A SIMPLE ANALYTIC APPROXIMATION FORMULA FOR THE BOND PRICE IN THE CHAN-KAROLYI-LONGSTAFF-SANDERS MODEL

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Abstract. We propose an analytic approximation formula for pricing zero-coupon bonds in the case when the short-term interest rate is driven by a one-factor mean-reverting process with a volatility proportional to the power the interest rate itself. We derive its order of accuracy. Afterwards, we suggest its use in calibration and show that it can be reduced to a simple optimization problem. To test the calibration methodology, we use the simulated data from the Cox-Ingersoll-Ross model where the exact bond prices can be computed. We show that using the approximation in the calibration recovers the parameters with a high precision.

Key words. one-factor interest rate model, Vasicek model, bond price, analytical approximation formula, order of accuracy, calibration

1. Introduction

Interest rate is a rate charged for the use of the money. As an example we show Euribor interest rates on the interbank market; Figure 1 shows the evolution of the 1-week interest rate in 2012, as well as the interest rates with different maturities (so called term structure) on a selected day.

The interest rate itself is not a tradable asset. It can be derived from the bond prices which are traded on the market. The discount bond is a secure paper which pays the unit amount of money at its maturity. A popular way of modelling bond prices is through short rate models. They are formulated in terms of a stochastic differential equation for the instantaneous interest rate (short rate). The bond prices are then given by a solution to a parabolic partial differential equation.

It is often assumed that the short rate \( r \) evolves according to the stochastic differential equation

\[
\frac{dr}{dt} = (\alpha + \beta r)dt + \sigma r^{\gamma} dw,
\]

where \( w \) is a Wiener process. If the equation (1) is considered in the so called risk neutral measure, then the price \( P(\tau, r) \) of the discount bond, when the current level of the short rate is \( r \) and time remaining to maturity is \( \tau \), is given by the solution to the partial differential equation

\[
-\frac{\partial P}{\partial \tau} + \frac{1}{2} \sigma^2 r^{2\gamma} \frac{\partial^2 P}{\partial r^2} + (\alpha + \beta r) \frac{\partial P}{\partial r} - rP = 0, \quad r > 0, \quad \tau \in (0, T),
\]

satisfying the initial condition \( P(0, r) = 1 \) for all \( r > 0 \), see, e.g., [12], [2]. This includes the Vasicek model from [21] with \( \gamma = 0 \) and the Cox-Ingersoll-Ross (CIR hereafter) model proposed in [6] with \( \gamma = 1/2 \), in which the explicit solutions to bond pricing partial differential equations are known (cf. the original papers [21], [6] or current books on the topic of interest rate modelling, e.g., [2], [12]).

With the exception of these two models, such an explicit solution is not available. However, the empirical analysis of the market data suggests that models with

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The pioneering paper [3] started the discussion on the correct form of the volatility. Authors used proxy for the short rate process and estimated the parameters using the generalized method of moments. They found the parameter $\gamma$ to be significantly different from the values indicated by Vasicek and CIR models. A modification of generalized method of moments (so called robust generalized method of moments), which is robust to a presence of outliers, was developed in [8]. Another contribution to this class of estimators is for example indirect robust estimation by [7]. Another popular method for parameter estimation are Nowman’s Gaussian estimates [13], based on approximating the likelihood function. They were used in [9] for a wide range of interest rate markets. There are several other calibration methods for the short rate process, such as quasi maximum likelihood, maximum likelihood based on series expansion of likelihood function by Ait-Sahalia [1], Bayesian methods such as Markov chain Monte Carlo and others.

