MULTISCALE FEM-FVM HYBRID METHOD FOR
CONVECTION-DIFFUSION EQUATIONS WITH PERIODIC
DISCONTINUOUS COEFFICIENTS IN GENERAL CONVEX
DOMAINS

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Abstract. This paper presents the multiscale analysis and numerical algorithms for the convection-
diffusion equations with rapidly oscillating periodic discontinuous coefficients. The multiscale
asymptotic expansions are developed and an explicit rate of convergence is derived for the con-

vex domains. An efficient multiscale hybrid FEM-FVM algorithm is constructed, and numerical
experiments are reported to validate the predicted convergence results.

Key words. Convection-diffusion equation, convex domain, periodic discontinuous coefficients,
composite materials, porous media, multiscale asymptotic expansion, finite element method, finite
volume element method.

1. Introduction

In this paper, we consider the convection-diffusion equations with rapidly oscillating periodic discontinuous coefficients given as follows:

\[
\begin{align*}
\mathcal{L}_\varepsilon(u^\varepsilon) & \equiv \frac{\partial u^\varepsilon(x,t)}{\partial t} - \frac{\partial}{\partial x_i} \left( a_{ij}^\varepsilon(x,t) \frac{\partial u^\varepsilon(x,t)}{\partial x_j} \right) + \frac{\partial}{\partial x_i} \left( b_{ij}^\varepsilon(x,t) u^\varepsilon(x,t) \right) \\
& \quad + a_0^\varepsilon(x,t) u^\varepsilon(x,t), \quad (x,t) \in \Omega \times (0,T), \\
\sigma_\varepsilon(u^\varepsilon) & \equiv \nu_i \left( a_{ij}^\varepsilon(x,t) \frac{\partial u^\varepsilon(x,t)}{\partial x_j} - b_{ij}^\varepsilon(x,t) u^\varepsilon(x,t) \right) = g_1(x,t), \quad (x,t) \in \Gamma_1 \times (0,T), \\
u^\varepsilon(x,0) & = \bar{u}_0(x), \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^d \) (\( d \geq 1 \)) is a bounded convex Lipschitz domain with the boundary \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \), \( \Gamma_0 \cap \Gamma_1 = \emptyset \), \( \text{meas}(\Gamma_0) > 0 \), where \( \text{meas}(\Gamma_0) \) denotes the Lebesgue’s measure of \( \Gamma_0 \); \( u^\varepsilon(x,t) \) is the unknown function, \( f(x,t) \), \( g_0(x,t) \), \( g_1(x,t) \) and \( \bar{u}_0(x) \) are known functions. Here, we focus on the following specified cases: \( a_{ij}^\varepsilon(x,t) = a_{ij}(\xi, \tau), b_{ij}^\varepsilon(x,t) = b_{ij}(\xi, \tau) \) and \( a_0^\varepsilon(x,t) = a_0(\xi, \tau) \), \( k = 0, 1 \); \( \nu = (\nu_1, \cdots, \nu_d) \) is the outward unit normal to \( \Gamma_1 \). Throughout the paper, the Einstein summation convention on repeated indices is adopted. By \( C \) we denote a positive constant independent of \( \varepsilon \).

Let \( \xi = \varepsilon^{-1} x, \tau = \varepsilon^{-k} t, k = 0, 1 \). We make the following assumptions:

(A_1) For \( k = 0 \), \( a_{ij}(\xi, t), b_{ij}(\xi, t) \) and \( a_0(\xi, t) \) are 1-periodic in \( \xi \); For \( k = 1 \), \( a_{ij}(\xi, t) \) \( b_{ij}(\xi, t) \) and \( a_0(\xi, t) \) are 1-periodic in \( \xi \) and \( \tau_0 \)-periodic in \( \tau \).

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In this topic, we refer to [31, 21, 29, 40].

error estimates for the triangular and tetrahedral elements. For other recent results
on the error estimates for 3-D are not available. Under the assumption that the term
for triangular elements under the assumption \( \Delta \)
and the stabilized FVEM for the Navier-Stokes equations in 2-D, and obtained \( L \)
the estimates of triangular elements. Li et al. [25] and Luo et al. [26] developed FVEM
and nonconforming elements, and the DG for 2-D elliptic equations with the error
estimates for triangular elements. For example, we refer to [3, 7, 27, 11, 14, 15, 18, 19, 20] for the early important
finite volume element methods (FVEM) for the elliptic and parabolic equations.

By introducing a cutoff function, Bensoussan, Lions and Papanicolaou [4] obtained \( L \)
the strong convergence result without an explicit rate for the first-order corrector of the linear parabolic equations in \( L^2(0,T; H^1(\Omega)) \). Braham-Otismane, Francfort and Murat [5] extended this result to \( L^2(0,T; W^{1,1}(\Omega)) \). Ming and Zhang [30] derived the convergence result with an explicit rate \( \varepsilon^{1/2} \) for the case \( k = 0 \) under the assumption \( u^0 \in H^{1/2}(\Omega \times (0,T)) \), where \( u^0(x,t) \) is the solution of the linear homogenized parabolic equation. Allegretto, Cao and Lin [2] investigated the higher-order multiscale method for the linear parabolic equations in the four specific cases \( k = 0, 1, 2, 3 \), and obtained the convergence results with an explicit rate \( \varepsilon^{1/2} \) under the assumption \( u^0 \in H^{s+1}(\Omega \times (0,T)), s = 1, 2 \). It is well known that, for a bounded convex polygonal Lipschitz domain \( \Omega \), the assumptions \( u^0 \in H^{s+1}(\Omega \times (0,T)), s = 1, 2 \) may be invalid. Thus the error estimates presented in [2] are not valid. On the other hand, many results are now available for the finite volume element methods (FVEM) for the elliptic and parabolic equations. For example, we refer to [3, 7, 27, 11, 14, 15, 18, 19, 20] for the early important results. Chou and Ye [13] presented the unified variational form of the conforming and nonconforming elements, and the DG for 2-D elliptic equations with the error estimates of triangular elements. Li et al. [25] and Luo et al. [26] developed FVEM and the stabilized FVEM for the Navier-Stokes equations in 2-D, and obtained \( L \) the error estimates for triangular elements under the assumption \( \Delta t = O(h) \). However, the error estimates for 3-D are not available. Under the assumption that the term of right side \( f(x,t) \equiv 0 \) is indispensable, Sinha and Geiser [35] investigated the FVEM for the 2-D and 3-D convection-diffusion equations, and derived the optimal error estimates for the triangular and tetrahedral elements. For other recent results in this topic, we refer to [31, 21, 29, 40].

In this paper, we present the following new results:

(i) The interior error estimates for the multiscale asymptotic solutions of the original problem (1) are obtained under the weaker assumptions (see Theorem 3.1).
By defining the boundary layer solutions, the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale approximate solutions of (1) in a bounded convex domain are derived (see Theorem 3.2).

(ii) The optimal error estimates of FVEM for solving the modified homogenized convection-diffusion equation (44) in a bounded convex Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ are given (see Theorem 4.2).

(iii) A multiscale hybrid FEM-FVM algorithm is developed and numerical examples modeling 2-D and 3-D problems in composite materials are reported.

The remainder of this paper is organized as follows. In section 2, we present the formal multiscale asymptotic expansions for the original problem (1). In section 3, we derive the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale approximate solutions in a bounded convex domain based on the boundary layer solutions. In section 4, we discuss numerical algorithms and the error estimates for the related problems. Finally, a multiscale hybrid FEM-FVM algorithm is presented, and numerical simulations for problems in 2-D and 3-D composite materials are reported.

2. Multiscale asymptotic expansions

In this section, we present the formal multiscale asymptotic expansions for the original problem (1). First, we introduce two sets of variables: $x$, $\xi = \varepsilon^{-1} x$ and $t$, $\tau = \varepsilon^{-k} t$, where $k = 0, 1$. We define the multiscale asymptotic expansions of (1) in the following forms:

\begin{align}
  u_1^\varepsilon(x, t) &= u^0(x, t) + \varepsilon N_m(\xi, \tau) \frac{\partial u^0(x, t)}{\partial x} + \varepsilon R_0(\xi, \tau) u^0(x, t), \\
  u_2^\varepsilon(x, t) &= u_1^\varepsilon(x, t) + \varepsilon^2 N_{ml}(\xi, \tau) \frac{\partial^2 u^0(x, t)}{\partial x^2} \\
  &+ \varepsilon^2 M(\xi, \tau) \frac{\partial u^0(x, t)}{\partial t} + \varepsilon^2 R_1(\xi, \tau) u^0(x, t).
\end{align}

In the case of $k = 0$, the cell functions are defined by

\begin{align}
  \begin{cases}
    \frac{\partial}{\partial \xi_i} (a_{ij}(\xi, \tau) \frac{\partial N_m(\xi, \tau)}{\partial \xi_j}) = -\frac{\partial}{\partial \xi_i} (a_{im}(\xi, t)), & \xi \in Q \\
    \frac{\partial}{\partial \xi_i} (a_{ij}(\xi, \tau) \frac{\partial R_0(\xi, \tau)}{\partial \xi_j}) = \frac{\partial}{\partial \xi_i} (b_i(\xi, \tau)), & \xi \in Q,
  \end{cases}
  \int_Q N_m(\xi, t) d\xi = 0, \quad \int_Q R_0(\xi, t) d\xi = 0,
\end{align}

and

\begin{align}
  \begin{cases}
    \frac{\partial}{\partial \xi_i} (a_{ij}(\xi, \tau) \frac{\partial N_{ml}(\xi, t)}{\partial \xi_j}) = -\frac{\partial}{\partial \xi_i} (a_{im}(\xi, t) N_i(\xi, t)) \\
    -a_{mj}(\xi, \tau) \frac{\partial N_i(\xi, t)}{\partial \xi_j} = a_{mi}(\xi, t) + \hat{a}_{mi}(t), & \xi \in Q, \\
    N_{ml}(\xi, t) \text{ is 1-periodic in } \xi, \quad \int_Q N_{ml}(\xi, t) d\xi = 0,
  \end{cases}
\end{align}

where the elements of the homogenized matrix ($\hat{a}_{ml}(t)$) are calculated by

\[ \hat{a}_{ml}(t) = \int_Q [a_{ml}(\xi, \tau) + a_{mq}(\xi, \xi, \tau) \frac{\partial N_i(\xi, t)}{\partial \xi_q}] d\xi. \]
where the elements of the homogenized vector \( \hat{b}_l(t) \) are calculated by
\[
\hat{b}_l(t) = \int_Q [b_l(\xi, t) - a_{ij}(\xi, t)\frac{\partial R_0(\xi, t)}{\partial \xi_j}] d\xi.
\]

For the case \( t \in (0, T) \) plays the role of a parameter.

For the case \( k = 1 \), we define the cell functions by
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi, \tau) \frac{\partial M_i(\xi, \tau)}{\partial \xi_j} \right) = -\frac{\partial}{\partial \xi_i} \left[ a_{im}(\xi, \tau) R_0(\xi, \tau) - b_i(\xi, \tau) M_l(\xi, \tau) \right]
\end{array} \right.
\]

\[
\frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi, \tau) \frac{\partial R_i(\xi, \tau)}{\partial \xi_j} \right) = \frac{\partial}{\partial \xi_i} \left( b_i(\xi, \tau) R_0(\xi, \tau) \right) + a_0(\xi, \tau) - \langle a_0 \rangle (t), \quad \xi \in Q,
\]

\[
R_l(\xi, \tau) \text{ is } 1\text{-periodic in } \xi, \quad \int_Q R_l(\xi, \tau) d\xi = 0,
\]

\[
\langle a_0 \rangle (t) = \int_Q a_0(\xi, \tau) d\xi,
\]

where \( t \in (0, T) \) plays the role of a parameter.

