INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, Series B
Volume 3, Number 2, Pages 168–184

SCHEME OF THE SOLUTION DECOMPOSITION METHOD FOR A SINGULARLY PERTURBED REACTION-DIFFUSION EQUATION; APPROXIMATION OF SOLUTIONS AND DERIVATIVES

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Abstract. A Dirichlet problem is considered for a singularly perturbed ordinary differential reaction-diffusion equation. For this problem, a new approach is developed to construct difference schemes convergent uniformly with respect to a perturbation parameter ε for $\varepsilon \in (0, 1]$, i.e., ε-uniformly. This approach is based on a *decomposition of the discrete solution* into the regular and singular components which are solutions of *discrete subproblems* considered on *uniform grids*. Using an *asymptotic construction technique*, a *difference scheme* of the *solution decomposition method* is constructed that converges ε-uniformly in the maximum norm at the rate $\mathcal{O}\left(N^{-2}\ln^2 N\right)$, where $N + 1$ is the number of nodes in the grids used; for fixed values of the parameter ε, the scheme converges at the rate $\mathcal{O}\left(N^{-2}\right)$. For the constructed scheme, approximations of the regular and singular components to the solution and their derivatives up to the second order are studied. A modified scheme of the solution decomposition method is constructed for which the regular component of the solution and its discrete derivatives converge ε-uniformly in the maximum norm at the rate $\mathcal{O}\left(N^{-2}\right)$ for $\varepsilon = o(\ln^{-1} N)$.

Key words. singularly perturbed boundary value problem, ordinary differential reaction-diffusion equation, discrete solution decomposition, asymptotic construction technique, difference scheme of a solution decomposition method, uniform grids, ε-uniform convergence, maximum norm, approximation of derivatives.

1. Introduction

At present, for singularly perturbed boundary value problems, methods for constructing ε-uniformly convergent difference schemes *on grids condensing in a neighborhood of the boundary layer* (using the classical schemes on the Bakhvalov and/or Shishkin grids) are well developed. Such methods have widespread application due to their simplicity and convenience (see, e.g., [1]–[6] and the bibliography therein). A drawback of these numerical methods is the necessity to solve discrete equations on grids in which step-sizes change sharply in a neighborhood of the boundary layer. Such type grids do not allow one to apply high-efficiency computational methods developed for solving grid equations in the case of regular boundary value problems (see, e.g., [7, 8]), and also give rise to difficulties in the construction of high-order accurate schemes (see, e.g., [9, 10]) and in approximation of derivatives.

Note that fitted operator methods (see the description in [11]–[13], and also [3], Ch. 4; [4], Ch. 2; [5], Part I, Ch. 4, and the bibliography therein) have advantage in the simplicity of uniform grids used; however, the coefficients of grid equations in these methods depend on the explicit form of the main term in the singular component of the solution. For this reason, fitted operator methods have a restricted applicability in constructing ε-uniformly convergent numerical methods, in particular, for problems with parabolic initial or boundary layers (see [14, 15] and also [2], Ch. II, § 1; [4], Ch. 6; [6], Ch. 1, 9 and the bibliography therein).

Received by the editors March 25, 2011 and, in revised form, December 12, 2011.
2000 *Mathematics Subject Classification.* 35B25, 35B45, 35C20, 65L10, 65L12, 65L20, 65L70.

In the present paper, for a model singularly perturbed boundary value problem for an ordinary differential reaction-diffusion equation, a *new approach* based on the asymptotic construction technique is proposed to construct special schemes, namely, the *method of decomposition of a grid solution* using the *classical approximations of subproblems* on *uniform grids* for the regular and singular components of the solution. Unlike known approaches to the construction of ε-uniformly convergent difference schemes — the *fitted operator method* and the *method of special condensing grids* — discrete subproblems *in the new approach* are solved on *uniform grids*; moreover, the *coefficients* of grid equations *do not depend on the explicit form of the singular component* of the solution. Construction of difference schemes of the new solution decomposition method have been developed in the papers [16]–[18], where approximation of the solution have been studied only. In the present paper, approximations of the regular and singular components of the solution and their derivatives up to the second order are examined for schemes based on the solution decomposition method. Earlier, such a problem was not considered.

Contents of the paper. The formulation of the boundary value problem for a singularly perturbed reaction-diffusion equation and the aim of the research are presented in Section 2. Standard difference schemes on uniform and piecewise-uniform grids are given in Section 3. A difference scheme of the solution decomposition method is constructed in Section 4. This approach applies the decomposition of a grid solution into its regular and singular component considered on uniform grids. The scheme of the solution decomposition method converges ε-uniformly at the rate $\mathcal{O}\left(N^{-2}\ln^2 N\right)$, which is the same as for the standard scheme on a piecewise uniform grid. Here, $N+1$ is the number of nodes in the grids used; note that the number of nodes is the same in the grids on which grid approximations of the regular component, as well as the singular components, are considered in neighborhoods of the left and right boundaries of the domain. Approximations of the regular and singular components of the solution and their derivatives are discussed in Section 5. "Standard" a priori estimates used in the constructions are considered in Section 6. A modified scheme of the solution decomposition method for which the regular component of the solution and its discrete derivatives converge ε-uniformly in the maximum norm at the rate $\mathcal{O}\left(N^{-2}\right)$ for $\varepsilon = o(\ln^{-1} N)$ is constructed in Section 7.

2. Problem Formulation. Aim of Research

On the set $\overline{D}$

$$\overline{D} = D \bigcup \Gamma, \quad D = (0,d), \quad \Gamma = \Gamma_1 \cup \Gamma_2, \tag{2.1}$$

where Γ_1 and Γ_2 are the left and right parts of the boundary Γ, we consider the Dirichlet problem for the ordinary differential reaction-diffusion equation

$$L\,u(x) \equiv \left\{\varepsilon^2\, a(x)\frac{d^2}{dx^2} - c(x)\right\} u(x) = f(x), \quad x \in D, \quad u(x) = \varphi(x), \quad x \in \Gamma. \tag{2.2}$$

The functions $a(x)$, $c(x)$, $f(x)$ are assumed to be sufficiently smooth on $\overline{D}$; moreover,[†]

$$a_0 \le a(x) \le a^0, \quad c_0 \le c(x) \le c^0, \quad x \in \overline{D}; \quad a_0,\, c_0 > 0; \tag{2.3}$$
$$|f(x)| \le M, \quad x \in \overline{D}; \qquad |\varphi(x)| \le M, \quad x \in \Gamma;$$

[†] We denote by M (by m) sufficiently large (small) positive constants that do not depend on the value of the parameter ε. In the case of grid problems, these constants are also independent of the stencils of the difference schemes.

the parameter ε takes arbitrary values in the open-closed interval $(0,1]$. For small values of ε, a boundary layer appears in a neighbourhood of the set Γ.

Solutions of singularly perturbed boundary value problems are characterized by multiscale nature (for small values of ε), which leads to difficulties in their analysis and numerical solution. In such problems for partial differential equations, a technique of the asymptotic construction is well known for obtaining a priori estimates for the solution of the problem on the basis of the *decomposition of the solution* into regular and singular components (see, e.g., [2], Ch. 1, Sect. 3; [3], Ch. 12; [6], Ch. 3, 6] and references therein). If the also subjected to decomposition. Each component of the solution is a solution of the corresponding “canonical” boundary value problem, which has not more than one type of singularity in the solution. This allows us to obtain sufficiently effective *a priori* estimates for the components of the solution. For the problem under study, a similar technique of asymptotic constructions, under a suitable *decomposition of a discrete solution* into regular and singular components, allows us to construct the corresponding “canonical” difference scheme on a uniform grid for each grid component and to investigate the grid approximation of these components and their derivatives.

Note that the approach using the asymptotic construction method based on a *domain decomposition* and the *asymptotics of a solution with respect to the parameter* ε was applied in [19]–[21] and [6, Ch. 10] to construct ε-uniformly convergent schemes on piecewise uniform grids with higher-order accuracy.

