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NUMERICAL SOLUTIONS OF A HYPERSINGULAR INTEGRAL EQUATION WITH APPLICATION TO PRODUCTIVITY FORMULAE OF HORIZONTAL WELLS PRODUCING AT CONSTANT WELLBORE PRESSURE

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Abstract. The performance of horizontal wells producing at constant wellbore pressure is a critical problem in petroleum engineering. But few articles on the well performance under constant wellbore pressure can be found in the literature due to the difficulty of hypersingular integral equations, which are needed for this problem. This article proposes and studies a new model using a hypersingular integral equation for the productivity of horizontal wells producing at constant wellbore pressure. An efficient numerical method is developed for this hypersingular integral equation based on a new expansion with respect to the singularity at arbitrary points. And numerical examples are provided to illustrate the convergence of the numerical methods. By using fluid potential superposition principle, productivity equations for a line sink model are derived from the point sink solution to the diffusivity equation. By solving the hypersingular integral equation, the authors obtain the productivity formulae of a horizontal well producing at constant wellbore pressure, which provide fast analytical tools to evaluate production performance of horizontal wells. Numerical examples are provided to illustrate the features of the model and the numerical method.

Key words. Hypersingular Integral Equation, Quadrature method, Horizontal Well, Constant Wellbore Pressure.

1. Introduction

For horizontal wells, steady-state and unsteady-state pressure transient testings are critical tools to evaluate in-situ reservoir and wellbore parameters that describe the production characteristics of a well. The use of transient well testings for determining reservoir parameters and productivity of horizontal wells has become common because of the upsurge in horizontal drilling. Hence the determination of transient pressure behavior and productivity for horizontal wells has aroused considerable interest recently [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

During the last two decades, numerous analytic solutions have been presented for the pressure behavior of horizontal wells producing at constant flow rates. Interpretation of well tests from horizontal wells is much more difficult than interpretation of those from vertical wells due to a considerable wellbore storage effect, the three dimensional nature of the flow geometry, the lack of radial symmetry, and the strong correlations between certain parameters, see [3, 4, 5, 6, 7, 8, 9, 10] and references therein. Joshi [11] presented a steady state productivity formula for horizontal wells based on a pseudo three-dimensional reservoir model which was splitted into two two-dimensional models. Babu [12] presented a pseudo-steady state productivity formula for a horizontal wells in a closed box-shaped reservoir. By solving three dimensional Laplace equation and using fluid potential superposition principle, Lu [13, 14, 15, 16] presented steady state productivity formulae

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and pressure transient formulae of horizontal wells in infinite slab reservoirs and circular cylinder reservoirs.

Most well test analysis methods assume constant rate production since constant wellbore pressure production conditions are common. Examples of conditions under which constant pressure is maintained at a well include production into a constant pressure separator or pipeline, open flow to the atmosphere, and production from a low permeability reservoir. Particularly, in order to keep a steady water cone, the constant wellbore pressure production is required for a reservoir with bottom water.

In order to solve the problem of constant wellbore pressure production, one often needs the solution to a hypersingular integral equation [17]. In recent years, different types of singular integral equations have been utilized to study the problems in acoustics, fracture mechanics, elastic mechanics, electromagnetics, traffic flow, and so on [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. And different efficient numerical methods for hypersingular integral equations have been also studied, such as Galerkin methods [33, 34, 35, 36, 37, 38], quadrature methods [39, 40, 41], additive Schwarz method [42, 43, 44], multilevel methods [42, 45], collocation methods [46, 47], Newton's method [48], hybrid method [49] and others [50, 51, 52, 53]. See [54] and references therein for a survey on the hypersingular integral equations.

This article proposes and studies a new model, which uses the hypersingular integral equation (1) for the productivity of horizontal wells producing at constant wellbore pressure. To our best knowledge, there are few papers discussing the numeric solution of this type of hypersingular integral equation even though a number of numerical methods have been developed to solve other types of hypersingular integral equations. Galerkin method can be used to solve equation (1). But the computational cost is relatively high. Without non-trivial extension, other existing methods may not be directly valid for equation (1) due to the difficulties caused by the term $2\psi\left(\frac{1}{2}\right) - 2\ln\left(\frac{1}{h}\right)$ in (28).

In this article, an efficient numerical method is proposed to solve the equation (1) based on a new expansion with respect to the singularity at arbitrary points instead of the endpoints and then the results are applied in the new productivity model of horizontal wells producing at constant wellbore pressure. The hypersingular integral equation is studied in Section 2 and the numerical method for this equation is developed in Section 3. In Section 4, three numerical examples are presented to illustrate the convergence of this proposed method and solve the hypersingular integral equation arising from the new model. In Section 5, the new model is presented and the numerical solution of the hypersingular integral equation is used to obtain a universal productivity formula of horizontal wells producing at constant wellbore pressure.

2. A hypersingular integral equation for the new model

In order to obtain the productivity formulae of horizontal wells under the constant wellbore pressure in Section 5, we first consider the following new model of an integral equation which will be derived in Section 5:

$$\int_{-1}^{1} \frac{g(x)}{|x-y|} dx = -1, \forall y \in [-1,1], \tag{1}$$

where the unknown function $g(x)$ is called the point convergence intensity in petroleum engineering.

This is a hypersingular integral equation since the integral is divergent in Cauchy principal value. It means that we need to understand the left of (1) as a Hadamard finite-part integral [17, 55]. In the following we will recall the definition and fundamental properties of Hadamard finite-part of a divergent integral. First, it is well known that the following integral of $f(x) = x^\alpha g(x)$ is divergent when $\alpha \leq -1$ even if $g(x)$ is a smooth function:

$$\int_0^b f(x)dx = \int_0^b x^\alpha g(x)dx. \tag{2}$$

Then the Hadamard finite-part of divergent integral is defined as follows for a function $f(x)$ which is hypersingular nearby the origin of coordinates.