The paper [20] provides an extensive testing of the robustness of the estimation results. The authors examine the robustness over different data sets (they consider eight countries), time periods, sampling frequencies (it is important to note that the estimation procedures are based on the discretized process and the discretization error may not be negligible if the sampling interval is not sufficiently small), and estimation techniques (they consider quasi maximum likelihood and generalized method of moments). In general, the results are not robust. The highly cited result from [3] that models with $\gamma > 1$ outperform those with $\gamma \leq 1$ is not generally confirmed neither. However, the necessity to go beyond the Vasicek and CIR models is clear: using daily data (which should minimize the bias coming from discretization error) for the eight countries and both estimation methods considered, the Vasicek
model is rejected in all the cases and CIR is not rejected only in one case (on the usual 5 percent significance level).

The common feature of the cited approaches is taking a certain market rate as a proxy to the short rate and using the econometric techniques of time series analysis to estimate the parameters of the model. These parameters can be afterwards used to price the bonds and other derivatives. For example in [15], the parameters of the CKLS process were first estimated using the Nowman’s methodology and afterwards derivatives prices were computed by numerically solving the partial differential equation using the Box method. For more results of this kind see [14], [16].

An alternative would be using the derivatives prices to calibrate the parameters of the model. This, however requires quick computation of the prices, since they have to be computed many times with different parameters during the calibration procedure. Exact solution to the bond pricing equation available for Vasicek and CIR model made this possible in the case of these two models, cf. [18], [19]. In general, when the exact solution is not available, approximate analytical solution provides a convenient alternative.

In [5] an approximate analytical solution for the general bond pricing equation (2) was suggested. Its derivation is based on approximating the integral in the probabilistic representation of the solution. The order of accuracy was derived in [17] and an approximation of higher order was suggested using the partial differential equations approach. In the recent paper [4] the approximation from [5] was used in the context of numerical methods. The idea of deriving the formula in [5] was adopted in [11], where the authors derived approximation formulae for processes with non-Gaussian innovations.

The approximations from [5] and [17] are, however, complicated functions of model parameters. This can cause difficulties when calibrating the model from real data. Therefore we propose another approximation and show its convenient form for calibration.

The paper is organized as follows. In Section 2 we define the approximation of the bond price in the model (1) and we derive its order of accuracy. In Section 3 we compare this approximation with the exact solution from the CIR model. We present the empirical order of convergence, as well as comparison of the interest rates computed from the approximate formula and from the exact bond prices. In Section 4 we propose a calibration method based on the new approximation formula and we end the paper with concluding remarks in Section 5.

2. Approximation formula and order of accuracy

Consider the stochastic differential equation (1) in the risk neutral measure for the evolution of the short rate $r$ and the corresponding partial differential equation (2) for the bond price $P(t, r)$.

The main result of the paper [5] is the following approximation $P^{ap}$ for the exact solution $P^{ex}$:

**Theorem 1.** [5, Theorem 2] The approximate analytical solution $P^{ap}$ is given by

$$
\ln P^{ap}(t, r) = -rB + \frac{\alpha}{\beta}(r - B) + \left(\frac{\gamma^2}{2} + qr\right) \frac{\sigma^2}{4\beta} \left[ B^2 + \frac{2}{\beta}(r - B) \right]
$$

(3)

$$
- q^\frac{\sigma^2}{8\beta^2} \left[ B^2(2\beta\tau - 1) - 2B \left( 2\tau - \frac{3}{\beta} \right) + 2\tau^2 - \frac{6\tau}{\beta} \right]
$$

where

$$
q(r) = \gamma(2\gamma - 1)\sigma^2 + 2\gamma r^{2\gamma - 1} + 2\gamma r^{2\gamma - 1}(\alpha + \beta r)
$$

(4)
and
\begin{equation}
B(\tau) = (e^{\beta \tau} - 1)/\beta.
\end{equation}
Derivation of the formula (3) is based on calculating the price as an expected value in the risk neutral measure. The tree property of conditional expectation was used and the integral appearing in the exact price was approximated to obtain closed form approximation. The reader is referred to [5] for more details of derivation of (3).