Our aim for boundary value problem (2.2), (2.1) is to construct an ε-uniformly convergent (in the maximum norm) scheme based on the method of decomposition of the solution into regular and singular components and using grid approximations of the components on the related uniform grids. For the constructed scheme, convergence of the solution, of the regular and singular components to the solution and also their discrete derivatives are studied.

3. Difference schemes on uniform and piecewise-uniform grids

We consider difference schemes constructed on the basis of classical approximations of the problem (2.2), (2.1) on uniform and piecewise uniform grids.

3.1. First we give a difference scheme on a uniform grid. On the set $\overline{D}$, we introduce the grid

$$\overline{D}_h = \overline{\omega}, \tag{3.1}$$

where $\overline{\omega}$ is an arbitrary, in general, nonuniform mesh on $\overline{D}$. Let $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$, $h = \max_i h^i$. We assume that the condition $h \le M\,N^{-1}$ is fulfilled, where $N+1$ is the number of nodes in the mesh $\overline{\omega}$.

We approximate problem (2.2), (2.1) by the difference scheme [22]

$$\Lambda z(x) = f(x), \quad x \in D_h, \qquad z(x) = \varphi(x), \quad x \in \Gamma_h. \tag{3.2}$$

Here $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, $\Lambda \equiv \varepsilon^2\, a(x)\, \delta_{\overline{x}\widehat{x}} - c(x)$, $x \in D_h$, $\delta_{\overline{x}\widehat{x}}\, z(x)$ is the second central difference derivative on a nonuniform grid:

$$\delta_{\overline{x}\widehat{x}}\, z(x) = 2(h^i + h^{i-1})^{-1}[\delta_x\, z(x) - \delta_{\overline{x}}\, z(x)], \quad x = x^i \in D_h;$$

$\delta_x\, z(x)$ and $\delta_{\overline{x}}\, z(x)$ are the first (forward and backward) difference derivatives.

The scheme (3.2), (3.1) is ε-uniformly monotone [22]. The following variant of the comparison theorem is valid.

Theorem 3.1. *Let the functions $z^1(x)$, $z^2(x)$, $x \in \overline{D}_h$, satisfy the condition*

$$\Lambda z^1(x) < \Lambda z^2(x), \quad x \in D_h, \quad z^1(x) > z^2(x), \quad x \in \Gamma_h.$$

Then, $z^1(x) > z^2(x), \quad x \in \overline{D}_h$.

In the case of uniform grids

$$\overline{D}_h = \overline{D}_h^{\,u} \equiv \overline{\omega}, \tag{3.3}$$

using the maximum principle, we obtain the estimate

$$|u(x) - z(x)| \le M \, \left(\varepsilon + N^{-1}\right)^{-2} N^{-2}, \quad x \in \overline{D}_h. \tag{3.4}$$

The scheme (3.2), (3.3) converges under the condition

$$N^{-1} = o(\varepsilon), \quad N = N_{(3.1)}. \tag{3.5}$$

3.2. Now we construct a difference scheme on a piecewise-uniform grid that converges ε-uniformly (see, e.g., [2], Ch. 1, § 1; [3], Ch. 6). On the set $\overline{D}$, we introduce the grid

$$\overline{D}_h = \overline{D}_h^{\,*} \equiv \overline{\omega}^{\,*}, \tag{3.6a}$$

where $\overline{\omega}^{\,*}$ is a piecewise-uniform mesh, which is constructed as follows. The interval $[0, d]$ is divided into three intervals $[0, \sigma]$, $[\sigma, d - \sigma]$ and $[d - \sigma, d]$, the step-sizes on these intervals are constant and equal to $h^{(1)} = 4\,\sigma\, N^{-1}$ on the intervals $[0, \sigma]$, $[d - \sigma, d]$ and to $h^{(2)} = 2(d - 2\sigma)N^{-1}$ on the interval $[\sigma, d - \sigma]$. The parameter σ is defined by the relation

$$\sigma = \sigma(\varepsilon,\, N,\, l) = \min\left[4^{-1}\, d,\, l\, m^{-1}\, \varepsilon\, \ln N\right], \tag{3.6b}$$

where m is an arbitrary number from $(0, m_0)$, $m_0 = m_{(6.9)}$, and $l = 2$. The grid $\overline{D}_h^{\,*}$ is constructed.

For the solutions of the difference scheme (3.2), (3.6), we obtain the estimate

$$|u(x) - z(x)| \le M\, N^{-2} \min{}^2 \left[\varepsilon^{-1},\, \ln N\right], \quad x \in \overline{D}_h, \tag{3.7}$$

and also the ε-uniform estimate

$$|u(x) - z(x)| \le M\, N^{-2} \ln^2 N, \quad x \in \overline{D}_h. \tag{3.8}$$

The estimates (3.7) and (3.8) are unimprovable with respect to the values N, ε and N, respectively; the scheme (3.2), (3.6) converges ε-uniformly with the second order, up to a logarithmic factor; this scheme converges with the second order for fixed values of ε.

Theorem 3.2. *Let the solution* $u(x)$ *of the problem* (2.2), (2.1) *satisfy the estimates of Theorem* 6.1, *where* $K = 4$ *and* $n = 0$. *Then, the difference scheme* (3.2), (3.3) *converges under the condition* (3.5)*; the scheme* (3.2), (3.6) *converges* ε*-uniformly. The solutions of the difference schemes* (3.2), (3.3) *and* (3.2), (3.6) *satisfy the estimates* (3.4) *and* (3.7), (3.8), *respectively.*

4. Scheme of the asymptotic construction method

Using the decomposition of the solution of the differential problem (2.2), (2.1), we construct an ε-uniformly convergent difference scheme based on the method of decomposition of the discrete solution, in which the discrete regular and singular components of the solution are computed on uniform grids.

4.1. Here we consider a decomposition of the solution to the problem (2.2), (2.1), which is slightly different from that given in Section 7 below. We write the solution of the boundary value problem as the sum of its regular and singular components

$$u(x) = U(x) + V(x), \quad x \in \overline{D}. \tag{4.1a}$$

The function $U(x)$ is decomposed into the sum of functions

$$U(x) = U_0(x) + v_U(x), \quad x \in \overline{D}. \tag{4.1b}$$

Here, $U_0(x)$ is the main term of the "expansion" of the regular component, and $v_U^0(x)$ is the remainder term. The functions $U_0(x)$ and $v_U^0(x)$ in (4.1b) are solutions of the problems

$$\text{(4.2a)}\quad L_{(4.2)} U_0(x) = f(x), \qquad x \in \overline{D};$$

$$\text{(4.2b)}\quad L_{(2.2)} v_U(x) = -\varepsilon^2 a(x) \frac{d^2}{dx^2} U_0(x), \quad x \in D, \qquad v_U(x) = 0, \quad x \in \Gamma.$$

Here $L_{(4.2)}$ is the operator $L_{(2.2)}$ for $\varepsilon = 0$, i.e., $L_{(4.2)} \equiv -c(x)$, $x \in \overline{D}$.

The function $V(x)$, $x \in \overline{D}$, is the solution of the problem

$$L_{(2.2)}\, V(x) = 0, \quad x \in D, \qquad V(x) = \varphi_V(x), \quad x \in \Gamma, \tag{4.3a}$$

$$\varphi_V(x) = \varphi(x) - U(x), \quad x \in \Gamma, \qquad U(x) = U_{(4.1)}(x), \quad x \in \overline{D}.$$

We represent the function $V(x)$ as the sum of functions

$$V(x) = V_1(x) + V_2(x), \quad x \in \overline{D}, \tag{4.1c}$$

where the functions $V_1(x)$ and $V_2(x)$ are solutions of the problems

$$\begin{aligned} L_{(2.2)}\, V_j(x) &= 0, && x \in D, \\ V_j(x) &= \varphi_V(x), && x \in \Gamma_j, \\ V_j(x) &= 0, && x \in \Gamma \setminus \Gamma_j, \quad j = 1, 2. \end{aligned} \tag{4.3b}$$

Thus, the components of the solution of the problem (2.2), (2.1) in the representation (4.1) are solutions of problems (4.2a), (4.2b), (4.3a), and (4.3b).