Definition 1[17, 55]**:** Let $f(x)$ be integrable over (ϵ, b) for any ϵ satisfying $0 < \epsilon < b < \infty$. Suppose that there exists a strictly monotonic increasing sequence $\alpha_0 < \alpha_1 < \alpha_2 < ...$, and a non-negative integer J such that the following expansion

$$\int_\varepsilon^b f(x)dx = \sum_{i=0}^{\infty}\sum_{j=0}^{J} I_{i,j}\varepsilon^{\alpha_i}\ln^j(\varepsilon) \tag{3}$$

converges for any $\varepsilon \in (0, h)$ for some $h > 0$. Then the Hadamard finite-part integral of (3) is defined as follows:

$$f.p. \int_0^b f(x)dx = \begin{cases} 0, & (when\ \alpha_i \neq 0,\ for\ all\ i) \\ I_{I,0}(b) & others. \end{cases} \tag{4}$$

where

$$I_{I,0}(b) = \sum_{\alpha_i=0} I_{i,0}\ \ for\ all\ i.$$

Remark: The right side of the equation (3) is an expansion of the integral $\int_\varepsilon^b f(x)dx$ in term of ε where $I_{i,j}$ are the coefficients of this expansion. In the following, a simple example is provided to illustrate the definition of the Hadamard finite-part integral:

$$\begin{aligned}
\int_\varepsilon^b \frac{e^t}{t^2}dt &= \int_\varepsilon^b \frac{\sum_{i=0}^{\infty}\frac{1}{i!}t^i dt}{t^2}dt = \sum_{i=0}^{\infty}\int_\varepsilon^b \frac{1}{i!}t^{i-2}dt \\
&= \int_\varepsilon^b t^{-2}dt + \int_\varepsilon^b t^{-1}dt + \sum_{i=2}^{\infty}\int_\varepsilon^b \frac{1}{i!}t^{i-2}dt \\
&= \varepsilon^{-1} - b^{-1} + \ln b - \ln\varepsilon + \sum_{i=2}^{\infty}\frac{1}{i!}\cdot\frac{1}{i-1}\cdot(b^{i-1} - \varepsilon^{i-1}) \\
&= \ln b - b^{-1} + \sum_{i=2}^{\infty}\frac{1}{i!}\cdot\frac{1}{i-1}\cdot b^{i-1} + \varepsilon^{-1} - \ln\varepsilon - \sum_{i=2}^{\infty}\frac{1}{i!}\cdot\frac{1}{i-1}\varepsilon^{i-1} \\
&= (\ln b - b^{-1} + \sum_{i=2}^{\infty}\frac{1}{i!}\cdot\frac{1}{i-1}\cdot b^{i-1})\varepsilon^0 + \varepsilon^{-1} - \ln\varepsilon - \sum_{i=2}^{\infty}\frac{1}{i!}\cdot\frac{1}{i-1}\varepsilon^{i-1}
\end{aligned}$$

Then by Definition 1, we have

$$f.p. \int_0^b \frac{e^t}{t^2}dt = \ln b - b^{-1} + \sum_{i=2}^{\infty}\frac{1}{i!}\cdot\frac{1}{i-1}\cdot b^{i-1}.$$

It can be found that both Riemman integral and Cauchy primary value integral satisfy Definition 1. In particular, the following formulaes are valid in the sense of Definition 1 [56, 57, 58]:

$$f.p.\int_a^b \frac{g(x)}{(x-t)^2}dx = \lim_{\epsilon\to 0}[\int_a^{t-\epsilon} \frac{g(x)}{(x-t)^2}dx + \int_{t+\epsilon}^b \frac{g(x)}{(x-t)^2}dx - \frac{2g(t)}{\epsilon}] \tag{5}$$

$$f.p.\int_a^b \frac{g(x)}{(x-t)^3}dx = \lim_{\epsilon\to 0}[\int_a^{t-\epsilon} \frac{g(x)}{(x-t)^3}dx + \int_{t+\epsilon}^b \frac{g(x)}{(x-t)^3}dx - \frac{2g^{'}(t)}{\epsilon}] \tag{6}$$

Therefore, by treating the left of equation (1) as a Hadamard finite-part integral, we obtain

$$f.p.\int_{-1}^1 \frac{g(x)}{|x-y|}dx = -1. \tag{7}$$

Now we recall the following two lemmas from [55] for the fundamental properties of the Hadamard finite-part integral, which will be needed for Example 1 in Section 4.

Lemma 1 [55]**:** For any $b > 0$, we have

$$f.p.\int_0^b x^\alpha dx = \begin{cases} \ln(b), & (\alpha = -1), \\ b^{\alpha+1}/(\alpha+1), & (\alpha \neq -1). \end{cases}$$

Lemma 2 [55]**:** Assume $\alpha < -1$ and $m > -\alpha - 2$ and $g(x) \in C^{m+1}[0, b)$. Then for any $b > 0$, we have

$$\begin{aligned} f.p.\int_0^b g(x)x^\alpha dx &= \int_0^b x^\alpha \left[g(x) - \sum_{k=0}^m g^{(k)}(0)x^k/k!\right] dx \\ &\quad + \sum_{k=0}^m \left[\frac{g^{(k)}(0)}{k!}\right] f.p.\int_0^b x^{\alpha+k}dx. \end{aligned} \tag{8}$$

Furthermore, we consider the following integral

$$\int_0^1 f(x)dx = \int_0^1 [x^\alpha(1-x)^\gamma g(x)]dx, \tag{9}$$

where $g(x) \in C^m[0, 1]$. When $\min(\alpha, \gamma) = -1$, (9) is a singular integral; when $\min(\alpha, \gamma) < -1$, (9) is a hypersingular integral. Clearly, both singular integral and hypersingular integral are divergent, thus (9) needs to be understood in the meaning of Hadamard finite-part integral. And the following theorem, which is recalled from [55] and will be needed later, is a generalization of the Euler-Maclaurin expansion to the hypersingular integral.

Theorem 1[55]**:** Assume $f(x) = x^\alpha(1-x)^\gamma g(x)$, $g(x) \in C^{n+1}[0, 1]$, $\alpha = -l-1$, $\gamma = -m-1$, α and γ are negative integers, and N is a sufficiently large positive integer. Then for any $\beta \in (0, 1)$, the following Euler-Maclaurin expansion of offset trapezoidal rule holds:

$$\begin{aligned}
& \frac{1}{N}\sum_{k=0}^{N-1} f(\frac{\beta+k}{N}) \\
= & \; f.p.\int_0^1 f(x)dx - \frac{g_0^{(l)}(0)}{l!}\psi(\beta) + \frac{g_0^{(l)}(0)}{l!}\ln(N) \\
& -(-1)^m \frac{g_1^{(l)}(1)}{m!}\psi(\beta) + (-1)^m \frac{g_1^{(l)}(1)}{m!}\ln(N) \\
& + \sum_{k=0,k\neq l}^{n}\left[\frac{1}{N^{k+1+\alpha}k!}\right]\zeta(-k-\alpha,\beta)g_0^{(k)}(0) \\
& + \sum_{k=0,k\neq m}^{n}\left[\frac{(-1)^k}{N^{k+1+\gamma}k!}\right]\zeta(-k-\gamma,1-\beta)g_1^{(k)}(1) + O(N^{-n-\omega-1}),
\end{aligned} \tag{10}$$

where

$$g_0(x) = (1-x)^\gamma g(x), \qquad g_1(x) = x^\alpha g(x), \tag{11}$$

$$\psi(z) = \Gamma'(z)/\Gamma(z), \qquad \omega = \min(\alpha,\gamma). \tag{12}$$

ζ is the Riemann-zeta function and Γ is the Gamma function[59] .