Authors furthermore showed that such an approximation coincides with the exact solution in the case of the Vasicek model [21]. Moreover, they compared the above approximation with the exact solution of the CIR model which is also known in a closed form (cf. [6]). Graphical and tabular descriptions of the relative error in the approximation with the exact solution of the CIR model which is also known in a

**Theorem 2.** [17, Theorem 4] Let \( P^{ap} \) be the approximative solution given by (3) and \( P^{ex} \) be the exact bond price given as a unique complete solution to (2). Then
\begin{equation}
\ln P^{ap}(\tau, r) - \ln P^{ex}(\tau, r) = c_5(r)\tau^5 + o(\tau^5)
\end{equation}
as \( \tau \to 0^+ \) where
\begin{equation}
c_5(r) = \frac{1}{120}r^2(\gamma - 2)^2\sigma^2 \left[ 2\alpha^2 (-1 + 2\gamma) r^2 + 4\beta^2 \gamma r^4 - 8r^3 + 2\gamma \sigma^2 \\
+ 2\beta (1 - 5\gamma + 6\gamma^2) r^2 (1 + \gamma) \sigma^2 + \sigma^4 (2\gamma - 1)^2 (4\gamma - 3) + 2\sigma (\beta (-1 + 4\gamma) r^2 + (2\gamma - 1)(3\gamma - 2) r^2 \sigma^2) \right].
\end{equation}

**Theorem 3.** [17, Theorem 4] Let \( P^{ex} \) be the exact bond price. Let us define an improved approximation \( P^{ap^2} \) by the formula
\begin{equation}
\ln P^{ap^2}(\tau, r) = \ln P^{ap}(\tau, r) - c_5(r)\tau^5 - c_6(r)\tau^6
\end{equation}
where \( \ln P^{ap} \) is given by (3), \( c_5(\tau) \) is given by (6) in Theorem 2 and
\begin{equation}
c_6(r) = \frac{1}{6} \left( \frac{1}{2} \sigma r^2 \gamma c_5''(r) + (\alpha + \beta r) c_5'(r) - k_5(r) \right)
\end{equation}
where \( c_5' \) and \( c_5'' \) stand for the first and second derivative of \( c_5(r) \) w. r. to \( r \) and \( k_5 \) is defined by
\begin{equation}
k_5(r) = \frac{\gamma \sigma^2}{120} r^{2(-2+\gamma)} \left[ 6\alpha^2 \beta (-1 + 2\gamma) r^2 + 12\beta^2 \gamma r^4 - 10(1 - 2\gamma)^2 r^{1+4\gamma} \sigma^4 + 6\beta^2 \sigma^2 (1 - 5\gamma + 6\gamma^2) r^{2(1+\gamma)} + \beta r^2 \gamma \sigma^2 \left( -10 (5 + 2\gamma) r^3 + 3(1 - 2\gamma)^2 (-3 + 4\gamma) r^2 \sigma^2 \right) + 2\alpha r \left( 3\beta^2 (-1 + 4\gamma) r^2 + 3\beta (2 - 7\gamma + 6\gamma^2) r^2 \gamma \sigma^2 \right) - 5 (-1 + 2\gamma) r^{1+2\gamma} \sigma^2 \right].
\end{equation}
Then the difference between the higher order approximation \( \ln P^{ap^2} \) given by (7) and the exact solution \( \ln P^{ex} \) satisfies
\begin{equation}
\ln P^{ap^2}(\tau, r) - \ln P^{ex}(\tau, r) = o(\tau^6)
\end{equation}
as $\tau \to 0^+$. 

