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ENTROPY STABLE SCHEMES FOR COMPRESSIBLE EULER EQUATIONS

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Abstract. A novel numerical flux for the Euler equations which is consistent for kinetic energy and entropy condition was proposed recently [1]. This flux makes use of entropy variable based matrix dissipation which can be shown to satisfy an entropy inequality. For hypersonic flows a blended scheme is proposed which gives carbuncle free solutions for blunt body flows while still giving accurate resolution of boundary layers. Several numerical results on standard test cases using high order accurate reconstruction schemes are presented to show the performance of the new schemes.

Key words. Euler equation, finite volume method, kinetic energy preservation, entropy stability

1. Introduction

The finite volume method for hyperbolic problems requires the specification of a numerical flux function. For scalar problems there is a well developed mathematical theory which provides a route to develop stable and accurate schemes. For systems of conservation laws like Euler equations, the mathematical theory is not complete. Usually the schemes are developed to satisfy certain additional properties like entropy condition and kinetic energy stability which can be important for turbulent flows. Tadmor [17] proposed the idea of entropy conservative numerical fluxes which can then be combined with some dissipation terms using entropy variables to obtain a scheme that respects the entropy condition, i.e., the scheme must produce entropy in accordance with the second law of thermodynamics. However some of these entropy conservative numerical fluxes have to be computed with quadrature rules since the integrals involved in the definition of the flux cannot be evaluated explicitly. For the Euler equations, Roe proposed explicit entropy conservative numerical fluxes [13, 6] which are augmented by Roe-type dissipation terms using entropy variables. These schemes do not suffer from entropy violating solutions that are observed in the original Roe scheme. However for strong shocks, even the first order schemes can produce oscillations indicating that the amount of numerical dissipation is not sufficient. Roe [6] proposed modifying the eigenvalues of the dissipation matrix which lead to non-oscillatory solutions. The modification of the eigenvalues is such that the amount of entropy production is of the correct order of magnitude for weak shocks. The availability of cheap entropy conservative fluxes allows us to use the procedure of [9] to develop high order accurate entropy conservative schemes. Matrix dissipation can be added following the ENO procedure of [2] to develop arbitrarily high order accurate entropy stable schemes for the Euler equations on structured grids.

Faithful representation of kinetic energy evolution is another desirable property of a numerical scheme [8]. This is important for direct numerical simulation (DNS) of turbulent flows where the kinetic energy balance plays an important role in the evolution of turbulence [10, 14, 11]. The scheme is also stable in the sense that

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spurious kinetic energy is not produced by the numerical fluxes. The essential feature for a numerical flux in a semi-discrete finite volume method to correctly capture the kinetic energy balance is that the momentum flux should be of the form $f^m_{j+\frac{1}{2}} = \tilde{p}_{j+\frac{1}{2}} + \overline{u}_{j+\frac{1}{2}} f^\rho_{j+\frac{1}{2}}$ where $\overline{u}_{j+\frac{1}{2}} = (u_j + u_{j+1})/2$ and $\tilde{p}_{j+\frac{1}{2}}, f^\rho_{j+\frac{1}{2}}$ are *any* consistent approximations of the pressure and the mass flux. This scheme thus leaves most terms in the numerical flux unspecified and various authors have used simple averaging. Subbareddy and Candler [16] have proposed a fully discrete finite volume scheme for the compressible Euler equations which preserves kinetic energy but the resulting scheme is implicit. All of these kinetic energy preserving schemes are however not entropy conservative, while on the other hand, the entropy conservative schemes do not have the kinetic energy preservation property. It is thought that for DNS of compressible flows, a numerical scheme which preserves kinetic energy and satisfies entropy condition is desirable since such schemes would be non-linearly stable. Schemes which satisfy entropy condition are found to lead to stable density fluctuations in compressible isotropic turbulence simulations, while schemes which do not have this property can be unstable with respect to density fluctuations [4, 11].

In [1] explicit centered numerical fluxes for the compressible Euler equations which are entropy conservative and also preserve kinetic energy in the case of the semi-discrete finite volume scheme were developed. Due to lack of upwinding, the schemes are not stable for discontinuous solutions and for Navier-Stokes equations on coarse meshes for which shocks may not be well resolved. They yield stable solutions for Navier-Stokes equations when used on very fine meshes where the physical viscosity is enough to stabilize the scheme. However for Euler equations and for Navier-Stokes equations on coarse meshes, the centered fluxes are unstable and must be augmented with dissipation terms. Matrix dissipation similar to Roe scheme but using entropy variables was used to developed entropy stable schemes as in [17]. The eigenvalue modification of Roe [6] is used to compute strong shocks without oscillations. All the schemes are shown to give entropy consistent solutions in cases where the Roe scheme would give entropy violating shocks. The entropy stable schemes with matrix dissipation preserve stationary contacts exactly but also suffer from 1-D shock instability and the carbuncle phenomenon [1]. A modification of the eigenvalues in the dissipation flux based on a blending of the Roe and Rusanov schemes is used which avoids these problems but is still able to accurately compute shear flows like boundary layers. The new schemes are tested here on several standard problems to study their performance. Second order schemes are tested using the MUSCL reconstruction approach and minmod limiter. The blended scheme is shown to give good performance on all the test cases while the basic scheme performs well on problems with weak shocks.

The rest of the paper is organized as follows. Section (2) introduces the 1-D Navier-Stokes equations and finite volume method. This is followed by a discussion of the kinetic energy preservation property and the entropy condition. The new entropy conservative and kinetic energy consistent fluxes are introduced in section (2.5). Matrix dissipation flux is treated in section (3) and modifications to ensure monotone solutions are discussed including a scheme which blends a more accurate scheme with Roe-type eigenvalues, with the Rusanov form of the eigenvalues. Section (4) presents a range of test problems for 1-D shock case involving shocks, expansions and rarefaction solutions. The schemes are compared with other entropy stable schemes and the classical Roe scheme and their performance in the second order version is also demonstrated including under grid refinement.

2. 1-D Navier-Stokes equations and finite volume scheme

The one dimensional Navier-Stokes equations can be written in vector conservation form as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \frac{\partial \mathbf{g}}{\partial x} \tag{1}$$

where $\mathbf{u}$ is the set of conserved variables and $\mathbf{f}$, $\mathbf{g}$ are the inviscid and viscous fluxes whose expressions are given by

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} \rho u \\ p + \rho u^2 \\ (E+p)u \end{bmatrix} = \begin{bmatrix} m \\ p + um \\ (E+p)u \end{bmatrix}, \qquad \mathbf{g} = \begin{bmatrix} 0 \\ \tau \\ u\tau - q \end{bmatrix} \tag{2}$$

In the above equations ρ is the density, u is the velocity, p is the pressure and E is the total energy per unit volume. For a perfect gas is given by $E = p/(\gamma-1)+\rho u^2/2$, where γ is the ratio of specific heats which is taken to be constant. Moreover, τ, q are the shear stress and heat flux for which we take the Newtonian and Fourier laws respectively, leading to

$$\tau = \frac{4}{3}\mu \frac{\partial u}{\partial x}, \qquad q = -\kappa \frac{\partial T}{\partial x}$$

where μ, κ are the coefficient dynamic viscosity and heat conduction respectively. In the absence of the viscous fluxes $\mathbf{g}$, the resulting Euler equations form a hyperbolic system of conservation laws. Consider a partition of the one dimensional domain into uniform finite volumes of size Δx and the j'th cell is the interval $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$. The semi-discrete finite volume scheme is given by

$$\Delta x \frac{\mathrm{d}\mathbf{u}_j}{\mathrm{d}t} + \mathbf{f}_{j+\frac{1}{2}} - \mathbf{f}_{j-\frac{1}{2}} = \mathbf{g}_{j+\frac{1}{2}} - \mathbf{g}_{j-\frac{1}{2}} \tag{3}$$

where $\mathbf{u}_j$ is the cell average value in the j'th cell and $\mathbf{f}_{j+\frac{1}{2}}$, $\mathbf{g}_{j+\frac{1}{2}}$ are numerical inviscid and viscous fluxes respectively at the interface $x_{j+\frac{1}{2}}$. In the numerical computations, the above set of ordinary differential equations will be solved using a strong stability preserving Runge-Kutta scheme [15].