We readily extend the formula (10) to a general interval $[a,b]$. Consider the following singular integral

$$f.p.\int_a^b (x-a)^\alpha (b-x)^\gamma g(x)dx = f.p.\int_a^b f(x)dx, \qquad (\min(\alpha,\gamma)\leq -1), \tag{13}$$

we take integral transform $y = x-a, d = b-a$, consequently,

$$f.p.\int_a^b (x-a)^\alpha (b-x)^\gamma g(x)dx = f.p.\int_0^d y^\alpha (d-y)^\gamma \tilde{g}(y)dy = f.p.\int_0^d \tilde{f}(y)dy, \tag{14}$$

where

$$\tilde{g}(y) = g(a+y), \tag{15}$$

$$\tilde{f}(y) = y^\alpha (d-y)^\gamma \tilde{g}(y) = f(a+y). \tag{16}$$

Let $h = (b-a)/N$. If $\alpha = -l-1$ and $\gamma = -m-1$ are negative integers, then using the quadrature formula of offset trapezoidal rule, we obtain [55]

$$\begin{aligned}
& \sum_{j=0}^{N-1} hf(a+(j+\beta)h) \\
= & \; f.p.\int_a^b f(x)dx - \left[\frac{g_0^{(l)}(a)}{l!}\right]\psi(\beta) + \left[\frac{g_0^{(l)}(a)}{l!}\right]\ln\left(\frac{1}{h}\right) - (-1)^m\left[\frac{g_1^{(m)}(b)}{m!}\right]\psi(\beta) \\
& +(-1)^m\left[\frac{g_1^{(m)}(b)}{m!}\right]\ln\left(\frac{1}{h}\right) + \sum_{k=0,k\neq l}^{n}\left[\frac{h^{k+1+\alpha}}{k!}\right]\zeta(-k-\alpha,\beta)g_0^{(k)}(a) \\
& + \sum_{k=0,k\neq m}^{n}\left[\frac{(-1)^k h^{k+1+\alpha}}{k!}\right]\zeta(-k-\gamma,1-\beta)g_1^{(k)}(b) + O(h^{n+\omega+1}),
\end{aligned} \tag{17}$$

where

$$g_0(x) = (b-x)^{\gamma} g(x), \qquad g_1(x) = (x-a)^{\alpha} g(x). \tag{18}$$

3. Approximation method

In this section we will consider the following hypersingular integral:

$$f.p. \int_a^b G(x)dx = f.p. \int_a^b \frac{g(x)}{|x-t|} dx, \tag{19}$$

and develop the approximation method for the corresponding hypersingular integral equation

$$f.p. \int_a^b G(x)dx = -1. \tag{20}$$

In Theorem 1, the Euler-Maclaurin expansion has been generalized for the singularity at the endpoints, which are often the partition nodes in practice. In order to overcome the difficulty arising from the hypersingular integral equation, the basic idea is to build up a proper expansion with respect to the singularity at arbitrary points.

Assume $x_j = a + jh$, $j = 0, 1, ..., N$, $h = (b-a)/N$, N is a positive integer, and $t \in \{x_j : 1 \leq j \leq N-1\}$ is a fixed parameter. Choose $\beta = 1/2$ and construct the integral formula of mid-rectangle quadrature as follows

$$I_h(G) = \sum_{j=0}^{N-1} hG(a + (j+1/2)h). \tag{21}$$

Then we obtain the following theorem which provides an expansion for singularity at an arbitrary point t.

Theorem 2: If $g(x) \in C^{2n+1}[a,b]$, $G(x) = g(x)|x-t|^{\alpha}$, $t = x_i, 1 \leq i \leq N-1$, and $\alpha = -2l-1$ is an odd negative integer, then we obtain the following asymptotic expansion

$$\begin{aligned} & \sum_{j=0}^{N-1} hG(a + (j+1/2)h) \\ = \;& f.p. \int_a^b G(y)dy + \sum_{k=1}^{n} \left[\frac{h^{2k}}{(2k)!}\right] B_{2k}\left(\frac{1}{2}\right) \left[G^{(2k-1)}(b) - G^{(2k-1)}(a)\right] \\ & -2\left[\frac{g^{(2l)}(x_i)}{(2l)!}\right] \psi\left(\frac{1}{2}\right) + 2\left[\frac{g^{(2l)}(x_i)}{(2l)!}\right] \ln\left(\frac{1}{h}\right) \\ & +2 \sum_{k=0,k\neq l}^{n} \left[\frac{h^{2(k-l)}}{(2k)!}\right] \zeta(2l+1-2k, 1/2) g^{(k)}(x_i) + O(h^{2n+1}), \end{aligned} \tag{22}$$

where

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2 \approx -1.9635, \tag{23}$$

$\gamma \approx 0.5772$ is the Euler constant, $h = \frac{b-a}{N}$, and $B_{2k}(x)$ is the Bernoulli polynomials [59].

Remark: This is an important conclusion for approximating the hypersingular integrals. More general conclusions can be found in [60].