For the case \( k = 1 \), we define the cell functions by
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi, \tau) \frac{\partial N_{ml}(\xi, \tau)}{\partial \xi_j} \right) = -\frac{\partial}{\partial \xi_i} \left[ a_{im}(\xi, \tau) N_l(\xi, \tau) \right]
\end{array} \right.
\]

\[
\frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi, \tau) \frac{\partial N_{ml}(\xi, \tau)}{\partial \xi_j} \right) = \frac{\partial}{\partial \xi_i} \left[ a_{ml}(\xi, \tau) + \hat{a}_{ml} \right], \quad (\xi, \tau) \in Q \times (0, \tau_0),
\]

\[
N_{ml}(\xi, \tau) = 0, \quad (\xi, \tau) \in Q \times (0, \tau_0),
\]

where the elements of the homogenized matrix \( \hat{a}_{ml} \) are calculated by
\[
\hat{a}_{ml} = \frac{1}{\tau_0} \int_0^{\tau_0} \int_Q [a_{ml}(\xi, \tau) + a_{mq}(\xi, \tau) \frac{\partial N_l(\xi, \tau)}{\partial \xi_q}] d\xi d\tau.
\]

For the case \( k = 1 \), we define the cell functions by
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi, \tau) \frac{\partial M_l(\xi, \tau)}{\partial \xi_j} \right) = \frac{\partial}{\partial \xi_i} \left[ a_{il}(\xi, \tau) R_0(\xi, \tau) - b_i(\xi, \tau) M_l(\xi, \tau) \right]
\end{array} \right.
\]

\[
\frac{\partial}{\partial \xi_i} \left( a_{ij}(\xi, \tau) \frac{\partial R_i(\xi, \tau)}{\partial \xi_j} \right) = \frac{\partial}{\partial \xi_i} \left( b_i(\xi, \tau) R_0(\xi, \tau) \right) + a_0(\xi, \tau) - \langle a_0 \rangle (t), \quad (\xi, \tau) \in Q \times (0, \tau_0),
\]

\[
M_l(\xi, \tau) = 0, \quad (\xi, \tau) \in Q \times (0, \tau_0),
\]

where the elements of the homogenized vector \( \hat{b}_l \) are calculated by
\[
\hat{b}_l = \frac{1}{\tau_0} \int_0^{\tau_0} \int_Q [b_l(\xi, \tau) - a_{ij}(\xi, \tau) \frac{\partial R_0(\xi, \tau)}{\partial \xi_j}] d\xi d\tau.
\]
In a bounded convex Lipschitz domain $\Omega$. Letting the idea of [8], we first define a boundary layer solution. Then we derive the domain $\Omega$, the assumptions $u$
olinebreak[4]
olinebreak[4]main. As already discussed in the Introduction, for a bounded convex Lipschitz results for the multiscale asymptotic method in a bounded convex Lipschitz domains (2) for the original problem (1). In this section, we derive the convergence

3. Convergence for the multiscale asymptotic method

In the previous section, we present the formal multiscale asymptotic expansions (2) for the original problem (1). In this section, we derive the convergence results for the multiscale asymptotic method in a bounded convex Lipschitz domain $\Omega$. As already discussed in the Introduction, for a bounded convex Lipschitz domain $\Omega$, the assumptions $u^0 \in H^{s+2,1}(\Omega \times (0, T))$, $s = 1, 2$ may be invalid. Thus the error estimates reported in [2] cannot be accepted. In this study, following the idea of [8], we first define a boundary layer solution. Then we derive the convergence results with an explicit rate $\varepsilon^{1/2}$ for the multiscale asymptotic solutions in a bounded convex Lipschitz domain $\Omega$. Let $\Omega_0 = \bigcup_{x \in I_\varepsilon} \varepsilon(z + \mathbb{Q})$, where $I_\varepsilon = \{z = (z_1, \cdots, z_d) \in \mathbb{Z}^d : \varepsilon(z + \mathbb{Q}) \subset \Omega\}$. The interior subdomain $\Omega_0$ and the boundary layer $\Omega_1 = \Omega \setminus \Omega_0$ are shown in Figs.3.1 and 3.2. Let $\text{dist}(\partial\Omega_0, \partial\Omega) > 2\varepsilon$. Since $\Omega$ is a bounded convex domain, this condition can be satisfied.

The boundary layer solutions $u^b_s(x, t)$ are defined as follows:

$$
\mathcal{L}_\varepsilon(u^b_s(x, t)) = f(x, t), \quad (x, t) \in \Omega_1 \times (0, T),
$$

$$
u^b_s(x, t) = g_0(x, t), \quad (x, t) \in \Gamma_0 \times (0, T),
$$

$$\sigma^b_s(x, t) = g_1(x, t), \quad (x, t) \in \Gamma_1 \times (0, T),
$$

$$u^b_s(x, t) = u^b_s(x, t), \quad (x, t) \in \Gamma^s \times (0, T),
$$

$$u^b_s(x, 0) = u_0(x), \quad x \in \Omega,
$$

where $\mathcal{L}_\varepsilon$ and $\sigma^b$ have been given in (1), $u^s_1(x, t)$, $s = 1, 2$ are defined in (2), $\Gamma^s = \partial\Omega_0 \cap \partial\Omega_1$, $s = 1, 2$. The multiscale asymptotic solutions for problem (1) are

Remark 2.1. Under the assumptions $(A_1) - (A_2)$, the existence and uniqueness of the cell problems (3)-(10) can be established based on the Lax-Milgram lemma, see also [2].

The homogenized equation associated with problem (1) is written in the following unified form:

$$
\begin{aligned}
\frac{\partial}{\partial t}u^0(x, t) - \sum_{i, j} \frac{\partial}{\partial x_i} \left( \tilde{a}_{ij}(x, t) \frac{\partial u^0(x, t)}{\partial x_j} \right) + g_0(x, t) = f(x, t), \\
\end{aligned}
$$

$$
\begin{aligned}
\hat{\sigma}(u^0) = \nu_1(\tilde{a}_{ij}(x, t) \frac{\partial u^0(x, t)}{\partial x_j} - \frac{\partial u^0}{\partial x_j}(x, t)) = g_1(x, t), \\
\end{aligned}
$$

$$
\begin{aligned}
u^0(x, 0) = \nu_0(x),
\end{aligned}
$$

Remark 2.2. It can be proved that the homogenized coefficients matrix ($\tilde{a}_{ij}$) is symmetric and positive-definite (see [2, 4, 23, 32]). Existence and uniqueness of the solution for the homogenized parabolic equation (11) can be established.
defined as

\begin{align}
U^\varepsilon_s(x,t) = \begin{cases} 
    u^\varepsilon_s(x,t), & (x,t) \in \Omega_0 \times (0,T), \\
    u^{\varepsilon,b}_s(x,t), & (x,t) \in \Omega_1 \times (0,T),
\end{cases}
\end{align}

where \( u^{\varepsilon,b}(x,t), s = 1, 2 \) have been given in (12), \( u^\varepsilon_s(x,t), s = 1, 2 \) denote the restrictions in a sub-domain \( \Omega_0 \subset \subset \Omega \).

**Theorem 3.1.** Suppose that \( \Omega \subset \mathbb{R}^d, d \geq 1 \) is a bounded convex domain with a Dirichlet’s boundary \( \partial \Omega \). Let \( \bar{u}^\varepsilon(x,t) \) be the unique weak solution of problem (1), and let \( \bar{u}^0(x,t) \) be the solution of the homogenized convection-diffusion equation (11), and \( \bar{u}^1(x,t), \bar{u}^2(x,t) \) be the first-order and the second-order multiscale asymptotic solutions defined in (2), respectively. Assume that \( f \in L^2(0,T; L^2(\Omega)) \cap H^1(0,T; \mathcal{H}^s(\Omega'')) \), \( g_0 \in L^2(0,T; \mathcal{H}^{s/2}(\partial \Omega)) \), \( \bar{u}_0 \in H^1(\Omega) \cap H^{s+1}(\Omega''), s = 1, 2 \), where \( \Omega_0 \subset \subset \Omega'' \subset \subset \Omega \). For the specified case \( k = 0 \), we assume that \( a_{ij}(\frac{x}{\varepsilon}, \cdot) \in C^1(0,T) \) for any fixed \( x \in \Omega \). Under the assumptions (A_1) – (A_4), then it holds

\begin{align}
\sup_{0 \leq t \leq T} \int_{\Omega_0} (u^\varepsilon(x,t) - u^\varepsilon_s(x,t))^2 dx + \int_0^T \|\bar{u}^\varepsilon(\cdot,t) - u^\varepsilon_s(\cdot,t)\|^2_{H^1(\Omega_0)} dt \leq C\varepsilon,
\end{align}

for \( k = 0 \) and \( s = 1, 2 \), or \( k = 1 \) and \( s = 2 \).

**Proof.** Let us introduce the following sub-domains:

\[
\begin{align*}
\Omega' &= \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \varepsilon/2 \}, \\
K_\varepsilon &= \{ x \in \Omega : \text{dist}(x, \partial \Omega) \leq \varepsilon \}, \\
K'_\varepsilon &= \{ x \in \Omega : \varepsilon \leq \text{dist}(x, \partial \Omega) \leq 2\varepsilon \}.
\end{align*}
\]

Observe that the homogenized equation (11) is the initial-boundary value problem with smooth coefficients. Under the assumptions of this theorem, using the interior regularity estimates for the parabolic equations (see, e.g., [24, 37]), we infer that \( \bar{u}^0 \in H^1(0,T; H^{s+1}(\Omega')) \), \( s = 1, 2 \). Now, introduce the cutoff function \( m_\varepsilon(x) \) given by

\begin{align}
\begin{cases}
    m_\varepsilon \in \mathcal{D}(\Omega) \\
    m_\varepsilon = 0, & \text{if dist}(x, \partial \Omega) \leq \varepsilon \\
    m_\varepsilon = 1, & \text{if dist}(x, \partial \Omega) \geq 2\varepsilon \\
    \varepsilon \left| \frac{\partial m_\varepsilon}{\partial x_i} \right| \leq C, & i = 1, 2, \ldots, d.
\end{cases}
\end{align}
Define
\[
\begin{align*}
\theta_1^e(x,t) &= u_0^e(x,t) + \varepsilon m_x(x) \left\{ N_m^e(\xi, \tau) \frac{\partial u_0^e(x,t)}{\partial x_m} + R_0^e \frac{\partial u_0^e(x,t)}{\partial t} \right\}, \\
\theta_2^e(x,t) &= \theta_1^e(x,t) + \varepsilon^2 m_x(x) \left\{ N_{ml}^e(\xi, \tau) \frac{\partial^2 u_0^e(x,t)}{\partial x_m \partial x_l} + M_l^e \left( \frac{\partial u_0^e(x,t)}{\partial x_l} \right) + R_l^e \frac{\partial u_0^e(x,t)}{\partial t} \right\}.
\end{align*}
\]

(16)