2.1. Kinetic energy preserving scheme. The kinetic energy is an important quantity in fluid flows and it is destroyed by the physical viscosity. In turbulent flows, the kinetic energy injected into the fluid at large scales cascades to smaller scales and is eventually destroyed by viscosity. Hence, it is desirable that the numerical scheme faithfully represent the kinetic energy balance consistent with the Navier-Stokes equations. The kinetic energy per unit volume $K = \frac{1}{2}\rho u^2$ satisfies the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega K \mathrm{d}x = \int_\Omega p \frac{\partial u}{\partial x}\mathrm{d}x - \frac{4}{3}\int_\Omega \mu \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \tag{4}$$

where we have ignored boundary conditions. The first term on the right is the rate at which work is done by pressure forces and is present only for compressible flows. The second term represents the irreversible destruction of kinetic energy which is converted into internal energy due to viscous dissipation. We would like the numerical scheme to also satisfy this equation in a discrete sense which will then be referred to as kinetic energy preserving scheme. Consider the following

approximation for the inviscid and viscous fluxes

$$\mathbf{f}_{j+\frac{1}{2}} = \begin{bmatrix} f^\rho \\ f^m \\ f^e \end{bmatrix}_{j+\frac{1}{2}} = \begin{bmatrix} f^\rho \\ \tilde{p} + \overline{u} f^\rho \\ f^e \end{bmatrix}_{j+\frac{1}{2}}, \qquad g_{j+\frac{1}{2}} = \begin{bmatrix} 0 \\ \tau \\ \tilde{u}\tau - q \end{bmatrix}_{j+\frac{1}{2}} \tag{5}$$

where

$$\overline{u}_{j+\frac{1}{2}} = \frac{1}{2}(u_j + u_{j+1}), \qquad \tau_{j+\frac{1}{2}} = \frac{4}{3}\mu \frac{u_{j+1} - u_j}{\Delta x}, \qquad q_{j+\frac{1}{2}} = -\kappa \frac{T_{j+1} - T_j}{\Delta x}$$

Throughout the paper, we will use the overbar to denote the arithmetic average. The quantities $\tilde{p}, f^\rho, \tilde{u}, f^e$ are assumed to be consistent approximations but are yet to be specified. The global kinetic energy balance equation for the finite volume scheme can be shown to be

$$\begin{aligned} \sum_j \Delta x \frac{\mathrm{d}K_j}{\mathrm{d}t} &= \sum_j \left[-\frac{1}{2} u_j^2 \frac{\mathrm{d}\rho_j}{\mathrm{d}t} + u_j \frac{\mathrm{d}m_j}{\mathrm{d}t} \right] \Delta x \\ &= \sum_j \left[\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \tilde{p}_{j+\frac{1}{2}} - \frac{4}{3}\mu \left(\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \right)^2 \right] \Delta x \end{aligned}$$

This is consistent with the continuous kinetic energy equation. The crucial property used in the above proof was that the momentum flux has the form $f^m = \tilde{p} + \overline{u} f^\rho$ which leads to the disappearance of the convective flux from the kinetic energy equation. However, we still have freedom in the choice of $f^\rho, \tilde{p}, \tilde{u}, f^e$.

2.2. Entropy condition. The concept of entropy condition is borrowed from the second law of thermodynamics and generalized to any arbitrary system of hyperbolic conservation laws [3]. Consider the hyperbolic conservation law by taking $g = 0$ in equation (1). Assume the existence of a pair $(U(\mathbf{u}), F(\mathbf{u}))$ such that $U(\mathbf{u})$ is a strictly convex function and $U'(\mathbf{u})\mathbf{f}'(\mathbf{u}) = F'(\mathbf{u})$. Such a pair is said to be an entropy-entropy flux pair. Then smooth solutions of the hyperbolic system satisfy an additional conservation law

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

But for discontinuous solutions we can only satisfy the entropy inequality

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} \leq 0 \tag{6}$$

where the inequality is satisfied in a weak sense. Define the entropy variables as $\mathbf{v}(\mathbf{u}) = U'(\mathbf{u})$ and since $U(\mathbf{u})$ is strictly convex, we can invert the above equation to obtain $\mathbf{u} = \mathbf{u}(\mathbf{v})$. Define the quantity $\psi(\mathbf{v})$ which is the dual of the entropy flux $F(\mathbf{u})$, by the relation $\psi(\mathbf{v}) = \mathbf{v} \cdot \mathbf{f}(\mathbf{u}(\mathbf{v})) - F(\mathbf{u}(\mathbf{v}))$. Tadmor [17, 18] introduced the idea of an *entropy conservative numerical flux* which should satisfy the following condition

$$(\mathbf{v}_{j+1} - \mathbf{v}_j) \cdot \mathbf{f}_{j+\frac{1}{2}} = \psi_{j+1} - \psi_j \tag{7}$$

Consider the finite volume scheme for the hyperbolic conservation law

$$\Delta x \frac{\mathrm{d}\mathbf{u}_j}{\mathrm{d}t} + \mathbf{f}_{j+\frac{1}{2}} - \mathbf{f}_{j-\frac{1}{2}} = 0 \tag{8}$$

Taking the dot product of the scheme (8) with $\mathbf{v}_j$, we obtain the entropy equation

$$\Delta x \frac{\mathrm{d}U_j}{\mathrm{d}t} + F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} = 0, \qquad F_{j+\frac{1}{2}} = \overline{\mathbf{v}}_{j+\frac{1}{2}} \cdot \mathbf{f}_{j+\frac{1}{2}} - \overline{\psi}_{j+\frac{1}{2}}$$

where $F_{j+\frac{1}{2}}$ is a consistnt numerical entropy flux. An entropy conservative flux is given by Tadmor [17] but it requires the computation of an integral in entropy state-space, which usually requires some numerical quadrature to approximate the flux. Once an entropy conservative flux has been constructed, we can add dissipative terms to the flux which leads to the satisfaction of the entropy condition as given by equation (6), i.e., the dissipative flux must lead to generation of entropy.

2.3. Entropy condition for Euler equations. For the Euler equations we can take the entropy-entropy flux pair to be

$$U = -\frac{\rho s}{\gamma - 1}, \qquad F = -\frac{\rho u s}{\gamma - 1} \tag{9}$$

where s is the physical entropy given by

$$s = \ln(p) - \ln(\rho^\gamma) + \text{const} = -(\gamma - 1)\ln(\rho) - \ln(\beta) + \text{const}, \qquad \beta = \frac{1}{2RT} \tag{10}$$

and the constant term can be ignored. There are many other possible choices for the entropy function, such as the family of functions of the form $U(\mathbf{u}) = \rho\eta(s)$ where η is any convex function [3], but the above choice is the only one which is consistent with the entropy condition from thermodynamics [5] in the presence of heat transfer. Since we work with the correct choice of the entropy function, the schemes we develop will satisfy entropy condition for the Navier-Stokes equations as well. The entropy variables $\mathbf{v}$ and the Legendre transform ψ are given by

$$\mathbf{v} = \begin{bmatrix} \dfrac{\gamma - s}{\gamma - 1} - \beta u^2 & 2\beta u & -2\beta \end{bmatrix}^\top, \qquad \psi = m = \rho u \tag{11}$$

Hence, an entropy conservative numerical flux for the Euler equations has to satisfy the following condition

$$(\mathbf{v}_{j+1} - \mathbf{v}_j) \cdot \mathbf{f}_{j+\frac{1}{2}} = m_{j+1} - m_j \tag{12}$$

This provides only one equation whereas the flux $\mathbf{f}$ has more than one component; we can expect that there are many possible entropy conservative fluxes.

