A NUMERICAL STUDY FOR A VELOCITY-VORTICITY-HELICITY FORMULATION OF THE 3D TIME-DEPENDENT NSE

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Abstract. We study a finite element method for the 3D Navier-Stokes equations in velocity-vorticity-helicity formulation, which solves directly for velocity, vorticity, Bernoulli pressure and helical density. Moreover, the algorithm strongly enforces solenoidal constraints on both the velocity (to enforce the physical law for conservation of mass) and vorticity (to enforce the mathematical law that \( \text{div} (\text{curl} v) = 0 \)). We prove unconditional stability of the velocity, and with the use of a (consistent) penalty term on the difference between the computed vorticity and curl of the computed velocity, we are also able to prove unconditional stability of the vorticity in a weaker norm. Numerical experiments are given that confirm expected convergence rates, and test the method on a benchmark problem.

Key words. Navier-Stokes equations, Finite element method, Velocity-Vorticity-Helicity formulation.

1. Introduction

Approximating solutions to the 3D Navier-Stokes equations (NSE) is an important subtask in many engineering applications, and improvements in methods to do so remains an active and important area of research. To this end, we study an accurate and efficient numerical method for approximating solutions to the incompressible NSE that is based on a natural 2-step linearization of a finite element discretization of the velocity-vorticity-helicity (VVH) formulation, together with an element choice and mesh condition that leads to efficient and optimally accurate solves of the resulting saddle point systems as well as strong enforcement of the solenoidal constraints on the velocity and vorticity. Recent work by the authors has shown in [16] that the incompressible Navier-Stokes system in a bounded, connected domain \( \Omega \subset \mathbb{R}^3 \), with a piecewise smooth boundary \( \partial \Omega \), end time \( T \), and force field \( f : Q \to \mathbb{R}^3 \) (where \( Q := (0, T) \times \Omega \)), can be equivalently written in VVH form as:

Find \( u : Q \to \mathbb{R}^3, p : Q \to \mathbb{R} \) satisfying

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + w \times u + \nabla p &= f \quad \text{in} \quad Q, \\
\nabla \cdot u &= 0 \quad \text{in} \quad Q, \\
u_0 &= u_0 \quad \text{in} \quad \Omega, \\
u &= \phi \quad \text{on} \quad \partial \Omega \times (0, T),
\end{align*}
\]

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and find \( \mathbf{w} : Q \rightarrow \mathbb{R}^3, \eta : Q \rightarrow \mathbb{R} \) satisfying

\[
\begin{align*}
\frac{\partial \mathbf{w}}{\partial t} - \nu \Delta \mathbf{w} + 2 \mathbf{D}(\mathbf{w}) \mathbf{u} - \nabla \eta &= \text{curl } \mathbf{f} \quad \text{in } Q, \\
\nabla \cdot \mathbf{w} &= 0 \quad \text{in } \overline{Q}, \\
\mathbf{w}|_{t=0} &= \text{curl } \mathbf{u}_0 \quad \text{in } \Omega, \\
\mathbf{w} &= \text{curl } \mathbf{u} \quad \text{on } (0,T) \times \partial \Omega,
\end{align*}
\]

where \( \mathbf{u} \) represents the velocity, \( P \) is the Bernoulli pressure, \( \mathbf{f} \) an external force, \( \nu > 0 \) the kinematic viscosity coefficient, \( \phi \) is a Dirichlet boundary condition for velocity satisfying \( \int_{\Omega} \phi \cdot \mathbf{n} = 0 \ \forall t \in (0,T) \), \( \mathbf{w} := \text{curl } \mathbf{u} \) is the vorticity, \( \eta := \mathbf{u} \cdot \mathbf{w} \) is the helical density, and \( \mathbf{D}(\mathbf{w}) := \frac{1}{2} (\nabla \mathbf{w} + (\nabla \mathbf{w})^T) \) the symmetric part of the vorticity gradient. Since its initial development, VVH has been studied in several applications, including numerical methods for solving steady incompressible flow [11], the Boussinesq equations [15], and as a selection criterion for the filtering radius in the NS-\( \omega \) turbulence model [12], all with excellent results. The work herein will extend numerical study of VVH to the time-dependent case.