Next we prove Theorem 3.1 for the case \(k = 1\). Other cases are similar. For simplicity, we assume that \(g_0(x,t) \equiv 0\). Let \(\Omega_t = \Omega \times (0,t)\) and \((u,v)_{\Omega_T} = \int_0^t \int_\Omega u \, v \, d\xi \, d\tau\). For simplicity, let \(a_{ij}^e = a_{ij}(\xi, \tau, \varepsilon)\), \(b_l^e = b_l(\xi, \tau, \varepsilon)\), \(a_0^e = a_0(\xi, \tau, \varepsilon)\). For all \(v \in L^2(0,T; H^1(\Omega))\), from (2) and (16), we obtain the following equation which holds in the sense of distribution:
\[
(J^e(u^e - \theta_2^e), v)_{\Omega_T} = J^e(v),
\]

where
\[
J^e(v) = (K_1 + K_2 + K_3 + K_4,v)_{\Omega_T}
\]

and
\[
K_1 = -m_x \frac{\partial N_m}{\partial t} \frac{\partial u_0}{\partial x_m} - m_x R_0 \frac{\partial u_0}{\partial t} - \varepsilon m_x N_m \frac{\partial^2 u_0}{\partial x_m \partial t} - \varepsilon m_x R_0 \frac{\partial u_0}{\partial t},
\]

\[
K_2 = (m_x - 1) \left[ (a_{ij}^e - a_{ij}^e) \frac{\partial^2 u_0}{\partial x_i \partial x_j} + (b_l^e - \hat{b}_l^e) \frac{\partial u_0}{\partial x_i} + (a_0^e - \langle a_0^e \rangle) u_0 \right],
\]

\[
K_3 = -\varepsilon^{-1} \frac{\partial}{\partial \xi_i} \left( a_{ij}^e \frac{\partial N_m}{\partial \xi_j} \right) (m_x - 1) \frac{\partial u_0}{\partial x_m} + \varepsilon^{-1} \frac{\partial}{\partial \xi_i} \left( a_{ij}^e R_0 \right) (m_x - 1) u_0,
\]

\[
K_4 = \varepsilon a_{ij}^e \frac{\partial N_m}{\partial \xi_i} \frac{\partial u_0}{\partial x_m} + \varepsilon a_{ij}^e \frac{\partial M_l}{\partial \xi_i} \frac{\partial u_0}{\partial x_l} + \varepsilon a_{ij}^e \frac{\partial R_0}{\partial \xi_i} \frac{\partial u_0}{\partial x_l}.
\]
Under the assumptions (A₁), (A₂) and (A₄), and using Theorem 1.1 of [28], we can prove that

\[
\begin{align*}
N_m, R_0, N_{ml}, M_t, R_t & \in L^2(0, \tau_0; W^{1,\infty}(Q)), \\
\frac{\partial N_m}{\partial \tau}, \frac{\partial R_0}{\partial \tau}, \frac{\partial N_{ml}}{\partial \tau}, \frac{\partial M_t}{\partial \tau}, \frac{\partial R_t}{\partial \tau} & \in L^2(0, \tau_0; L^\infty(Q)).
\end{align*}
\]

Since \(\int_Q \frac{\partial N_m}{\partial \tau} d\xi = 0\), applying Lemma 1.6 and Lemma 1.5 of [32, p. 7-8], we get

\[
(18)
\]

\[
(24)
\]

Thus,

\[
(19)
\]

\[
(20)
\]

Combining (18), (19) and (20) gives

\[
(21)
\]

Applying Lemma 1.5 of [32, p. 7] again, we get

\[
(22)
\]

and consequently

\[
(23)
\]

We observe that

\[
(24)
\]

Thus,

\[
(25)
\]
where (see, e.g., [24, 37]), we can show that
\begin{equation}
(27) \quad \sup_{\text{the specified case}} u(x, t) \leq C \varepsilon^2 \|u^0\|_{L^2(0, T; H^4(\Omega))} \|v\|_{L^2(0, T; H^1(\Omega))}.
\end{equation}

Using (18) and the regularity of the solution $u(x, t)$, we can prove that
\begin{equation}
(28) \quad \sup_{\text{the specified case}} u(x, t) \leq C \varepsilon^2 \|u^0\|_{L^2(0, T; H^4(\Omega))} \|v\|_{L^2(0, T; H^1(\Omega))}.
\end{equation}

We can similarly estimate other terms in $K_4$, and obtain
\begin{equation}
(29) \quad |(K_4, v)_{\Omega_T}| \leq C \varepsilon^3/2 \|u^0\|_{L^2(0, T; H^2(\Omega))} \|v\|_{L^2(0, T; H^1(\Omega))}.
\end{equation}

Using (18) and Lemma 1.5 of [32, p. 7], we can prove that
\begin{equation}
(30) \quad |(K_4, v)_{\Omega_T}| \leq C \varepsilon^3/2 \|u^0\|_{L^2(0, T; H^2(\Omega))} \|v\|_{L^2(0, T; H^1(\Omega))}.
\end{equation}

Combining (21), (23), (29) and (30), it gives
\begin{equation}
(31) \quad J^f(v) \leq C \varepsilon^3/2 \|v\|_{L^2(0, T; H^1(\Omega))}.
\end{equation}

For the initial condition, we have
\begin{equation}
(32) \quad u^\varepsilon(x, 0) - U_0^\varepsilon(x, 0) = \varepsilon \Psi_\varepsilon(x),
\end{equation}

where
\begin{equation}
\Psi_\varepsilon(x) = m_\varepsilon(x) \left\{ N_m(x, t) \frac{\partial u^0(x, t)}{\partial x_m} + R_0(x, t) u^0(x, t) + \varepsilon N_m(x, t) \frac{\partial^2 u^0(x, t)}{\partial x_m \partial x_l} + \varepsilon M_l(x, t) \frac{\partial u^0(x, t)}{\partial x_l} + \varepsilon R_1(x, t) u^0(x, t) \right\} \big|_{t=0}.
\end{equation}

By using (18) and the regularity of the solution $u^0(x, t)$, i.e. $u^0 \in L^2(0, T; H^2(\Omega))$ (see, e.g., [24, 37]), we can show that $\|\Psi_\varepsilon\|_{L^2(\Omega)} \leq C$, where $C$ is a constant independent of $\varepsilon$. In a standard way, applying the Gronwall’s inequality, it completes the proof of Theorem 3.1. □

**Remark 3.1.** It should be emphasized that, in order to derive the convergence results with an explicit rate $\varepsilon^{3/2}$ for the first-order and the second-order multiscale asymptotic solutions of problem (1) in general convex domains, Theorem 3.1 plays a key role. Allaire and Amar [1] obtained a similar convergence result in an interior subdomain $\Omega_0 \subset \subset \Omega$ for elliptic equations. However, their method fails for parabolic equations.

**Theorem 3.2.** Suppose that $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded convex domain with a pure Dirichlet’s boundary $\partial \Omega$. Let $u^\varepsilon(x, t)$ be the unique weak solution of problem (1), and let $u^0(x, t)$ be the solution of the homogenized convection-diffusion equation (11), and $U^\varepsilon_1(x, t)$, $U^\varepsilon_2(x, t)$ be the first-order and the second-order multiscale approximate solutions defined in (13), respectively. Assume that $f \in L^2(0, T; L^2(\Omega)) \cap H^1(0, T; H^1(\Omega))$, $g_0 \in L^2(0, T; H^1(\partial \Omega))$, $\psi_0 \in H^1(\Omega) \cap H^{s+1}(\Omega_0)$, $s = 1, 2$. For the specified case $k = 0$, we assume that $a_{ij}(x, \cdot) \in C^{s}(0, T)$ for any fixed $x \in \Omega$. Under the assumptions (A1) – (A4), then it holds
\begin{equation}
(32) \quad \sup_{0 \leq t \leq T} \int_\Omega (u^\varepsilon(x, t) - U^\varepsilon_1(x, t))^2 \, dx + \int_0^T \|u^\varepsilon(\cdot, t) - U^\varepsilon_2(\cdot, t)\|_{H^1(\Omega)}^2 \, dt \leq C \varepsilon.
\end{equation}
for \( k = 0 \) and \( s = 1, 2 \), or \( k = 1 \) and \( s = 2 \).

**Proof.** We prove this theorem only for the case \( k = 1, s = 2 \), and other cases are similar. Recalling the boundary layer solution \( u_2^{ε,b}(x, t) \) defined in (12), and using (14) and the trace theorem, we get

\[
\|u^ε - u_2^{ε,b}\|_{L^2(0,T;H^1(Ω_1))} \leq C\|u^ε - u_2^{ε}\|_{L^2(0,T;H^1/2(Γ^1))}
\]

\[
\leq C\|u^ε - u_2^{ε}\|_{L^2(0,T;H^1(Ω_2))} \leq Cε^{1/2}.
\]

We thus have

\[
\|u^ε - U_2^ε\|_{L^2(0,T;H^1(Ω))} \leq \|u^ε - u_2^{ε}\|_{L^2(0,T;H^1(Ω_2))}
\]

\[
+ \|u^ε - u_2^{ε}\|_{L^2(0,T;H^1(Ω_1))} \leq Cε^{1/2}.
\]

\[\square\]

**Remark 3.2.** Assume that \( ∂Ω = \overline{T}_0 \cup \overline{T}_1 \) is the mixed boundary. Under the assumptions of Theorem 3.1 or Theorem 3.2, if the condition \( u^0 \in L^2(0,T;H^3(Ω)) \) is imposed, then we can derive a similar convergence results to Theorem 3.1 or Theorem 3.2 for the case \( k = 0 \).

**Remark 3.3.** It should be emphasized that, in order to obtain the convergence results with an explicit rate \( ε^{1/2} \) for the case \( k = 1 \), we need to apply the second-order multiscale asymptotic expansion defined in (2) and (13). The numerical results presented in Section 5 show that the second-order correctors are necessary regardless of \( k = 0 \) or \( k = 1 \).

### 4. Numerical algorithms for related problems and error estimates

By adopting the standard notation \( W^{m,q}(Ω) \) for the Sobolev spaces in \( Ω \) with norm \( \|·\|_{m,q,Ω} \) and semi-norm \( |·|_{m,q,Ω} \), we set \( W_0^{m,q}(Ω) \equiv \{ w ∈ W^{m,q}(Ω) : w|_{∂Ω} = 0 \} \) and denote \( W^{m,2}(Ω) \) (\( W_0^{m,2}(Ω) \)) by \( H^m(Ω) \) (\( H_0^m(Ω) \)) with norm \( \|·\|_{m,Ω} \) and semi-norm \( |·|_{m,Ω} \).

#### 4.1. Adaptive finite element method (AFEM) for cell problems

For simplicity, we only consider the cell problems (7) for the case \( k = 1 \). As for cell problems (3)-(6), (8)-(10), and other cases, they are similar. Since the elements \( a_{ij}(ξ, τ) \) of the coefficients matrix of (7) are discontinuous, we employ an adaptive finite element method (see, e.g., [12, 39]). For convenience, we present a *posteriori* error estimate for solving the cell problems (7). Let \( K_μ \) be a sequence of tetrahedrons for the reference cell \( Q \), and let \( F_μ \) be the set of faces not lying on \( ∂Q \), \( μ ≥ 0 \), i.e. \( F_μ \cap ∂Q = ∅ \). Note that the tetrahedrons must be aligned with the boundary of \( Q \) to employ the periodic boundary conditions on the boundary \( ∂Q \). The linear finite element space \( W_μ \) in \( Q \) is defined by

\[
W_μ(Q) = \{ v ∈ C^0(Ω) \mid v|_K ∈ P_1(K), \ v \ \text{takes the same value on the opposite faces of} \ Q, \ ∀K ∈ K_μ \},
\]

where \( P_1(K) \) is the space of all polynomials of degree, not exceeding 1, on each element \( K \). For any \( K ∈ K_μ \) and \( F ∈ F_μ \), we denote the diameters of \( K \) and \( F \) by \( h_K \) and \( h_F \), respectively.