2.4. Roe's entropy conservative flux. Roe [13] has constructed explicit numerical fluxes for the Euler equations which satisfy condition (12) for the entropy given in equation (9) but do not have the kinetic energy preservation property. Roe introduces the set of independent state variables or parameter vector

$$z = [z_1 \quad z_2 \quad z_3]^\top = \sqrt{\frac{\rho}{p}}[1 \quad u \quad p]^\top$$

and the logarithmic average $\hat{\varphi}$ of any strictly positive quantity φ_l, φ_r which is defined as

$$\hat{\varphi} = \frac{\varphi_r - \varphi_l}{\ln \varphi_r - \ln \varphi_l} = \frac{\Delta\varphi}{\Delta \ln \varphi}$$

A numerically stable procedure to compute the average when $\varphi_l \approx \varphi_r$ is given in [6]. Then the entropy conservative numerical flux at any cell face $j + \frac{1}{2}$ is given by

$$\mathbf{f}^* = \begin{bmatrix} \tilde{\rho}\tilde{u} \\ \tilde{p}_1 + \tilde{u} f^\rho \\ \tilde{H} f^\rho \end{bmatrix}, \quad \tilde{\rho} = \overline{z}_1 \hat{z}_3, \quad \tilde{u} = \frac{\overline{z}_2}{\overline{z}_1}, \quad \tilde{p}_1 = \frac{\overline{z}_3}{\overline{z}_1}, \quad \tilde{p}_2 = \frac{\gamma + 1}{2\gamma}\frac{\hat{z}_3}{\hat{z}_1} + \frac{\gamma - 1}{2\gamma}\frac{\overline{z}_3}{\overline{z}_1}$$

$$\tilde{a} = \left(\frac{\gamma \tilde{p}_2}{\tilde{\rho}}\right)^{\frac{1}{2}}, \quad \tilde{H} = \frac{\tilde{a}^2}{\gamma - 1} + \frac{1}{2}\tilde{u}^2$$

and all the averages are evaluated using the state in cell j and $j+1$. This flux is entropy conservative and it can be made entropy stable [13] by adding entropy variable based matrix dissipation terms which is described in a later section. The resulting numerical flux will be referred to as ROE-ES flux. Note that the momentum flux contains the weighted average velocity $\tilde{u}$ while kinetic energy preservation requires the presence of the arithmetic average $\overline{u}$. Hence the above entropy conservative flux is not kinetic energy preserving. In the following sections, we construct numerical fluxes by approximately or exactly satisfying the entropy condition for the Euler equations which also preserves kinetic energy.

2.5. Kinetic energy preserving and entropy conservative flux. A novel central flux which preserves kinetic energy and also conserves entropy has been proposed in [1]. The form of the numerical flux is given by (5) which ensures the kinetic energy preservation property. The remaining terms in the flux are determined by satisfying the entropy conservation condition of Tadmor as given by equation (12). This leads to the following expressions for the numerical flux function.

$$f^{*,\rho}_{j+\frac{1}{2}} = \hat{\rho}_{j+\frac{1}{2}} \overline{u}_{j+\frac{1}{2}} \tag{13}$$

$$f^{*,m}_{j+\frac{1}{2}} = \tilde{p}_{j+\frac{1}{2}} + \overline{u}_{j+\frac{1}{2}} f^{*,\rho}_{j+\frac{1}{2}}, \qquad \tilde{p}_{j+\frac{1}{2}} = \frac{\overline{\rho}_{j+\frac{1}{2}}}{2\overline{\beta}_{j+\frac{1}{2}}} \tag{14}$$

$$f^{*,e}_{j+\frac{1}{2}} = \left[\frac{1}{2(\gamma-1)\hat{\beta}_{j+\frac{1}{2}}} - \frac{1}{2}\overline{u^2}_{j+\frac{1}{2}} \right] f^{*,\rho}_{j+\frac{1}{2}} + \overline{u}_{j+\frac{1}{2}} f^{*,m}_{j+\frac{1}{2}} \tag{15}$$

Now we can derive the global entropy balance equation for the semi-discrete finite volume scheme for the Navier-Stokes equations. Taking dot product of equation (3) with the entropy variables $\mathbf{v}_j$ and summing up over all the finite volumes yields

$$\sum_j \frac{\mathrm{d}U_j}{\mathrm{d}t} \Delta x + \sum_j [F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}] = -\sum_j \Delta \mathbf{v}_{j+\frac{1}{2}} \cdot \mathbf{g}_{j+\frac{1}{2}}$$

The terms involving $F_{j+\frac{1}{2}}$ represent convection of entropy and cancel one another when we sum over all cells. The terms on the right which consist of viscous shear stress and heat flux can be shown to lead to entropy generation. Hence the entropy equation becomes

$$\sum_j \Delta x \frac{\mathrm{d}U_j}{\mathrm{d}t} = -\sum_j \left[\frac{8\mu\overline{\beta}_{j+\frac{1}{2}}}{3} \left(\frac{\Delta u_{j+\frac{1}{2}}}{\Delta x} \right)^2 + \frac{\kappa}{RT_jT_{j+1}} \left(\frac{\Delta T_{j+\frac{1}{2}}}{\Delta x} \right)^2 \right] \Delta x \leq 0$$

which is consistent with the entropy condition from the second law of thermodynamics.

3. Matrix dissipation flux

The central numerical flux can be useful for DNS of compressible flows on highly resolved meshes where the physical dissipation would be enough to stabilize the numerical scheme. But for unresolved simulations and especially those in which shocks might be present, some numerical dissipation is necessary. In [1] both scalar and matrix dissipation schemes have been developed which maintain kinetic energy and entropy stability. The matrix dissipation is more appropriate for problems with strong shock waves. The dissipation terms are motivated by the form of the

dissipation in the classical Roe scheme but makes use of entropy variables and is given by

$$\mathbf{f}_{j+\frac{1}{2}} = \mathbf{f}^*_{j+\frac{1}{2}} - \frac{1}{2} R_{j+\frac{1}{2}} |\Lambda_{j+\frac{1}{2}}| S_{j+\frac{1}{2}} R^\top_{j+\frac{1}{2}} \Delta \mathbf{v}_{j+\frac{1}{2}} \tag{16}$$

where R is the matrix of right eigenvectors and Λ is the diagonal matrix containing the eigenvalues, both of which are evaluated at an average state.

$$R = \begin{bmatrix} 1 & 1 & 1 \\ u-a & u & u+a \\ H-ua & \frac{1}{2}u^2 & H+ua \end{bmatrix}, \qquad |\Lambda| = |\Lambda|^{Roe} = \text{diag}\left[|u-a|,\ |u|,\ |u+a|\right]$$

and the matrix $S = \text{diag}\left[\frac{\rho}{2\gamma},\ \frac{(\gamma-1)\rho}{\gamma},\ \frac{\rho}{2\gamma}\right]$ provides the appropriate scaling from conserved to entropy variables. The above numerical flux together with $|\Lambda| = |\Lambda|^{Roe}$ will be called the KEP-ES flux. We note that the dissipation matrix $Q = R|\Lambda|SR^\top$ is positive definite which allows us to derive the entropy inequality as follows. By taking the dot product of the semi-discrete finite volume scheme for Euler equations with $\mathbf{v}_j$ we obtain the entropy equation

$$\Delta x \frac{\mathrm{d}U_j}{\mathrm{d}t} + F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}} = -\frac{1}{4}\left[\Delta \mathbf{v}^\top_{j+\frac{1}{2}} Q_{j+\frac{1}{2}} \Delta \mathbf{v}_{j+\frac{1}{2}} + \Delta \mathbf{v}^\top_{j-\frac{1}{2}} Q_{j-\frac{1}{2}} \Delta \mathbf{v}_{j-\frac{1}{2}}\right] \leq 0$$

where the numerical entropy flux is given by

$$F_{j+\frac{1}{2}} = \overline{\mathbf{v}}_{j+\frac{1}{2}} \cdot \mathbf{f}^*_{j+\frac{1}{2}} - \overline{\psi}_{j+\frac{1}{2}} - \frac{1}{2} \bar{\mathbf{v}}^\top_{j+\frac{1}{2}} Q_{j+\frac{1}{2}} \Delta \mathbf{v}_{j+\frac{1}{2}}$$

Hence the matrix dissipation flux leads to generation of entropy in each cell. Numerical experiments in [1] show that this scheme gives entropy satisfying solutions to the shock tube problem where the original Roe scheme would give entropy violating shocks in the expansion region. This consistency is obtained due to the entropy conservation of the central flux $\mathbf{f}^*$ and the entropy generation mechanism of the dissipative flux.

