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UNIFORM CONVERGENCE VIA PRECONDITIONING

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Abstract. The linear singularly perturbed convection-diffusion problem in one dimension is considered and its discretization on the Shishkin mesh is analyzed. A new, conceptually simple proof of pointwise convergence uniform in the perturbation parameter is provided. The proof is based on the preconditioning of the discrete system.

Key words. singular perturbation, convection-diffusion, boundary-value problem, Shishkin mesh, finite differences, uniform convergence, preconditioning.

1. Introduction

We consider the following one-dimensional singularly perturbed problem of convection-diffusion type,

$$\mathcal{L}u := -\varepsilon u'' - b(x)u' + c(x)u = f(x),\ x \in (0,1),\ u(0) = u(1) = 0, \tag{1}$$

with a small positive perturbation parameter ε and $C^1[0,1]$-functions b, c, and f, where b and c satisfy

$$b(x) \geq \beta > 0, \quad c(x) \geq 0 \ \text{ for } x \in I := [0,1].$$

It is well known, see [6, 9] for instance, that (1) has a unique solution u in $C^3(I)$, which in general has an exponential boundary layer near $x = 0$.

Singular perturbation problems arise in various applications, see [3, 4]. Typical of them are boundary and/or interior layers, regions whose size decreases as $\varepsilon \to 0$ and where the solution changes abruptly. This is why these problems require special numerical methods [5, 10, 4, 12, 7]. One of the most popular methods is to use an appropriate finite-difference scheme on the layer-adapted meshes of Shishkin [10, 4, 12, 7] or Bakhvalov [13, 12, 7] types.

We consider here the standard upwind discretization of (1) on the Shishkin mesh with N mesh steps. It is shown in [11] that for the matrix of the resulting system the condition number in the maximum norm is of magnitude $\mathcal{O}(\varepsilon^{-1}(N/\ln N)^2)$. Since this is unsatisfactory when $\varepsilon \to 0$, a simple preconditioning is proposed in the same paper. This behavior of the condition number is contrasted in [11] to that of the singularly perturbed reaction-diffusion problem, which can be described as (1) with $b \equiv 0$ and $c > 0$ on I. When the reaction-diffusion problem is discretized using the standard central scheme on the Shishkin mesh, there is no need for preconditioning because the condition number behaves like $\mathcal{O}((N/\ln N)^2)$.

We note that there is another difference between the two types of the singularly perturbed problems, viz. the difference in the proofs of ε-uniform convergence of the numerical solution to the discretized continuous solution. One of the ways to prove that a finite-difference discretization yields ε-uniform convergence is to use the following principle, which originated from non-perturbed problems (cf. [2, 13, 5]):

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Principle 1. *ε-uniform stability and ε-uniform consistency imply ε-uniform convergence.*

Moreover, ε-uniform pointwise convergence is desired when solving singular perturbation problems. For reaction-diffusion problems, this can be achieved by using the following version ([13]) of the above principle:

Principle 2. *ε-uniform stability and ε-uniform consistency, both in the maximum norm, imply ε-uniform pointwise convergence.*

However, Principle 2 does not work for convection-diffusion problems (1) because ε-uniform pointwise consistency is not present, although it is easy to show that the upwind scheme is ε-uniformly stable in the maximum norm. For these problems, ε-uniform consistency can be proved in a discrete L^1 norm and this is why the proofs based on Principle 1 have to rely on some kind of hybrid stability inequality [5, 1, 8, 7], an approach that typically involves the discrete Green's function. Other ε-uniform convergence proofs also exist, like those that use barrier functions [10, 4, 12, 7].

