INSTABILITY OF CRANK-NICOLSON LEAP-FROG FOR NONAUTONOMOUS SYSTEMS

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Abstract. The implicit-explicit combination of Crank-Nicolson and Leap-Frog methods is widely used for atmosphere, ocean and climate simulations. Its stability under a CFL condition in the autonomous case was proven by Fourier methods in 1962 and by energy methods for autonomous systems in 2012. We provide an energy estimate showing that solution energy can grow with time in the nonautonomous case, with worst case rate proportional to time step size. We present two constructions showing that this worst case growth rate is attained for a sequence of timesteps $\Delta t \to 0$. The construction exhibiting this growth for leapfrog is for a problem with a periodic coefficient.

Key words. partitioned methods, energy stability.

1. Introduction

Stability of CNLF, the Crank-Nicolson Leap-Frog method (CNLF) below, is considered for systems with nonautonomous $A(t), \Lambda(t)$:

\begin{equation}
\frac{du}{dt} + A(t)u + \Lambda(t)u = 0, \quad \text{for } t > 0, \text{ and } u(0) = u_0.
\end{equation}

Here $A(t), \Lambda(t)$ are $d \times d$ matrices and $u(t)$ is a $d$ vector. $A(t)$ is positive semi-definite symmetric and $\Lambda(t)$ is skew symmetric. Let $|\cdot|_2$ denote the euclidean norm. The CNLF discretization of (1.1) is expressed as follows. Let $t^n = n\Delta t$; given $u^0, u^1$ find $u^n \in X$ for $n \geq 2$ satisfying

\begin{equation}
(CNLF) \quad \frac{u^{n+1} - u^{n-1}}{2\Delta t} + A(t^n)\frac{u^{n+1} + u^{n-1}}{2} + \Lambda(t^n)u^n = 0,
\end{equation}

with approximations to appropriate accuracy, [23], at the first two time steps. CNLF is the implicit-explicit (IMEX) method used for the dynamic core of most current atmosphere, ocean and climate codes, e.g., [6], [15], [22], [13] and other geophysics problems, [16].

Stability was shown for the scalar, autonomous case under the timestep condition

\begin{equation}
\Delta t|\Lambda|_2 \leq \alpha < 1,
\end{equation}

in 1963 by Johansson and Kreiss [14] and for (non-commuting) autonomous systems in 2012 [17], see also [23], [6] for background. We prove herein weak instability in the nonautonomous case.

Remark 1. CNLF is often used in geophysical fluid dynamics codes together with a time filter. The general strategy used is to split the (nonlinear) equations of motion into terms corresponding to high speed/low energy waves and low speed/high energy waves. The respective terms are discretized by CN and LF with time filters. Williams [24], for example, lists 20 atmosphere codes, 15 ocean codes and 24 coupled / geophysics codes based on this approach. (The precise realization varies with...

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the effects included and the implementation. For one detailed development of one
such splitting see Section 3 in [9].) Linearization of the split system leads to nonau-
tonomous systems of the form (1.1) above. Time filters are used with CNLF, e.g.,
[3], [19], [24], [13], with the usual explanation that the latter controls the unstable
mode of CNLF. However, the (so-called) unstable mode (or computational mode)
of CNLF has recently been proven in [12] to be asymptotically stable under (1.2).
We do not study time filters herein. However, the instability result does suggest
that one positive contribution of time filters may be to control the weak instability
identified herein that arises from nonautonomous and periodic \( \Lambda(t) \).

The extension of stability theory for linear multistep methods from autonomous
to nonautonomous (with test problem \( y' = \lambda(t)y \)) has a rich history. Dahlquist [5]
proved that an A-stable method is similarly stable for \( y' = \lambda(t)y \) when \( \text{Re}(\lambda(t)) \leq 0 \),
further developed in [18]. For the corresponding AN-stability theory for Runge-
Kutta methods, see Hundsdorfer and Stetter [10]. For non-A-stable multi-step
methods, nonautonomous stability theory was recently developed in [4] with both
stability and instability conditions for \( y' = \lambda(t)y \). Given a linear multistep method
for \( y' = \lambda(t)y \), let \( \rho(z), \sigma(z) \) be the complex polynomials associated with the method
in a standard way and form

\[ A := \text{Re} \left[ \frac{\rho(z)}{\sigma(z)} \right]_{z = i} . \]

Even if \( \Delta t \) is small enough to be in the stability region of the method, if \( A < 0 \) then there exists a \( \lambda(t) < 0 \) for which the method is unstable [4].