Proof: Suppose that $\alpha = -2l - 1$. Obviously we have

$$f.p.\int_a^b G(y)dy = f.p.\int_{x_i}^b G(y)dy + f.p.\int_a^{x_i} G(y)dy. \tag{24}$$

Then from equation (17), we obtain

$$\begin{aligned}
&\sum_{j=0}^{i-1} hG(a+(j+1/2)h) \\
= \;& f.p.\int_a^{x_i} G(y)dy - \sum_{k=1}^{n}\left[\frac{h^{2k}}{(2k)!}\right] B_{2k}\left(\frac{1}{2}\right) G^{(2k-1)}(a) - \left[\frac{g^{(2l)}(x_i)}{(2l)!}\right]\psi\left(\frac{1}{2}\right) \\
&+ \left[\frac{g^{(2l)}(x_i)}{(2l)!}\right]\ln(\frac{1}{h}) + \sum_{k=0,k\neq 2l}^{2n}\left[\frac{(-1)^k h^{k+1+\alpha}}{k!}\right]\zeta(-k-\alpha,1/2)g^{(k)}(x_i) + O(h^{2n+1}).
\end{aligned} \tag{25}$$

Note that $\zeta(-2k,1/2) = 0, k = 0,1,...,n; \zeta(-(2k-1),1/2) = B_{2k}(\frac{1}{2}), k = 1,2,...,n.$ Similarly, we can prove that

$$\begin{aligned}
&\sum_{j=i}^{N-1} hG(a+(j+1/2)h) \\
= \;& f.p.\int_{x_i}^b G(y)dy + \sum_{k=1}^{n}\left[\frac{h^{2k}}{(2k)!}\right] B_{2k}\left(\frac{1}{2}\right) G^{(2k-1)}(b) - \left[\frac{g^{(2l)}(x_i)}{(2l)!}\right]\psi\left(\frac{1}{2}\right) \\
&+ \left[\frac{g^{(2l)}(x_i)}{(2l)!}\right]\ln\left(\frac{1}{h}\right) + \sum_{k=0,k\neq 2l}^{2n}\left[\frac{h^{k+1+\alpha}}{k!}\right]\zeta(-k-\alpha,1/2)g^{(k)}(x_i) + O(h^{2n+1}).
\end{aligned} \tag{26}$$

Combining equations (25) and (26) yields

$$\begin{aligned}
&\sum_{j=0}^{N-1} hG(a+(j+1/2)h) \\
= \;& f.p.\int_a^b G(y)dy + \sum_{k=1}^{n}\left[\frac{h^{2k}}{(2k)!}\right] B_{2k}\left(\frac{1}{2}\right)[G^{(2k-1)}(b) - G^{(2k-1)}(a)] \\
&-2\left[\frac{g^{(2l)}(x_i)}{(2l)!}\right]\psi\left(\frac{1}{2}\right) + 2\left[\frac{g^{(2l)}(x_i)}{(2l)!}\right]\ln\left(\frac{1}{h}\right) \\
&+ \sum_{k=0,k\neq 2l}^{2n}[1+(-1)^k]\left(\frac{h^{k+1+\alpha}}{k!}\right)\zeta(-k-\alpha,1/2)g^{(k)}(x_i) + O(h^{2n+1}) \\
= \;& f.p.\int_a^b G(y)dy + \sum_{k=1}^{n}\left[\frac{h^{2k}}{(2k)!}\right] B_{2k}\left(\frac{1}{2}\right)[G^{(2k-1)}(b) - G^{(2k-1)}(a)] \\
&-2\left[\frac{g^{(2l)}(x_i)}{(2l)!}\right]\psi\left(\frac{1}{2}\right) + 2\left[\frac{g^{(2l)}(x_i)}{(2l)!}\right]\ln\left(\frac{1}{h}\right) \\
&+2\sum_{k=0,k\neq l}^{n}\left[\frac{h^{2(k-l)}}{(2k)!}\right]\zeta(2l+1-2k,1/2)g^{(k)}(x_i) + O(h^{2n+1}).
\end{aligned} \tag{27}$$

This completes the proof.

Recall that $\zeta(-2m, 1/2) = 0$ $(m = 0, 1, \cdots)$ [59]. If $k > 0$ is an odd number and $-k - \alpha \leq 0$, then $\zeta(-k - \alpha, 1/2) = 0$. On the other hand, if $k > 0$ is an even number, then $[-1 + (-1)^k] = 0$. Therefore, using the above theorem, we can obtain the following conclusion.

Corollary: If $\alpha = -1$, i.e $l = 0$, $G(x) = g(x)|x - t|^{-1}$,$t = x_i, 1 \leq i \leq N - 1$,

$$I_h = \sum_{j=0}^{N-1} hG(a + (j + 1/2)h) + 2\psi\left(\frac{1}{2}\right) - 2\ln\left(\frac{1}{h}\right), \tag{28}$$

then we have

$$f.p. \int_a^b G(x)dx = I_h + O(h^2). \tag{29}$$

In the following, a numerical algorithm for solving equation (20) will be proposed based on the above conclusion, which is valid for singularity at any point t. Let

$$\begin{aligned} h &= 1/N, \\ x_i &= -1 + ih, \qquad (i = 0, ..., 2N), \\ y_i &= x_{i+1/2} = -1 + (i + 1/2)h, \qquad (i = 0, ..., 2N - 1). \end{aligned} \tag{30}$$

Let u_i be the approximation of $g(x_{i+1/2})$ $(i = 0, ..., 2N - 1)$. By using (29), the discrete equation for solving equation (20) can be constructed by the following steps:

(1) Let the singular point $t = y_{2i} = x_{2i+1/2}, (i = 0, 1, 2, ..., N - 1)$. Then

$$\sum_{j=0}^{N-1} 2h\frac{u_{2j+1}}{| y_{2j+1} - y_{2i} |} + \left[2\psi\left(\frac{1}{2}\right) - 2\ln\left(\frac{1}{2h}\right)\right] u_{2i} = -1, \qquad (i = 0, 1, 2, ..., N - 1). \tag{31}$$

(2) Let the singular point $t = y_{2i+1} = x_{2i+1+1/2}, (i = 0, 1, 2, ..., N - 1)$. Then

$$\sum_{j=0}^{N-1} 2h\frac{u_{2j}}{| y_{2j} - y_{2i+1} |} + \left[2\psi\left(\frac{1}{2}\right) - 2\ln\left(\frac{1}{2h}\right)\right] u_{2i+1} = -1, \qquad (i = 0, 1, 2, ..., N-1). \tag{32}$$

It can be denoted by

$$A_N \bar{u} = -I_{2N}, \tag{33}$$

where

$$I_{2N} = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} \tag{34}$$

and

$$A_N = [a_{ij}]_{i,j=0}^{2N-1}, \tag{35}$$

with

$$
\begin{aligned}
a_{ii} &= 2\psi\left(\frac{1}{2}\right) - 2\ln\left(\frac{1}{2h}\right), \qquad (0 \le i \le 2N-1),\\
a_{ij} &= 0, \qquad (i \ne j, \quad mod(|i-j|,2)=0),\\
a_{ij} &= \frac{2}{|i-j|}, \qquad (i \ne j, \quad mod(|i-j|,2)=1),
\end{aligned} \tag{36}
$$

$$\bar{u} = (u_0, ..., u_{2N-1})^T. \tag{37}$$

Once the solution $\bar{u}$ of equation (33) is obtained, we can calculate the flow rate of horizontal wells under the constant wellbore pressure by the following quadrature formula:

$$Q = Q(h) = h \sum_{i=0}^{2N-1} u_i. \tag{38}$$

4. Numerical examples

In this section we provide three numerical examples. First, we numerically verify the theoretical accuracy order of (29) in Example 1. Second, we validate the proposed algorithm numerically by using a hypersingular integral equation with analytic solution in Example 2. Last, we solve the target equation in Example 3.