The dissipative flux still needs the specification of the manner in which the terms in R and Λ are calculated. It was shown in [1] that stationary contact waves can be exactly resolved provided the sound speed in the enthalpy H is calculated as $a_{j+\frac{1}{2}} = \sqrt{(\gamma/2\hat{\beta}_{j+\frac{1}{2}})}$. The choice $|\Lambda| = |\Lambda|^{Roe}$ leads to an accurate scheme since it is close to the Roe scheme but one can choose the eigenvalues in other consistent ways. Kinetic energy stability can be obtained if we choose the eigenvalues such that $|\lambda_1| = |\lambda_3|$ but this leads to a more dissipative scheme as compared to the Roe type dissipation. However it is still possible to retain the property of exactly resolving stationary contact waves [1] provided $|\lambda_2| = |u|$. A still more dissipative but stable choice is the Rusanov scheme given by

$$|\Lambda| = |\Lambda|^{Rus} = \lambda I, \qquad \lambda = |u| + a \tag{17}$$

3.1. Monotone resolution of shocks. An important desirable property of a numerical scheme for hyperbolic problems is that it should yield monotone discontinuous solutions. It is desirable that the first order scheme should yield monotone solutions which can be used to construct higher order schemes using the TVD principle. The entropy variable based fluxes of Roe and also the scheme developed in [1] yield non-monotone solutions when the Roe eigenvalues are used. Ismail and Roe attribute the non-monotonicity to insufficient entropy production and hence insufficient dissipation at the shock. For a weak shock, the entropy production is $O(\Delta\rho)^3$ whereas the entropy production due to the dissipative flux is $O(\Delta\rho)^2$.

Hence they propose modifying the acoustic eigenvalues so that the eigenvalue matrix $|\Lambda|$ becomes

$$|\Lambda| = |\Lambda|^{EC1} = \text{diag}\,[|u-a| + \beta|\Delta\lambda_1|,\ |u|,\ |u+a| + \beta|\Delta\lambda_3|] \tag{18}$$

where $\Delta\lambda_1$ and $\Delta\lambda_3$ are the jumps in the corresponding eigenvalues across the cell face. If $\beta = \frac{1}{6}$ then the entropy production across a weak shock due to matrix dissipation corresponds to the correct entropy production of $O(\Delta\rho)^3$ as shown in [6]. With this modification of the eigenvalues, the entropy conservative Roe flux is called ROE-EC1 flux where EC stands for "entropy consistent". We will adopt the same eigenvalue modifications as in equation (18) in combination with the new entropy conservative flux referred to as KEP-EC1. The ROE-EC1 scheme was however found to give non-monotone solutions for strong shocks so that the authors proposed to further enhance the dissipation by increasing the acoustic eigenvalues which they refer to as the ROE-EC2 scheme.

3.2. Carbuncle and a hybrid scheme. The ROE-EC2 flux which has increased dissipation compared to the basic Roe scheme can still suffer from the carbuncle problem in multi-dimensions [7]. The KEP-EC1 scheme also produces carbuncle effect on certain types of meshes where the shock is aligned with the mesh. Increasing the value of β in equation (18) even up to $\beta = 1$ does not seem to eliminate this problem. The kinetic energy stable scheme which has more dissipation in the acoustic waves also produces the carbuncle but the Rusanov scheme of equation (17) does not produce the carbuncle. In fact what we observe is that all the schemes which resolve grid aligned stationary contacts exactly seem to suffer from the carbuncle effect, which is consistent with what is noticed in other schemes in the literature. The usual fix in such cases is to increase the amount of dissipation in the numerical scheme which however causes poor resolution of boundary layers. There is also the idea of switching the numerical scheme to a more dissipative one only near shocks and using a high resolution Riemann solver type scheme in smooth parts of the flow [12]. In the framework of the entropy conservative/stable scheme as discussed in this paper, we have the freedom in designing the eigenvalues which essentially control the amount of dissipation in the scheme. Since the Rusanov scheme is free of carbuncles, we propose a blending of the usual Roe scheme with the Rusanov scheme, i.e., the matrix $|\Lambda|$ appearing in the entropy dissipation flux is of the form

$$|\Lambda| = |\Lambda|^{Hyb} = (1-\phi)|\Lambda|^{Roe} + \phi|\Lambda|^{Rus}$$

where the switching function ϕ is based on the pressure jump

$$\phi = \left|\frac{\Delta p}{2\overline{p}}\right|^{\frac{1}{2}}$$

Note that $0 \le \phi \le 1$; for a strong shock $\phi \approx 1$ and the scheme is close to the Rusanov scheme, while near a contact discontinuity $\phi \approx 0$ and the scheme is close to the more accurate Roe scheme. This hybrid scheme does not produce the 1-D shock instability problem. In the results section of [1], it is shown that this scheme avoids the carbuncle effect and still gives good resolution of boundary layers and shear layers. The new entropy conservative flux together with the above hybrid dissipation will be referred to as KEP-ES(Hyb) scheme. We also remark that these modifications of the dissipation term still retain the entropy stability property of the scheme.

4. Numerical results

In all the test problems, we consider an ideal gas with $\gamma = 1.4$ except when indicated otherwise. The method of lines approach is followed in which the space discretization is performed first followed by time integration using a strong stability preserving Runge-Kutta scheme. To achieve second order accuracy in space, we follow the MUSCL approach and reconstruct the solution inside each cell by piecewise linear functions. This reconstruction is performed on the primitive variables $\mathbf{w} = (\rho, u, p)$ instead of the conserved variables in order to ensure positivity of pressure also. The numerical flux for the second order scheme is evaluated as

$$\mathbf{f}_{j+\frac{1}{2}} = \mathbf{f}(\mathbf{w}^-_{j+\frac{1}{2}}, \mathbf{w}^+_{j+\frac{1}{2}})$$

where

$$\mathbf{w}^{\pm}_{j+\frac{1}{2}} = \mathbf{w}_j \pm \frac{1}{2}\Delta\mathbf{w}_j,$$

$$\Delta\mathbf{w}_j = \text{minmod}(\theta(\mathbf{w}_j - \mathbf{w}_{j-1}), \frac{1}{2}(\mathbf{w}_{j+1} - \mathbf{w}_{j-1}), \theta(\mathbf{w}_{j+1} - \mathbf{w}_j))$$

Here minmod is the standard function which selects the argument with least absolute value if all of them have the same sign and returns zero otherwise. The parameter θ is chosen between $[1, 2]$, the lower limit giving more monotone solutions.