Let the data of the boundary value problem (2.2), (2.1) satisfy the hypothesis of Theorem 6.1 for $l_0 = 4$, $n = 0$. Then, the components in the representations (4.1a,c) satisfy the following estimates:

$$\left.\begin{aligned} \left| \frac{d^k}{dx^k} U(x) \right| &\le M \left[1 + \varepsilon^{2-k}\right] \\ \left| \frac{d^k}{\partial x^k} V_1(x) \right| &\le M\, \varepsilon^{-k} \exp(-m\, \varepsilon^{-1}\, x) \\ \left| \frac{d^k}{dx^k} V_2(x) \right| &\le M\, \varepsilon^{-k} \exp\big(- m\, \varepsilon^{-1}\, (d - x)\big) \end{aligned}\right\}, \quad x \in \overline{D}, \quad k \le K, \tag{4.4}$$

where m is an arbitrary number in $(0, m_0)$, $m_0 = \min_{\overline{D}}^{1/2} \left[a^{-1}(x)\, c(x)\right]$; $K = 4$.

For the components in (4.1b), we have the estimates

$$\left.\begin{aligned} \left| \frac{d^k}{dx^k} U_0(x) \right| &\le M, && k \le K + 2 \\ \left| \frac{d^k}{dx^k} v_U(x) \right| &\le M\, \varepsilon^{2-k}, && k \le K \end{aligned}\right\}, \quad x \in \overline{D}. \tag{4.5}$$

Theorem 4.1. *Let the data of boundary value problem* (2.2), (2.1) *satisfy the hypothesis of Theorem* 6.1 *for* $l_0 = 4$, $n = 0$. *Then, the components in the representations* (4.1a, c) *and* (4.1b) *satisfy the estimates* (4.4) *and* (4.5), *respectively, where* $K = 4$.

4.2. We construct a difference scheme for the problem (2.2), (2.1) by approximating problems (4.2) and (4.3b). For not too small values of the parameter ε, namely, under the condition

$$\varepsilon \geq \varepsilon_0(N), \quad \varepsilon_0(N) = m\, l^{-1}\, d\, \ln^{-1} N, \tag{4.6}$$

where $m = m_{(4.4)}$, $l = 2$, the problem (2.2), (2.1) is approximated by the standard difference scheme (3.2) on the uniform grid (3.3); $N = N_{(3.3)}$.

For sufficiently small values of the parameter ε, namely, under the condition

$$\varepsilon < \varepsilon_0(N), \tag{4.7}$$

we approximate the problems for the components in the representation (4.1b) of the function $U(x)$ by schemes on the uniform grid (3.3) and the problems for the singular components $V_j(x)$ from (4.3b) by schemes on uniform grids, which are constructed on the subdomains $\overline{D}_j^{\,\sigma}$ of $\overline{D}$ adjacent to the boundaries Γ_j, $j = 1, 2$:

$$\overline{D}_j^{\,\sigma} = D_j^\sigma \cup \Gamma_j^\sigma, \quad j = 1, 2, \quad D_1^\sigma = (0, \sigma), \quad D_2^\sigma = (d - \sigma, d); \tag{4.8a}$$

$$\sigma = \sigma(\varepsilon, N, l) = \min \left[d,\ m^{-1}\, l\, \varepsilon\, \ln N \right]; \tag{4.8b}$$

here $m = m_{(4.6)}$, $l = l_{(4.6)}$, $N = N_{(4.6)}$ and $\varepsilon = \varepsilon_{(4.7)}$.

4.2.1. We construct a difference scheme under the condition (4.7). We approximate the differential problems (4.2), (2.1) by the following discrete problems on the uniform grid (3.3):

$$\Lambda_{(4.9)} z_{U_0}(x) \ = \ f(x), \qquad x \in \overline{D}_h; \tag{4.9}$$

$$\Lambda_{(3.2)} z_{v_U}(x) \ = \ -\varepsilon^2\, a(x)\, \delta_{x\overline{x}}\, z_{U_0}(x), \quad x \in D_h, \quad z_{v_U}(x) = 0, \quad x \in \Gamma_h. \tag{4.10}$$

Here $D_h = D \cap \overline{D}_h$, $\Gamma_h = \Gamma \cap \overline{D}_h$, $\Lambda_{(4.9)} \equiv -c(x)$, $x \in \overline{D}_h$. Set

$$z_U(x) = z_{U_0}(x) + z_{v_U}(x), \quad x \in \overline{D}_h. \tag{4.11}$$

By $\overline{z}_U(x)$, $x \in \overline{D}$, we denote the linear interpolant constructed using the values of $z_U(x)$ at the nodes of the grid $\overline{D}_h$ on elementary partitions of the set $\overline{D}$ generated by the grid $\overline{D}_h$.

The function $z_U(x)$, $x \in \overline{D}_h$, and also its interpolant $\overline{z}_U(x)$, $x \in \overline{D}$, are called the solutions (discrete and continual) of the difference scheme $\{$(4.9) — (4.10), (3.3); (4.7)$\}$, approximating differential problems (4.2), (2.1) under the condition (4.7).

Now we approximate the problem (4.3), (2.1). On the set $\overline{D}_{j\,(4.12)}^{\,\sigma}$, we introduce the uniform grid

$$\overline{D}_{j\,h}^{\,\sigma} = \overline{D}_{j\,h}^{\,\sigma u} \equiv \overline{\omega}_j^{\,\sigma}, \quad j = 1, 2, \tag{4.12}$$

where $\overline{\omega}_j^{\,\sigma}$ is a mesh on $\overline{D}_{j\,(4.8)}^{\,\sigma}$ with the step-size $h^\sigma = \sigma N^{-1}$, $N+1$ is the number of nodes in the mesh $\overline{\omega}_j^{\,\sigma}$, $N = N_{(4.6)}$; $\overline{D}_{j\,h}^{\,\sigma} = D_{j\,h}^\sigma \cup \Gamma_{j\,h}^\sigma$. On the grid $\overline{D}_{j\,h}^{\,\sigma}$, we solve the discrete problem

$$\Lambda_{(3.2)} z_{V_j}(x) = 0, \quad x \in D_{j\,h}^\sigma, \tag{4.13}$$

$$z_{V_j}(x) = \left\{ \begin{array}{ll} \varphi(x) - z_U(x), & x \in \Gamma_{j\,h}^\sigma \cap \Gamma \\ 0, & x \in \Gamma_{j\,h}^\sigma \setminus \Gamma \end{array} \right\}, \quad x \in \Gamma_{j\,h}^\sigma, \quad j = 1, 2.$$

Using the function $z_{V_j}(x)$, $x \in \overline{D}_{j\,h}^{\,\sigma}$, we construct the interpolant $\overline{z}_{V_j}(x)$, $x \in \overline{D}_j^{\,\sigma}$. The functions $z_{V_j}(x)$ and $\overline{z}_{V_j}(x)$ outside the set $\overline{D}_j^{\,\sigma}$ are assumed equal to zero. We set

$$\overline{z}_V(x) = \overline{z}_{V_1}(x) + \overline{z}_{V_2}(x), \quad x \in \overline{D}.$$

The function $\overline{z}_V(x)$, $x \in \overline{D}^{\,\sigma}$, is called the solution of the difference scheme {(4.13), (4.12); (4.7)} approximating the differential problem (4.3a), (2.1) under the condition (4.7). The function

$$\overline{z}_u(x) = \overline{z}_U(x) + \overline{z}_V(x), \quad x \in \overline{D}, \quad \textit{under the condition} \quad (4.7), \tag{4.14a}$$

is called the solution of the difference scheme {{(4.9)–(4.10), (3.3); (4.13) (4.12)}; (4.7)} approximating the differential problem (2.2), (2.1) under the condition (4.7).

4.2.2. In the case of condition (4.6), we solve difference scheme (3.2), (3.3). The interpolant

$$\overline{z}_u(x), \quad x \in \overline{D}, \quad \textit{under the condition} \quad (4.6), \tag{4.14b}$$

constructed from the solution of the scheme (3.2), (3.3) is called the solution of the difference scheme {(3.2), (3.3); (4.6)} approximating the differential problem (2.2), (2.1) under the condition (4.6).

