
[29] J.L. Lv and H.H. Li, L^2 error estimates and superconvergence of the finite volume element methods on quadrilateral meshes, Adv. Comp. Math., 37 (2012) 393–416.
[30] P.B. Ming and P.W. Zhang, Analysis of the heterogeneous multiscale method for parabolic homogenization problems, Math. Comp., 51 (2007) 153–177.
[31] C.Y. Nie and M. Tan, A symmetric finite volume element scheme on tetrahedron grids, J.Korean Math. Soc., 49 (2012) 765–778.
[32] O.A. Oleinik, A.S. Shamaev and G.A. Yosifian, Mathematical Problems in Elasticity and Homogenization, North Holland, 1992.
[33] A.A. Pankov, G-convergence and Homogenization of Nonlinear Parabolic Partial Differential Operators, Kluwer Academic Publishers, Dordrecht, 1997.
[34] E. Sanchez-Palencia, Nonhomogeneous Media and Vibration Theory, Lecture Notes in Physics, 127 (1980).
[35] R. K. Sinha and J. Geiser, Error estimates for finite volume element methods for convection-diffusion-reaction equations, Appl. Numer. Math., 57 (2007) 59–72.
[36] N.Svanstedt, G-convergence of parabolic operators, Nonlinear Anal., 36 (1999) 807–842.
[37] V. Thomee, Galerkin Finite Element Methods for Parabolic Problems, Springer, Berlin, 1997.
[38] X. Wang and L.Q. Cao, The hole-filling method and the uniform multiscale computation of the elastic equations in perforated domains, Internat. J. Numer. Anal. Model., 5 (2008) 612–634.
[39] N.N. Yan, Superconvergence Analysis and a Posteriori Error Estimation in Finite Element Methods, Science Press, Beijing, 2008.
[40] T. Zhang, The semidiscrete finite volume element method for the nonlinear convection-diffusion problem, Appl. Math. Comput., 217 (2011) 7546–7556.
[41] V.V. Zhikov, S.M. Kozlov and O.A. Oleinik, Averaging of parabolic operators with almost periodic coefficients, Math. Sb. (N.S.), 117 (1982) 69–85.

Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China
E-mail: shenlr@lsec.cc.ac.cn

LSEC,NCMIS, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China
E-mail: clq@lsec.cc.ac.cn

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada
E-mail: yaushu.wong@ualberta.ca