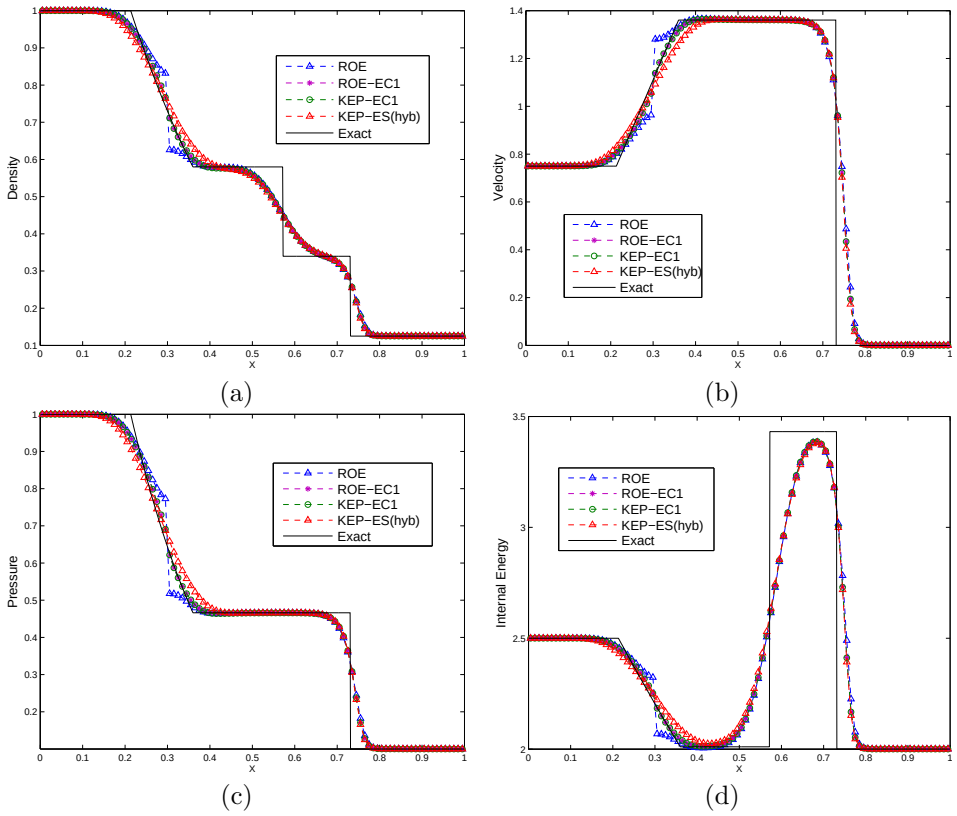


FIGURE 1. Modified Sod: Solution at time t=0.2, (a) density, (b) velocity, (c) pressure, (d) internal energy

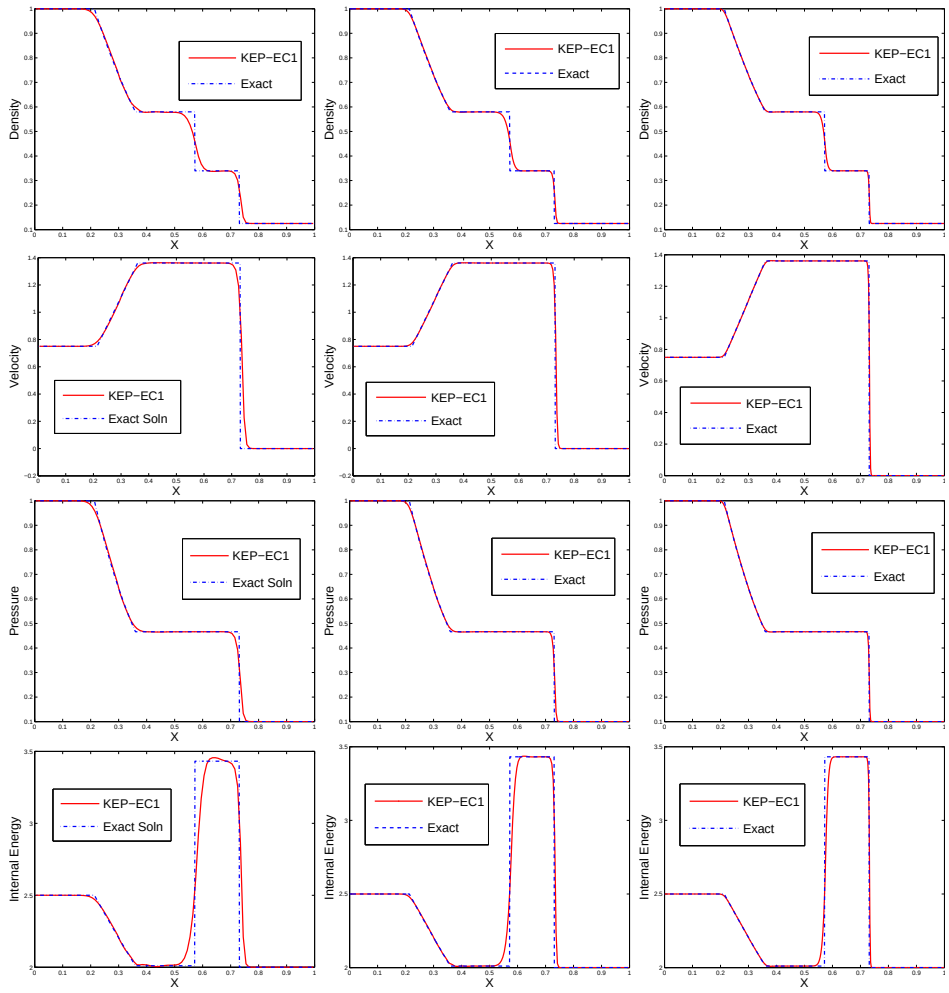


FIGURE 2. Modified Sod: KEP-EC1, using MUSCL scheme and minmod limiter: (left column) $\Delta x = 0.01$, (middle column) $\Delta x = 0.005$, (right column) $\Delta x = 0.0025$

4.1. Modified Sod test case. This is a shock tube problem and is one of the most commonly used test cases for checking accuracy of schemes for the Euler equations. The initial condition has the left state $(\rho_L, u_L, p_L) = (1.0, 0.75, 1.0)$ and the right state $(\rho_R, u_R, p_R) = (0.125, 0.0, 0.1)$. The domain is [0,1] with the initial discontinuity at $x_0 = 0.3$. The computations are made with CFL=0.4 up to a final time of t=0.2 units. The original Roe scheme gives an entropy violating jump in the expansion region where the flow becomes sonic, as shown in figure (1). This can be attributed to the vanishing of one of the eigenvalues at the sonic point, which leads to insufficient dissipation. The ROE-EC1, KEP-EC1 and KEP-ES(hyb) schemes give solutions which do not suffer from this problem. Note that even the entropy stable schemes have a vanishing eigenvalue in the expansion fan. However, due to the entropy conservative form of the central part of the flux, they do not give rise to entropy violating shocks, unlike the ROE scheme which is not entropy conservative.

Figure (2) shows the solutions for different values of Δx with the KEP-EC1 flux, using the MUSCL scheme and minmod limiter.

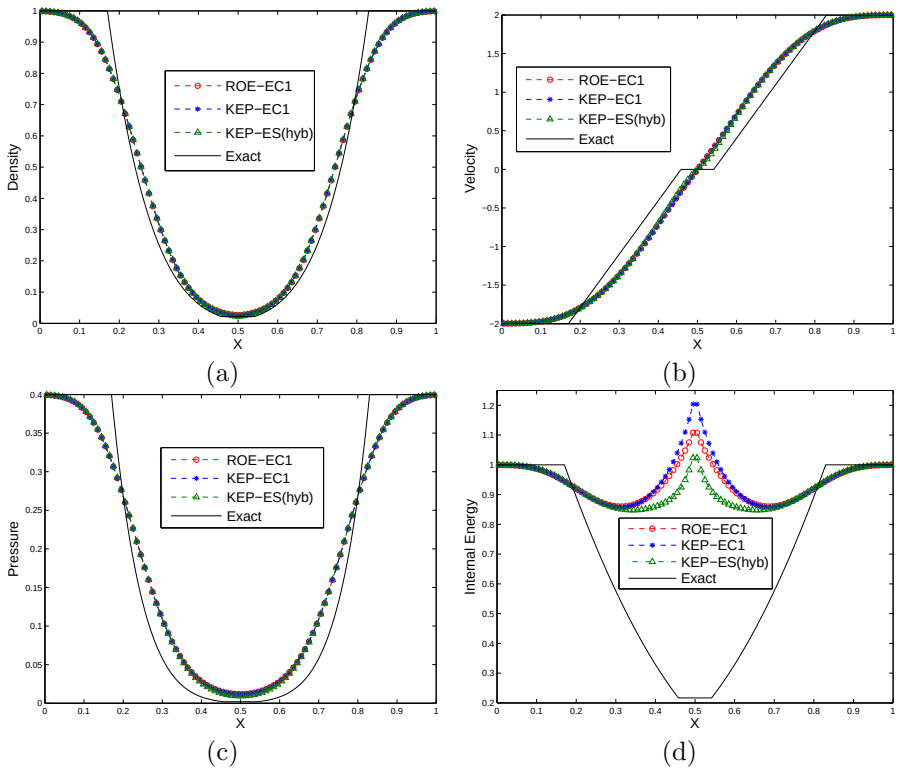


FIGURE 3. Low Density: Solution at time t=0.15, (a) density, (b) velocity, (c) pressure, (d) internal energy