Thus, the constructed function $\overline{z}_{u\,(4.14\mathrm{a,b})}(x)$, $x \in \overline{D}$, approximates the solution of the problem (2.2), (2.1). This function and also the grid functions $z_{U_0}(x)$, $z_{v_U}(x)$, $x \in \overline{D}_h$ and $z_{V_i}(x)$, $x \in \overline{D}^{\,\sigma}_{i\,h}$, $i = 1, 2$, are called the solutions (respectively, continual and discrete) of the difference scheme {{(3.2), (3.3)}; {(4.9)–(4.10), (3.3); (4.13), (4.12)}}, i.e., of the *scheme of the solution decomposition method*.

The scheme of the solution decomposition method is ε-uniformly monotone. The following variant of the comparison theorem is valid.

Theorem 4.2. *Let the condition* (4.7) *be fulfilled, and let the functions* $z^k_{U_0}(x)$, $z^k_{v_U}(x)$, $x \in \overline{D}_h$ *and* $z^k_{V_j}(x)$, $x \in \overline{D}^{\,\sigma}_{j\,h}$, $j = 1, 2$, *for* $k = 1, 2$, *satisfy the conditions*

$$\Lambda_{(4.9)} z^1_{U_0}(x) < \Lambda_{(4.9)} z^2_{U_0}(x), \quad x \in \overline{D}_h;$$

$$\Lambda_{(3.2)} z^1_{v_U}(x) < \Lambda_{(3.2)} z^2_{v_U}(x), \quad x \in D_h, \quad z^1_{v_U}(x) > z^2_{v_U}(x), \quad x \in \Gamma_h;$$

$$\Lambda_{(3.2)} z^1_{V_j}(x) < \Lambda_{(3.2)} z^2_{V_j}(x), \quad x \in D^{\sigma}_{j\,h}, \quad z^1_{V_j}(x) > z^2_{V_j}(x), \quad x \in \Gamma^{\sigma}_{j\,h}, \quad j = 1, 2.$$

Then

$$z^1_{U_0}(x) > z^2_{U_0}(x),\ z^1_{v_U}(x) > z^2_{v_U}(x),\ x \in \overline{D}_h; \quad z^1_{V_j}(x) > z^2_{V_j}(x),\ x \in \overline{D}^{\,\sigma}_{j\,h},\ j = 1, 2.$$

4.3. Let us estimate $u(x) - \overline{z}_u(x)$, $x \in \overline{D}$, assuming that the hypothesis of Theorem 4.1 is fulfilled in order to ensure the inclusions $U(x)$, $V(x) \in C^4(\overline{D})$, and $U_0(x)$, $v_U \in C^4(\overline{D})$, where $U_0(x)$ and $v_U(x)$ are the components from the expansion (4.1b), and also assuming that the estimates (4.4), (4.5) hold.

Taking into account *a priori* estimates for the components $U_0(x)$, $v_U(x)$ and $V_j(x)$ we find estimates for $U_0(x) - \overline{z}_{U_0}(x)$, $v_U(x) - \overline{z}_{v_U}(x)$ and $V_j(x) - \overline{z}_{V_j}(x)$. From these estimates, we obtain an estimate for $u(x) - \overline{z}_{u\,(4.14\mathrm{a})}(x)$ under the condition (4.7). Estimating $u(x) - \overline{z}_{u\,(4.14\mathrm{b})}(x)$ under the condition (4.6), we use the estimate (3.4), where $\varepsilon > \varepsilon_0(N)$.

For the solution of the difference scheme {{(3.2), (3.3)}; {(4.9)–(4.10), (3.3); (4.13), (4.12)}} of the solution decomposition method, we obtain the estimate

$$|u(x) - \overline{z}_u(x)| \le M\, N^{-2} \min{}^2\, [\varepsilon^{-1}, \ln N], \quad x \in \overline{D}, \tag{4.15}$$

and also the ε-uniform estimate

$$|u(x) - \overline{z}_u(x)| \le M\, N^{-2} \ln^2 N, \quad x \in \overline{D}. \tag{4.16}$$

Theorem 4.3. *Let the components of the solution of the boundary value problem* (2.2), (2.1) *in the representation* (4.1) *satisfy the estimates from Theorem* 4.1. *Then the solution of the difference scheme of the solution decomposition method*

$\{\{(3.2),\ (3.3)\};\ \{(4.9)\text{–}(4.10),\ (3.3);\ (4.13),\ (4.12)\}\}$ *converges ε-uniformly at the rate $\mathcal{O}\left(N^{-2}\ln^2 N\right)$. The solution of the difference scheme satisfy the estimates* (4.15), (4.16).

Remark 4.1. The estimates (4.15), (4.16) for the scheme of the solution decomposition method $\{\{(3.2),\ (3.3)\};\ \{(4.9)\text{–}(4.10),\ (3.3);\ (4.13),\ (4.12)\}\}$ are the same as the estimates (3.7), (3.8) for the scheme (3.2) on the piecewise-uniform grid (3.6). □

Remark 4.2. *The estimates* (4.15), (4.16) for the scheme of the solution decomposition method (as the estimates (3.7), (3.8) for the scheme (3.2) on the piecewise-uniform grid (3.6)) *do not allow to reveal distinctions in the convergence rate of the functions* $z_U(x)$ and $z_{V_j}(x)$ which are discrete regular and singular components of the solution to the solution decomposition method. To estimate the convergence rate of the functions $z_U(x)$ and $z_{V_j}(x)$ is of intrinsic interest. □

5. Grid approximation of normalized derivatives

For the problem (2.2), (2.1), we consider approximations of derivatives up to the second order for the scheme (3.2) on the uniform grid (3.3) and on the piecewise-uniform grid (3.6), and also for the scheme of the solution decomposition method $\{\{(3.2),\ (3.3)\};\ \{(4.9)\text{–}(4.10),\ (3.3);\ (4.13),\ (4.12)\}\}$.

From the estimate (6.2) it follows that for sufficiently smooth data of the problem, the derivatives of its solution satisfy the estimates

$$\left|\frac{d^k}{dx^k}\,u(x)\right| \le M\,\varepsilon^{-k}, \quad x \in \overline{D}, \quad 0 \le k \le 2.$$

We call the following values

$$q^k(x) = \varepsilon^k\,\frac{d^k}{dx^k}\,u(x), \quad x \in \overline{D}, \quad k \le 2,$$

which are ε-uniformly bounded on $\overline{D}$, *normalized derivatives.*

In the case of the scheme (3.2) on the grids (3.3) and (3.6), the discrete normalized derivatives

$$q_h^1(x) = \varepsilon\,\delta_x\,z(x), \quad x \in \overline{D}_h \setminus \{x = d\}, \quad q_h^2(x) = \varepsilon^2\,\delta_{\overline{x}\,\widehat{x}}\,z(x), \quad x \in D_h,$$

correspond to the normalized derivatives $q^k(x)$, $k = 1, 2$. Discrete normalized derivatives for the scheme of the solution decomposition method are defined in Section 5.3.

5.1. Consider approximation of normalized derivatives in the case of the difference scheme (3.2) on the uniform grid (3.3).

By virtue of the differential and discrete equations from the problems (2.2), (2.1) and (3.2), (3.3) and taking into account the estimate (3.4), we find the estimate

$$\left|\varepsilon^2\left(\frac{d^2}{dx^2}u(x) - \delta_{\overline{x}\widehat{x}}z(x)\right)\right| \le M\,(\varepsilon + N^{-1})^{-2}\,N^{-2}, \quad x \in D_h. \tag{5.1}$$

Taking into account the estimates

$$|u(x) - z(x)| \le M\,(\varepsilon + N^{-1})^{-2}\,N^{-2}, \qquad x \in \overline{D}_h,$$

$$|\Lambda\ (u(x) - z(x))| \le M\,(\varepsilon + N^{-1})^{-2}\,N^{-2}, \quad x \in D_h,$$

we find the estimate

$$|\varepsilon\,\delta_x\ (u(x) - z(x))| \le M\,(\varepsilon + N^{-1})^{-2}\,N^{-2}, \quad x \in \overline{D}_h \setminus \{x = d\}.$$

Thus, we have

$$\left|\varepsilon\left(\frac{d}{dx}u(x^{(1)}(x)) - \delta_x\, z(x)\right)\right| \le M\,(\varepsilon + N^{-1})^{-2}\,N^{-2}, \quad x \in \overline{D}_h \setminus \{x = d\}, \tag{5.2}$$

$$x^{(1)}(x) = 2^{-1}(x^i + x^{i+1}), \quad x = x^i,\ x^i, x^{i+1} \in \overline{D}_h.$$

From the estimates (5.1), (5.2), it follows that normalized discrete derivatives converge to corresponding differential derivatives under the condition (3.5).