While the theory in [4] does not apply to IMEX methods like CNLF, it can be ap-
plied to the special case \( A(t) = 0 \). The leapfrog method \( (A(t) = 0) \) is an important
wedge example. Indeed, for leapfrog, we calculate \( \rho(z) = \frac{1}{2} z^2 - \frac{1}{2} \), and \( \sigma(z) = z \).

Thus \( A = 0 \) and the theory of [4] is inconclusive. Many interesting behaviors
are possible between exponential asymptotic stability and exponential instability.
One hint is that there is a rich catalog (e.g., [1],[20],[21],[25] ) of exotic behavior
of leapfrog for Burgers equation starting (to our knowledge) with Fornberg’s 1973
paper [8].

The results are clearest for the case that \( \Lambda(t) \) is Lipschitz,
\begin{equation}
|\Lambda(t^n) - \Lambda(t^{n-1})|_2 \leq a_0 \Delta t .
\end{equation}

We prove in Theorem 1 that any instability is, at worst, a weak one:
\begin{equation}
|w^{n+1}|^2 + |w^n|^2 \leq C(\alpha, u^0, u^1) \exp \left[ \Delta t \frac{a_0}{1 - \alpha} t^n \right] .
\end{equation}

The rate constant \( \Delta t a_0/(1 - \alpha) \to 0 \) as \( \Delta t \to 0 \) but \( \exp \left[ \Delta t \frac{a_0}{1 - \alpha} t^n \right] \to \infty \) as \( t^n \to \infty \). However, the true solution of (1.1) is uniformly bounded and if \( A(t) > 0 \),
\( u(t) \to 0 \) as \( t \to \infty \). Section 3 presents give two constructions that show that (1.4)
is best possible for the leapfrog case \( A(t) = 0 \).

**Remark 2.** Theorem 2.5 in [4] shows that for \( A < 0 \), then there is an alternating
pair of states that together lead to growth. The first construction in Section 3 shows
similar behavior for \( A = 0 \).

In Section 3 we give two constructions that show that LF (and thus CNLF)
is exponentially unstable for arbitrarily small timesteps when \( \Lambda(t) \) is a bounded
function that changes sign periodically. A numerical study is also given in Sec-
tion 3. The numerical data suggest that the instability occurs for a sparse set of
timesteps. The two constructions in Section 3 show conclusively that the upper estimate in Theorem 1 is attained and that it persists for arbitrarily small timesteps. Naturally, upper estimates (in Theorem 1) follow more readily than non-generic lower estimates in Section 3 that prove instability. Thus, the proof of instability in Construction 2 in Section 3 is necessarily technical.

2. CNLF when $\Lambda = \Lambda(t)$

We prove the claimed stability bound for the CNLF method. In the Lipschitz case, we then show that the upper estimate implies one slightly sharper than the bound claimed in the introduction.

**Theorem 1.** Assume for every $t^n$ that (1.2) holds. Then CNLF satisfies: for every $N \geq 2$

\[
|u^{n+1}|^2 + |u^n|^2 \leq C(\alpha, u^0, u^1) \exp \left[ \frac{\Delta t}{1 - \alpha} \sum_{n=1}^{N} |\Lambda(t^{n+1}) - \Lambda(t^n)|^2 \right],
\]

where $C(\alpha, u^0, u^1) = \left[ |u^1|^2 + |u^0|^2 + 2\Delta t (u^1)^T \Lambda(t^0) u^0 \right] / (1 - \alpha)$.