Let \( N_{m,μ}(ξ, τ) \) denote the approximate solution of \( N_m(ξ, τ) \), \( m = 1, 2, \ldots, d \) in the linear finite element space \( W_μ(Q) \), \( μ ≥ 0 \), respectively. By means of Theorem 5.2.1 of [39], a *posteriori* error estimate for \( N_m(ξ, τ) \) is given by

\[
\|N_m - N_{m,μ}\|_{H^1(Q)}^2 \leq C \left( \sum_{K ∈ K_μ} η_K^2 + \sum_{F ∈ F_μ} η_F^2 \right),
\]
where
\[
\eta_K^2 = h_K \int_K \left( \frac{\partial a_{m,ij}(\xi, \tau)}{\partial \xi_i} + \frac{\partial}{\partial \xi_i} (a_{ij}(\xi, \tau) \frac{\partial N_m(\xi, \tau)}{\partial \xi_j}) \right)^2 \, d\xi,
\]
\[
\eta_F^2 = h_F \int_F \left( \nu^F (a_{m,ij}(\xi, \tau) + a_{ij}(\xi, \tau) \frac{\partial N_m(\xi, \tau)}{\partial \xi_j}) \right)^2 \, ds,
\]
where \( \nu^F = (\nu_F^1, \cdots, \nu_F^F) \) is the outward unit normal to the element face \( F \in F_\mu \).

Here, with a careful design of the mesh, \( a_m(\cdot, \tau) \) is smooth on \( K \) for any fixed \( \tau \).

Under the assumptions \((A_1) - (A_2)\), for any fixed \( \tau \in (0, \tau_0) \), it can be proved that \( N_m(\cdot, \tau) \in H^1(Q) \), \( m = 1, 2, \cdots, d \) (see, e.g., [4, p. 13]). Furthermore, if the assumption \((A_1)\) is imposed, it follows from Theorem 1.1 of [28] that \( N_m(\cdot, \tau) \in W^{1, \infty}(Q) \) for any fixed \( \tau \in (0, \tau_0) \), \( m = 1, 2, \cdots, d \). However, until now one cannot show that \( N_m(\cdot, \tau) \in H^{1, \sigma}(Q) \), for some \( \sigma > 0 \). This is still an unsolved problem.

In order to obtain the error estimates of \( N_m(\cdot, \tau) - N_m^0(\cdot, \tau) \), \( \|N_m(\cdot, \tau) - N_m^0(\cdot, \tau)\|_{H^1(Q)} \), where \( N_m^0(\cdot, \tau) \) is the finite element solution of \( N_m(\xi, \tau) \) and \( h_0 \) is the final mesh parameter of the adaptive finite element method, following the idea of [9], we introduce the approximate technique of smooth coefficients for \( a_{ij}^\beta \). It is well known that the smooth functions are not dense in \( L^\infty(Q) \). To this end, we choose a number \( q > 2 \) such that \( \frac{1}{q} + \frac{1}{p} = \frac{1}{2} \) (see Theorem 4.2 of [4, p. 38]). We can find a sequence of smooth functions \( a_{ij}^{\beta_j} \in C^\infty(Q) \) such that
\[
\|a_{ij} - a_{ij}^{\beta_j}\|_{L^\infty(Q)} \to 0, \quad \text{as} \quad \beta \to \infty.
\]

Similarly to (7), let \( N_{m}^{(\beta_j)}(\xi, \tau), m = 1, 2, \cdots, d \), be the solutions of the following equations:
\[
\begin{align*}
&\frac{\partial}{\partial \xi_i} \left( a_{ij}^{(\beta_j)}(\xi, \tau) \frac{\partial N_{m}^{(\beta_j)}(\xi, \tau)}{\partial \xi_j} \right) = -\frac{\partial a_{m,ij}^{(\beta_j)}(\xi, \tau)}{\partial \xi_i}, \quad (\xi, \tau) \in Q \times (0, \tau_0), \\
&N_{m}^{(\beta_j)}(\xi, \tau) \text{ is 1-periodic in } \xi \text{ and } \tau_0 \text{ periodic in } \tau,
\end{align*}
\]
where \( \tau \in (0, \tau_0) \subset (0, +\infty) \) plays the role of a parameter.

Subtracting (3) from (38), for \( v(\xi, \tau) = N_m(\xi, \tau) - N_{m}^{(\beta_j)}(\xi, \tau) \), we get
\[
\|N_m - N_{m}^{(\beta_j)}\|_{H^1(Q)}^2 \leq C \int_Q a_{ij}(\xi, \tau) \frac{\partial(N_m(\xi, \tau) - N_{m}^{(\beta_j)}(\xi, \tau))}{\partial \xi_j} \, d\xi
\]
\[
\leq C \{ \|a_{ij} - a_{ij}^{(\beta_j)}\|_{L^\infty(Q)}\|N_{m}^{(\beta_j)}\|_{W^{1, \infty}(Q)} + \|a_{ij} - a_{ij}^{(\beta_j)}\|_{L^2(Q)}\|v\|_{H^1(Q)} \},
\]
where \( C \) is a constant independent of \( \beta, \varepsilon \).

Since \( a_{ij}^{\beta_j} \in L^\infty(0, \tau_0; C^\infty(Q)) \), we have \( N_{m}^{(\beta_j)} \in L^\infty(0, \tau_0; W^{1, \infty}(Q)) \). From (38), for any fixed \( \tau \in (0, \tau_0) \), we get
\[
\|N_m - N_{m}^{(\beta_j)}\|_{H^1(Q)} \leq C\|a_{ij} - a_{ij}^{(\beta_j)}\|_{L^\infty(Q)} \to 0 \quad \text{as} \quad \beta \to +\infty.
\]

For any fixed \( \tau \in (0, \tau_0) \), Eq.(38) is an elliptic equation with smooth coefficients in a bounded convex domain \( Q \), we deduce that \( N_{m}^{(\beta_j)} \in H^2(Q) \). Let \( N_{m, h_0} \) be the adaptive finite element solution of \( N_{m}^{(\beta_j)} \) in \( W_h \), then we have (see, e.g., [16])
\[
\|N_{m}^{(\beta_j)} - N_{m, h_0}\|_{H^1(Q)} \leq C h_0\|N_{m}^{(\beta_j)}\|_{H^2(Q)}, \quad \text{for any fixed } \tau \in (0, \tau_0).
\]
Let $N_{m,h_0}(\xi, \tau)$ be the adaptive finite element solution of $N_m(\xi, \tau)$ in $\mathcal{W}_m$ for any fixed $\tau \in (0, \tau_0)$. Similarly to (40), we have

$$
\|N_{m,h_0} - N_{m,h_0}^{(\beta)}\|_{H^1(Q)} \leq C\|a_{ij} - a_{ij}^{(\beta)}\|_{L^2(Q)} \to 0 \quad \text{as} \quad \beta \to +\infty.
$$

From (40), (41) and (42), using the triangle inequality and choosing a sufficiently large $\beta > 0$, we get

$$
\|N_m - N_{m,h_0}\|_{H^1(Q)} \leq \|N_m^{(\beta)} - N_{m,h_0}^{(\beta)}\|_{H^1(Q)} + \|N_m - N_m^{(\beta)}\|_{H^1(Q)}
$$

$$
+ \|N_{m,h_0} - N_{m,h_0}^{(\beta)}\|_{H^1(Q)} \leq C h_0 \|N_m^{(\beta)}\|_{H^1(Q)},
$$

where $C$ is a constant independent of $h_0, \varepsilon$. Therefore, we obtain the error estimates of the adaptive finite element method for solving the cell problems.

**Theorem 4.1.** Let $N_m(\xi, \tau), m = 1, 2, \cdots, d$, be the weak solutions of the cell problems (7), and let $N_{m,h_0}(\xi, \tau)$ be the associated adaptive finite element solutions. Under the assumptions $(A_1) - (A_2)$, it holds

$$
\|N_m - N_{m,h_0}\|_{H^1(Q)} \leq C h_0, \quad m = 1, 2, \cdots, d,
$$

where $C$ is a constant independent of $h_0, \varepsilon$; $h_0$ is the final mesh parameter of the adaptive finite element method.

**Remark 4.1.** The error estimates of the adaptive finite element method for computing other cell functions can also be obtained similarly.

### 4.2. Finite volume element method (FVEM) for the homogenized convection-diffusion equation.

In this section, we employ FVEM to solve the homogenized convection-diffusion equation (11) over a whole domain $\Omega$ using a coarse mesh and at a larger time step. The associated error estimates for FVEM are derived. In real simulations, we need to solve the modified homogenized convection-diffusion equation, namely we replace $\hat{a}_{ij}, \hat{b}_i$ in (11) with $\hat{a}_{ij}^{h_0}, \hat{b}_i^{h_0}$, respectively. For example, in the case of $k = 1$, we have

$$
\hat{a}_{ij}^{h_0} = \frac{1}{\tau_0} \int_0^{\tau_0} \int_Q (a_{ij}(\xi, \tau) + a_{i\ell}(\xi, \tau) \frac{\partial N_j^{h_0}(\xi, \tau)}{\partial \xi}) d\xi d\tau,
$$

$$
\hat{b}_i^{h_0} = \frac{1}{\tau_0} \int_0^{\tau_0} \int_Q (b_i(\xi, \tau) - a_{ij}(\xi, \tau) \frac{\partial R_j^{h_0}(\xi, \tau)}{\partial x_j}) d\xi d\tau,
$$

where $N_j^{h_0}(\xi, \tau), R_j^{h_0}(\xi, \tau)$ are the finite element solutions of $N_j(\xi, \tau), R_0(\xi, \tau)$, respectively; $h_0$ is the final mesh parameter of the adaptive finite elements for computing $N_j(\xi, \tau)$ and $R_0(\xi, \tau)$.

**Remark 4.2.** If the mesh parameter $h_0 > 0$ is sufficiently small, then it holds

$$
\hat{a}_{ij}^{h_0} = \hat{a}_{ij}^{h_0}, \quad \forall i, j = 1, 2, \cdots, d, \quad \max |\hat{a}_{ij} - \hat{a}_{ij}^{h_0}| \leq Ch_0^2,
$$

$$
\max |\hat{b}_i - \hat{b}_i^{h_0}| \leq Ch_0^2, \quad \hat{\gamma}_0 |\eta|^2 \leq \hat{a}_{ij}^{h_0} \eta_i \eta_j \leq \hat{\gamma}_1 |\eta|^2, \quad |\hat{b}_i^{h_0}| \leq \hat{\beta}_1,
$$

$$
\forall \eta = (\eta_1, \cdots, \eta_d) \in \mathbb{R}^d, \quad |\eta|^2 = \eta_i \eta_i,
$$

where $C, \hat{\gamma}_0, \hat{\gamma}_1, \hat{\beta}_1$ are constants independent of $h_0, \varepsilon$. Furthermore, we can show that $\|u^0 - u^{0,h_0}\|_{L^2(\Omega; H^1(\Omega))} \leq Ch_0^2$, where $u^0(x, t)$ and $u^{0,h_0}(x, t)$ are respectively the solutions of the homogenized convection-diffusion equation (11) and the associated modified equation, $C$ is a constant independent of $h_0, \varepsilon$. Here, we omit the detailed proof and we refer the interested reader to [8, 9, 10].
For the sake of convenience, we define \( u(x,t) \) is the solution of the modified homogenized convection-diffusion equation as follows:

\[
\begin{aligned}
\begin{cases}
\begin{align*}
    u_t - \nabla \cdot (A \nabla u - bu) + ru &= f, & (x,t) \in \Omega \times (0,T), \\
    u &= g_0, & (x,t) \in \Gamma_0 \times (0,T), \\
    \sigma(u) &\equiv \nu_i (\hat{e}^0_{ij} \frac{\partial u}{\partial x_j} - \hat{b}^0_{ij} u) = g_1, & (x,t) \in \Gamma_1 \times (0,T), \\
    u(x,0) &= \bar{u}_0(x), & x \in \Omega,
\end{align*}
\end{cases}
\end{aligned}
\]  

(44)

where \( A = (\hat{e}^0_{ij}), b = (\hat{b}^0_{ij}) \) and \( f = (a_0) \), \( \nabla \cdot \nabla \) are the divergence and the gradient operators, respectively.