In the case of Vasicek model, i.e., for $\gamma = 0$, the solution $P_{vas}$ can be expressed in the closed form. In particular, see [21], we have

$$
(9) \quad \ln P_{vas}(\tau, r) = \left( \frac{\alpha}{\beta} + \frac{\sigma^2}{2\beta^2} \right) \left( 1 - \frac{e^{\beta \tau}}{\beta} + \tau \right) + \frac{\sigma^2}{4\beta^3} (1 - e^{\beta \tau})^2 + \frac{1 - e^{\beta \tau}}{\beta} r.
$$

Now, let us consider a general model (1) and the approximation of the bond price obtained by substituting the instantaneous volatility $\sigma r$ for $\sigma$ in the Vasicek price (9), i.e.,

$$
(10) \quad \ln P^{ap}(\tau, r) = \left( \frac{\alpha}{\beta} + \frac{\sigma^2 r^2 \gamma}{2\beta^2} \right) \left( 1 - \frac{e^{\beta \tau}}{\beta} + \tau \right) + \frac{\sigma^2 r^2 \gamma}{4\beta^3} (1 - e^{\beta \tau})^2 + \frac{1 - e^{\beta \tau}}{\beta} r.
$$

**Theorem 4.** Let $P^{ap}$ be the approximate solution given by (10) and $P^{ex}$ be the exact bond price given as a solution to (2). Then

$$
\ln P^{ap}(\tau, r) - \ln P^{ex}(\tau, r) = c_4(r) \tau^4 + o(\tau^4)
$$

as $\tau \to 0^+$ where

$$
c_4(r) = -\frac{1}{24} \gamma r^{2\gamma - 2} \sigma^2 [2\alpha r + 2\beta r^2 + (2\gamma - 1)r^{2\gamma} \sigma^2]\n$$

**Proof:**

Let us define the auxiliary function $f^{ex}(\tau, r) = \ln P^{ex}(\tau, r)$. Then, since $P^{ex}(\tau, r)$ is a a solution of (2), the PDE for the function $f^{ex}$ reads as

$$
(11) \quad -\partial_\tau f^{ex} + \frac{1}{2} \sigma^2 r^{2\gamma} \left[ (\partial_r f^{ex})^2 + \partial^2_r f^{ex} \right] + (\alpha + \beta r) \partial_r f^{ex} - r = 0.
$$

Substitution of $f^{ap} = \ln P^{ap}$ into equation (11) yields a nontrivial right-hand side $h(\tau, r)$ in the equation for the approximate solution $f^{ap}$:

$$
(12) \quad -\partial_\tau f^{ap} + \frac{1}{2} \sigma^2 r^{2\gamma} \left[ (\partial_r f^{ap})^2 + \partial^2_r f^{ap} \right] + (\alpha + \beta r) \partial_r f^{ap} - r = h(\tau, r).
$$

If we insert the approximate solution (10) into (12), we obtain that

$$
(13) \quad h(\tau, r) = k_3(r) \tau^3 + o(\tau^3),
$$

where $k_3$ is given by

$$
k_3(r) = \frac{1}{6} \gamma r^{2\gamma - 2} \sigma^2 [2\alpha r + 2\beta r^2 + (2\gamma - 1)r^{2\gamma} \sigma^2].
$$

Let us consider a function $g(\tau, r) = f^{ap} - f^{ex}$. It follows from (11) and (12) that the function $g$ satisfies the following PDE:

$$
(14) \quad -\partial_\tau g + \frac{1}{2} \sigma^2 r^{2\gamma} \left[ (\partial_r g)^2 + \partial^2_r g \right] + (\alpha + \beta r) \partial_r g = h(\tau, r) - \sigma^2 r^{2\gamma} (\partial_r f^{ex})(\partial_r g),
$$

where $h(\tau, r)$ satisfies (13). We expand the solution of (14) into a Taylor series with respect to $\tau$ with coefficients depending on $r$ in the form $g(\tau, r) = \sum_{i=0}^\infty c_i(r)\tau^i = \sum_{i=0}^\infty c_i(r)\tau^i$, i.e., the first nonzero term in the expansion is $c_3(r)\tau^3$. Then $\partial_\tau g = \omega c_3(r)\tau^{\omega - 1} + o(\tau^{\omega - 1})$ and we recall that $h(\tau, r) = k_3(r) \tau^3 + o(\tau^3)$ as $\tau \to 0^+$. The remaining terms in (14) are of the order $o(\tau^{\omega - 1})$ as $\tau \to 0^+$. Hence