**Proof.** Define $E^{n+1/2} := |u^{n+1}|^2 + |u^n|^2$. Take the dot product of (CNLF) with $(u^{n+1} + u^{n-1})$ and add and subtract $|u^n|^2$. This gives

\[
\begin{align*}
\left[ E^{n+1/2} - E^{n-1/2} \right] + \Delta t (u^{n+1} + u^{n-1})^T A(t^n) (u^{n+1} + u^{n-1}) + \\
+ 2\Delta t \left( (u^{n+1} + u^{n-1})^T \Lambda(t^n) u^n \right) = 0.
\end{align*}
\]

Since $A$ is positive semi-definite $(u^{n+1} + u^{n-1})^T A(t^n) (u^{n+1} + u^{n-1}) \geq 0$ and can be dropped. From skew symmetry of $\Lambda(t^n)$ we can write

\[
\left[ E^{n+1/2} - E^{n-1/2} \right] + 2\Delta t \left[ (u^{n+1})^T \Lambda(t^n) u^n - (u^n)^T \Lambda(t^n) u^{n-1} \right] \leq 0.
\]

Rewrite the $\Lambda$ terms as $C^{n+1/2} - C^{n-1/2} - Q^n$ with $Q^n$ the extra term that now occurs due to time dependence

\[
\begin{align*}
(u^{n+1})^T \Lambda(t^n) u^n - (u^n)^T \Lambda(t^n) u^{n-1} &= C^{n+1/2} - C^{n-1/2} - Q^n, \\
C^{n+1/2} := (u^{n+1})^T \Lambda(t^n) u^n, \quad C^{n-1/2} := (u^n)^T \Lambda(t^{n-1}) u^{n-1}, \\
Q^n := (u^n)^T \left[ \Lambda(t^n) - \Lambda(t^{n-1}) \right] u^{n-1}.
\end{align*}
\]

Note that by the usual inequalities $2Q^n \leq |\Lambda(t^n) - \Lambda(t^{n-1})|_2 E^{n-1/2}$, giving, for $n \geq 1$,

\[
E^{n+1/2} + 2\Delta t C^{n+1/2} \leq E^{n-1/2} + 2\Delta t C^{n-1/2} + \Delta t |\Lambda(t^n) - \Lambda(t^{n-1})|_2 E^{n-1/2}.
\]

The timestep condition (1.2) implies $E^{n+1/2} + 2\Delta t C^{n+1/2} \geq (1 - \alpha) E^{n+1/2}$. Summing, we have

\[
E^{N+1/2} \leq \frac{1}{1 - \alpha} \left[ E^{1/2} + \Delta t C^{1/2} \right] + \frac{\Delta t}{1 - \alpha} \sum_{n=1}^{N} |\Lambda(t^n) - \Lambda(t^{n-1})|_2 E^{n-1/2}.
\]

Thus by the discrete Gronwall lemma we have, for every $N \geq 1$

\[
E^{N+1/2} \leq \frac{1}{1 - \alpha} \left[ E^{1/2} + \Delta t C^{1/2} \right] + \\
\left[ E^{1/2} + \Delta t C^{1/2} \right] \frac{\Delta t}{(1 - \alpha)^2} \sum_{n=1}^{N} |\Lambda(t^n) - \Lambda(t^{n-1})|_2 \cdot M_N,
\]
It is easily proven by induction that \( \Lambda(t) \) is Lipschitz continuous (1.3) then

\[
\exp \left[ \frac{\Delta t}{1 - \alpha} \sum_{n=1}^{N} |\Lambda(t^n) - \Lambda(t^{n-1})|_2 \right] \leq \exp \left[ \frac{\Delta t}{1 - \alpha} a_0 t^N \right].
\]

Returning to the proof, the growth rate arises from

\[
\left( 1 + \frac{a_0 \Delta t^2}{1 - \alpha} \right)^N \leq \left( 1 + \frac{a_0 \Delta t^2}{1 - \alpha} + \left( \frac{a_0 \Delta t^2}{1 - \alpha} \right)^2 + \cdots \right)^N \leq \exp \left( \frac{\Delta t}{1 - \alpha} a_0 N \Delta t \right),
\]