For simplicity, we assume that \( \Gamma_1 = \emptyset \), \( g_0(x,t) \equiv 0 \) without loss of generality. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded convex polygonal domain. Denote by \( \mathcal{T}_h = \{e\} \) a family of some quasi-uniform tetrahedrons of \( \Omega \), by \( \mathcal{T}_h^* = \{e^*\} \) the dual partition shown in Fig. 3. We introduce two finite dimensional spaces as follows:

\[
\begin{aligned}
\mathcal{V}_h &= \{ v_h \in C^0(\Omega) : v_h|_e \in P_1(e), \ \forall e \in \mathcal{T}_h, \ v_h|_{\partial \Omega} = 0 \}, \\
\mathcal{Q}_h^* &= \{ w_h \in L^2(\Omega) : w_h|_{e^*} \in P_0(e^*), \ \forall e^* \in \mathcal{T}_h^*, \ w_h|_{\partial \Omega} = 0 \}.
\end{aligned}
\]

The functions spaces \( C^0(\Omega) \), \( P_1 \) and \( P_0 \) can be found in [16].

\[\text{Figure 3. Dual partition on the triangular and tetrahedral mesh.}\]

Define an interpolation operator \( \mathcal{I}_h^* : H^2(\Omega) \to \mathcal{Q}_h^* \) by

\[
\mathcal{I}_h^* v = \sum_{x^{(i)} \in \mathcal{N}} v(x^{(i)}) \chi_i(x), \ \forall v \in H^2(\Omega),
\]

(45)

where \( \chi_i(x) = 1 \), if \( x \in e_i^* \); \( \chi_i(x) = 0 \), if \( x \notin e_i^* \); \( \mathcal{N} \) is the set of all nodes in \( \mathcal{T}_h \), and \( e_i^* \) is the control volume which contains node \( x^{(i)} \).

For any \( v \in H^2(\Omega) \cap H_0^1(\Omega) \), multiplying Eq. (44) by \( \mathcal{I}_h^* v \), integrating and then summing over \( e^* \in \mathcal{T}_h^* \), we get

\[
\sum_{e^* \in \mathcal{T}_h^*} \int_{e^*} u_t \mathcal{I}_h^* v dx - \sum_{e^* \in \mathcal{T}_h^*} \int_{\partial e^*} (A \nabla u - bu) \cdot \nu^*_e \mathcal{I}_h^* v ds
\]

\[
+ \sum_{e^* \in \mathcal{T}_h^*} \int_{e^*} ru \mathcal{I}_h^* v dx = \sum_{e^* \in \mathcal{T}_h^*} \int_{e^*} f \mathcal{I}_h^* v dx,
\]

(46)

where \( \nu^*_e \) is the outward unit normal to \( \partial e^* \). Let \( M \geq 1 \) be an integer and \( \Delta t = T/M \) be the time step. Assume that \( u_h^n \) is the approximation in \( \mathcal{V}_h \) of \( u_h(x,t) \) at
where \( t = t_n = n\Delta t \). The fully finite volume element scheme is given by

\[
\begin{align*}
\sum_{e \in \mathcal{T}_h} \int_{\partial e} (A^n \nabla u - b^n u) \cdot \nu_e^* v_h d s + \int_{\Omega} r^n u I_h^* v_h d x, \\
\end{align*}
\]

where \( \cdot \) denotes the inner product in \( L^2(\Omega) \), \( \partial e \) is the diameter of \( e \), and make the assumption:

\[
(\ref{eq:euler_difference}) = (u^n_h, v_h) + \Delta t a^n_h(u^n_h, v_h) = (\Delta t f^n + u^{n-1}_h, I_h^* v_h), \quad \forall v_h \in V_h.
\]

For the interpolation operator \( I_h^* \), we have the following lemma:

**Lemma 4.1.** (see \cite{[3, 7, 13, 18]}) For all \( e \in \mathcal{T}_h \), \( \forall F_e \in \partial e \), \( \forall v_h \in V_h \), it holds

\[
\int_e (v_h - I_h^* v_h) d x = 0, \quad \int_{\partial e} (v_h - I_h^* v_h) d s = 0, \quad \| I_h^* v_h \|_{0, e} \leq C \| v_h \|_{0, e},
\]

where \( h_e \) is the diameter of \( e \), and \( C \) is a constant independent of \( h_e \).

We define another bilinear form as follows

\[
a^n(u, v) = \int_{\Omega} (A^n \nabla u - b^n u) \cdot \nabla v d x + \int_{\Omega} r^n u v d x,
\]

and make the assumption:

\[
(\ref{eq:gamma_prime}) = a^n(v, v) \geq \gamma' \| v \|_{1, \Omega}, \quad \forall v \in H^1(\Omega),
\]

where \( \gamma' > 0 \) is a constant, then we obtain the following proposition:

**Proposition 4.1.** We can prove that

\[
a^n(u, v) = a^n(u, v) + a^n(u, v), \quad \forall u, v \in H^2(\mathcal{T}_h) \cap H^1(\Omega),
\]

where \( H^2(\mathcal{T}_h) = \{ v \in L^2(\Omega) : \forall e \in \mathcal{T}_h \}, \) and that

\[
a^n(u, v) = \int_{\Omega} (A^n \nabla u - b^n u) \cdot \nabla v d x - \sum_{e \in \mathcal{T}_h} \int_{\partial e} (A^n \nabla u - b^n u)(I_h^* v - v) d x + \sum_{e \in \mathcal{T}_h} \int_{\partial e} (A^n \nabla u - b^n u) \cdot \nu_e (I_h^* v - v) d x,
\]

where \( \nu \) and \( \nu_e \) denote the outward unit normals to \( \Omega \) and \( e \), respectively. Furthermore, if \( A^n \in (W^{1,\infty}(\Omega))^{3 \times 3}, b^n \in (W^{1,\infty}(\Omega))^3, r^n \in L^\infty(\Omega), \) then

\[
|a^n(u_h, v_h)| \leq C h |u_h|_{1,\Omega} |v_h|_{1,\Omega}, \quad \forall u_h, v_h \in V_h.
\]

If \( A^n \in Q_h^{3 \times 3}, b^n \in (W^{2,\infty}(\Omega))^3, r^n \in W^{1,\infty}(\Omega), \) then

\[
|a^n(u_h, v_h)| \leq C h^2 |u_h|_{1,\Omega} |v_h|_{1,\Omega}, \quad \forall u_h, v_h \in V_h,
\]

where \( Q_h = \{ v_h \in L^2(\Omega) : \forall e \in P_0(\mathcal{T}_h), \forall e \in \mathcal{T}_h \}. \)

**Proof.** For any vertex \( P \) of a given \( e \in \mathcal{T}_h \), denote by \( K_P \) the control volume containing the node \( P \) and by \( K_P|_e = D_P \) the part of \( K_P \) restricted onto \( e \) (see Fig. 3). Then the tetrahedron \( e \) can be divided into four parts (denoted by \( D_i, i = 1, 2, 3, 4 \)). The boundary of \( D_i \) denoted by \( \partial D_i \) consists of two parts: \( \Gamma_i \)
and $\Gamma_i$, $\Gamma_i = \partial D_i \cap \partial e$ and $\Gamma_i = \partial D_i \setminus \Gamma_i$. Let $G := A^u \nabla u - b^u u$. For any $u, v \in H^2(\mathcal{T}_h) \cap H^1(\Omega)$, we have

$$
\sum_{e \in \mathcal{T}_h} \int_{\partial e}^* G \cdot \nu_e^* I_h^e v ds = \sum_{e \in \mathcal{T}_h} \sum_{i=1}^4 \int_{\Gamma_i^e} G \cdot \nu_e^* I_h^e v ds + \int_{\partial \Omega} G \cdot \nu v ds,
$$

and

$$
\sum_{e \in \mathcal{T}_h} \int_{\partial e}^* G \cdot \nu_e^* I_h^e v ds = \sum_{e \in \mathcal{T}_h} \sum_{i=1}^4 \left\{ \int_{D_i} \nabla \cdot G I_h^e v dx - \int_{\Gamma_i} G \cdot \nu_e^* I_h^e v ds \right\}
$$

$$
= \sum_{e \in \mathcal{T}_h} \int_{\partial e}^* G \nu_e^* I_h^e v ds - \sum_{e \in \mathcal{T}_h} \int_{\partial e} G \cdot \nu_e I_h^e v ds = \sum_{e \in \mathcal{T}_h} \int_{\partial e}^* G (I_h^e v - v) ds dx
$$

$$
- \sum_{e \in \mathcal{T}_h} \int_{\partial e} G \cdot \nu_e (I_h^e v - v) ds + \sum_{e \in \mathcal{T}_h} \int_{\partial e} \nabla \cdot G v dx - \sum_{e \in \mathcal{T}_h} \int_{\partial e} G \cdot \nu v ds
$$

$$
= \sum_{e \in \mathcal{T}_h} \int_{\partial e}^* G (I_h^e v - v) ds - \sum_{e \in \mathcal{T}_h} \int_{\partial e} G \cdot \nu_e (I_h^e v - v) ds - \int_{\Omega} G \cdot \nabla v dx.
$$

From (48), (50), (55) and (56), we get $a_h^n(u, v) = a_h^n(u, v) + a^n(u, v)$. For any $u_h, v_h \in V_h$, it is obvious that

$$
\left| \sum_{e \in \mathcal{T}_h} \int_{\partial e}^* (A^u \nabla u_h)(I_h^e v_h - v_h) dx \right| \leq \sum_{e \in \mathcal{T}_h} Ch_{\mathcal{T}} \left| A^u \nabla u_h \right|_{1, \infty, e} \left| \nabla u_h \right|_{0, e, \partial \Omega_{1, e}} v_h \leq Ch \left| A^u \nabla u_h \right|_{1, \infty} v_h \left| v_h \right|_{1, \Omega}.
$$

$$
\left| \sum_{e \in \mathcal{T}_h} \int_{\partial e}^* (b^u u_h)(I_h^e v_h - v_h) dx \right| \leq Ch \left| b^u u_h \right|_{1, \infty} v_h \left| v_h \right|_{1, \Omega}.
$$

For any given $e \in \mathcal{T}_h$, it follows from Lemma 4.1 that

$$
\left| \int_{\partial e}^* A^u \nabla u_h \cdot \nu_e (I_h^e v_h - v_h) ds \right| \leq Ch_{\mathcal{T}} \left| \nabla u_h \right|_{0, \partial e, v_h} \left| I_h^e v_h - v_h \right|_{0, \partial e} \leq Ch_{\mathcal{T}} \left| \nabla u_h \right|_{0, \partial e, \partial \Omega_{1, e}} v_h \left| v_h \right|_{1, \Omega}.
$$

Define $Lip(\Omega) := \left\{ v \in L^\infty(\Omega) : \sup \left\{ \left| \frac{v(x) - v(y)}{x - y} \right| : x, y \in \Omega; x \neq y \right\} < \infty \right\}$. Since $\Omega$ is a bounded convex domain, one can deduce that $W^{1, \infty}(\Omega) \subset Lip(\Omega)$, see [6, p. 30]. We thus have

$$
\sum_{e \in \mathcal{T}_h} \int_{\partial e}^* b^u u_h \cdot \nu_e (I_h^e v_h - v_h) ds = \sum_{F_e \in \mathcal{E}_h^e} \int_{F_e} \left[ b^u u_h (I_h^e v_h - v_h) \right]_{F_e} ds = 0,
$$

where $\mathcal{E}_h^e = \left\{ F_e : F_e \subset \partial e, F_e \notin \partial \Omega, \forall e \in \mathcal{T}_h \right\}$, $[w]_{F_e} = w|_{\partial e_1} \cdot \nu_e^{(1)} + w|_{\partial e_2} \cdot \nu_e^{(2)}$, $F_e = e_1 \cap e_2$, $\nu_e^{(i)}$ is the unit normal vector on $F_e$ outward to $\partial e_i$, $i = 1, 2$. For the
last term in \(a^n_\partial (u_h, v_h)\), we have
\[
|\sum_{e \in T_h} \int_e r^n u_h (I_h^e v_h - v_h) dx| \leq C \sum_{e \in T_h} \|r^n\|_{0, \infty, e} h_e \|u_h\|_{0, e} \|v_h\|_{1, e} \\
\leq Ch \|r^n\|_{0, \infty} \|u_h\|_{1, \Omega} \|v_h\|_{1, \Omega}.
\]
Estimate (53) follows from (57)-(61).