The VVH formulation has four important characteristics that make it attractive for use in simulations. First, numerical methods based on finding velocity and vorticity tend to be more accurate (usually for an added cost, but not necessarily with VVH) [20, 21, 18, 19, 13], and especially in the boundary layer [6]. Second, it solves directly for the helical density \( \eta \), which may give insight into the mysterious quantity helicity, \( H = \int_{\Omega} \eta \; dx \), which is not well understood but is believed to play a fundamental role in turbulence [1, 14, 9, 2, 3, 4, 8, 7]. VVH is the first formulation to directly solve for this helical quantity. Third, the use of \( \nabla \eta \) in the vorticity equation enables \( \eta \) to act as a Lagrange multiplier corresponding to the divergence free constraint for the vorticity, analogous to how the pressure relates to the conservation of mass equation. VVH is the first velocity-vorticity method to naturally enforce the incompressibility of the vorticity. Finally, the structure of the VVH system allows for a natural splitting of the system into a 2-step linearization, since freezing vorticity in the velocity equation linearizes the equation, and similarly freezing velocity in the vorticity equation linearizes this equation as well. A numerical method based on such a splitting was proposed in [16], and when coupled with a finite element discretization, was shown to be accurate on some simple test problems. In the present study, we will study further this discretization of VVH (defined precisely in Section 3), by providing a rigorous stability analysis (including for the vorticity), testing it with different boundary conditions for the vorticity, and testing the method on benchmark problems.

This paper is arranged as follows. In Section 2, we provide the necessary notation and preliminaries to allow for a smooth analysis to follow. In Section 3, we present the 2-step method, and provide a stability analysis for it. Lastly, in Section 4, we test the proposed method on benchmark problems.

2. Preliminaries

We present now preliminary results and notation, for the function spaces to be used, and describe the discrete setting.

2.1. Function spaces. We will use bold font to denote vector function spaces,

\[
H^1(\Omega) := (H^1(\Omega))^3, \quad H_0^1(\Omega) := (H_0^1(\Omega))^3,
\]
and other vector function spaces will be denoted analogously. The divergence-free subspace of $H^1$ will be denoted by $V$:

$$V := \{ v \in H^1(\Omega) : \nabla \cdot v = 0 \},$$

and its dual space by $V^*$.

We will denote the $L^2(\Omega)$ norm and inner product by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The $H^k(\Omega)$ norms will be denoted by $\| \cdot \|_k$. All other norms will be clearly labeled with subscripts. The zero-mean subspace of $L^2(\Omega)$ will be denoted by

$$L^2_0(\Omega) = \{ q \in L^2(\Omega) : \int_{\Omega} q \, d\mathbf{x} = 0 \}.$$

We will also employ the Poincare inequality, $\forall \phi \in H^1(\Omega)$ satisfying $\phi = 0$ on a set of positive measure on $\partial \Omega$:

$$\| \phi \| \leq C_{PF} \| \nabla \phi \|. \quad (2.1)$$

2.2. The discrete setting. We assume a mesh $\tau_h$ to be a regular, conforming tetrahedralization of $\Omega$, and we will choose the finite element spaces $(X_h, Q_h) \subset (H^1(\Omega), L^2(\Omega))$ on $\tau_h$ to be the $((P_k)^3, P_{k-1}^{\text{disc}})$ Scott-Vogelius (SV) element pair. This pair is a natural choice for both the velocity-pressure and vorticity-helicity systems, as this will enforce pointwise the solenoidal constraints of velocity and vorticity, respectively. For this element pair to be LBB stable, we assume any of the following sufficient conditions hold: i) $k \geq 6$ on a quasi-uniform tetrahedral mesh [23], ii) $k \geq 3$ and the mesh is generated as a barycenter refinement of a regular conforming tetrahedral mesh [22], or iii) $k \geq 2$ and the mesh is of Powell-Sabin type [24]. A complete classification of conditions for LBB stability for SV elements, including the minimum degree for general meshes without special refinements, is an open question.