in which the first step obviously is not sharp. If we rescale by \( s = \frac{a_0}{1 - \alpha} (n \Delta t)^2 = \frac{a_0}{1 - \alpha} (t^m)^2, m = n^2, \) we have \( \left( 1 + \frac{a_0 \Delta t^2}{1 - \alpha} \right)^n = \left( 1 + \frac{a_0 \Delta t^2}{1 - \alpha} \right)^{\sqrt{n^2}}. \) Sharp double asymptotic limits \( m \to \infty \) and \( s \to \infty \) can be obtained using calculus giving a slight improvement for large timesteps:

\[
\left( 1 + \frac{a_0 \Delta t^2}{1 - \alpha} \right)^N \leq \begin{cases} \exp \left( \frac{\Delta t a_0 t^N}{1 - \alpha} \right), & \text{for } \Delta t < \sqrt{0.6117a_0^{-1}(1 - \alpha)} \\ \exp \left( \frac{0.807a_0 t^N}{1 - \alpha} \right), & \text{for } \Delta t > \sqrt{0.6117a_0^{-1}(1 - \alpha)} \end{cases}
\]

3. Exponential Instabilities

In this section, we present one construction exhibiting a disastrous exponential instability. This is followed by a numerical study which suggests strongly that the leapfrog may be generically stable and that the set of instability may even be stable. This is not the case. Our second construction shows that for a periodic \( \Lambda(t) \) there is a sequence of \( \Delta t \to 0 \) for which exponential growth is exhibited.

**Construction 1: Exponential instability under (1.2).** Take \( A(t) = 0 \) (so CNLF reduces to LF), \( \Delta t = 1/2, u = (u_1, u_2)^T, y^0 = (1, 1)^T, y^1 = (1, -1)^T \) and choose \( \ell(t) = \cos(2\pi t) \). Consider LF for the \( 2 \times 2 \) system:

\[
(3.1) \quad u' + \Lambda(t)u = 0, \quad \Lambda(t) = \ell(t) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

For this choice of \( \Delta t \) and \( \Lambda(t) \), \( \Delta t|\Lambda(t^n)|_2 = \frac{1}{2} < 1 \), inside the stability region for autonomous systems. LF is given by: \( u^0, u^1 \) given, \( u^{n+1} = u^n - 2\Delta t\Lambda(t^n)u^n \), where

\[
\Lambda(t^n) = (-1)^{n+1} \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}.
\]

It is easily proven by induction that \( u^n \) is given by \( u^n = F_n(1, (-1)^{n+1})^T \) where \( \{F_n\} \) is the Fibonacci sequence. Thus \( |u^n| \to \infty \), as \( n \to \infty \).

The instability in this construction is so dramatic that it is surprising that in the next numerical investigation stability seemed (incorrectly) to be the generic behavior.
Numerical Study of Stability: For smaller $\Delta t$ instability occurs on a sparse set. We computed the growth rates of CNLF for smaller time steps in the previous example with $\ell(t) = \cos(2\pi t)$, and (smaller) timesteps $\Delta t = P/K$ for integers $K$, $P \leq K$, $K \leq 1400$. We applied LF, (CNLF) with $A(t) = 0$.

The average growth rate $\rho$ was computed by constructing the $2 \times 2$ matrix given in (3.7) below and computing its eigenvalues $\lambda_j$. The value $\sigma = \max\{|\lambda_j|\}$ is the largest possible net growth after $K$ steps of LF. Thus, for some starting values, the $n$th LF step satisfies $|u^n| \approx |u^0|\sigma^n/K = |u^0|e^{\rho n}$, with $\rho = \frac{1}{K}\log \sigma$. When $\rho = 0$, (we shall take $10^{-8}$ as numerically zero) the iteration is stable while if $\rho > 0$ the LF iterates grow exponentially.

![Figure 1. Average growth rate $\rho$ vs. $\Delta t$ for $\ell(t) = \cos(2\pi t)$](image)

Figure 1 plots the largest growth rate, $\rho$, of LF vs. $\Delta t$. The vertical axis is $\rho$ scaled logarithmically and labelled with $\rho$ values and the horizontal axis is $\Delta t$. Values of $\rho$ smaller than $10^{-8}$ can be regarded as numerically zero because it would take $10^6$ steps before $u^n$ would increase 1\% in size.