The following theorem holds.

Theorem 5.1. *Let the solution of the boundary value problem* (2.2)*,* (2.1) *satisfy the estimate* (6.2) *for* $K = 4$ *from Theorem* 6.1*. Then the normalized discrete derivatives* $q_h^k(x)$ *for* $k = 1, 2$ *in the case of the difference scheme* $\{(2.2), (3.3)\}$ *converge to the normalized differential derivatives under the condition* (3.5)*. The discrete derivatives satisfy the estimates* (5.1) *and* (5.2)*.*

5.2. Consider approximation of normalized derivatives in the case of the difference scheme (3.2) on the piecewise-uniform grid (3.6). Analysis of convergence of the discrete derivatives is carried out analogously to the case of the uniform grid.

Taking into account the estimate (3.8), and by virtue of the differential and discrete equations from the problems (2.2), (2.1) and (3.2), (3.6), we find the estimate

$$\left|\varepsilon^2\left(\frac{d^2}{dx^2}u(x) - \delta_{\overline{x}\widehat{x}}z(x)\right)\right| \le M\,N^{-2}\,\ln^2 N, \quad x \in D_h. \tag{5.3}$$

Further, taking into account this estimate and the estimate (3.8), we obtain the estimate

$$\left|\varepsilon\left(\frac{d}{dx}u(x^{(1)}(x)) - \delta_x\, z(x)\right)\right| \le M\,N^{-2}\,\ln^2 N, \quad x \in \overline{D}_h \setminus \{x = d\}, \tag{5.4}$$

$$x^{(1)}(x) = x^{(1)}_{(5.2)}(x).$$

Theorem 5.2. *Let the solution of the boundary value problem* (2.2)*,* (2.1) *satisfy the estimate* (6.9) *for* $K = 4$ *from Theorem* 6.1*. Then the normalized discrete derivatives* $q_h^k(x)$ *for* $k = 1, 2$ *in the case of the difference scheme* $\{(3.2), (3.6)\}$ *converge to the normalized differential derivatives* ε*-uniformly. The discrete derivatives satisfy the estimates* (5.3) *and* (5.4)*.*

Thus, the normalized discrete derivatives converge to the normalized differential derivatives ε-uniformly at the same rate $\mathcal{O}\left(N^{-2}\,\ln^2 N\right)$ as the discrete solution.

5.3. Now we consider approximation of normalized derivatives for the difference scheme $\{\{(3.2), (3.3)\}; \{(4.9)–(4.10), (3.3); (4.13), (4.12)\}\}$ of the solution decomposition method.

Solving the problem $\{\{(3.2), (3.3)\}; \{(4.9)–(4.10), (3.3); (4.13), (4.12)\}\}$ for relatively large values of ε, we find the discrete function

$$z(x), \quad x \in \overline{D}_h \textit{ under condition } (4.6), \tag{5.5a}$$

and for relatively small values of ε, we find the components

$$z_U(x), x \in \overline{D}_h,\ z_{V_1}(x), x \in \overline{D}^{\sigma}_{1\,h}, z_{V_2}(x), x \in \overline{D}^{\sigma}_{2\,h} \textit{ under condition } (4.7). \tag{5.5b}$$

The functions $z_{(5.5)}(x)$, $z_{U\,(5.5)}(x)$ and $z_{V_j\,(5.5)}(x)$ correspond to the normalized derivatives $q_z^k(x)$, $q_{z_U}^k(x)$ and $q_{z_{V_j}}^k(x)$ with $k, j = 1, 2$, which are defined by the relations

$$q_g^1(x) = \varepsilon\,\delta_x g(x), \quad q_g^2(x) = \varepsilon^2\,\delta_{\overline{x}\,\widehat{x}}g(x),$$

where $g(x)$ is one of the functions $z_{(5.5)}(x)$, $z_{U\,(5.5)}(x)$ and $z_{V_j\,(5.5)}(x)$.

For the functions $z(x)$, $z_U(x)$, $z_{V_j}(x)$ and their discrete derivatives, under the condition (4.6), we obtain the estimates

$$(5.6a)\qquad |u(x) - z(x)| \le M\,N^{-2}\ln^2 N, \quad x \in \overline{D}_h,$$

$$(5.6b)\qquad \left|\varepsilon\left(\frac{d}{dx}u(x^{(1)}(x)) - \delta_x\,z(x)\right)\right| \le M\,N^{-2}\ln^2 N, \quad x \in \overline{D}_h \setminus \{x = d\},$$

$$(5.6c)\qquad \left|\varepsilon^2\left(\frac{d^2}{dx^2}u(x) - \delta_{\overline{x}\widehat{x}}z(x)\right)\right| \le M\,N^{-2}\ln^2 N, \quad x \in D_h,$$

and under the condition (4.7), the following estimates hold

$$(5.7a)\qquad |U(x) - z_U(x)| \le M\,N^{-2}, \quad x \in \overline{D}_h,$$

$$(5.7b)\qquad \left|\varepsilon\left(\frac{d}{dx}U(x^{(1)}(x)) - \delta_x\,z_U(x)\right)\right| \le M\,N^{-2}, \quad x \in \overline{D}_h \setminus \{x = d\},$$

$$(5.7c)\qquad \left|\varepsilon^2\left(\frac{d^2}{dx^2}U(x) - \delta_{\overline{x}\widehat{x}}z_U(x)\right)\right| \le M\,N^{-2}, \quad x \in D_h;$$

$$(5.8a)\quad |V_1(x) - z_{V_1}(x)| \le M\,N^{-2}\ln^2 N\,\exp(-m\,\varepsilon^{-1}x), \quad x \in \overline{D}^{\sigma}_{1\,h},$$

$$(5.8b)\quad \left|\varepsilon\left(\frac{d}{dx}V_1(x^{(1)}(x)) - \delta_x\,z_{V_1}(x)\right)\right| \le M\,N^{-2}\ln^2 N\,\exp(-m\,\varepsilon^{-1}x),$$
$$x \in \overline{D}^{\sigma}_{1\,h} \setminus \{x = \sigma\},$$

$$(5.8c)\quad \left|\varepsilon^2\left(\frac{d^2}{dx^2}V_1(x) - \delta_{\overline{x}\widehat{x}}z_{V_1}(x)\right)\right| \le M\,N^{-2}\ln^2 N\,\exp(-m\,\varepsilon^{-1}x),\ x \in D^{\sigma}_{1\,h};$$

$$(5.9a)\quad |V_2(x) - z_{V_2}(x)| \le M\,N^{-2}\ln^2 N\,\exp(-m\,\varepsilon^{-1}(d-x)), \quad x \in \overline{D}^{\sigma}_{2\,h},$$

$$(5.9b)\quad \left|\varepsilon\left(\frac{d}{dx}V_2(x^{(1)}(x)) - \delta_x\,z_{V_2}(x)\right)\right| \le M\,N^{-2}\ln^2 N\,\exp(-m\,\varepsilon^{-1}(d-x)),$$
$$x \in \overline{D}^{\sigma}_{2\,h} \setminus \{x = d\},$$

$$(5.9c)\quad \left|\varepsilon^2\left(\frac{d^2}{dx^2}V_2(x) - \delta_{\overline{x}\widehat{x}}z_{V_2}(x)\right)\right| \le M\,N^{-2}\ln^2 N\,\exp(-m\,\varepsilon^{-1}(d-x)),$$
$$x \in D^{\sigma}_{2\,h};$$
$$x^{(1)}(x) = x^{(1)}_{(5.2)}(x), \quad m = m_{(4.8)}.$$

The following theorem is valid.