Our main result is that we show that essentially the same preconditioning (we appropriately modify the method from [11]), which eliminates the difference in the condition numbers of simple finite-difference discretizations for the convection-diffusion and reaction-diffusion problems, can also be used to eliminate the difference in the proofs of ε-uniform pointwise convergence for these two problem types. In other words, a suitable preconditioning technique enables the use of Principle 2 for the convection-diffusion problem. Using this approach, we prove an almost (up to logarithmic factors) first-order pointwise ε-uniform convergence for the upwind scheme discretizing the problem (1) on the Shishkin mesh. This result, however, is not the main contribution of this paper, because the same has already been proved elsewhere (see the above references). Rather, we feel that the main contribution is this conceptually simple proof which points out that there is a connection between conditioning and ε-uniform pointwise convergence for convection-diffusion problems.

The rest of the paper is organized as follows. We give the properties of the continuous solution in Section 2. Then, in Section 3, we introduce the finite-difference scheme on the Shishkin mesh and discuss the conditioning of the discrete problem. Section 4 provides the proof of ε-uniform pointwise convergence. Finally, some concluding remarks are given in Section 5.

2. Properties of the continuous solution

The solution u of (1) can be decomposed into the smooth and boundary-layer parts. We present here Linß's [7, Theorem 3.48] version of such a decomposition:

$$u(x) = s(x) + y(x), \tag{2}$$

$$|s^{(k)}(x)| \le C\left(1+\varepsilon^{2-k}\right), \quad |y^{(k)}(x)| \le C\varepsilon^{-k}e^{-\beta x/\varepsilon}, \tag{3}$$
$$x \in I, \quad k = 0,1,2,3.$$

Above and throughout the paper, C denotes a generic positive constant which is independent of ε. For the construction of the function s, see [7], since the details are not of interest here. As for y, it solves the problem

$$\mathcal{L}y(x) = 0, \quad x \in (0,1), \quad y(0) = -s(0), \quad y(1) = 0. \tag{4}$$

It is important to note that y satisfies a homogeneous differential equation. We shall use this fact later on in the paper.

3. The discrete problem and conditioning

We first define a finite-difference discretization of the problem (1) on a general mesh I^N with mesh points x_i, $i = 0, 1, \ldots, N$, such that $0 = x_0 < x_1 < \cdots < x_N = 1$. Throughout the rest of the paper, the constants C are also independent of N.

Let $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$, and $\hbar_i = (h_i + h_{i+1})/2$, $i = 1, 2, \ldots, N-1$. Mesh functions on I^N are denoted by W^N, U^N, etc. If g is a function defined on I, we write g_i instead of $g(x_i)$ and g^N for the corresponding mesh function. Any mesh function W^N is identified with an $(N+1)$-dimensional column vector, $W^N = [W_0^N, W_1^N, \ldots, W_N^N]^T$, and its maximum norm is given by

$$\|W^N\| = \max_{0 \le i \le N} |W_i^N|.$$

For the matrix norm, which we also denote by $\|\cdot\|$, we take the norm subordinate to the above maximum vector norm.

We discretize the problem (1) on I^N using the upwind finite-difference scheme:

$$U_0^N = 0,$$

$$\mathcal{L}^N U_i^N := -\varepsilon D'' U_i^N - b_i D' U_i^N + c_i U_i^N = f_i, \quad i = 1, 2, \ldots, N-1, \tag{5}$$

$$U_N^N = 0,$$

where

$$D'' W_i^N = \frac{1}{\hbar_i} \left(\frac{W_{i+1}^N - W_i^N}{h_{i+1}} - \frac{W_i^N - W_{i-1}^N}{h_i} \right),$$

and

$$D' W_i^N = \frac{W_{i+1}^N - W_i^N}{h_{i+1}}.$$

The linear system (5) can be written down in matrix form,

$$A_N U^N = \hat{f}^N, \tag{6}$$

where $A_N = [a_{ij}]$ is a tridiagonal matrix with $a_{00} = 1$ and $a_{NN} = 1$ being the only nonzero elements in the 0th and Nth rows, respectively, and where $\hat{f}^N = [0, f_1, f_2, \ldots, f_{N-1}, 0]^T$.