$$
-\omega c_3(r)\tau^{\omega - 1} = k_3(r)\tau^3,
$$

from which we deduce $\omega = 4$ and

$$
c_4(r) = -\frac{1}{4} k_3(r) = -\frac{1}{24} \gamma r^{2\gamma - 2} \sigma^2 [2\alpha r + 2\beta r^2 + (2\gamma - 1)r^{2\gamma} \sigma^2].
$$
It means that 
\[ g(\tau, r) = \ln P^{ap}(\tau, r) - \ln P^{ex}(\tau, r) = -12\gamma^2 \sigma^2 r^2 + 2\beta r^2 + (2\gamma - 1)\sigma^2 r^4 + o(\tau^4), \] 
which completes the proof.

Corollary 1. It follows from the formula 
\[ R(\tau, r) = -\ln \frac{P^{ap}(\tau, r)}{\tau} \] 
for calculating interest rates and Theorem 4 that the error in interest rate curves can be expressed as
\[ R^{ap}(\tau, r) - R^{ex}(\tau, r) = -c_4(r)\tau^3 + o(\tau^3) \] 
as \( \tau \to 0^+ \).

Remark 1. It would be possible to use a similar procedure to obtain the approximation based on the Cox-Ingersoll-Ross model by replacing \( \sigma_{CIR} \) in the closed form formula for the CIR bond price with \( \sigma r^{\gamma - 1/2} \) (i.e., equating the instantaneous volatilities in these two models), which would make the formula exact in the case of the CIR model. However, the resulting formula would lose its simplicity which is the main advantage of the current approach. The usefulness of this form will be seen better in the section dealing with calibration.

Remark 2. The idea to use the Vasicek-type model as a base to approximate the bond prices in more general model can be extended also to multifactor models. We have successfully applied it in the context of the convergence models modeling the evolution of the interest rates in a country before entering the monetary union, cf. [22]. The one-factor model considered in this paper is needed to model the European interest rate. Application for the models where the short rate is a sum of two unobservable factors is in preparation [10].

3. Comparison of the approximation to the exact solution for the CIR model

In this section we present a comparison of the approximation and the exact solution in the case of the CIR model where the exact solution is known (cf. [6]). We use the parameter values from [5] and [17], i.e., \( \alpha = 0.00315 \), \( \beta = -0.0555 \) and \( \sigma = 0.0894 \).

In Table 1 we show \( L_\infty \) and \( L_2 \) norms of the error in \( \ln P \) (we considered \( r \in [0, 0.15] \)). We also compute the experimental order of convergence (EOC) in these norms. The results confirm the order of convergence stated in Theorem 4.

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( | \ln P^{ap} - \ln P^{ex} |_2 )</th>
<th>EOC</th>
<th>( | \ln P^{ap} - \ln P^{ex} |_\infty )</th>
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<td>1.9446 \times 10^{-6}</td>
<td>4.0930</td>
</tr>
<tr>
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<td>6.9637 \times 10^{-9}</td>
<td>–</td>
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</tbody>
</table>

For the practical usage of the approximate formula, besides the order of accuracy also the absolute value of the error is significant. In Table 2 we compare the exact and approximate interest rates for several maturities and several values of the short rate. Note that for shorter maturities the differences are less than the accuracy to which the market data are quoted. Euribor, for example, is quoted in percentage points rounded to three decimal places. Moreover, Figure 2 shows that even though the accuracy of this approximation is one order lower to that of the approximation from [5], it gives numerically comparable results for the real set of parameters.
Table 2. The exact and approximate interest rates for CIR model for several maturities and values of the short rate (given in the table as the value corresponding to $\tau = 0$) in percentage points.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>exact interest rate</th>
<th>approximate interest rate</th>
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</thead>
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<td>0</td>
<td>2.00000</td>
<td>2.00000</td>
</tr>
<tr>
<td>0.25</td>
<td>2.02522</td>
<td>2.02522</td>
</tr>
<tr>
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<tr>
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<tr>
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<tr>
<td>2</td>
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<td>2.18684</td>
</tr>
<tr>
<td>3</td>
<td>2.26718</td>
<td>2.26850</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau$</th>
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<th>approximate interest rate</th>
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<td>0</td>
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<td>5.00000</td>
</tr>
<tr>
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<td>5.00425</td>
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<td>5.00766</td>
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<td>5.01024</td>
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<td>5.01202</td>
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</table>