4.2. Low density problem. This problem is used to test the ability of schemes to preserve positivity of density and pressure. It has initial conditions with the left state $(\rho_L, u_L, p_L) = (1.0, -0.2, 0.4)$ and the right state $(\rho_R, u_R, p_R) = (1.0, 0.2, 0.4)$. The domain is [0,1] with the initial discontinuity at $x_0 = 0.5$. We use CFL=0.4 and evaluate the solution at t=0.12 units. The exact solution consists of two symmetric rarefaction waves. The region between the two non-linear waves is close to vacuum. The original Roe scheme fails for this test case. ROE-EC1, KEP-EC1 and KEP-ES(hyb) give results similar to each other, and are able to preserve the positivity of density and pressure, as seen in figure (3). The schemes give good solutions for ρ, u, and p. However, they fail to do a good job in predicting the internal energy $e = \frac{p}{(\gamma-1)\rho}$. In this test, both ρ and p are close to zero and thus small errors get exaggerated by their ratio. We have also computed the solution with KEP-EC1 scheme with the MUSCL approach for different values of Δx, as shown in figure (4).

4.3. Left half of blast wave problem. The solution of this problem contains a strong shock wave of Mach number 198, a contact surface and a left rarefaction wave. This test is actually the left half of the blast wave problem of Woodward and Colella [19]. The left state is $(\rho_L, u_L, p_L) = (1.0, 0.0, 1000.0)$ while the right state is $(\rho_R, u_R, p_R) = (1.0, 0.0, 0.01)$. The domain is [0,1.4] with the initial discontinuity

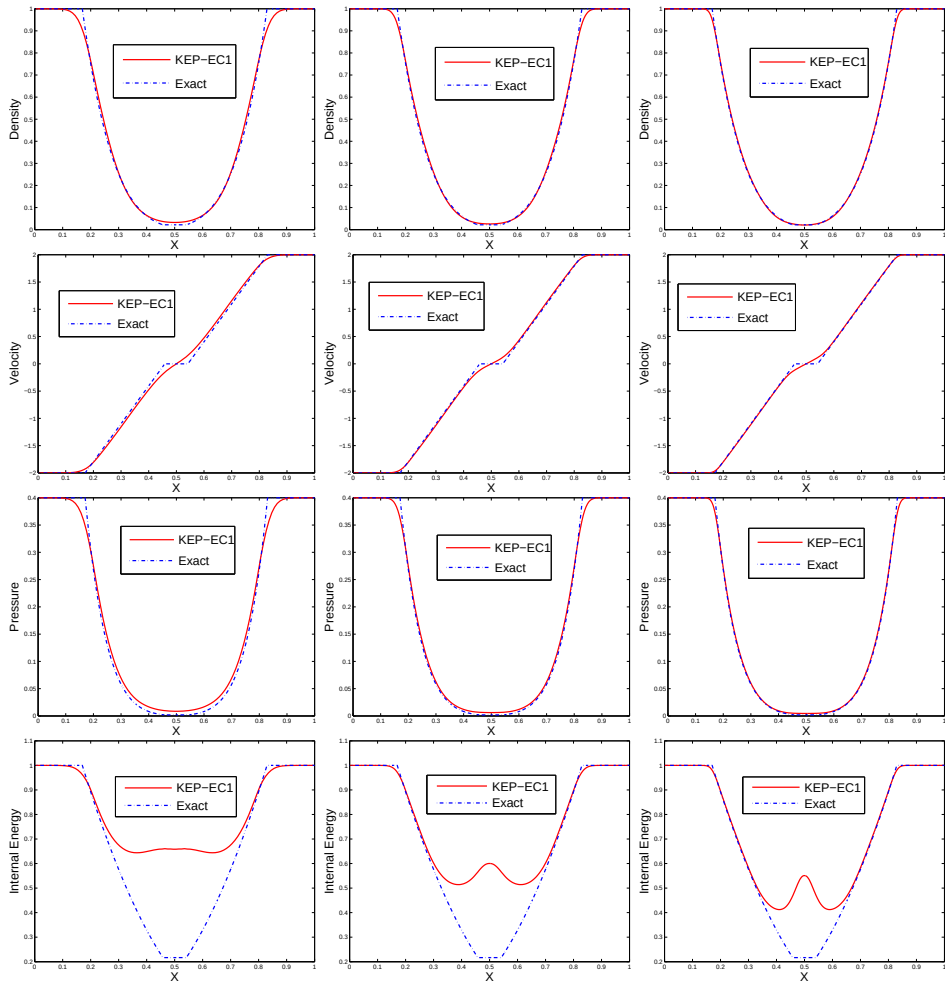


FIGURE 4. Low Density: KEP-EC1, using MUSCL scheme and minmod limiter: $\Delta x = 0.01$, (middle column) $\Delta x = 0.005$, (right column) $\Delta x = 0.0025$

at $x_0 = 0.7$. The solution is evaluated at t=0.012 units. ROE, KEP-EC1 and KEP-ES(hyb) schemes give similar monotone solutions at CFL=0.1 (see figure (5)), while ROE-EC1 scheme blows up. Note that schemes are unable to compute density accurately even on meshes with N=200 cells. This could be attributed to the insufficient number of grid points in the region of the strong shock. However, the solution converges as we refine the mesh, as can be seen in figure (6)

4.4. Shock Collision. The exact solution of this test represents the collision of two strong shocks and consists of a left facing shock travelling very slowly to the right, a right travelling contact and a right travelling shock wave. The left state is $(\rho_L, u_L, p_L) = (5.99924, 19.5975, 460.894)$ and the right state is $(\rho_R, u_R, p_R) = (5.99242, -6.19633, 46.0950)$. The domain is [0,1] with the initial discontinuity at $x_0 = 0.4$. The solution is evaluated at t=0.035 units. ROE, ROE-EC1, KEP-EC1 and KEP-ES(hyb) all give monotone solution as shown in figure (7). Like the previous test case, we require more grid points in the strong shock regions to