It is easy to see that A_N is an L-matrix, i.e., $a_{ii} > 0$ and $a_{ij} \le 0$ if $i \ne j$, for all $i, j = 0, 1, \ldots, N$. The matrix A_N is also inverse monotone, which means that it is non-singular and that $A_N^{-1} \ge 0$ (inequalities involving matrices and vectors should be understood component-wise), and therefore an M-matrix (inverse monotone L-matrix). This can be proved using the following M-criterion, see [2] for instance.

Theorem 1. *Let A be an L-matrix and let there exist a vector w such that $w > 0$ and $Aw \ge \gamma$ for some positive constant γ. A is then an M-matrix and it holds that $\|A^{-1}\| \le \gamma^{-1}\|w\|$.*

To see that A_N is an M-matrix, just set $w_i = 2 - x_i$, $i = 0, 1, \ldots, N$ in Theorem 1 to get that $A_N w \ge \min\{1, \beta\}$. This also implies that the discrete problem (6) is stable uniformly in ε,

$$\|A_N^{-1}\| \le \frac{2}{\min\{1, \beta\}} \le C. \tag{7}$$

Of course, the system (6) has a unique solution U^N.

From this point on, we take the standard Shishkin mesh for the discretization mesh I^N. However, our results equally hold true for the slightly generalized

Shishkin mesh considered in [16]. Let N be even and let $J = N/2$. Let also $L = \ln N$ and let

$$\sigma = \min\left\{\frac{1}{2}, \frac{a\varepsilon L}{\beta}\right\}, \quad a \geq 2.$$

The Shishkin mesh is constructed by forming a fine equidistant mesh with J mesh steps of size h in the interval $[0, \sigma]$ and a coarse equidistant mesh with J mesh steps of size H in $[\sigma, 1]$. We only consider the case when $\sigma = a\varepsilon L/\beta$, since N is otherwise unrealistically large. We have that

$$h = \frac{\sigma}{J} \leq C\varepsilon\frac{L}{N} \quad \text{and} \quad H = \frac{1-\sigma}{J} \leq CN^{-1},$$

and we define $\hbar = (h + H)/2$.

When the discrete problem (5) is formed on the Shishkin mesh, it is shown in [11] that the condition number of A_N,

$$\kappa(A_N) := \|A_N^{-1}\|\|A_N\|,$$

satisfies the following sharp estimate:

$$\kappa(A_N) \leq C\frac{N^2}{\varepsilon L^2}.$$

Therefore, the system is ill-conditioned when $\varepsilon \to 0$. This unpleasant behavior is eliminated in [11] using the preconditioning by the diagonal matrix $D := [\mathrm{diag}(A_N)]^{-1}$. When the system (5) is multiplied by D, the resulting matrix DA_N satisfies

$$\|DA_N\| \leq C \quad \text{and} \quad \|(DA_N)^{-1}\| \leq C\frac{N^2}{L},$$

so that

$$\kappa(DA_N) \leq C\frac{N^2}{L}. \tag{8}$$

Note, however, that the matrix DA_N no longer satisfies that $\|(DA_N)^{-1}\| \leq C$, thus the original stability estimate $\|A_N^{-1}\| \leq C$ in (7) is not preserved. We modify below the preconditioning by a diagonal matrix so that the same estimate as in (8) holds true, while the stability of type (7) is retained.

Let $M = \mathrm{diag}\ (m_0, m_1, \ldots, m_N)$ be a diagonal matrix with the entries

$$m_0 = 1, \quad m_i = \frac{h}{H}, \ i = 1, 2, \ldots J-1, \quad \text{and} \quad m_i = 1, \ i = J, J+1, \ldots, N.$$