Let $V_h$ be the discretely divergence-free subspace of $X_h$. It is easy to see that for SV element pair, this space is also the pointwise divergence-free subspace of $X_h$,

$$V_h = \{ v_h \in X_h : (\nabla \cdot v_h, q_h) = 0 \ \forall q_h \in Q_h \} = \{ v_h \in X_h : \nabla \cdot v_h = 0 \}.$$ 

Define $X_{h0} := X_h \cap H^1_0(\Omega)$ and $V_{h0} := V_h \cap H^1_0(\Omega)$.

To simplify the analysis, we require the mesh is sufficiently regular so that the inverse inequality holds,

$$\| \nabla u_h \| \leq C_i h^{-1} \| u_h \|. \quad (2.2)$$

Define the operator $A_h^{-1} : L^2(\Omega) \to V_{h0}$ as the solution operator to the discrete Stokes problem:

$$\langle \nabla A_h^{-1} \psi, \nabla v_h \rangle = \langle \psi, v_h \rangle, \ \forall v_h \in V_{h0}. \quad (2.3)$$

This operator will not be used in computations, but is used in the analysis of the proposed algorithm. The following lemma was proven in [11].

**Lemma 2.1.** Assume $\Omega$ is such that the Stokes problem is $H^2$-regular. For any $\psi \in L^2(\Omega)$ it holds

$$\| A_h^{-1} \psi \|_{L^\infty} + \| \nabla A_h^{-1} \psi \|_{L^2} \leq C_0 \| \psi \|, \quad (2.4)$$

and for any $f \in L^2(\Omega)$, $q \in L^2(\Omega)$, and $\phi \in H^1(\Omega)$

$$\| (f, \nabla \times A_h^{-1} \psi) \| \leq C(\| f \|_{-1} + h \| f \|) \| \psi \|, \quad (2.5)$$

$$\| (q, \nabla \cdot A_h^{-1} \psi) \| \leq C(\| q \|_{-1} + h \| q \|) \| \psi \|, \quad (2.6)$$

$$\| (\nabla \phi, \nabla A_h^{-1} \psi) \| \leq C(\| \phi \| + \| \phi \|_{-1, \partial \Omega} + h \| \nabla \phi \|) \| \psi \|. \quad (2.7)$$
3. Discrete VVH Formulation

We have now given the necessary preliminaries to define the finite element scheme for VVH we propose to study. The chosen time discretization is trapezoidal, and the linearization uses second order extrapolation. Denoting \( \phi^{n+1/2} := \frac{1}{2}(\phi^n + \phi^{n+1}) \), the fully discrete 2-step version of \((1.1)-(1.8)\) we study is: find \((u_h, w_h, P_h, \eta_h) \in (X_{h0}, X_h, Q_h, Q_h)\) satisfying \(\forall (v_h, \chi_h, q_h, r_h) \in (X_{h0}, X_h, Q_h, Q_h)\),

Step 1:

\[
\frac{1}{\Delta t} (u_h^{n+1} - u_h^n, v_h) + \nu(\nabla u_h^{n+1/2}, \nabla v_h) - (P_h^{n+1}, \nabla \cdot v_h) \\
+ (\frac{3}{2} w_h^n - \frac{1}{2} w_h^{n-1}) \times u_h^{n+1/2}, v_h) - (f^{n+1/2}, v_h) = 0,
\]

(3.1)

\[
(\nabla \cdot u_h^{n+1}, q_h) = 0,
\]

(3.2)

Step 2:

\[
\frac{1}{\Delta t} (w_h^{n+1} - w_h^n, \chi_h) + \nu(\nabla w_h^{n+1/2}, \nabla \chi_h) \\
+ (\eta_h^{n+1}, \nabla \cdot \chi_h) + \gamma \nu^{-1}((\text{curl} A_h^{-1} w_h^{n+1/2}) \times u_h^{n+1/2}, (\text{curl} \chi_h) \times u_h^{n+1/2}) \\
+ 2(D(w_h^{n+1/2})u_h^{n+1/2}, \chi_h) - (\text{curl} f_h^{n+1/2}, \chi_h) = 0,
\]