Notice that the values of $\Delta t$ which give $\rho = 0$ (to numerical precision) are overwhelmingly more common than values of $\Delta t$ for which $\rho$ is of significant size. Thus, a randomly-chosen $\Delta t$ along with randomly-chosen initial values would be unlikely to produce an instability. Notice also that as $\Delta t$ becomes smaller, values of $\Delta t$ which produce an instability become even less common. It even appears (incorrectly) that there are no values of $\Delta t$ which produce an instability for $\Delta t < 0.05$. (We shall show that this is not true for a different $\ell(t)$ in the construction below.)

While Figure 1 suggests considerable structure, especially for larger $\Delta t$, the authors have been unable to observe a pattern in $\rho$ for small $\Delta t$. It is clear that values of $\Delta t$ that give rise to significant growth rates are not easy for the authors to guess.
Construction 2: Exponential instability for $\Delta t \to 0$.

We give next a construction of a $2 \times 2$ system (1.1) that is equivalent to a complex scalar case and for which LF is unstable for arbitrarily small timesteps. The coefficient $\ell(t)$ is chosen as a square wave with unit amplitude and unit period (see (3.8) below), so that

$$\Lambda(t) = \ell(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

satisfies $|\Lambda(t)|_2 \equiv 1$ and $|\Lambda(t^n) - \Lambda(t^{n-1})|_2$ takes the value 0 except twice per period, when it is 2.

A sequence $\Delta t_k \to 0$ is exhibited for which the solution to (CNLF) with $A(t) = 0$ grows as $e^{\Delta t_k c t}$. This example shows that the RHS of (2.1) has asymptotic behavior that cannot be improved and thus LF is unconditionally unstable when $A(t) = 0$.

The construction is performed in several steps:

1. Define a sequence of complex $z^n$ derived from the original sequence $v^n$, thus reducing the original $2 \times 2$ system to a complex scalar system.
2. Rewrite the nonautonomous recursion (CNLF) as an autonomous recursion $A_1 Z^m = A_2 Z^{m-1}$, where the complex vectors $Z^m$ are defined by grouping the iterates $z^n$ according to the period of $\ell(t)$.
3. Write an explicit recursive expression for $Z^m = A_1^{-1} A_2 Z^{m-1}$.
4. Construct a $2 \times 2$ matrix $B_{2 \times 2}$ whose eigenvalues agree with the nontrivial eigenvalues of $A_1^{-1} A_2$.
5. Based on the choice of $\ell(t)$, show that the eigenvalues of $B_{2 \times 2}$ are real and one of them has magnitude $1 + 2\Delta t + O(\Delta t^2) > 1$.

Step 1. Equivalent complex recursion. Rewrite the vector $(u_1, u_2)$ as a complex scalar $z = u_1 + iu_2$. Since $A = 0$, (CNLF) becomes

$$z^{k+1} = z^{k-1} - 2i\Delta t \ell(k\Delta t) z^k.$$  \hspace{1cm} (3.2)

Step 2. Autonomous recursion. Assume that $\ell(k\Delta t)$ is periodic with period $K\Delta t$. Set $a_k = 2\Delta t \ell(k\Delta t)$. Periodicity implies that $a_{k+K} = a_k$. Define a complex vector $Z^m$ with components $Z^m_k$, for $k = 1, \ldots, K$ as

$$Z^m_k = z^{mK+k}.$$  \hspace{1cm} (3.3)

Substituting $m-1$ for $m$ gives $Z^{m-1}_k = z^{mK+k-K}$.

Writing (3.2) in terms of $Z^m_k$ gives

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ ia_1 & 1 & 0 & \cdots & 0 \\ -1 & ia_2 & 1 & \cdots & 0 \\ 0 & -1 & ia_3 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & ia_{K-1} \\
\end{bmatrix} \begin{bmatrix} Z^m_1 \\ Z^m_2 \\ Z^m_3 \\ \vdots \\ Z^m_K \\
\end{bmatrix} = \begin{bmatrix} Z^{m-1}_1 \\ Z^{m-1}_2 \\ Z^{m-1}_3 \\ \vdots \\ Z^{m-1}_K \\
\end{bmatrix}.$$  \hspace{1cm} (3.4)
Denoting the two matrices in (3.4) as $A_1$ and $A_2$,
\begin{equation}
A_1 Z^m = A_2 Z^{m-1}
\end{equation}
where $Z^m$ is the vector of $Z_k^m$ and $A_1$ and $A_2$ are the above matrices. The recursion (3.5) is autonomous.