Example 1: Consider the hypersingular integral

$$I(y) = \int_{-1}^{1} \frac{g(x)}{|x-y|} dx, \; y \in (-1,1).$$

When $g(x) = e^x$, by the definition and fundamental properties of Hadamard finite-part integral, we have

$$
\begin{aligned}
&I(y)\\
&= \int_{-1}^{1} \frac{e^x}{|x-y|} dx = \int_{-1}^{1} \frac{e^x - e^y + e^y}{|x-y|} dx\\
&= \int_{-1}^{1} \frac{e^x - e^y}{|x-y|} dx + e^y \int_{-1}^{1} \frac{1}{|x-y|} dx\\
&= \int_{-1}^{1} \frac{\sum_{k=0}^{\infty} e^y \frac{(x-y)^k}{k!}}{|x-y|} dx + e^y \int_{-1}^{1} \frac{1}{|x-y|} dx\\
&= e^y \sum_{k=0}^{\infty} \int_{-1}^{1} \frac{(x-y)^k}{k!|x-y|} dx + e^y \int_{-1}^{1} \frac{1}{|x-y|} dx\\
&= e^y \sum_{k=1}^{\infty} \int_{-1}^{1} \frac{(x-y)^k}{k!|x-y|} dx + 2e^y \int_{-1}^{1} \frac{1}{|x-y|} dx\\
&= e^y \sum_{k=1}^{\infty} \left(\int_{-1}^{y} \frac{(x-y)^k}{k!(y-x)} dx + \int_{y}^{1} \frac{(x-y)^k}{k!(x-y)} dx \right) + 2e^y \left(\int_{-1}^{y} \frac{1}{(y-x)} dx + \int_{y}^{1} \frac{1}{(x-y)} dx \right)\\
&= e^y \sum_{k=1}^{\infty} \frac{1}{k \cdot k!} [(-1-y)^k + (1-y)^k] + 2e^y (\ln|1+y| + \ln|1-y|).
\end{aligned}
$$

Using the formula (28) , we obtain the numerical approximation I_h for I in Table (1) and Table (2). The rate $r_h^{(1)} = \log_2 \frac{|I-I_{2h}|}{|I-I_h|}$ shows that (28) has the second order

TABLE 1. Numerical results for $I = \int_{-1}^{1} \frac{e^x}{|x-y|} dx \approx -12.7662064272$ at y=0.984375.

h	$1/2^6$	$1/2^7$	$1/2^8$	$1/2^9$
I_h	-12.6685343292	-12.7394120880	-12.7593066033	-12.7644650613
$\|I - I_h\|$	0.0976720980	0.0267943393	0.0068998240	0.0017413659
$r_h^{(1)}$		1.8660182462	1.9572967830	1.9863401617

TABLE 2. Numerical results $I = \int_{-1}^{1} \frac{e^x}{|x-y|} dx \approx 0.0542082332$ at y=-0.984375.

h	$1/2^7$	$1/2^8$	$1/2^9$	$1/2^{10}$
I_h	0.0678390449	0.0579477440	0.0551712102	0.0544512688
$\|I - I_h\|$	0.0136308118	0.0037395108	0.0009629770	0.0002430356
$r_h^{(1)}$		1.8659500189	1.9572763360	1.9863337479

accuracy, which is consistent with the theoretical conclusion in (29).

Example 2: Consider the following integral equation whose analytic solution is $g(x) = e^x, x \in [-1, 1]$:

$$\int_{-1}^{1} \frac{g(x)}{|x-y|} dx = I(y), \;\; y \in [-1, 1]. \tag{39}$$

The numerical results with $h = 1/8$ are listed in Table (3). And the average errors of the numerical solutions on all nodes for varying meshes are listed in Table (4). Moreover, Figure 1 gives the plots of the curve of the analytic solution and the dotted curves of the numerical solution with h=1/8 and $h = 1/64$ respectively. All of these numerical results and graphs illustrate the convergence of the proposed algorithm.

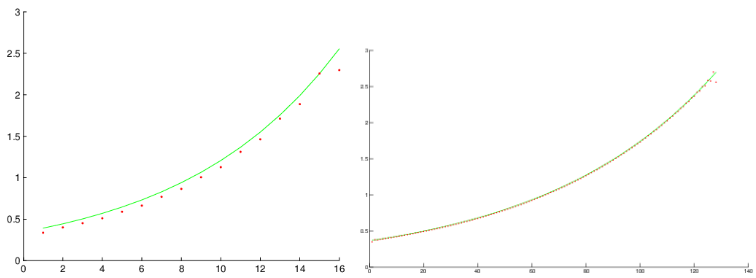

FIGURE 1. Comparison between the analytic curve of equation (39) and numerical solutions (in dots) with $h = 1/8$ (left) and $h = 1/64$ (right).

Example 3: Consider the hypersingular integral equation (1). Table 5 shows the numerical solutions by using (33). We can see that when a horizontal well is producing at constant wellbore pressure, the point convergence intensity at the endpoints is smaller than that of the midpoint. Moreover, the point convergence intensity is symmetric, i.e. $u_i = u_{31-i}$, i=0,1,...,16.