(3.3)

\[
(\nabla \cdot w_h^{n+1}, r_h) = 0,
\]

(3.4)

\[
w_h^{n+1}|_{\partial \Omega} - I_h(\nabla \times u_h^{n+1})|_{\partial \Omega} = 0,
\]

(3.5)

where \( I_h \) is an appropriate interpolant. As is common practice in trapezoidal schemes for fluid flow, the Lagrange multiplier terms are solved for directly at their \( n + 1/2 \) time levels, i.e. no splitting into time \( n \) and \( n + 1 \) pieces is necessary, and so \( P_h^{n+1} \) and \( \eta_h^{n+1} \) are really approximations to their continuous counterparts at \( t = t^{n+1/2} \). Note also that we have assumed a homogeneous Dirichlet boundary condition for velocity, and a Dirichlet condition for vorticity that it be equal to an appropriate interpolant of the curl of the velocity on the boundary. This is the simplest case for analysis, which is quite formidable, even for this case. Extension to other common boundary conditions will lead to additional technical details, and need considered on case by case basis. The authors plan to address some of these issues in later works.

Due to the difficulties associated with any analysis involving the vorticity equation, there are two components in the above scheme that are for the purposes of analysis only. The unconditional stability of the velocity does not depend on either of these components of the numerical scheme, but proving unconditional stability of the vorticity requires both of them.

First, the boundary condition for the discrete vorticity (3.5) is given in terms of the true velocity, which is not practical. In computations, we use instead the condition

\[
w_h^{n+1}|_{\partial \Omega} - I_h(\nabla \times u_h^{n+1})|_{\partial \Omega} = 0,
\]

(3.6)

however analyzing the system with such a boundary condition does not appear possible in this particular formulation. Developing improved formulations for which such a vorticity boundary condition does allow analysis is an important open question. We will consider two possibilities of interpolants in our computations: i) a nodal interpolant of the \( L^2 \) projection of the curl of the velocity into \( \mathbf{V}_h \), and ii) a nodal interpolant of a local averaging of the curl of the velocity.
The second part of the scheme that is not used in computations is the penalty term in (3.3), i.e. we choose $\gamma = 0$ in our computations. In the continuous case this term is consistent for the homogeneous or periodic boundary conditions on a rectangular box: for sufficiently regular solutions, $w = \nabla \times u$, and $A^{-1}$ the continuous Stokes solution operator, since $\nabla \cdot u = 0$,

$$
(\text{curl } A^{-1}w) \times u = (\text{curl } A^{-1}(\nabla \times u)) \times u \\
= (A^{-1}(\nabla \times (\nabla \times u))) \times u \\
= (A^{-1}(-\Delta u - \nabla(\nabla \cdot u))) \times u \\
= (A^{-1}(-\Delta u)) \times u \\
= (A^{-1}(A\Delta)) \times u \\
= 0.
$$

Outside the periodic case, the differential operators will not commute and thus errors will arise at the boundary from this term; hence the term appears to damp vorticity creation at the boundary, and we do not use it in our computations. However, it does not appear possible to prove a vorticity stability bound without it.

**Lemma 3.1** (Stability). Assume $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$. Then velocity solutions to (3.1)-(3.3) are unconditionally stable, and satisfy

$$
\|u_h^M\|^2 + \Delta t \sum_{n=0}^{M-1} \nu \left\| \nabla u_h^{n+\frac{1}{2}} \right\|^2 \leq \Delta t \sum_{n=0}^{M-1} \nu^{-1} \left\| f^{n+\frac{1}{2}} \right\|_{-1}^2 + C \|u_0\|^2 := C_4.
$$