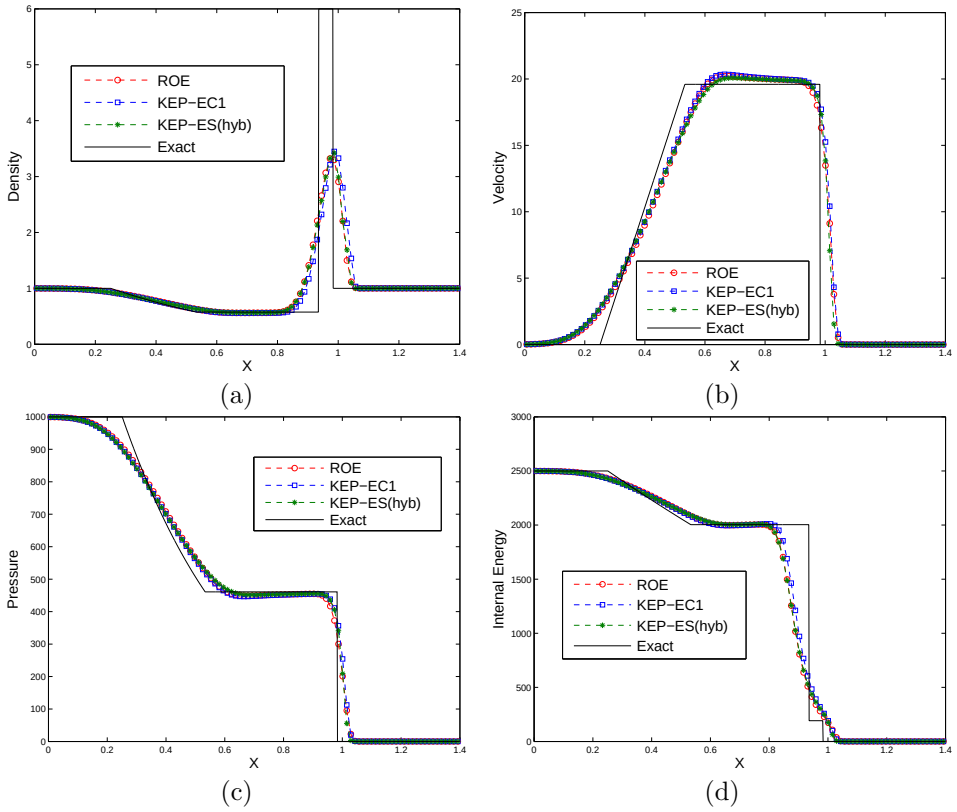


FIGURE 5. Left half of blast wave: Solution at time t=0.012, (a) density, (b) velocity, (c) pressure, (d) internal energy

capture the density more accurately. We compute the solution of KEP-EC1, using the MUSCL scheme for different values of Δx, as shown in figure (8).

4.5. Slowly moving contact wave. This test assesses the ability of numerical methods to resolve slowly moving contact discontinuities. The initial condition has the left state $(\rho_L, u_L, p_L) = (1.0, -19.59745, 1000.0)$ and the right state $(\rho_R, u_R, p_R) = (1.0, -19.59745, 0.01)$. The domain used is [0,2] with the initial discontinuity at $x_0 = 1.0$. The solution is evaluated at t=0.012 units. ROE scheme gives monotone solution, while ROE-EC1 and KEP-EC1 give solutions with spurious oscillations, as can be seen in figure (9). These oscillations do not disappear even if we increase the value of the β parameter used to modify the eigenvalues of the dissipation matrix. This is perhaps due to the existence of a strong shock. As the Rusanov scheme can handle strong shocks well, the KEP-ES(hyb) scheme is able to give monotone solutions, which behaves very similar to the solution with ROE scheme. We compute the solution with KEP-ES(hyb) (using MUSCL) for different values of Δx, as shown in figure (10).

4.6. CPU run-times. Due to the structure of the dissipation and the presence of the logarithmic averages, the KEPES flux can be expected to be more costly than the Roe flux. To test the computational efficiency of the KEPES flux, in terms of the CPU time, we compared it with the original Roe flux with the entropy fix and

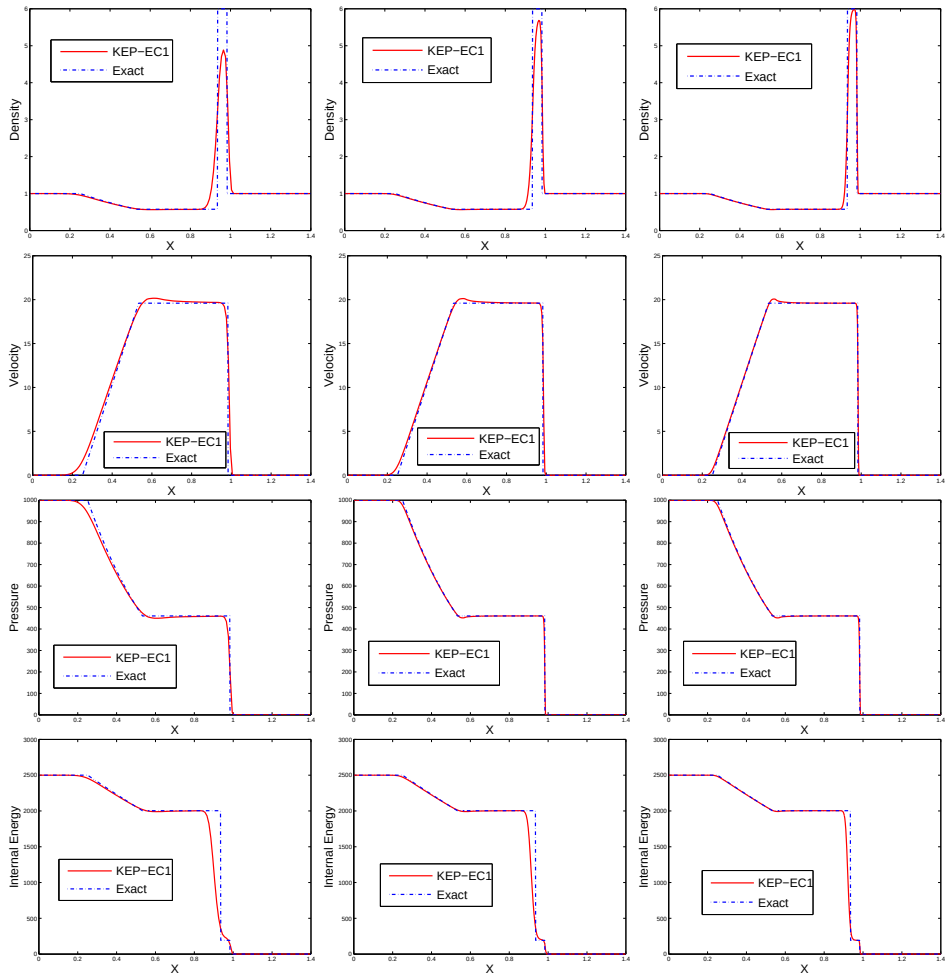


FIGURE 6. Left half of blast wave: KEP-EC1, using MUSCL scheme and minmod limiter: (left column) $\Delta x = 0.01$, (middle column) $\Delta x = 0.005$, (right column) $\Delta x = 0.0025$

TABLE 1. CPU run-times

Flux	**Number of calls**	**Average time per call (sec)**	**Ratio with ROE**
ROE (entropy fix)	5×10^6	5.343e-8	1
KEP-ES	5×10^6	9.803e-8	1.83
ROE-ES	5×10^6	1.115e-7	2.09

the ROE-ES flux. Each flux was called 5×10^6 times and the average time per call was evaluated. These have been displayed in Table 1. The cost of the KEPES scheme is about 1.83 times that of the Roe scheme with entropy fix. However, we feel that the superiority of the KEPES scheme over the Roe scheme in terms of entropy stability makes the increased computational cost acceptable. Moreover,

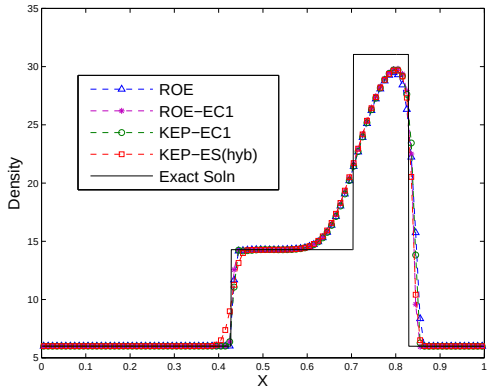
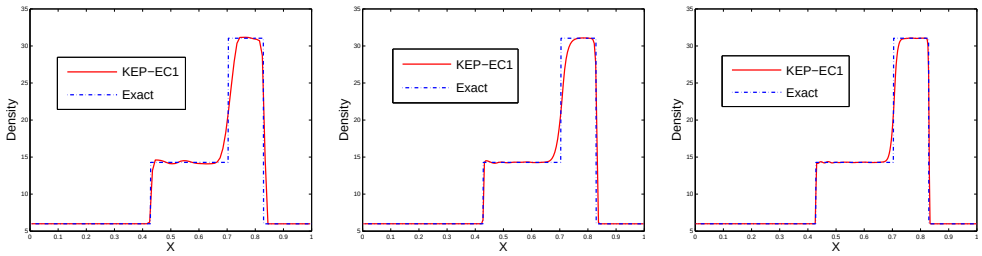


FIGURE 7. Shock collision: Density at time t=0.035



(a) (b) (c)

FIGURE 8. Shock collision: KEP-EC1, using MUSCL scheme and minmod limiter, (left) $\Delta x = 0.01$, (middle) $\Delta x = 0.005$, (right) $\Delta x = 0.0025$

KEPES scheme is slightly better than the entropy stable ROE-ES scheme, which is about 2.09 times as costly as the Roe scheme.