TABLE 3. Numerical results of equation (39) with $h = 1/8$.

i	x_i	g_i	$err = \lvert g_i - g(x_i)\rvert$
0	-0.9375	0.3362465316	0.0553590951
1	-0.8125	0.4010365005	0.0427108096
2	-0.6875	0.4514588196	0.0513727583
3	-0.5625	0.5107423617	0.0590404630
4	-0.4375	0.5892967793	0.0563517471
5	-0.3125	0.6636359587	0.0679796703
6	-0.1875	0.7697138490	0.0593152691
7	-0.0625	0.8651292961	0.0742837667
8	0.0625	1.0051102551	0.0593842038
9	0.1875	1.1267742816	0.0794559678
10	0.3125	1.3113735005	0.0554644407
11	0.4375	1.4629705831	0.0858597155
12	0.5625	1.7112674492	0.0437872078
13	0.6875	1.8857365314	0.1030009382
14	0.8125	2.2557471990	0.0022124118
15	0.9375	2.2963489502	0.2572405079

TABLE 4. Average errors of the numerical solutions at all nodes.

h	1/8	1/16	1/32	1/64
average error	0.1450787814	0.0666594772	0.0311315183	0.0147031037

TABLE 5. Numerical results of equation (1) with $h = 1/16$.

i	0	1	2	3	4	5
u_i	1.21416	1.54719	1.67501	1.82569	1.91855	2.02123
u_{i+6}	2.09241	2.16767	2.22191	2.27682	2.31630	2.35404
u_{i+12}	2.37989	2.40202	2.41482	2.42213	2.42213	2.41482
u_{i+18}	2.40202	2.37989	2.35404	2.31630	2.27682	2.22191
u_{i+24}	2.16767	2.09241	2.02123	1.91855	1.82569	1.67501
u_{i+30}	1.54719	1.21416				

5. Productivity formulae of horizontal wells producing at constant wellbore pressure

In this section we will propose a new model using the hypersingular integral equation (1) and then derive the productivity formulae of horizontal wells producing under constant wellbore pressure based on the following assumptions:

(a) The reservoir is infinite extension, i.e., the boundaries of the reservoir are so far away that the pressure disturbance does not travel far enough to reach the boundaries during the well production. The reservoir is homogeneous, anisotropic, and has constant K_x, K_y, K_z permeabilities, and porosity ϕ.

(b) The reservoir pressure is initially a constant. The pressure remains constant and equal to the initial value at an infinite distance from the well.

(c) The production occurs through a well of radius R_w, represented in the model by a line sink, the producing length of the well is L. The wellbore pressure remains constant and equal to P_0 during production.

(d) A single-phase fluid, of small and constant compressibility C_f, constant viscosity μ, and formation volume factor B, flows from the reservoir to the well. Fluids properties are independent of pressure.

Remark: In practice the assumption of the infinite extension in z direction is ideal. However, this assumption can dramatically reduce the computation complexity and simplify the model while keeping the most critical and difficult part of the problem, which is the hypersingular integral. The results also provide a significant theoretical insight for the further works on this problem. And it is an interesting future work to consider a limited domain in z direction.

Define the porous media domain:

$$\Omega = \{(x, y, z)|(-\infty, \infty) \times (-\infty, \infty) \times (-\infty, \infty)\}. \tag{40}$$

Assume

$$K_x = K_y = K_h, \quad K_z = K_v, \tag{41}$$

and define the following average permeability

$$K_a = (K_x K_y K_z)^{1/3} = K_h^{2/3} K_v^{1/3}. \tag{42}$$

The horizontal well is taken as a line sink in the three dimensional space, and the coordinates of the two ends are $(-L/2, 0, 0)$ and $(L/2, 0, 0)$. Suppose that point $(x', 0, 0)$ is on the horizontal producing portion, and its point convergence intensity is $q(x')$. In order to obtain the pressure at point (x, y, z) caused by the point $(x', 0, 0)$, we need to obtain the basic solution of the following diffusivity equation in Ω,

$$K_h \frac{\partial^2 P}{\partial x^2} + K_h \frac{\partial^2 P}{\partial y^2} + K_v \frac{\partial^2 P}{\partial z^2} = -\mu B q(x') \delta(x - x') \delta(y) \delta(z), \tag{43}$$

where $\delta(x - x')$, $\delta(y)$, and $\delta(z)$ are Dirac functions.

The initial pressure in the reservoir, denoted by P_{ini}, is uniform,

$$P(r, t)|_{t=0} = P_{ini}, \quad in \quad \Omega, \tag{44}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{45}$$

The pressure remains constant and equals to the initial value at an infinite distance from the well,

$$P(r, t)|_{r \to \infty} = P_{ini}. \tag{46}$$

In order to simplify the above equations, we take the following dimensionless transforms:

$$x_D = \left(\frac{2x}{L}\right) \sqrt{\frac{K_a}{K_h}}, \quad y_D = \left(\frac{2y}{L}\right) \sqrt{\frac{K_a}{K_h}}, \quad z_D = \left(\frac{2z}{L}\right) \sqrt{\frac{K_a}{K_v}}, \tag{47}$$

where K_a has the same meaning as in equation (42). Consequently, we have

$$R_{wD} = \left[\left(\frac{K_h}{K_v}\right)^{1/4} + \left(\frac{K_v}{K_h}\right)^{1/4}\right] \left(\frac{R_w}{2L}\right), \tag{48}$$

$$L_D = 2\sqrt{\frac{K_a}{K_h}}. \tag{49}$$

Furthermore, there hold the following simplifications:

$$\frac{\partial x_D}{\partial x} = \left(\frac{2}{L}\right)\sqrt{\frac{K_a}{K_h}}, \tag{50}$$

$$\frac{\partial P}{\partial x} = \left(\frac{\partial P}{\partial x_D}\right)\left(\frac{\partial x_D}{\partial x}\right) = \left(\frac{2}{L}\right)\sqrt{\frac{K_a}{K_h}}\left(\frac{\partial P}{\partial x_D}\right), \tag{51}$$

$$\begin{aligned}
\frac{\partial^2 P}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial P}{\partial x}) \\
&= \frac{\partial}{\partial x}\left[\left(\frac{2}{L}\right)\sqrt{\frac{K_a}{K_h}}\left(\frac{\partial P}{\partial x_D}\right)\right] \\
&= \frac{\partial}{\partial x_D}\left[\left(\frac{2}{L}\right)\sqrt{\frac{K_a}{K_h}}\left(\frac{\partial P}{\partial x_D}\right)\right]\left(\frac{\partial x_D}{\partial x}\right) \\
&= \left(\frac{4K_a}{L^2K_h}\right)\frac{\partial^2 P}{\partial x_D^2}. \qquad (52)
\end{aligned}$$

In the same manner,

$$\frac{\partial^2 P}{\partial y^2} = \left(\frac{4K_a}{L^2K_h}\right)\frac{\partial^2 P}{\partial y_D^2}, \tag{53}$$

$$\frac{\partial^2 P}{\partial z^2} = \left(\frac{4K_a}{L^2K_v}\right)\frac{\partial^2 P}{\partial z_D^2}. \tag{54}$$