If \(A^n \in Q_h^{3 \times 3}\) and \(b^n \in (W^{2, \infty}(\Omega))^3\), applying Lemma 4.1, we get
\[
a^n_\partial (u_h, v_h) = \sum_{e \in T_h} \int_e (\nabla \cdot (b^n u_h) + r^n u_h)(I_h^e v_h - v_h) dx, \quad \forall u_h, v_h \in V_h.
\]
For any \(\varphi \in H^1(\Omega), v_h \in V_h\), it follows from Lemma 4.1 that
\[
|\sum_{e \in T_h} \int_e \varphi (I_h^e v_h - v_h) dx| = \left| \sum_{e \in T_h} \int_e (\varphi - M_e(\varphi))(I_h^e v_h - v_h) dx \right|
\leq \sum_{e \in T_h} \|\varphi - M_e(\varphi)\|_{0, e} \|I_h^e v_h - v_h\|_{0, e}
\leq Ch^2 \sum_{e \in T_h} |\varphi|_{1, e} \|v_h\|_{1, e} \leq Ch^2 |\varphi|_{1, \Omega} \|v_h\|_{1, \Omega},
\]
where \(M_e(\varphi) = \frac{1}{|e|} \int_e \varphi(x) dx\). (54) follows from (62) and (63). Therefore, the proof of Proposition 4.1 is complete.

We define a Ritz projection \(R_h : H^1_0(\Omega) \to V_h\) as follows:
\[a^n_\partial (R_h \varphi, v_h) = a^n_\partial (\varphi, v_h), \quad \forall v_h \in V_h, \quad \varphi \in H^1_0(\Omega).
\]
One can directly check that (see also[12, p. 80])
\[
|\varphi - R_h \varphi|_{0, \Omega} + h |\varphi - R_h \varphi|_{1, \Omega} \leq Ch^2 |\varphi|_{2, \Omega}, \quad \forall \varphi \in H^2(\Omega) \cap H^1_0(\Omega),
\]
(64)
\[
|\varphi - R_h \varphi|_{0, \Omega} \leq Ch |\varphi|_{1, \Omega}, \quad \varphi \in H^1_0(\Omega).
\]
For \(t = t_n\), we split the error into two parts: \(u(\cdot, t_n) - u_h^n = \rho^n + \theta_h^n\), where \(\rho^n = u(\cdot, t_n) - R_h u(\cdot, t_n), \theta_h^n = R_h u(\cdot, t_n) - u_h^n\). It follows from (64) that
\[
|\rho^n|_{0, \Omega} + h |\rho^n|_{1, \Omega} \leq Ch^2 |u(\cdot, t_n)|_{2, \Omega}.
\]
Observe that
\[
(\partial \theta_h^n, v_h) + a^n_\partial (\theta_h^n, v_h) = F^n_h (v_h), \quad \forall v_h \in V_h,
\]
where
\[
F^n_h (v_h) = (\partial R_h u(\cdot, t_n) - u(\cdot, t_n), v_h) + X^n_h (v_h; u_h^n, f^n),
\]
\[
X^n_h (v_h; u_h^n, f^n) = (\partial u_h^n, I_h^e v_h - v_h) + a^n_\partial (u_h^n, v_h) - (f^n, I_h^e v_h - v_h).
\]
We first note that
\[
\partial R_h u(\cdot, t_n) - u(\cdot, t_n) = (R_h - 1) \partial u(\cdot, t_n) + (\partial u(\cdot, t_n) - u(\cdot, t_n)) = \omega_1^n + \omega_2^n.
\]
Following (7.40) of [12, p. 84], we obtain
\[
\Delta t \|\omega_1^n\|_{0, \Omega} \leq Ch \int_{t_{n-1}}^{t_n} |u(\cdot, s)|_{2, \Omega} ds,
\]
(67)
\[
\Delta t \|\omega_1^n\|_{0, \Omega} \leq Ch \int_{t_{n-1}}^{t_n} \|u(\cdot, s)|_{1, \Omega} ds,
\]
\[
\|\omega_2^n\|_{0, \Omega} \leq C \int_{t_{n-1}}^{t_n} \|u(\cdot, s)\|_{0, \Omega} ds.
\]
where $C$ is a constant independent of $h$, $\Delta t$, $\varepsilon$; $h$ and $\Delta t$ are the mesh parameter and the time step for solving the modified homogenized convection-diffusion equation (44), respectively.

In order to estimate the term $X_h^n(v_h; u_h^n, f^n)$, we introduce the following hypotheses

(H$_1$) : $A$ consists of elements $a_{ij} \in L^\infty(0, T; W^{1,\infty}(\Omega))$,

$b$ has entries $b_i \in L^\infty(0, T; W^{1,\infty}(\Omega))$, $0 \leq r \in L^\infty(0, T; L^\infty(\Omega))$;

(H$_2$) : $A$ consists of elements $a_{ij} \in W^{1,\infty}(0, T; Q_h)$,

$b$ has entries $b_i \in W^{1,\infty}(0, T; W^{2,\infty}(\Omega))$, $0 \leq r \in W^{1,\infty}(0, T; W^{1,\infty}(\Omega))$,

and give the following proposition:

**Proposition 4.2.** For a given $u_h^n \in V_h$ and $f^n = f(\cdot, t_n)$, $\forall v_h \in V_h$, we can prove that under the hypotheses (H$_1$):

\[
|X_h^n(v_h; u_h^n, f^n)| \leq ChJ_0^n|v_h|_{1,\Omega};
\]

under the hypotheses (H$_2$):

\[
|X_h^n(v_h; u_h^n, f^n)| \leq \begin{cases} Ch^2J_0^n|v_h|_{1,\Omega}; \\ ChJ_0^n|v_h|_{0,\Omega}. \end{cases}
\]

where $J_0^n = |u_h^n|_{1,\Omega} + \|\hat{\partial}u_h^n\|_{0,\Omega} + \|f^n\|_{0,\Omega}$ and $J_1^n = |u_h^n|_{1,\Omega} + |\hat{\partial}u_h^n|_{1,\Omega} + |f^n|_{1,\Omega}$.

**Proof.** Applying Lemma 4.1, for $\varphi \in H^1(\Omega)$, we get

\[
|\varphi, I_h^n v_h - v_h| = \left| \sum_{e \in T_h} \int_e \varphi(I_h^n v_h - v_h) dx \right|
\]

\[
= \left| \sum_{e \in T_h} \int_e (\varphi - M_e(\varphi))(I_h^n v_h - v_h) dx \right|
\]

\[
\leq \sum_{e \in T_h} \|\varphi - M_e(\varphi)\|_{0,e} \|I_h^n v_h - v_h\|_{0,e}
\]

\[
\leq \sum_{e \in T_h} Ch_e |\varphi|_{1,e} h_e |v_h|_{1,e} \leq Ch^2|\varphi|_{1,\Omega} |v_h|_{1,\Omega},
\]

where $M_e(\varphi) = \frac{1}{|e|} \int_e \varphi(x) dx$. On the other hand, we have

\[
|\varphi, I_h^n v_h - v_h| = \left| \sum_{e \in T_h} \int_e \varphi(I_h^n v_h - v_h) dx \right|
\]

\[
\leq \left| \sum_{e \in T_h} \int_e (\varphi - M_e(\varphi))(I_h^n v_h - v_h) dx \right|
\]

\[
\leq \sum_{e \in T_h} \|\varphi - M_e(\varphi)\|_{0,e} \left( \|I_h^n v_h\|_{0,e} + \|v_h\|_{0,e} \right)
\]

\[
\leq C \sum_{e \in T_h} h_e |\varphi|_{1,e} \|v_h\|_{0,e} \leq Ch |\varphi|_{1,\Omega} \|v_h\|_{0,\Omega}.
\]

Similarly, we can show that

\[
|\varphi, I_h^n v_h - v_h| \leq Ch |\varphi|_{0,\Omega} |v_h|_{1,\Omega}.
\]
From (70)-(72), applying Proposition 4.1, we can complete the proofs of (68) and (69). □

Remark 4.3. If $h^2 \leq \min\left(\frac{\gamma^2}{4\mu\epsilon}, \frac{\Delta t^2}{2(c_1)^2}\right)$ and $\Delta t \leq \frac{T}{2}$ one can show that under the hypotheses $(H_1)$

$$\Delta t \sum_{i=1}^{n} |J(t_i)|^2 \leq C \left\{ \|u_h^0\|^2_{1, \Omega} + \Delta t \sum_{i=1}^{n} \|f^i\|^2_{0, \Omega} \right\}$$

under the hypotheses $(H_2)$

$$\Delta t \sum_{i=1}^{n} |J(t_i)|^2 \leq C \left\{ \|u_h^0\|^2_{1, \Omega} + \Delta t \sum_{i=1}^{n} \|f^i\|^2_{1, \Omega} + \Delta t \sum_{i=1}^{n} \|\partial f^i\|^2_{0, \Omega} \right\}.$$  

Here, $M_r = \|r\|_{L^\infty(0,T;L^\infty(\Omega))}$, $C_1$ and $C_T$ are constants satisfying

$$\|v_h - T_h v_h\|_{0,e} \leq C_1 h \|v_h\|_{1,e}, \quad \forall \ v_h \in V_h$$

$$\|v_h - \frac{1}{|e|} \int v_h(x) dx\|_{0,e} \leq C_T h \|v_h\|_{1,e}, \quad \forall \ v_h \in V_h.$$  

Next we give the error estimates of the finite volume element method for solving the modified homogenized convection-diffusion equation (44) in a whole domain $\Omega$.  

Theorem 4.2. Suppose $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded convex Lipschitz domain. Let $u(x,t)$ be the unique weak solution of the modified homogenized convection-diffusion equation (44), and let $u_h^0 = u_h(x, t_n)$ be the associated fully discrete finite volume element solution at time $t = t_n$ of $u(x,t)$ in a coarse mesh and at a larger time step. Let $h$ be the mesh size and $\Delta t(\leq \frac{T}{2})$ be the time step. Set $E_n = u(x, t_n) - u_h^0$. The assumptions $(A_1)$ - $(A_3)$ and the initial conditions

$$\|u_h^0 - u_0\|_{0, \Omega} + h \|u_h^0 - \tilde{u}_0\|_{1, \Omega} \leq C h^2 \|u_0\|_{2, \Omega}$$  

are satisfied, where $\tilde{u}_0^0(x)$ is the interpolation function of the initial function $\hat{u}_0(x)$ on $V_h$. Under the hypothesis $(H_1)$, it holds

$$\|E_n\|_{0, \Omega} + \left( \sum_{i=1}^{n} \Delta t |E_i|^2_{0, \Omega} \right)^{\frac{1}{2}} \leq C h \left\{ |u_0|_{2, \Omega} + h |\tilde{u}_0|_{2, \Omega} + \left( \sum_{i=1}^{n} \Delta t |u_i|_{2, \Omega}^2 \right)^{\frac{1}{2}} \right\}$$

$$+ h \left( \int_0^{t_n} |u_i|_{2, \Omega}^2 ds \right)^{\frac{1}{2}} + \left( \sum_{i=1}^{n} \Delta t |J_i|^2 \right)^{\frac{1}{2}} + C \Delta t \left( \int_0^{t_n} |u_i|_{0, \Omega}^2 ds \right)^{\frac{1}{2}}.$$

Under the hypothesis $(H_2)$, it holds

$$\|E_n\|_{0, \Omega} \leq C h \left\{ |u_0|_{2, \Omega} + h |\tilde{u}_0|_{2, \Omega} + \left( \int_0^{t_n} |u_i|_{2, \Omega}^2 ds \right)^{\frac{1}{2}} \right\}$$

$$+ \left( \sum_{i=1}^{n} \Delta t |J_i|^2 \right)^{\frac{1}{2}} + C \Delta t \left( \int_0^{t_n} |u_i|_{0, \Omega}^2 ds \right)^{\frac{1}{2}},$$

$$\|E_n\|_{1, \Omega} \leq C h \left\{ |u_0|_{2, \Omega} + h |\tilde{u}_0|_{2, \Omega} + \left( \int_0^{t_n} |u_i|_{2, \Omega}^2 ds \right)^{\frac{1}{2}} \right\}$$

$$+ \left( \sum_{i=1}^{n} \Delta t |J_i|^2 \right)^{\frac{1}{2}} + C \Delta t \left( \int_0^{t_n} |u_i|_{0, \Omega}^2 ds \right)^{\frac{1}{2}}.$$  

where $C$ is a constant independent of $h$ and $\Delta t$, but dependent of $T$.  