5. Summary and conclusions

Novel entropy conservative and entropy stable fluxes for the Euler equations have been proposed which are also consistent with the kinetic energy equation. They give accurate solutions on smooth and transonic flows where shocks are not very strong and also for boundary layers [1]. These schemes need careful selection of the dissipation terms to ensure monotone solution at strong shocks. However they can still lead to pathological problems like the carbuncle phenomenon which cannot be avoided even with entropy stable schemes which exactly preserve stationary contact waves. The blended scheme which switches between an accurate Roe-type eigenvalues and the more dissipative Rusanov eigenvalues is able to avoid these problems and still give accurate resolution of shocks and other discontinuities as demonstrated in a wide range of test cases. Several standard problems have been solved using second order accurate schemes to show the performance and stability of the proposed schemes including their behaviour under grid refinement. These results indicate that the blended scheme gives accurate and monotone solutions on a wide range of test problems.

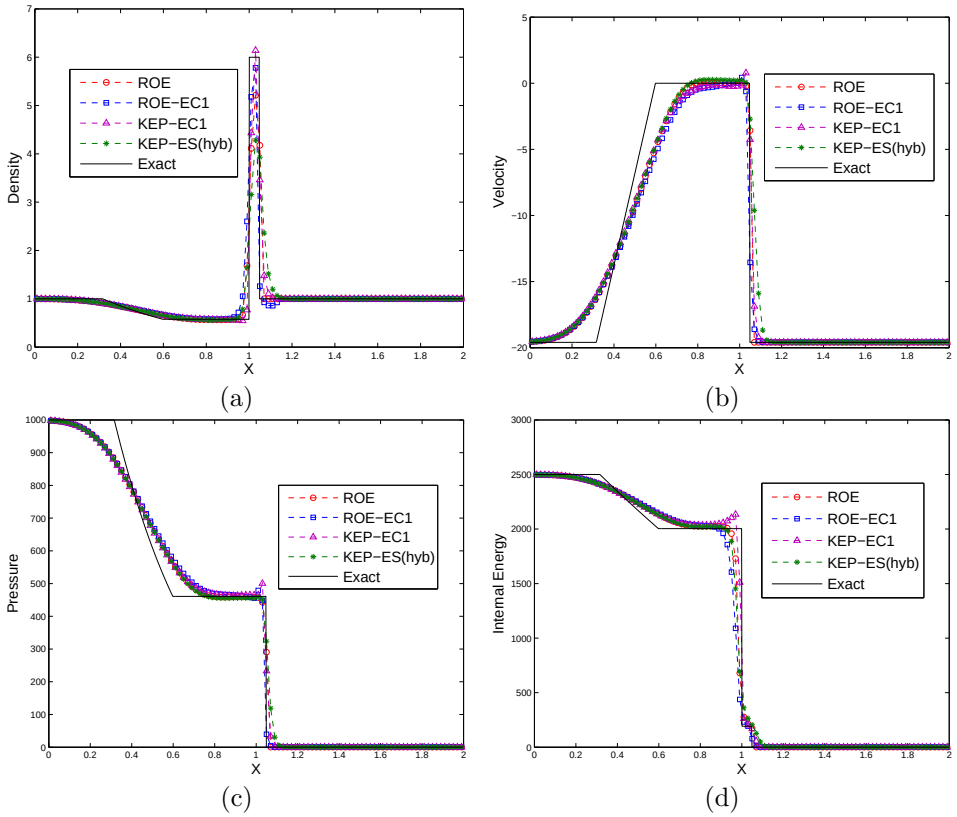


FIGURE 9. Slow moving contact: Solution at time t=0.012, (a) density, (b) velocity, (c) pressure, (d) internal energy

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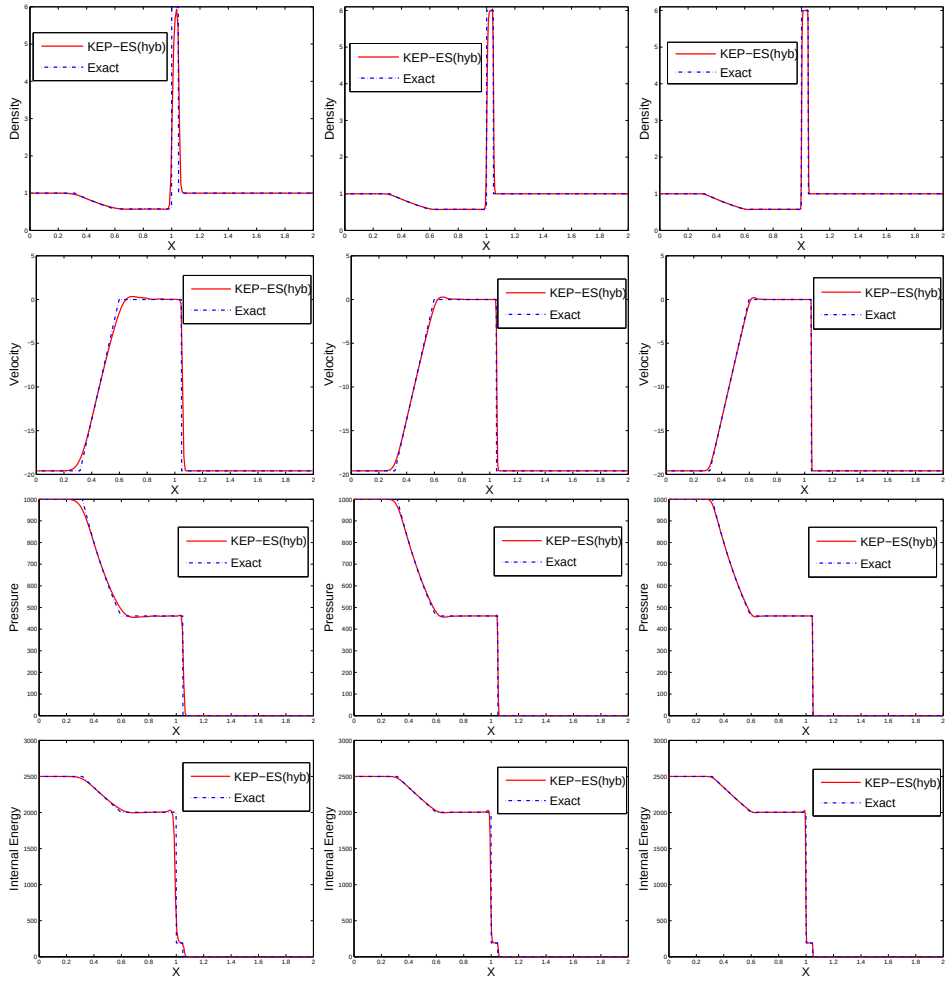

FIGURE 10. Slow moving contact: KEP-ES(hyb), using MUSCL scheme and minmod limiter: (left column) $\Delta x = 0.01$, (middle column) $\Delta x = 0.005$, (right column) $\Delta x = 0.0025$

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