When the system (6) is multiplied by M, this is equivalent to multiplying the equations 1, 2, ..., $J-1$ of the system (5) by h/H. The modified system is

$$\tilde{A}_N U^N = M\hat{f}^N, \tag{9}$$

where $\tilde{A}_N = MA_N$. Let the entires of $\tilde{A}_N$ be denoted by $\tilde{a}_{ij}$, the nonzero ones being

$$l_i := \tilde{a}_{i-1,i} = \begin{cases} -\dfrac{\varepsilon}{hH}, & 1 \leq i \leq J-1, \\[2ex] -\dfrac{\varepsilon}{h\hbar}, & i = J, \\[2ex] -\dfrac{\varepsilon}{H^2}, & J+1 \leq i \leq N-1, \end{cases}$$

$$r_i := \tilde{a}_{i,i+1} = \begin{cases} -\dfrac{\varepsilon}{hH} - \dfrac{b_i}{H}, & 1 \le i \le J-1, \\ -\dfrac{\varepsilon}{H\hbar} - \dfrac{b_i}{H}, & i = J, \\ -\dfrac{\varepsilon}{H^2} - \dfrac{b_i}{H}, & J+1 \le i \le N-1, \end{cases}$$

and

$$d_i := \tilde{a}_{ii} = \begin{cases} 1, & i = 0 \\ -l_i - r_i + \dfrac{h}{H}c_i, & 1 \le i \le J-1, \\ -l_i - r_i + c_i, & J \le i \le N-1, \\ 1, & i = N. \end{cases}$$

It is easy to see that $\tilde{A}_N$ is an L-matrix. The next lemma shows that $\tilde{A}_N$ is an M-matrix and that the modified discretization (9) is stable uniformly in ε.

Lemma 1. *The matrix $\tilde{A}_N$ of the system* (9) *satisfies*

$$\left\| \tilde{A}_N^{-1} \right\| \le C.$$

Proof. We construct a vector $v = [v_0, v_1, \ldots, v_N]^T$ such that

(a) $v_i \ge \delta$, $i = 0, 1, \ldots, N$, where δ is a positive constant independent of both ε and N,
(b) $v_i \le C$, $i = 0, 1, \ldots, N$,
(c) $l_i v_{i-1} + d_i v_i + r_i v_{i+1} \ge \delta$, $i = 1, 2, \ldots, N-1$.

Then, according to Theorem 1,

$$\|\tilde{A}_N^{-1}\| \le \delta^{-1}\|v\| \le C.$$

The vector v can be constructed as follows:

$$v_i = \alpha - Hi + \lambda \min\{(1+\rho)^{J-i}, 1\},$$

where α and λ are fixed positive constants and $\rho = \beta H/\varepsilon$. This construction is motivated by the proof of Lemma 4 in [11].

Since $v_i \ge \alpha - HN$, we see that there exists a constant α such that $\alpha \le C$ and that condition (a) is satisfied. Then, because of $v_i \le \alpha + \lambda$, condition (b) holds true if we show that $\lambda \le C$. We do this next as we verify condition (c).

When $1 \le i \le J-1$, we have $v_j = \alpha - Hj + \lambda$, $j = i-1, i, i+1$, and

$$\begin{aligned} l_i v_{i-1} + d_i v_i + r_i v_{i+1} &= (l_i + d_i + r_i)v_i + l_i H - r_i H \\ &= \frac{h}{H} c_i v_i - \frac{\varepsilon}{h} + \frac{\varepsilon}{h} + b_i \\ &\ge \beta. \end{aligned}$$

For $i = J$, condition (c) is verified as follows:

$$\begin{aligned}
l_J v_{J-1} + d_J v_J + r_J v_{J+1} &= c_J v_J + l_J H - r_J \left(H + \frac{\lambda\rho}{1+\rho} \right) \\
&\geq -r_J \frac{\lambda\rho}{1+\rho} + (l_J - r_J)H \\
&= \left(\frac{\varepsilon}{\hbar H} + \frac{b_J}{H} \right) \frac{\lambda\rho}{1+\rho} - \frac{\varepsilon H}{h\hbar} + \frac{\varepsilon}{\hbar} + b_J \\
&\geq \frac{\varepsilon + \beta\hbar}{\hbar H} \cdot \frac{\lambda\beta H}{\varepsilon + \beta H} - \frac{\varepsilon H}{h\hbar} + \beta \\
&= \frac{1}{\hbar} \left(\frac{\varepsilon + \beta\hbar}{\varepsilon + \beta H} \beta\lambda - \frac{\varepsilon H}{h} \right) + \beta \\
&\geq \frac{1}{\hbar} \left(\frac{\beta\lambda}{2} - \frac{\varepsilon H}{h} \right) + \beta \\
&\geq \beta,
\end{aligned}$$