Proof. By setting $\theta^h_0 = R_h \tilde{u}_0 - \tilde{u}_0 + \tilde{u}_0 - a^0_h$, and using (73), we have

$$\|\theta^h_0\|_{0,\Omega} + b\|\theta^h_0\|_{1,\Omega} \leq C h^2 |u_0|_{2,\Omega}.$$ 

Next we only prove the estimates (75) and (76). The proof of (74) is similar. Setting $v_h = \theta^n_h$ in (66) and using (51), we thus get

$$(\partial_\theta^h, \theta^n_h) + \gamma |\theta^n_h|^2_{1,\Omega} \leq \|\omega^n_1 + \omega_2^n\|_{0,\Omega} \|\theta^n_h\|_{0,\Omega} + |X^n_h(\theta^n_h; u^n_h, f^n)|.$$ 

It is obvious that

$$\Delta t (\partial_\theta^h, \theta^n_h) = \frac{1}{2}\|\theta^n_h\|^2_{0,\Omega} + \frac{1}{2}\|\theta^n_h - \theta^{n-1}_h\|^2_{0,\Omega} - \frac{1}{2}\|\theta^{n-1}_h\|^2_{0,\Omega}.$$ 

It follows from Proposition 4.2 that

$$|X^n_h(\theta^n_h; u^n_h, f^n)| \leq \frac{Ch^4}{\gamma} |J^n_1|^2 + \gamma |\theta^n_h|^2_{1,\Omega}.$$ 

Hence we have

$$\|\theta^n_h\|^2_{0,\Omega} \leq \|\theta^{n-1}_h\|^2_{0,\Omega} + Ch^4 \int_{t_{n-1}}^{t_n} |u_t(\cdot, s)|^2_{2,\Omega} ds + C(\Delta t)^2 \int_{t_{n-1}}^{t_n} \|u_t(\cdot, s)\|^2_{0,\Omega} ds$$

$$+ \frac{Ch^4}{\gamma} \Delta t |J^n_1|^2 + \Delta t \|\theta^n_h\|^2_{0,\Omega}.$$ 

and consequently

$$\|\theta^n_h\|^2_{0,\Omega} \leq \|\theta^{n-1}_h\|^2_{0,\Omega} + Ch^4 \int_{0}^{t_n} |u_t(\cdot, s)|^2_{2,\Omega} ds + C(\Delta t)^2 \int_{0}^{t_n} \|u_t(\cdot, s)\|^2_{0,\Omega} ds$$

$$+ \frac{Ch^4}{\gamma} \sum_{i=1}^{n} \tau |J^n_i|^2 + \Delta t \sum_{i=1}^{n} \|\theta^n_h\|^2_{0,\Omega}.$$ 

Since $\Delta t < \frac{1}{2}$, using the discrete Gronwall’s inequality, we get

$$\|\theta^n_h\|^2_{0,\Omega} \leq C \{h^4 |u_0|_{2,\Omega} + h^4 \int_{0}^{t_n} |u_t(\cdot, s)|^2_{2,\Omega} ds + h^4 \sum_{i=1}^{n} \Delta t |J^n_i|^2$$

$$+ (\Delta t)^2 \int_{0}^{t_n} \|u_t(\cdot, s)\|^2_{0,\Omega} ds \}.$$ 

(77)

where $C$ is a constant independent of $h$ and $\Delta t$. Therefore, (75) follows from (65), (73) and (77).

It remains to prove (76). Taking $v_h = \partial_\theta^h$ in (66) and using Proposition 4.2, we obtain

$$|\partial_\theta^h|_{0,\Omega}^2 + a(\theta^n_h, \partial_\theta^h) \leq |\omega^n_1 + \omega_2^n|_{0,\Omega} |\partial_\theta^h|_{0,\Omega} + ChJ^n_1 |\partial_\theta^h|_{0,\Omega}.$$ 

Define the following two bilinear forms

$$a^1_h(u, v) = \int_{\Omega} A^n \nabla u \nabla v dx + \int_{\Omega} r^n u v dx, \quad a_2^1(u, v) = - \int_{\Omega} b^n u v dx.$$ 

Hence we get

$$a^1_h(\theta^n_h, \partial_\theta^h) = a^1_h(\theta^n_h, \partial_\theta^h) + a_2^1(\theta^n_h, \partial_\theta^h)$$

$$= \frac{1}{\Delta t} a^1_h(\theta^n_h, \theta^n_h) - \frac{1}{\Delta t} a^1_h(\theta^n_h, \theta^{n-1}_h) + a_2^1(\theta^n_h, \partial_\theta^h).$$ 

(79)

For the last two terms in (79), we have

$$a^1_h(\theta^n_h, \theta^{n-1}_h) \leq \frac{1}{2} a_1^1(\theta^n_h, \theta^n_h) + \frac{1}{2} a_1^1(\theta^n_h, \theta^{n-1}_h),$$

$$|a^1_h(\theta^n_h, \partial_\theta^h)| \leq C h^4 |J^n_1|_{0,\Omega} |\partial_\theta^h|_{0,\Omega}.$$ 

(80)
Combining (78), (79) and (80) gives
\[
a^n_t(\theta_h^n, \theta_h^n) \leq a^n_1(\theta_h^{n-1}, \theta_h^{n-1}) + C\Delta t|\theta_h^n|_{1,\Omega}^2 + Ch^2\Delta t|J^n_t|^2 \\
+ Ch^4 \int_{t_{n-1}}^{t_n} |u_t(\cdot, s)|_{2,\Omega}^2 ds + C(\Delta t)^2 \int_{t_{n-1}}^{t_n} \|u_{tt}(\cdot, s)\|^2_{0,\Omega} ds,
\]
Thanks to the assumption (H2), we get
\[
|a^n_1(\theta_h^{n-1}, \theta_h^{n-1}) - a^n_1(\theta_h^{n-1}, \theta_h^{n-1})| \leq C\Delta t|\theta_h^n|_{1,\Omega}^2
\]
and consequently
\[
a^n_1(\theta_h^n, \theta_h^n) \leq a^n_1(\theta_h^{n-1}, \theta_h^{n-1}) + C\Delta t|\theta_h^n|_{1,\Omega}^2 + |\theta_h^n|_{2,\Omega}^2 + Ch^2\Delta t|J^n_t|^2 \\
+ Ch^4 \int_{t_{n-1}}^{t_n} |u_t(\cdot, s)|_{2,\Omega}^2 ds + C(\Delta t)^2 \int_{t_{n-1}}^{t_n} \|u_{tt}(\cdot, s)\|^2_{0,\Omega} ds.
\]
Applying the discrete Gronwall’s inequality again, we obtain
\[
|\theta_h^n|_{1,\Omega} \leq Ch^2 \left\{ |\theta_h^n|_{2,\Omega}^2 + \sum_{i=1}^{n} \Delta t|J_i^n|^2 + h^2 \int_{0}^{t_n} |u_t(\cdot, s)|_{2,\Omega}^2 ds \right\} \\
+ C(\Delta t)^2 \int_{0}^{t_n} \|u_{tt}(\cdot, s)\|^2_{0,\Omega} ds,
\]
where \( C \) is a constant independent of \( h, \Delta t \). Therefore (76) follows from (65), (73) and (81).

**Remark 4.4.** As we already mentioned in Section 3, in a bounded convex domain, we need to define the boundary layer solutions. We employ the finite volume element method (FVEM) to solve the boundary layer equations (12). Due to space limitation, we omit the detailed process.

5. Multiscale hybrid FEM-FVM algorithm and numerical examples

5.1. Multiscale hybrid FEM-FVM algorithm. To summarize the above results, the multiscale hybrid FEM-FVM method for the convection-diffusion equation with rapidly oscillating coefficients can now be described as follows:

**Step I:** The adaptive finite element method (AFEM) is first applied to compute the cell functions \( N_m(\xi, \tau), N_{m1}(\xi, \tau), M_l(\xi, \tau), R_0(\xi, \tau) \) and \( R_l(\xi, \tau) \) on \( Q \times (0, T) \) (or \( Q \times (0, \tau_0) \)) defined in (3)-(10), where \( m, l = 1, 2, \ldots, d \).

**Step II:** The finite volume element method (FVEM) is then used to solve the modified homogenized convection-diffusion equation (44) in a whole domain \( \Omega \times (0, T) \) using a coarse mesh and a larger time step.

**Step III:** The finite volume element method (FVEM) is applied to solve the boundary layer equations over a subdomain \( \Omega_1 \times (0, T) \) using a fine mesh and a small time step, see (12).

**Step IV:** The derivatives \( \frac{\partial u(x, t)}{\partial x_m} \) and \( \frac{\partial^2 u(x, t)}{\partial x_m \partial x_l} \), \( m, l = 1, 2, \cdots, d \) are computed by the finite difference method, where \( u(x, t) \) is the unique solution of the modified homogenized convection-diffusion equation (44). The computational formulas can be found in [8, 9, 10].

**Remark 5.1.** From Theorems 3.2, 4.1 and 4.2, we can derive the convergence results of the multiscale hybrid FEM-FVM method for the original problem (1).
5.2. Numerical examples. To verify the numerical accuracy for the finite volume element method (FVEM) introduced in this paper, we first consider Example 5.1.

Example 5.1. Consider the convection-diffusion equation (44) with Dirichlet boundary condition with constant coefficients, in which

\[ A = 0.1 I, \quad b = (1, 3, 1)^T, \quad r = 1, \]

where \( I \) is an identity matrix, \( \Omega = (0, 1)^3, T = 1 \), and the exact solution is given by

\[ u(x, t) = \varepsilon^{(1-0.5)t} \left( x - e^{-\frac{x}{\varepsilon^2}} \right) \left( y^2 - e^{-\frac{2(y-1)}{k^2}} \right) \left( z - e^{-\frac{z}{\varepsilon^2}} \right). \]

The initial and boundary conditions for problem (82) and the term of right side \( f(x, t) \) are determined by (83). As we already mentioned in (47), the finite volume element method in space and the backward Euler difference scheme in time are applied. We take the time step \( \Delta t = 0.01 \). Relative errors at \( t = 1.0 \) in norms \( \| \cdot \|_{L^2(\Omega)} \) and \( \| \cdot \|_{H^1(\Omega)} \) are displayed in Fig. 4, which support the error estimates obtained in Theorem 4.2. Here, \( h \) denotes the mesh size.