<table>
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<th>$\tau$</th>
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<th>approximate interest rate</th>
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</tr>
<tr>
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<td>7.73200</td>
</tr>
</tbody>
</table>

4. Application of the approximation formula to calibration

Let us consider the calibration of the one-factor model based on the comparison of theoretical and market interest rates, where the parameters are chosen to minimize the function

\[ F = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} (R(\tau_j, r_i) - R_{ij})^2, \]

where $r_i$ ($i = 1, \ldots, n$) is the short rate observed on the $i$-th day, $\tau_j$ ($j = 1, \ldots, m$) is the $j$-th maturity of the interest rates in the data set, $R_{ij}$ is the interest rate with maturity $\tau_j$ observed on $i$-th day, $R(\tau, r)$ is the interest rate with maturity $\tau$ corresponding to the short rate $r$ computed from the model with the given parameters and $w_{ij}$ are the weights. In [19] and [18], this approach was used with $w_{ij} = \tau_j^2$ (i.e., giving more weight to fitting longer maturities) to calibrate Vasicek and CIR models using the explicit solutions for interest rates. To achieve the global minimum of the objective function, the authors used evolution strategies.

If we attempted to use this method to estimate a model with different $\gamma$ without analytical approximation, it would become computationally demanding, since each evaluation of the objective function would require numerical solutions of the PDE
Figure 2. Comparison of the exact term structures in the CIR model (solid line), proposed approximation (crosses) and approximation from [5] (circles).

(2). Note that the evaluation is needed for every member of the population in the evolution strategy (see [18] for details). Using an analytical approximation simplifies the computation of the objective function, but in general the dimension of the optimization problem is unchanged. We show that using the approximation proposed in this paper, we are able to reduce the calibration to a one-dimensional optimization problem which can be quickly solved using simple algorithms.

Hence we consider the criterion (15) with replacing $R(\tau, r)$ by its approximation $R_{ap}(\tau, r)$ calculated from (10). Note that the approximation formula for $P_{ap}$ is a linear function of parameters $\alpha$ and $\sigma^2$; it can be written as

$$
\ln P_{ap}(\tau, r) = c_0(\tau, r) + c_1(\tau, r)\alpha + c_2(\tau, r)\sigma^2,
$$

where

$$
c_0 = \frac{1 - e^{\beta \tau}}{\beta} r, \quad c_1 = \frac{1}{\beta} \left( 1 - \frac{e^{\beta \tau}}{\beta} + \tau \right), \quad c_2 = \frac{\gamma^2}{2\beta^2} \left( \frac{1 - e^{\beta \tau}}{\beta} + \tau + \frac{(1 - e^{\beta \tau})^2}{2\beta} \right).
$$

Hence taking the derivatives of (15) with respect to $\alpha$ and $\sigma^2$ and setting them equal to zero leads to a system of linear equations for these two parameters:

$$
\alpha \sum_{i,j} \frac{w_{ij}}{r_{ij}} c_1^2 + \sigma^2 \sum_{i,j} \frac{w_{ij}}{r_{ij}} c_1 c_2 = - \sum_{i,j} \frac{w_{ij}}{r_{ij}} (c_0 + R_{ij} \gamma) c_1,
$$

$$
\alpha \sum_{i,j} \frac{w_{ij}}{r_{ij}} c_1 c_2 + \sigma^2 \sum_{i,j} \frac{w_{ij}}{r_{ij}} c_2^2 = - \sum_{i,j} \frac{w_{ij}}{r_{ij}} (c_0 + R_{ij} \gamma) c_2.
$$