where in the last step we choose λ so that $\lambda \leq C$ and

$$\frac{\beta\lambda}{2} \geq \frac{\varepsilon H}{h}.$$

This is possible to do because

$$\frac{\varepsilon H}{h} \leq \frac{C}{L} \leq C.$$

Finally, if $J + 1 \leq i \leq N - 1$, we have

$$\begin{aligned}
l_i v_{i-1} + d_i v_i + r_i v_{i+1} &= c_i v_i + l_i H - r_i H + l_i \left[\frac{\lambda}{(1+\rho)^{i-1-J}} - \frac{\lambda}{(1+\rho)^{i-J}} \right] \\
&\quad + r_i \left[\frac{\lambda}{(1+\rho)^{i+1-J}} - \frac{\lambda}{(1+\rho)^{i-J}} \right] \\
&\geq b_i + \frac{\rho(1+\rho)l_i - \rho r_i}{(1+\rho)^{i+1-J}} \lambda \\
&\geq \beta + \frac{(l_i - r_i + l_i\rho)\rho}{(1+\rho)^{i+1-J}} \lambda \\
&= \beta + \left(\frac{b_i}{H} - \frac{\beta}{H} \right) \lambda\rho(1+\rho)^{J-i-1} \\
&\geq \beta.
\end{aligned}$$

□

By examining the elements of the matrix $\tilde{A}_N$, we see that

$$\|\tilde{A}_N\| \leq C\frac{N^2}{L}.$$

When we combine this with Lemma 1, we get the following result.

Theorem 2. *The matrix* $\tilde{A}_N$ *of the system* (9) *satisfies*

$$\kappa(\tilde{A}_N) \leq C\frac{N^2}{L}.$$

To conclude this section, we re-iterate that both discrete systems (6) and (9) are stable uniformly in ε. Their corresponding stability inequalities are

$$\|W^N\| \leq \|A_N^{-1}\| \|A_N W^N\| \tag{10}$$

and

$$\|W^N\| \le \|\tilde{A}_N^{-1}\| \|\tilde{A}_N W^N\|, \tag{11}$$

where both $\|A_N^{-1}\|$ and $\|\tilde{A}_N^{-1}\|$ are bounded from above by a constant independent of ε.

4. Uniform convergence

Let τ_i, $i = 1, 2, \ldots, N-1$, be the consistency error of the finite-difference operator $\mathcal{L}^N$,

$$\tau_i = \tau_i[u] := \mathcal{L}^N u_i - (\mathcal{L}u)_i,$$

that is,

$$\tau_i = \mathcal{L}^N u_i - f_i = [A_N(u^N - U^N)]_i.$$

Convergence uniform in ε would follow from (10) if we could show that

$$|\tau_i| \to 0 \text{ uniformly in } \varepsilon \text{ when } N \to \infty. \tag{12}$$

However, this does not hold true, as the following simple numerical experiment indicates.

Consider the test problem taken from [7, p.1],

$$-\varepsilon u'' - u' = 1, \quad x \in (0,1), \quad u(0) = u(1) = 0,$$

where we know the exact solution. Table 1 clearly shows that (12) is not satisfied.

TABLE 1. The maximum norm of the consistency error $A_N u^N - \hat{f}^N$.