**Step 3. Explicit recursion expression.** The only nontrivial columns of the matrix $A_2$ are the final two, so the the only nontrivial columns in the product $B = A_1^{-1} A_2$ are its final two columns. Furthermore, the matrix $A_1$ is lower triangular with only three nontrivial diagonals, so $B$ can be written in recursive form as
\begin{equation}
\begin{align*}
B_{1,K-1} &= 1 \\
B_{2,K-1} &= -ia_k \\
B_{1,K} &= -ia_1 B_{1,K-1} \\
B_{2,K} &= 1 - ia_1 B_{2,K-1} \\
B_{k,j} &= B_{k-2,j} - ia_{k-1} B_{k-1,j} \text{ for } k = 3, \ldots, K \text{ and } j = K-1, K
\end{align*}
\end{equation}

**Step 4. Reduction to a $2 \times 2$ matrix.** Since the recursion (3.5) is autonomous, its stability is determined by the spectral radius of the matrix $B$. It turns out that the spectral radius of $B$ is equal to the spectral radius of a derived $2 \times 2$ matrix $B_{2\times2}$. We will choose a function $\ell(t)$ for which the spectral radius of $B_{2\times2}$ is larger than one for a sequence of values $K \to \infty$.

Because of its structure, there are necessarily $K-2$ null eigenvalues of $B$. If a vector $Z$ is an eigenvector of $B$ with eigenvalue $\lambda \neq 0$, then the following $2 \times 2$ system must be satisfied.
\begin{equation}
B_{2\times2} \begin{bmatrix} Z_{K-1} \\ Z_K \end{bmatrix} = \begin{bmatrix} B_{K-1,K-1} & B_{K-1,K} \\ B_{K,K-1} & B_{K,K} \end{bmatrix} \begin{bmatrix} Z_{K-1} \\ Z_K \end{bmatrix} = \lambda \begin{bmatrix} Z_{K-1} \\ Z_K \end{bmatrix}.
\end{equation}
Each eigenvector of $B_{2\times2}$ can be expanded into an eigenvector of $B$, by choosing its two components as the two initial values in (3.2), so the non-null eigenvalues of $B$ and $B_{2\times2}$ agree.

**Step 5. Spectral radius.** Choose $\ell(t)$ to be the periodic function of period 1
\begin{equation}
\ell(t) = \begin{cases} 
+1 & 0 \leq t < 1/4 \\
-1 & 1/4 \leq t < 3/4 \\
+1 & 3/4 \leq t < 1 \\
\text{periodic} & \text{otherwise}
\end{cases}
\end{equation}

For an even\(^1\) integer $k_0$, let $K = 2(2k_0 + 1)$, and $\Delta t = 1/K$. With this choice of $K$, the values $t^k = k\Delta t$ never exactly equal a point of discontinuity of $\ell(t)$, and $\Delta t$ can be chosen arbitrarily small.

A straightforward but tedious induction shows that
\begin{equation}
B_{2\times2} = \begin{bmatrix} 1 & -2i\Delta t \\ 2i\Delta t & 1 + 4\Delta t^2 \end{bmatrix} + O(\Delta t^3).
\end{equation}
The eigenvalues of $B_{2\times2}$ are
\[\lambda = 1 \pm 2\Delta t + 2\Delta t^2 + O(\Delta t^3).\]
Thus, the spectral radius is larger than 1
\[\sigma = \text{spr}(B_{2\times2}) = \text{spr}(A_1^{-1} A_2) = 1 + 2\Delta t + O(\Delta t^2) > 1.\]
Choose an initial vector $Z^0$ as the dominant eigenvector of $B$, $Z^k = \sigma Z^{k-1}$. Since each vector $Z^k$ represents $K = 1/\Delta t$ LF timesteps, the complex iterates satisfy

\(^1\)Odd integers work similarly and with the same spectral radius, but the formulæ are slightly different.
\[ |z^K_n| = |z^0| e^{(Kn\Delta t) \log \sigma} + O(\Delta t^2). \]  
On average, then, with \( t^n = n\Delta t \), \( |z^n| \approx |z^0| e^{(2\Delta t/K)t^n} + O(\Delta t^2) \). Denote the average growth rate per timestep as \( \rho = 2\Delta t + O(\Delta t^2) \).