Theorem 5.3. *Let the solution of the boundary value problem* (2.2), (2.1) *satisfy the estimate* (6.9) *for* $K = 4$ *from Theorem* 6.1. *Then, in the case of the difference scheme* $\{\{(3.2),\ (3.3)\};\ \{(4.9)–(4.10),\ (3.3);\ (4.13),\ (4.12)\}\}$ *of the solution decomposition method, the normalized discrete derivatives* $q^k_h(x)$, $k = 1, 2$, *corresponding to the functions* $z_{(5.5)}(x)$, $z_{U\,(5.5)}(x)$ *and* $z_{V_j\,(5.5)}(x)$, *converge* ε*-uniformly to normalized derivatives corresponding to the functions* $u_{(2.2),(2.1)}(x)$, $U_{(4.1)}(x)$ *and* $V_{j\,(4.1)}(x)$. *The discrete solutions* $z_{(5.5)}(x)$, *the components* $z_{U\,(5.5)}(x)$ *and* $z_{V_j\,(5.5)}(x)$, *and also their discrete derivatives satisfy the estimates* (5.6), (5.7), (5.8), (5.9).

Remark 5.1. The estimates (5.6) for the discrete solution and its normalized derivatives under the condition (4.6) in the case of the scheme of the solution decomposition method {{(3.2), (3.3)}; {(4.9)–(4.10), (3.3); (4.13), (4.12)}} are the same as the estimates (3.7), (3.8) for the scheme (3.2) on the piecewise-uniform grid (3.6).

But *under the condition* (4.7), i.e., for $\varepsilon < \varepsilon_0(N) = m\, l^{-1}\, d\, \ln^{-1} N$, *the estimates* (5.7) and (5.8), (5.9) for the discrete regular and singular components $z_U(x)$ and $z_{V_j}(x)$ and their normalized derivatives in the case of the scheme of the solution decomposition method even *essentially better than the estimates* (3.7), (3.8) for the discrete solution and better than *the estimates* (5.3), (5.4) for the normalized derivatives in the case of the scheme (3.2) on the piecewise-uniform grid (3.6). □

Remark 5.2. Note that the derivatives up to the second order of the regular component to the solution of the boundary value problem are ε-uniformly bounded (see the estimate (6.9) for $K = 4$ from Theorem 6.1 in the assumption of Theorem 5.3). In the case of the scheme of the solution decomposition method, the discrete regular component $z_U(x)$ and its discrete derivatives $\delta_x z_U(x)$, $\delta_{\overline{x}\widehat{x}} z_U(x)$ are also ε-uniformly bounded. Consideration of examples shows that *the second-order discrete derivative $\delta_{\overline{x}\widehat{x}} z_U(x)$ does not converge ε-uniformly*, and *the first-order discrete derivative $\delta_x z_U(x)$ converges ε-uniformly with order* of accuracy *not higher than one*. Only the regular component $z_U(x)$ itself *converges ε-uniformly* with the *second order* of accuracy. □

Remark 5.3. The estimates (5.6), (5.7), (5.8), (5.9) from Theorem 5.3 are preserved also in the case when the discrete solutions $z(x)$, $z_U(x)$, $z_{V_j}(x)$ and their discrete derivatives $\delta_x\, z(x)$, $\delta_{\overline{x}\,\widehat{x}}\, z(x)$, …, $\delta_x\, z_{V_j}(x)$, $\delta_{\overline{x}\,\widehat{x}}\, z_{V_j}(x)$ are replaced by the linear interpolants $\overline{z}(x)$, $\overline{z}_U(x)$, $\overline{z}_{V_j}(x)$ and $\overline{\delta_x\, z}(x)$, $\overline{\delta_{\overline{x}\,\widehat{x}}\, z}(x)$, ..., $\overline{\delta_x\, z_{V_j}}(x)$, $\overline{\delta_{\overline{x}\,\widehat{x}}\, z_{V_j}}(x)$, respectively, as it is in the case of the estimates (4.15), (4.16). □

6. *A priori* estimates for solutions and derivatives

We give estimates for solutions and derivatives of the boundary value problem (2.2), (2.1) used in constructions; the derivation of the estimates is similar to that in [2, 6].

6.1. Applying the majorant function technique (see, for example, [23]), we find the estimate

$$|u(x)| \le M, \quad x \in \overline{D}. \tag{6.1}$$

For sufficiently smooth data of the problem ensuring the inclusion $u \in C^K(\overline{D})$, $K > 0$, the following estimate is satisfied (see, for example, [2], Ch. I, § 3; [6], Ch. 2):

$$\left| \frac{d^k}{dx^k}\, u(x) \right| \le M\, \varepsilon^{-k}, \quad k \le K, \quad x \in \overline{D}. \tag{6.2}$$

However, such estimation of the derivatives of the problem solution does not allow us to establish the ε-uniform convergence of the constructed schemes.

6.2. Derive estimates for derivatives of the solution based on the decomposition of the solution. For this we write the solution of the problem as the sum of functions

$$u(x) = U(x) + V(x), \quad x \in \overline{D}, \tag{6.3a}$$

where $U(x)$ and $V(x)$ are the regular and singular components of the solution. The function $U(x)$, $x \in \overline{D}$, is the restriction to $\overline{D}$ of the function $U^e(x)$, $x \in D^e$,

$U(x) = U^e(x)$, $x \in \overline{D}$. The function $U^e(x)$, $x \in D^e$ is a bounded solution of the problem on the straight line D^e

$$L^e U^e(x) = f^e(x), \quad x \in D^e. \tag{6.4}$$

The straight line $D^e = (-\infty, \infty)$ is an extension of the domain $\overline{D}$ beyond the sides Γ_1 and Γ_2; the data of the problem (6.4) are smooth extensions of the data to the problem (2.2), (2.1) preserving on D^e properties (2.3). We assume that the function $f^e(x)$, $x \in D^e$, equals zero outside an m_1-neighbourhood of the set $\overline{D}$, moreover, the operator L^e is defined by the relation $L^e \equiv \varepsilon^2 a^0 \dfrac{d^2}{dx^2} - c_0$, where a^0, c_0 are constants from the condition (2.3). The function $V(x)$, $x \in \overline{D}$, is a solution of the problem

$$L_{(2.2)} V(x) = 0, \quad x \in D, \qquad V(x) = \varphi(x) - U(x), \quad x \in \Gamma. \tag{6.5}$$

We represent the function $U(x)$ in the form of the expansion

$$U(x) = U_0(x) + \varepsilon^2 U_1(x) + \cdots + \varepsilon^{2n} U_n(x) + v_U(x), \quad x \in \overline{D}, \tag{6.3b}$$

that corresponds to the restriction on $\overline{D}$ of the following representation of the function $U^e(x)$:

$$U^e(x) = U_0^e(x) + \varepsilon^2 U_1^e(x) + \cdots + \varepsilon^{2n} U_n^e(x) + v_U^e(x), \quad x \in D^e, \tag{6.6a}$$

i.e, of the expansion of the solution for the problem (6.4). Here $U_i^e(x)$, $i = 0, 1, \dots, n$ are current terms of the expansion and $v_U^e(x)$ is the remainder term;

$$U(x) = U^e(x), \dots, \; U_i(x) = U_i^e(x), \dots, \; v_U(x) = v_U^e(x), \quad x \in \overline{D}.$$

The functions $U_0^e(x)$, $U_i^e(x)$, $i = 1, \dots, n$, and $v_U^e(x)$ in (6.6a) are solutions of the problems

$$L^e_{(6.6)} U_0^e(x) = f^e(x), \quad x \in D^e; \tag{6.6b}$$

$$L^e_{(6.6)} U_i^e(x) = -a^e(x) \frac{d^2}{dx^2} U_{i-1}^e(x), \quad x \in D^e, \quad i = 1, \dots, n; \tag{6.6c}$$

$$L^e_{(2.2)} v_U^e(x) = -\varepsilon^{2n+2} a^e(x) \frac{d^2}{dx^2} U_n^e(x), \quad x \in D^e. \tag{6.6d}$$

Here, $L^e_{(6.6)}$ is the operator $L^e_{(6.4)}$ for $\varepsilon = 0$, i.e., $L^e_{(6.6)} \equiv -c^e(x)$, $x \in D^e$, the function $a^e(x)$ is the coefficient at the derivative d^2/dx^2 in the operator $L^e_{(6.4)}$; $a^e(x)$ is the extension of $a(x)$ on D^e with preservation of properties (2.3).