Consequently,

$$K_h\frac{\partial^2 P}{\partial x^2} + K_h\frac{\partial^2 P}{\partial y^2} + K_v\frac{\partial^2 P}{\partial y^2} = \left(\frac{4K_a}{L^2}\right)\left(\frac{\partial^2 P}{\partial x_D^2} + \frac{\partial^2 P}{\partial y_D^2} + \frac{\partial^2 P}{\partial z_D^2}\right). \tag{55}$$

And we have

$$\begin{aligned}
&\delta(x-x')\delta(y)\delta(z) \\
&= \delta\left[\left(\frac{2}{L}\sqrt{\frac{K_a}{K_h}}\right)^{-1}(x_D - x'_D)\right]\delta\left[\left(\frac{2}{L}\sqrt{\frac{K_a}{K_h}}\right)^{-1} y_D\right]\delta\left[\left(\frac{2}{L}\sqrt{\frac{K_a}{K_v}}\right)^{-1} z_D\right] \\
&= \left(\frac{2}{L}\sqrt{\frac{K_a}{K_h}}\right)^{2}\left(\frac{2}{L}\sqrt{\frac{K_a}{K_v}}\right)\delta(x_D - x'_D)\delta(y_D)\delta(z_D) \\
&= \left(\frac{8}{L^3}\right)\left(\frac{K_a^{3/2}}{K_hK_v^{1/2}}\right)\delta(x_D - x'_D)\delta(y_D)\delta(z_D) \\
&= \left(\frac{8}{L^3}\right)\delta(x_D - x'_D)\delta(y_D)\delta(z_D). \qquad (56)
\end{aligned}$$

Here we use the conclusion that if c is a positive constant, then [61]

$$\delta(cx) = \delta(x)/c, \tag{57}$$

and

$$\frac{K_a^{3/2}}{K_hK_v^{1/2}} = 1. \tag{58}$$

Therefore,

$$\begin{aligned} & \mu B q(x')\delta(x-x')\delta(y)\delta(z) \\ = & \mu B q(x')\left(\frac{8}{L^3}\right)\delta(x_D-x'_D)\delta(y_D)\delta(z_D) \\ = & \left(\frac{8\mu B}{L^3}\right)q(x')\delta(x_D-x'_D)\delta(y_D)\delta(z_D). \end{aligned} \tag{59}$$

Substitute equations (55) and (59) into equation (43), we obtain

$$\begin{aligned} & \left(\frac{4K_a}{L^2}\right)\left(\frac{\partial^2 P}{\partial x_D^2}+\frac{\partial^2 P}{\partial y_D^2}+\frac{\partial^2 P}{\partial z_D^2}\right) \\ = & \left(\frac{8\mu B}{L^3}\right)q(x')\delta(x_D-x'_D)\delta(y_D)\delta(z_D), \end{aligned} \tag{60}$$

which is equivalent to the following equation,

$$\begin{aligned} & \frac{\partial^2 P}{\partial x_D^2}+\frac{\partial^2 P}{\partial y_D^2}+\frac{\partial^2 P}{\partial y_D^2} \\ = & \left(\frac{2\mu B}{K_a L}\right)q(x')\delta(x_D-x'_D)\delta(y_D)\delta(z_D). \end{aligned} \tag{61}$$

Multiplying $-K_aL/(2\mu Bq_{ref})$ on both sides of equation (61), we obtain

$$\begin{aligned} & \left(\frac{K_aL}{2\mu Bq_{ref}}\right)\left[\frac{\partial^2(P_{ini}-P)}{\partial x_D^2}+\frac{\partial^2(P_{ini}-P)}{\partial y_D^2}+\frac{\partial^2(P_{ini}-P)}{\partial z_D^2}\right] \\ = & \left[\frac{q(x')}{q_{ref}}\right]\delta(x_D-x'_D)\delta(y_D)\delta(z_D), \end{aligned} \tag{62}$$

where q_{ref} is a constant reference flow rate, and the initial reservoir pressure P_{ini} is a constant. Thus

$$\frac{\partial^2(P_{ini})}{\partial x_D^2}=\frac{\partial^2(P_{ini})}{\partial y_D^2}=\frac{\partial^2(P_{ini})}{\partial z_D^2}=0. \tag{63}$$

Define the dimensionless pressure:

$$P_D=\frac{K_aL(P_{ini}-P)}{2\mu Bq_{ref}}, \tag{64}$$

and the dimensionless point convergence intensity:

$$q_D(x'_D)=\frac{q(x')}{q_{ref}}. \tag{65}$$

Consequently, the dimensionless form of equation (62) can be obtained as follows:

$$\frac{\partial^2 P_D}{\partial x_D^2}+\frac{\partial^2 P_D}{\partial y_D^2}+\frac{\partial^2 P_D}{\partial z_D^2}=q_D(x'_D)\delta(x_D-x'_D)\delta(y_D)\delta(z_D) \tag{66}$$

in Ω_D, where

$$\Omega_D=\{(x_D,y_D,z_D)|(-\infty,\infty)\times(-\infty,\infty)\times(-\infty,\infty)\}, \tag{67}$$

and

$$x'_D\in\left(-\sqrt{\frac{K_a}{K_h}},\sqrt{\frac{K_a}{K_h}}\right).$$

If x'_D does not belong to $\left(-\sqrt{\frac{K_a}{K_h}}, \sqrt{\frac{K_a}{K_h}}\right)$, we have

$$q_D(x'_D) = 0. \tag{68}$$

Note that after taking dimensionless transforms, horizontal well producing section $(-L/2, L/2)$ is changed to $\left(-\sqrt{\frac{K_a}{K_h}}, \sqrt{\frac{K_a}{K_h}}\right)$. Consequently,

$$P(r_D, t_D)|_{t_D=0} = 0, \tag{69}$$

and

$$P_D|_{r_D \to \infty = 0}, \tag{70}$$

where

$$r_D = \sqrt{x_D^2 + y_D^2 + z_D^2}. \tag{71}$$

The fundamental solution of the three-dimensional Laplace equation is as follows [59, 62]:

$$F(x, y, z; x', y', z') = \frac{-1}{4\pi[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}. \tag{72}$$

Thus, if $q_D(x'_D)$ is known, the solution to equation (66) is given by:

$$P_D(x_D, y_D, z_D) = \int_{-\xi}^{\xi} F(x_D, y_D, z_D; x'_D, 0, 0) q_D(x'_D) dx'_D, \tag{73}$$

where

$$\xi = \sqrt{\frac{K_a}{K_h}} = \left(\frac{K_v}{K_h}\right)^{1/6}. \tag{74}$$

If the horizontal well is producing at constant wellbore pressure P_0, then

$$P_D(x_D, 0, z_{wD}) = P_{0D} = \frac{K_a L (P_{ini} - P_0)}{2\mu B q_{ref}}, \tag{75}$$

where $x_D \in \left(-\sqrt{\frac{K_a}{K_h}}, \sqrt{\frac{K_a}{K_h}}\right)$ and P_{0D} is a constant.