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
1.e-2	1.33e+1	9.71e+0	6.42e+0	3.95e+0	2.32e+0	1.32e+0
1.e-3	1.33e+2	9.71e+1	6.42e+1	3.95e+1	2.32e+1	1.32e+1
1.e-4	1.33e+3	9.71e+2	6.42e+2	3.95e+2	2.32e+2	1.32e+2
1.e-5	1.33e+4	9.71e+3	6.42e+3	3.95e+3	2.32e+3	1.32e+3
1.e-6	1.33e+5	9.71e+4	6.42e+4	3.95e+4	2.32e+4	1.32e+4
1.e-7	1.33e+6	9.71e+5	6.42e+5	3.95e+5	2.32e+5	1.32e+5
1.e-8	1.33e+7	9.71e+6	6.42e+6	3.95e+6	2.32e+6	1.32e+6

However, for the preconditioned system (9), the consistency error is

$$\tilde{\tau}_i[u] = \begin{cases} \dfrac{h}{H}\tau_i[u], & 1 \le i \le J-1, \\ \\ \tau_i[u], & J \le i \le N-1, \end{cases}$$

and it tends to 0 uniformly in ε when $N \to \infty$, as Table 2 indicates. We prove this in the following lemma.

TABLE 2. The maximum norm of the preconditioned consistency error $\tilde{A}_N u^N - M\hat{f}^N$.

ε	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
1.e-2	1.e+0	8.91e-1	6.97e-1	4.99e-1	3.35e-1	2.16e-1
1.e-3	9.38e-1	8.23e-1	6.35e-1	4.48e-1	2.97e-1	1.88e-1
1.e-4	9.32e-1	8.17e-1	6.29e-1	4.43e-1	2.93e-1	1.86e-1
1.e-5	9.32e-1	8.16e-1	6.29e-1	4.43e-1	2.93e-1	1.86e-1
1.e-6	9.32e-1	8.16e-1	6.29e-1	4.43e-1	2.93e-1	1.85e-1
1.e-7	9.32e-1	8.16e-1	6.29e-1	4.43e-1	2.93e-1	1.85e-1
1.e-8	9.32e-1	8.16e-1	6.29e-1	4.43e-1	2.93e-1	1.85e-1

Lemma 2. *The following estimate holds true for all $i = 1, 2, \ldots, N-1$:*

$$|\tilde{\tau}_i[u]| \le CN^{-1}L^2.$$

Proof. By Taylor's expansion we have that

$$|\tau_i[u]| \le Ch_{i+1}(\varepsilon\|u'''\|_i + \|u''\|_i), \tag{13}$$

where $\|g\|_i := \max_{x_{i-1}\le x\le x_{i+1}} |g(x)|$ for any $C(I)$-function g. We use the decomposition (2) to get

$$\tilde{\tau}_i[u] = \tilde{\tau}_i[s] + \tilde{\tau}_i[y].$$

Then (13) and the derivative-estimates of s, given in (3), immediately imply that

$$|\tilde{\tau}_i[s]| \le CN^{-1}$$

and therefore the following remains to be proved:

$$|\tilde{\tau}_i[y]| \le CN^{-1}L^2. \tag{14}$$

For $1 \le i \le J-1$, we use (13) again, together with the derivative-estimates of y, see (3):

$$|\tilde{\tau}_i(y)| \le C\frac{h^2}{H}(\varepsilon\|y'''\|_i + \|y''\|_i) \le C\frac{h^2}{H}\varepsilon^{-2} \le CN^{-1}L^2.$$

Therefore, (14) is proved in this case.

When $J+2 \le i \le N-1$, (13) and (3) give

$$\begin{aligned}|\tilde{\tau}_i[y]| &\le CH(\varepsilon\|y'''\|_i + \|y''\|_i) \le C\frac{H}{\varepsilon^2}e^{-\beta x_{i-1}/\varepsilon} \le C\frac{H}{\varepsilon^2}e^{-\beta(\sigma+H)/\varepsilon}\\ &\le CN\frac{H^2}{\varepsilon^2}e^{-\beta H/\varepsilon}e^{-\beta\sigma/\varepsilon}.\end{aligned}$$