6.3. Let us estimate the components $U(x)$ and $V(x)$ in the representation (6.3). We assume that the data of the problem (2.2), (2.1) are sufficiently smooth that guarantees the smoothness of the solution $u(x)$ of the problem (2.2), (2.1) and of its components from decomposition (6.3). The following condition is assumed to be fulfilled:

$$a,\, c,\, f \in C^{l_1}(\overline{D}), \quad l_1 = l + \alpha, \quad l = l_0 + 2n, \tag{6.7}$$

where $n = n_{(6.6)} \geq 0$, $l_0 > 0$ is an even number, and $\alpha \in (0, 1)$. By virtue of this condition, we have the inclusions

$$u,\, U \in C^{l^1}(\overline{D}), \quad U_i \in C^{l^2}(\overline{D}), \quad i = 0, 1, \dots, n, \tag{6.8}$$

where $l^1 = l_0 + \alpha$, $l^2 = l - 2i$, $l = l_0 + 2n$, $n \geq 0$, $\alpha > 0$. In this case, $V \in C^{l^1}(\overline{D})$.

For $U(x)$ and $V(x)$, we obtain the estimates

$$(6.9a)\quad \left|\frac{d^k}{dx^k}\,U(x)\right| \le M\,[1+\varepsilon^{2n+2-k}],$$

$$(6.9b)\quad \left|\frac{d^k}{dx^k}\,V(x)\right| \le M\,\varepsilon^{-k}\,[\exp(-m\,\varepsilon^{-1}\,x)+\exp(-m\,\varepsilon^{-1}\,(d-x))],\quad x\in\overline{D},$$

where $k \le K$, m is an arbitrary number from $(0, m_0)$, $m_0 = \min_{\overline{D}}^{1/2}[a^{-1}(x)\,c(x)]$; $K = l_0$.

The function $u(x)$ satisfies the etimate (6.2), where $K = l_0$.

Theorem 6.1. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the condition* (6.7), *i.e.,* $a,\ c,\ f \in C^{l_1}(\overline{D})$, $l_1 = l+\alpha$, $l = l_0+2n$, $l_0 = l_{0\,(6.7)} > 0$, $n = n_{(6.6)} \ge 0$. *Then the solution of the boundary value problem and its components in the representation* (6.3) *satisfy estimates* (6.1), (6.2), (6.9), *where* $K = l_0$.

7. Generalizations

As it follows from Remark 5.2, in the case of the scheme of the solution decomposition method {{(3.2), (3.3)}; {(4.9)–(4.10), (3.3); (4.13), (4.12)}} *the second-order discrete derivative* $\delta_{\overline{x}\widehat{x}} z_U(x)$ of the regular component $z_U(x)$ *does not converge* ε*-uniformly*, and *the first-order discrete derivative* $\delta_x z_U(x)$ *converges* ε*-uniformly with order* of accuracy *not higher than one*, however, the component $z_U(x)$ itself converges ε-uniformly with the second order of accuracy. We consider a *modification* of the scheme to the solution decomposition method {{(3.2), (3.3)}; {(4.9)–(4.10), (3.3); (4.13), (4.12)}}, for which the regular component $z_U(x)$ and its discrete derivatives $\delta_x z_U(x)$, $\delta_{\overline{x}\widehat{x}} z_U(x)$ converge ε-uniformly with the *second order of accuracy.*

7.1. When constructing schemes with improved accuracy of approximation for derivatives of the regular component, we need more strong estimates on derivatives of the solution to the boundary value problem and its component as compared with the estimates (4.4), (4.5) of Theorem 4.1. Let us give such estimates.

We represent the solution of the boundary value problem as the sum of the regular and singular components to the solution (4.1a); we write the singular component as the sum of the form (4.1c), and the regular component in the form of the following sum with the larger number of terms than in (4.1b):

$$(7.1)\qquad U(x) = U_0(x) + \varepsilon^2\,U_1(x) + v_U(x),\quad x\in\overline{D}.$$

The functions $U_0(x)$, $U_1(x)$ and $v_U^0(x)$ in (7.1) are solutions of the problems

$$(7.2a)\quad L_{(4.2)}\,U_0(x) \;=\; f(x),\qquad x\in\overline{D};$$

$$(7.2b)\quad L_{(4.2)}\,U_1(x) \;=\; -a(x)\,\frac{d^2}{dx^2}\,U_0(x),\qquad x\in\overline{D};$$

$$(7.2c)\quad L_{(2.2)}\,v_U(x) \;=\; -\varepsilon^4 a(x)\,\frac{d^2}{dx^2}\,U_1(x),\quad x\in D,\quad v_U(x)=0,\quad x\in\Gamma.$$

The functions $V(x)$ and $V_j(x)$, $x \in \overline{D}$ are solutions of the problems (4.3a) and (4.3b), where

$$\varphi_V(x) = \varphi(x) - U(x),\quad x\in\Gamma,\qquad U(x) = U_{(7.1)}(x),\quad x\in\overline{D}.$$

We assume that the data of the boundary value problem (2.2), (2.1) satisfy the hypothesis of Theorem 6.1 for $l_0 = 6$ and $n = 1$. Then, the components in the representations (4.1a,c), (7.1) satisfy the estimates similar to (4.4), (4.5)

$$\left.\begin{aligned}&\left|\frac{d^k}{dx^k}\,U(x)\right| \le M\,[1+\varepsilon^{4-k}]\\&\left|\frac{d^k}{dx^k}\,V_1(x)\right| \le M\,\varepsilon^{-k}\,\exp(-m\,\varepsilon^{-1}\,x)\\&\left|\frac{d^k}{dx^k}\,V_2(x)\right| \le M\,\varepsilon^{-k}\,\exp(-m\,\varepsilon^{-1}\,(d-x))\end{aligned}\right\},\quad x\in\overline{D},\quad k\le K, \tag{7.3}$$

and the components in (7.1) satisfy the estimates

$$\left.\begin{aligned}&\left|\frac{d^k}{dx^k}\,U_0(x)\right| \le M, && k\le K+2\\&\left|\frac{d^k}{dx^k}\,U_1(x)\right| \le M, && k\le K\\&\left|\frac{d^k}{dx^k}\,v_U(x)\right| \le M\,\varepsilon^{4-k} && k\le K,\end{aligned}\right\},\quad x\in\overline{D}, \tag{7.4}$$

where $m = m_{(4.4)}$ and $K = 6$.

Theorem 7.1. *Let the data of the boundary value problem* (2.2), (2.1) *satisfy the hypothesis of Theorem* 6.1 *for* $l_0 = 6$ *and* $n = 1$. *Then, the components in the representations* (4.1a,c) *and* (7.1) *satisfy the estimates* (7.3) *and* (7.4), *respectively, where* $K = 6$.

7.2. When constructing schemes with improved accuracy of the regular component, we approximate the problem (2.2), (2.1) by the standard difference scheme (3.2), (3.3) in the case of the condition (4.6), and in the case of the condition (4.7) we apply grid constructs similar to {{(4.9)–(4.10), (3.3); (4.13), (4.12); (4.7) }}. Here we consider the representation (7.1) instead of (4.1b) for the regular component $U(x)$.

Let the condition (4.7) be fulfilled. When approximating the problems (7.2a) and (7.2b), we need an extension of the problem (7.2a) to the set $\overline{G}^{\,e}$

$$\overline{G}^{\,e} = \overline{D}^{\,e} \times [0,T], \quad \overline{D}^{\,e} = [-h, d+h], \tag{7.5}$$

where h is the step-size in the grid (3.3); D^e is an h-neighbourhood of the set $\overline{D}$. We extend the data of the problem (2.2), (2.1) to the set $\overline{D}^{\,e}$ preserving their smoothness and the condition (2.3); for the extended functions $a(x), \ldots, f(x)$, $x \in \overline{D}$, we use the notations $a^e(x)$, …, $f^e(x)$, $x \in \overline{D}^{\,e}$.