From equation (73), we obtain the following integral equation

$$\int_{-\xi}^{\xi} F(x_D, 0, 0; x'_D, 0, 0) q_D(x'_D) dx'_D = P_{0D}, \tag{76}$$

which can be simplified to

$$\left(\frac{-1}{4\pi}\right) \int_{-\xi}^{\xi} \frac{1}{|x_D - x'_D|} q_D(x'_D) dx'_D = P_{0D}. \tag{77}$$

Let $x'_D = \xi x$ and $x_D = \xi y$. Then equation (77) can be further simplified to

$$\left(\frac{-1}{4\pi}\right) \int_{-1}^{1} \frac{1}{|x-y|} q_D(x\xi) dx = P_{0D}. \tag{78}$$

Let

$$q_D(x\xi) = \bar{q}(x), \qquad P_{1D} = 4\pi P_{0D}. \tag{79}$$

Then

$$\int_{-1}^{1} \frac{1}{|x-y|} \bar{q}(x) dx = -P_{1D}. \tag{80}$$

Finally we obtain the new model using the following equation

$$\int_{-1}^{1} \frac{1}{|x-y|} g(x)dx = -1. \tag{81}$$

Note that equation (81) is a super-singular integral equation [17]. And we have studied the properties of this equation and provided the algorithm to obtain the approximation solution in section 2 and section 3.

Once the solution to equation (81) is obtained, the solution to equation (80) can be expressed as:

$$\bar{q}(x) = P_{1D} g(x), \tag{82}$$

and the total productivity of the horizontal well before the pressure transient reaches any reservoir limit is given by

$$Q = P_{1D} \int_{-1}^{1} g(x)dx. \tag{83}$$

In practice, by solving equation (33), we obtain the approximation values of $g(x)$ at semi-grid points $\{x_{i+1/2}, i = 0, ..., 2N-1\}$. Then using middle point rectangular quadrature formula in equation (83), the flow rate of horizontal wells producing at constant wellbore pressure is approximated by

$$Q_h = Q(h) = P_1 h \sum_{i=0}^{2N-1} u_i. \tag{84}$$

Choose $P_1 = 1$. The approximation results are given in Table 2 with different h. It can be found that the results are convergent when $h \to 0$.

TABLE 6. Flow rates of horizontal well with different h.

i	1	2	3	4	5	6
$Q(1/2^i)$	4.02467	4.12222	4.14911	4.15625	4.15811	4.15859
$Q(1/2^{i+6})$	4.15871	4.15874	4.15875	4.15875	4.15875	

From Table 2, we can find that $\int_{-1}^{1} g(x)dx \approx 4.15875$, which is independent of well length, permeability or other fluid properties. This result can be used to calculate flow rates in any case,

$$Q = 4.15875 P_{1D}, \tag{85}$$

where

$$P_{1D} = 4\pi \frac{K_a L(P_{ini} - P_0)}{2\mu B q_{ref}} = \frac{2\pi K_h^{2/3} K_v^{1/3} L(P_{ini} - P_0)}{\mu B q_{ref}}. \tag{86}$$

Equations (85) and (86) imply that when a horizontal well is producing at constant wellbore pressure, its flow rate is directly proportional to the well length, the pressure difference, $K_h^{2/3}$ and $K_v^{1/3}$.

If $P_{ini} - P_0$ is a constant, the productivity index is given by

$$J_v = 4.15875 \times \frac{2\pi (K_h^2 K_v)^{1/3} L}{\mu B q_{ref}} \approx \frac{26.13 K_h^{2/3} K_v^{1/3} L}{\mu B q_{ref}}. \tag{87}$$

According to equation (73) and the results in Table 2, we can calculate the pressure at point (x, y, z) as follows:

$$P(x, y, z) = \left(\frac{h}{4\pi}\right) \sum_{i=0}^{31} \frac{u_i}{\sqrt{(x - x_{i+1/2})^2 + y^2 + z^2}}, \tag{88}$$

Those points that satisfy $P(x, y, z) = constant$ form the equipotential surfaces when the well is producing at constant wellbore pressure. Figure 2 shows the equipotential surfaces of different fluid potential values when $z = 0$. It can be found that the equipotential surfaces in the two dimensional space can be approximately taken as a family of ellipses whose focuses are the two endpoints of the horizontal well.

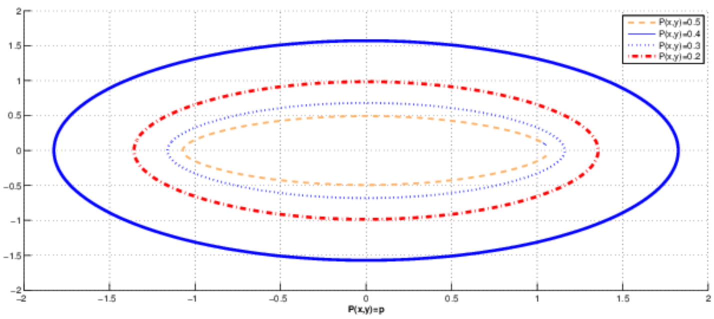

FIGURE 2. Profile of the equipotential surfaces of a horizontal Well.

6. Conclusions

This article proposes a new model using a hypersingular integral equation for the productivity of horizontal wells producing at constant wellbore pressure. Based on a new expansion with respect to singularity at any points, a numerical method for the hypersingular integral equation is developed and then used in the productivity formulae, which provide fast analytical tools to evaluate production performance of horizontal wells. Furthermore, the following conclusions are obtained:

(1) The numerical results for the hypersingular integral equation (1) are convergent. Moreover, the point convergence intensity is symmetric. And the point convergence intensity at the endpoints is smaller than that of the midpoint.

(2) The numerical approximation $Q(h)$ for the productivity is convergent when $h \to 0$.

(3) The equipotential surfaces in the three dimensional space are a family of ellipsoids of revolution whose focuses are the two endpoints of the horizontal wellbore.

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