It means that once we fix $\gamma$ and treat $\beta$ as parameter, we obtain the corresponding optimal values of $\alpha$ and $\sigma^2$ for each $\beta$. Substituting them into (15) then leads to a one-dimensional optimization problem. Doing this over a range of values of $\gamma$ allows us to find the optimal parameter $\gamma$ as well.
Table 3. Estimation of the parameter $\gamma$ using the approximate formula for interest rates. The data were simulated using the exact formula with the parameters $\alpha = 0.00315$, $\beta = -0.0555$, $\sigma = 0.0894$, $\gamma = 0.5$. Maturities used were 1, 2, $\ldots$, 12 months (above) and 1, 2, $\ldots$, 5 years (below).

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\sigma$</th>
<th>optimal value of $F$</th>
</tr>
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<tbody>
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<td>-0.0578</td>
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<td>0.25</td>
<td>0.00319</td>
<td>-0.0565</td>
<td>0.0403</td>
<td>$2.9 \times 10^{-13}$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.00315</td>
<td>-0.0555</td>
<td>0.0896</td>
<td>$1.1 \times 10^{-15}$</td>
</tr>
<tr>
<td>0.75</td>
<td>0.00312</td>
<td>-0.0548</td>
<td>0.1912</td>
<td>$6.3 \times 10^{-13}$</td>
</tr>
<tr>
<td>1</td>
<td>0.00310</td>
<td>-0.0548</td>
<td>0.3813</td>
<td>$2.5 \times 10^{-12}$</td>
</tr>
</tbody>
</table>

We show the proposed idea on simulated data. We consider the CIR model with parameters from [5] and [17], i.e., $\alpha = 0.00315$, $\beta = -0.0555$, $\sigma = 0.0894$. We generate the daily term structures - interest rates with maturities of 1, 2, 3, $\ldots$, 12 months using the exact formula for CIR model - for a period of year. In the objective function (15) we use the weights $w_{ij} = \tau^2_j$ as in [19] and [18]. Afterwards we repeat the same procedure with maturities of 1, 2, 3, 4 and 5 years.

Results of the estimation for several values of $\gamma$ are presented in Table 3, we show the estimated parameters and the optimal value of the objective function $F$. In Figure 3 we present the optimization with respect to $\gamma$. As we can see, we are able to identify its correct value quite precisely, even though we use the approximation formula and the data were simulated using the exact formula. Therefore we conclude that the accuracy of the proposed approximation is sufficiently good to be used for calibration in this way.

5. Conclusions

We have proposed an analytic approximation formula obtained by substituting the instantaneous volatility for the constant volatility in the Vasicek price. We have shown that the error in logarithms of the bond prices is of order $O(\tau^4)$. This is a lower accuracy compared to the approximation studied in [5] and [17] which has the order $O(\tau^5)$. However, comparing the approximation with the exact values in CIR model we have seen that for maturities up to several years, the error is less than the accuracy to which the interest rates are quoted on the market. Most importantly, the special form of the approximation allows an easy calibration. The quality of the approximation was confirmed by simulated data using the exact solution from CIR model. The proposed approximation and calibration method turned out to be accurate enough to recover the parameters with a high precision.

Acknowledgments

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Figure 3. Results of the estimation with respect to $\gamma$ using the approximate formula for interest rates. The data were simulated using the exact formula with the parameters $\alpha = 0.00315$, $\beta = -0.0555$, $\sigma = 0.0894$, $\gamma = 0.5$. Maturities used were 1, 2, $\ldots$, 12 months (left) and 1, 2, $\ldots$, 5 years (right).

References


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