We approximate the problems (7.2a), (2.1) and (7.2b), (7.2c), (2.1) by the difference schemes on the grids $\overline{D}^e_h$ and $\overline{D}_h$, respectively,

$$\Lambda^e_{(7.6)} z^e_{U_0}(x) = f^e(x), \qquad x \in \overline{D}^e_h; \tag{7.6}$$

$$\Lambda_{(4.9)} z_{U_1}(x) = -a(x)\,\delta_{x\,\overline{x}}\, z^e_{U_0}(x), \qquad x \in \overline{D}_h; \tag{7.7}$$

$$\Lambda_{(3.2)} z_{v_U}(x) = -\varepsilon^4\, a(x)\,\delta_{x\,\overline{x}}\, z_{U_1}(x), \quad x \in D_h, \quad z_{v_U}(x) = 0, \quad x \in \Gamma_h. \tag{7.8}$$

In the case of the problem on the grid $\overline{D}^e_h$, the operator $\Lambda^e_{(7.6)}$ is defined by the relation

$$\Lambda^e_{(7.6)} \equiv -c^e(x), \quad x \in \overline{D}^e_h.$$

Set

$$z_U(x) = z^e_{U_0}(x) + \varepsilon^2 z_{U_1}(x) + z_{v_U}(x), \quad x \in \overline{D}_h. \tag{7.9}$$

The problem (4.3b), (2.1) is approximated by the difference scheme

$$\Lambda_{(3.2)} z_{V_j}(x) = 0, \qquad x \in D^{\sigma}_{j\,h}, \tag{7.10}$$

$$z_{V_j}(x) = \left\{ \begin{array}{ll} \varphi(x) - z_U(x), & x \in \Gamma^{\sigma}_{j\,h} \bigcap \Gamma \\ 0, & x \in \Gamma^{\sigma}_{j\,h} \setminus \Gamma \end{array} \right\}, \quad x \in \Gamma^{\sigma}_{j\,h}, \quad j = 1, 2,$$

where $\overline{D}^{\sigma}_{j\,h} = \overline{D}^{\sigma}_{j\,h\,(4.12)}$; $z_U(x) = z_{U\,(7.9)}(x)$, $x \in \Gamma^{\sigma}_{j\,h} \bigcap \Gamma$.

According to the relation (4.14a), we define the function $\overline{z}_u(x)$:

$$\overline{z}_u(x) = \overline{z}_U(x) + \overline{z}_V(x), \quad x \in \overline{D}. \tag{7.11}$$

The function $\overline{z}_u(x)$ is called the solution of the *difference scheme* {{(7.6)–(7.8), (7.5)}; {(7.10), (4.12); (4.7)}} of the *solution decomposition method with the improved regular component* (or briefly, *the improved scheme of the solution decomposition method*) under the condition (4.7).

7.3. For sufficiently smooth data of the problem (2.2), (2.1), we obtain the following estimates for the component $z_{U\,(7.9)}(x)$ and its difference derivatives:

$$|U(x) - z_U(x)| \le M\,N^{-2}, \quad x \in \overline{D}_h, \tag{7.12a}$$

$$\left| \frac{d}{dx} U(x^{(1)}(x)) - \delta_x\, z_U(x) \right| \le M\,N^{-2}, \quad x \in \overline{D}_h \setminus \{x = d\}, \tag{7.12b}$$

$$\left| \frac{d^2}{dx^2} U(x) - \delta_{\overline{x}\widehat{x}} z_U(x) \right| \le M\,N^{-2}, \quad x \in D_h. \tag{7.12c}$$

Thus, *the regular component* $z_U(x)$ and its *discrete derivatives converge* ε-uniformly with the *second order of accuracy.*

The functions $z_{(5.5)}(x)$, $z_{V_j\,(7.10)}(x)$ and their normalized derivatives satisfy the estimates (5.6), (5.8), (5.9).

The following theorem is valid.

Theorem 7.2. *Let the components in the representations* (4.1*a, c*), (7.1) *of the solution to the boundary value problem* (2.2), (2.1) *satisfy the estimates* (7.3), (7.4) *for* $K = 6$ *from Theorem* 7.1. *Then, in the case of the difference scheme* {{(3.2), (3.3)}; {(7.6)–(7.8), (7.5); (7.10), (4.12)}} *of the solution decomposition method, the normalized discrete derivatives* $q^k_h(x)$, $k = 1, 2$, *corresponding to the functions* $z_{(5.5)}(x)$ *and* $z_{V_j\,(7.10)}(x)$, *converge* ε*-uniformly to normalized derivatives corresponding to the functions* $u_{(2.2),(2.1)}(x)$ *and* $V_{j\,(4.1)}(x)$*; the discrete derivatives of the function* $z_{U\,(7.9)}(x)$ *converge* ε*-uniformly to the derivatives of the function* $U_{(7.1)}(x)$*. The discrete solutions* $z_{(5.5)}(x)$*, the components* $z_{U\,(7.9)}(x)$ *and* $z_{V_j\,(7.10)}(x)$, $z_{V_j\,(7.10)}(x)$, *and also their discrete derivatives satisfy the estimates* (5.6), (7.12) *and* (5.8), (5.9), *respectively.*

Remark 7.1. The estimates (5.6), (7.12), (5.8), (5.9) from Theorem 7.2 are preserved also in that case when the discrete solutions $z(x)$, $z_U(x)$, $z_{V_j}(x)$ and their discrete derivatives $\delta_x\, z(x)$, $\delta_{\overline{x}\,\widehat{x}}\, z(x)$, …, $\delta_x\, z_{V_j}(x)$, $\delta_{\overline{x}\,\widehat{x}}\, z_{V_j}(x)$ are replaced by the linear interpolants $\overline{z}(x)$, $\overline{z}_U(x)$, $\overline{z}_{V_j}(x)$ and $\overline{\delta_x\, z}(x)$, $\overline{\delta_{\overline{x}\,\widehat{x}}\, z}(x)$, ..., $\overline{\delta_x\, z}_{V_j}(x)$, $\overline{\delta_{\overline{x}\,\widehat{x}}\, z}_{V_j}(x)$, respectively. □

8. Conclusions

8.1. For the Dirichlet problem for a singularly perturbed ordinary differential reaction-diffusion equation a new approach has been developed for constructing ε-uniformly convergent difference schemes, i.e., the *solution decomposition method*, based on the *asymptotic construction technique.* Unlike ε-uniformly convergent difference schemes of the *fitted operator method* and the *method of grids condensing in the boundary layer*, here, the discrete subproblems are solved on *uniform grids*; moreover, the *coefficients* in the grid equations *do not depend on the explicit form of the singular component* of the solution.

8.2. For the boundary value problem, a *difference scheme* of the *solution decomposition method* is constructed that converges ε-uniformly in the maximum norm at the rate $\mathcal{O}\left(N^{-2}\ln^2 N\right)$. This scheme allows us to approximate the *regular and singular components* of the solution and *their derivatives* up to the second order. A *modified scheme* of the solution decomposition method is constructed for which the *regular component of the solution and its discrete derivatives* converge ε-uniformly *at the rate* $\mathcal{O}\left(N^{-2}\right)$ *under the condition* $\varepsilon = o(\ln^{-1} N)$.

8.3. For the scheme of the solution decomposition method under the condition $\varepsilon = o(\ln^{-1} N)$, the convergence rate of the *regular and singular components* of the discrete solution and *their grid derivatives* up to the second order *are better* than the convergence rate of the discrete solution and its grid derivatives for the *scheme on piecewise-uniform grids.*

8.4. The technique to construct and study schemes of the *solution decomposition method approximating the solution* of the problem, its regular and singular components and also *their grid derivatives* that is considered in this paper for a singularly perturbed ordinary differential equation, can be applied for singularly perturbed *parabolic and elliptic* equations.

Acknowledgments

This research was supported by the Russian Foundation for Basic Research (project N 10-01-00726).

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