The estimate (14) follows from here because

$$\frac{H^2}{\varepsilon^2}e^{-\beta H/\varepsilon} \le C$$

and because the definition of σ and $a \ge 2$ imply that

$$e^{-\beta\sigma/\varepsilon} \le N^{-2}.$$

We finally prove (14) for $i = J, J+1$. In this case, we use the fact that $\mathcal{L}y = 0$ to work with

$$|\tilde{\tau}_i[y]| = |\tau_i[y]| \le P_i + Q_i + c_i|y_i|,$$

where

$$P_i = \varepsilon|D''y_i| \quad \text{and} \quad Q_i = b_i|D'y_i|.$$

We immediately have that

$$c_i|y_i| \le Ce^{-\beta x_i/\varepsilon} \le Ce^{-\beta\sigma/\varepsilon} \le CN^{-2}.$$

As for P_i and Q_i, it holds true that

$$P_i \le \hbar_i^{-1}\varepsilon \cdot 2\|y'\|_i \le CNe^{-\beta(\sigma-h)/\varepsilon} \le CN^{-1}$$

and

$$Q_i \le CH^{-1}\|y\|_i \le CNe^{-\beta(\sigma-h)/\varepsilon} \le CN^{-1}.$$

This technique can be found in [15]. □

Note that when the above proof technique is applied to the consistency error τ_i, this quantity cannot be estimated uniformly in ε. We can only get that

$$|\tau_i| \leq C\frac{L}{\varepsilon N}.$$

It is because we multiply equations 1, 2,. . . , $J-1$ of the system (5) by h/H that we get the extra ε-factor needed for the ε-uniform consistency on the fine part of the mesh.

When Lemmas 1 and 2 are combined, which amounts to the use of Principle 2, that is, of the stability inequality (11), we obtain the following result.

Theorem 3. *The solution U^N of the discrete problem* (6) *on the Shishkin mesh satisfies*

$$\|U^N - u^N\| \leq CN^{-1}L^2,$$

where u is the solution of the continuous problem (1).

5. Concluding remarks

The result of Theorem 3 is the same as in [10, Theorem 4 in Chapter 8], proved by the barrier-function technique for the case $c \equiv 0$, but with the mesh parameter $a > 1$. A finer, but more complicated, analysis in [4, Theorem 3.6] improves the above estimate to

$$\|U^N - u^N\| \leq CN^{-1}L, \tag{15}$$

with $a \geq 1$ and still for $c \equiv 0$. The same result as in (15) is proved in [7, Chapter 4] for the general problem (1), by using a finite-element approach to the discretization scheme, which is slightly different from $\mathcal{L}^N$, having h_{i+1} instead of $\hbar_i$ in D''.

Since $W^N = (A_N^{-1}M^{-1})(MA_NW^N)$, the stability inequality (11) can be represented as

$$\|W^N\| \leq \|A_N^{-1}\|'_M \|A_NW^N\|_M, \tag{16}$$

where for a matrix B, $\|B\|'_M = \|BM^{-1}\|$, and $\|W^N\|_M = \|MW^N\|$. Note that the matrix norm $\|\cdot\|'_M$ is not induced by the vector norm $\|\cdot\|_M$ (which is why we denote them differently), but the two norms are consistent in the sense that

$$\|BW^N\|_M \leq \|B\|'_M \|W^N\|_M.$$

The inequality (16) is a stability inequality of hybrid nature, having different vector norms on the two sides. However, the vector norm $\|\cdot\|_M$ is still essentially a maximum norm and this is completely different from the hybrid stability inequalities used in [5, 1, 8, 7], which have a discrete L^1 norm on the right-hand side. Moreover, these hybrid stability inequalities are derived by using the discrete Green's function, which we do not do here.

In conclusion, although the method presented here gives a slightly weaker result in some cases, it provides a straightforward proof, based on a simple principle, of ε-uniform pointwise convergence for the solution of the standard upwind scheme discretizing the singularly perturbed convection-diffusion problem (1). It is even more interesting that the proof is enabled by the preconditioning of the system arising from the discretization. Whether this can be used as a general approach when proving ε-uniform pointwise convergence for other types of singular perturbation problems, including multidimensional ones, remains to be seen, but the generalization to the semilinear problem of type (1) (with $c = c(x, u)$, $c_u \geq 0$) is straightforward.

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