For each even integer \( k_0 = 2, 4, \ldots \), letting \( K = 2(2k_0 + 1) \), and \( \Delta t = 1/K \to 0 \) results in an exponentially divergent LF iteration. The average rate of growth of the iterates is \( \rho = 2\Delta t + O(\Delta t^2) \). There is no limiting size of \( \Delta t \) below which the iteration does not diverge.

This choice of \( \Lambda(t) \) is not Lipschitz, but Theorem 1 does apply. The exponential factor in the theorem is
\[
\exp \left[ \frac{\Delta t}{1 - \alpha} \sum_{n=1}^{N} |\Lambda(t^{n+1}) - \Lambda(t^n)| \right].
\]
For this choice of \( \Lambda \), the difference \( |\Lambda(t^{n+1}) - \Lambda(t^n)| \) is nonzero (and then equal to 2) only twice per period. Replacing \( N \) with \( nK \), the exponential factor becomes
\[
\exp \left[ \frac{\Delta t}{1 - \alpha} 2nK(2/K) \right] = \exp \left[ \frac{\Delta t}{1 - \alpha} 4t \right]
\]
since \( nK\Delta t = t \). Thus the average growth rate in Theorem 1 is \( \rho = 4\Delta t/(1 - \alpha) \), and \( \alpha \) can be taken to be small.

**Remark 3.** It is interesting to note that (3.9) contains no explicit dependence on the value \( k_0 \). This is a consequence of the average of \( \ell(t) \) over a period being zero. Just after the point \( t = k_0/K \), where \( \ell(t) \) jumps from +1 to -1, the analogous matrix is
\[
\begin{pmatrix}
-ik_0\Delta t & 1 - k_0(k_0 + 2)/2\Delta t^2 \\
1 - k_0(k_0 + 2)/2\Delta t^2 & -i(k_0 + 2)\Delta t
\end{pmatrix}.
\]

**Remark 4.** The function \( \ell(t) \) was chosen because it has a Fourier series with terms related to the function in the first example, \( \ell(t) = (4/\pi) \sum_{n=0}^{\infty} \cos(2(2n + 1)\pi t)/(2n + 1) \). For \( K = 2(2k_0 + 1) \), and \( \Delta t = 1/K, \) as above, the term \( n = k_0 \) takes the values \( \pm 1 \), so that growth appears just as in the first example.

**Remark 5.** The leapfrog method for the differential equation
\[
\frac{du}{dt} - i\ell(t)u = 0
\]
is precisely (3.2), and the solution of this equation, starting with \( u(0) = u_o \) is
\[
u(t) = \begin{cases}
u_0e^{it} & 0 \leq t < 1/4 \\
u_0e^{-i(t-1/2)} & 1/4 \leq t < 3/4 \\
u_0e^{i(t-3/4)} & 3/4 \leq t < 1 \\
\text{periodic} & t > 1
\end{cases}
\]
This function is bounded, so the growth of the leapfrog approximation is due to the numerical approximation.

**4. Conclusions**

In contrast to the autonomous case, in the nonautonomous case we have shown through two constructions that CNLF (and, as a special case, leapfrog) computed solutions can grow even when the system itself has bounded or decaying solutions. In Theorem 1 we prove that the growth rate, however, is at worst proportional to \( \Delta t \) when \( \Lambda(t) \) is Lipschitz, a rate attained in the examples constructed.

One important open problem is related to Remark 1. Time filters are an important and proven method for GFD simulations. However, their computational
practice is far ahead of their analytic foundations. Understanding the impact of time filters on the nonautonomous instability delineated herein is an important open problem.

References