To validate the multiscale hybrid FEM-FVM algorithm and to confirm the theoretical analysis reported in this paper, numerical examples in 2-D and 3-D composite materials are presented.

Example 5.2. Consider the 2-D convection-diffusion equation with rapidly oscillating periodic discontinuous coefficients as follows:

\[
\begin{cases}
\frac{\partial u^\varepsilon(x, t)}{\partial t} - \frac{\partial}{\partial x_i} \left( a_{ij}(x \varepsilon^i \varepsilon) \frac{\partial u^\varepsilon(x, t)}{\partial x_j} \right) + \frac{\partial}{\partial x_1} \left( b_i(x \varepsilon^i \varepsilon) u^\varepsilon(x, t) \right) \\
+ a_0(x \varepsilon^i \varepsilon) u^\varepsilon(x, t) = f(x, t), \quad (x, t) \in \Omega \times (0, T), \\
u^\varepsilon(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u^\varepsilon(x, 0) = \bar{u}_0(x), \quad x \in \Omega,
\end{cases}
\]

For convenience, we choose \( \Omega = (0, 1)^2 \) and the reference cell \( Q = (0, 1)^2 \) which are shown in Fig. 5. We take \( \varepsilon = \frac{1}{8}, f(x, t) = 100, \bar{u}_0(x) = 0 \) and \( T = 1 \). Let \( \xi = \varepsilon^{-1} x \) and \( \tau = t/\varepsilon^k \), \( k = 0, 1 \).

Let \( S \) be a square inclusion of the reference cell \( Q \), assume that the coefficients \( a_{ij}(\xi, \tau), b_i(\xi, \tau) \) and \( a_0(\xi, \tau) \) are taken as follows:

\[
a_{ij}(\xi, \tau) = \begin{cases} 
10^{-2} a_{ij0}(\xi, \tau), & \xi \in S, \\
10^{-2} a_{ij0}(\xi, \tau), & \xi \in Q \setminus S,
\end{cases}
\]

\[
b_i(\xi, \tau) = \begin{cases} 
10^{-2} b_{i0}(\xi, \tau), & \xi \in S, \\
b_{i0}(\xi, \tau), & \xi \in Q \setminus S,
\end{cases}
\]
\[ a_0(\xi, \tau) = \begin{cases} 
10^{-2}a_{00}(\xi, \tau), & \xi \in S, \\
\quad a_{00}(\xi, \tau), & \xi \in Q \setminus S, 
\end{cases} \]

where
\[ a_{i0}(\xi, \tau) = 200(1 + 0.2\cos(2\pi\xi_1))(1 + 0.2\cos(2\pi\xi_2))(1 + 0.2\cos(2\pi\tau)) \delta_{ij}, \]
\[ b_{i0}(\xi, \tau) = 100(1 + 0.4\cos(2\pi\xi_1)\cos(2\pi\xi_2))(1 + 0.3\cos(2\pi\tau)), \]
\[ a_{00}(\xi, \tau) = 100(1 + 0.2\cos(2\pi\tau)), \quad \delta_{ij} \text{is the Kronecker symbol.} \]

In order to demonstrate the numerical accuracy of the present method, the exact solution \( u^\varepsilon(x, t) \) of the original problem (84) must be available. Since the exact solution is not available, we replace \( u^\varepsilon(x, t) \) by the numerical solution computed using a linear finite volume element method with a fine mesh and a small time step. This is a time-consuming task, and we only present the results for the case \( \varepsilon = \frac{1}{8} \).

The computational procedure is now briefly described. Firstly, we employ the linear triangle elements to solve cell problems (13)-(20). The linear finite volume elements are then applied to solve the modified homogenized convection-diffusion equation (44) in a coarse mesh using a larger time step. In particular, the computational cost are shown in Table 1.

**Table 1.** Computational costs for Example 5.2 with \( k = 1 \) and \( \varepsilon = 1/8 \).

<table>
<thead>
<tr>
<th>DOF</th>
<th>the original problem</th>
<th>the homogenized one</th>
<th>cell ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of nodes</td>
<td>9,409</td>
<td>1,369</td>
<td>1,369</td>
</tr>
<tr>
<td>number of elements</td>
<td>18,432</td>
<td>2,592</td>
<td>2,592</td>
</tr>
<tr>
<td>the time step</td>
<td>1.25e-3</td>
<td>1.0e-2</td>
<td>-</td>
</tr>
</tbody>
</table>

Set \( e_0 = u^\varepsilon - u^0, e_1 = u^\varepsilon - u_1^\varepsilon, e_2 = u^\varepsilon - u_2^\varepsilon \). For simplicity, denote by \( \| \cdot \|_{L^2(L^2)} \) the norm \( \| \cdot \|_{L^2(0,T;L^2(\Omega))} \) and by \( \| \cdot \|_{L^2(H^1)} \) the norm \( \| \cdot \|_{L^2(0,T;H^1(\Omega))} \). The relative errors for the homogenization method \( (s = 0) \), the first-order \( (s = 1) \) and the second-order \( (s = 2) \) multiscale methods in Example 5.2 are reported in Table 2.

The computational results for Example 5.2 with \( k = 0, 1 \) in line \( x_1 = x_2 \) at time \( t = 0.6 \) are displayed in Fig. 6. The figure shows that the exact solution exhibits oscillating behaviors at the interface between two materials and that the second-order multiscale solution makes a good agreement with the exact solution, while the
Table 2. Relative errors for Example 5.2 with $k = 0, 1$ and $\varepsilon = 1/8$.

<table>
<thead>
<tr>
<th></th>
<th>$s$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 0$</td>
<td>$\frac{|e_u|<em>{L^2(L^2)}}{|u|</em>{L^2(L^2)}}$</td>
<td>6.6931e-02</td>
<td>5.8982e-02</td>
<td>9.1313e-03</td>
</tr>
<tr>
<td></td>
<td>$\frac{|e_u|<em>{L^2(H^1)}}{|u|</em>{L^2(H^1)}}$</td>
<td>7.1000e-01</td>
<td>6.4740e-01</td>
<td>7.1110e-02</td>
</tr>
<tr>
<td>$k = 1$</td>
<td>$\frac{|e_u|<em>{L^2(L^2)}}{|u|</em>{L^2(L^2)}}$</td>
<td>1.6032e-01</td>
<td>1.5725e-01</td>
<td>1.4088e-01</td>
</tr>
<tr>
<td></td>
<td>$\frac{|e_u|<em>{L^2(H^1)}}{|u|</em>{L^2(H^1)}}$</td>
<td>7.1739e-01</td>
<td>6.5704e-01</td>
<td>1.2741e-01</td>
</tr>
</tbody>
</table>

Homogenized and first-order multiscale methods do not even capable of capturing the oscillatory behaviors.

Figure 6. Solutions on line $x_1 = x_2$; Left: $k = 0$. Right: $k = 1$.

Example 5.3. Now, consider the same equation as Example 5.2 but in 3-D case, where, for simplicity, a whole domain $\Omega = (0, 1)^3$ and the reference cell $Q = (0, 1)^3$ are shown in Fig. 7. We take $\varepsilon = \frac{1}{8}$, $f(x, t) = 100$, $u_0(x) = 0$ and $T = 1$. Let $\xi = \varepsilon^{-1}x$ and $\tau = t/\varepsilon^k$, $k = 0, 1$.

Figure 7. Left: A whole domain $\Omega$; Right: the reference cell $Q$.

Let $B$ be a cube inclusion of the reference cell $Q$, assume that the coefficients $a_{ij}(\xi, \tau)$, $b_i(\xi, \tau)$ and $a_0(\xi, \tau)$ are taken as follows:

$$a_{ij}(\xi, \tau) = \begin{cases} 10^{-2}a_{i,j,0}(\xi, \tau), & \xi \in B, \\ a_{ij,0}(\xi, \tau), & \xi \in Q \setminus B. \end{cases}$$

$$b_i(\xi, \tau) = \begin{cases} 10^{-2}b_{i,0}(\xi, \tau), & \xi \in B, \\ b_{i,0}(\xi, \tau), & \xi \in Q \setminus B. \end{cases}$$
\[ a_0(\xi, \tau) = \begin{cases} 10^{-2}a_{00}(\xi, \tau), & \xi \in B, \\ a_{00}(\xi, \tau), & \xi \in Q \setminus B, \end{cases} \]

where
\[ a_{ij0}(\xi, \tau) = 200(1 + 0.2\cos(2\pi\xi_1))(1 + 0.2\cos(2\pi\xi_2))(1 + 0.2\cos(2\pi\xi_3)) 
\]
\[ (1 + 0.2\cos(2\pi\tau))\delta_{ij}, \]

\[ b_{i0}(\xi, \tau) = 100(1 + 0.4\cos(2\pi\xi_1)\cos(2\pi\xi_2)\cos(2\pi\xi_3))(1 + 0.3\cos(2\pi\tau)), \]

\[ a_{00}(\xi, \tau) = 100(1 + 0.2\cos(2\pi\tau)). \]

Similar to the Example 5.2, we use the linear finite volume element method to solve the problem (84) using a fine mesh and at a small time step. The numerical solution is regarded as the reference solution of the original problem (84). Comparison of the computational costs and the relative errors are reported in Table 3 and 4, respectively.

**Table 3.** Computational costs for Example 5.3 with \( k = 1 \) and \( \varepsilon = 1/8 \).

<table>
<thead>
<tr>
<th>DOF</th>
<th>the original problem</th>
<th>the homogenized one</th>
<th>cell ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of nodes</td>
<td>887,753</td>
<td>87,598</td>
<td>88,127</td>
</tr>
<tr>
<td>number of elements</td>
<td>5,048,234</td>
<td>470,489</td>
<td>472,685</td>
</tr>
<tr>
<td>the time step</td>
<td>1.25e-3</td>
<td>1.0e-2</td>
<td>-</td>
</tr>
</tbody>
</table>

**Table 4.** Relative errors for Example 5.3 with \( k = 0, 1 \) and \( \varepsilon = 1/8 \).

<table>
<thead>
<tr>
<th>s</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 0 )</td>
<td>( |u - u_\varepsilon|_{L^2(\Omega)} )</td>
<td>( |u - u_\varepsilon|_{L^2(\Omega)} )</td>
<td>( |u - u_\varepsilon|_{L^2(\Omega)} )</td>
</tr>
<tr>
<td>( k = 0 )</td>
<td>( 5.0974e-02 )</td>
<td>( 4.7331e-02 )</td>
<td>( 1.2145e-02 )</td>
</tr>
<tr>
<td>( k = 1 )</td>
<td>( 5.7061e-01 )</td>
<td>( 5.4708e-01 )</td>
<td>( 7.8330e-02 )</td>
</tr>
<tr>
<td>( k = 1 )</td>
<td>( 1.5499e-01 )</td>
<td>( 1.5388e-01 )</td>
<td>( 1.4388e-01 )</td>
</tr>
</tbody>
</table>

**Figure 8.** Solutions on line \( x_1 = x_2 = x_3 \); Left: \( k = 0 \); Right: \( k = 1 \).

Similar to the Example 5.2, the second-order multiscale solution is more accurate than those resulted by the homogenized and first-order multiscale methods (see
Table 4 and Fig. 8). Moreover, the multiscale methods provide a tremendous saving in computing resource, in particular, for the three-dimensional cases and when the value of $\varepsilon$ is sufficiently small (see Table 3).

**Remark 5.2.** When solving the convection-diffusion equations with homogeneous Neumann boundary conditions on the surfaces of cavities in porous media, we usually fill these cavities with an almost degenerated phase, which is also called the hole-filling method, see [38]. It should be mentioned that the present method is suitable for solving this kind of problem.

**References**


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