**Proof.** Let $v_h = u_h^{n+\frac{1}{2}}$ in (3.1) and simplify to get

$$
\frac{1}{2\Delta t} \left( \|u_h^{n+1}\|^2 - \|u_h^n\|^2 \right) + \nu \left\| \nabla u_h^{n+\frac{1}{2}} \right\|^2 = (f^{n+\frac{1}{2}}, u_h^{n+\frac{1}{2}}).
$$

Using Cauchy-Schwarz, Young’s inequality, and simplifying yields

$$
\frac{1}{2\Delta t} \left( \|u_h^{n+1}\|^2 - \|u_h^n\|^2 \right) + \nu \left\| \nabla u_h^{n+\frac{1}{2}} \right\|^2 \leq \frac{\nu^{-1}}{2} \left\| f^{n+\frac{1}{2}} \right\|_{-1}^2.
$$

Multiplying by $2\Delta t$ and summing from 0 to $M - 1$ then gives

$$
\|u_h^M\|^2 + \Delta t \sum_{n=0}^{M-1} \nu \left\| \nabla u_h^{n+\frac{1}{2}} \right\|^2 \leq \Delta t \sum_{n=0}^{M-1} \nu^{-1} \left\| f^{n+\frac{1}{2}} \right\|_{-1}^2 + \|u_0^0\|^2 \\
\leq \Delta t \sum_{n=0}^{M-1} \nu^{-1} \left\| f^{n+\frac{1}{2}} \right\|_{-1}^2 + C \|u_0\|^2,
$$

which proves the estimate (3.8).

**Remark 3.1.** We note that the unconditional stability of the velocity solution is independent of both the vorticity boundary condition and the penalty term of the discrete vorticity equation.

**Lemma 3.2.** Assume $f \in L^2(0,T;L^2(\Omega))$, $u_0 \in H^1_0(\Omega)$, $u \in L^\infty(0,T;H^2(\Omega))$, $u_t \in L^\infty(0,T;H^1(\Omega))$, and $u_{tt} \in L^\infty(0,T;H^1(\Omega))$. Then vorticity solutions are also stable, in the sense of

$$
\left\| \nabla A_h^{-1}w_h^M \right\|^2 + \Delta t \sum_{n=0}^{M-1} \nu \left\| w_h^{n+\frac{1}{2}} \right\|^2 \leq C(\nu^{-2}, C_4, M, T, f, u) := C_5.
$$
Remark 3.2. It appears that the penalty parameter $\gamma$ needs to be an $O(1)$ constant for the proof to hold. When $\gamma = 0$, we are reduced to the non-penalty term case, for which we are unable to prove unconditional stability.

Proof. For the vorticity bound, let $w^{n+1}_h = I_h(\text{curl} u^n)$ where $I_h$ is a discretely div-free preserving interpolant. Note $w^{n+1}_h \in V_h$ and $w^{n+1}_h$ satisfies the vorticity boundary condition (3.5). The vorticity solution can then be decomposed as
\begin{equation}
\overline{w}^{n+\frac{1}{2}}_h = \overline{w}^{n+\frac{1}{2}}_h + \overline{w}^{n+\frac{1}{2}}_h,
\end{equation}
where $\overline{w}^{n+\frac{1}{2}}_h \in V_{h0}$. Letting $I_h(\nabla \times u) \leq C_u$ for all $t$, we have
\begin{equation}
(3.10) \quad \|w^{n+\frac{1}{2}}_h\| \leq C_u.
\end{equation}
Substituting (3.10) into the vorticity equation (3.3) yields, $\forall \chi_h \in V_{h0}$.

\begin{equation}
(3.11) \quad \frac{1}{\Delta t}(\overline{w}^{n+1}_h - \overline{w}^n_h, A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h) + \gamma \nu^{-1}(\text{curl} A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h \times u_h^{n+\frac{1}{2}}) + \nu(\nabla w^{n+\frac{1}{2}}_h, \nabla \chi_h) = (\text{curl} f^{n+\frac{1}{2}}, \chi_h) - 2(D(\overline{w}^{n+\frac{1}{2}}_h)u_h^{n+\frac{1}{2}}, \chi_h) - \gamma \nu^{-1}(\text{curl} A_h^{-1}\overline{w}^{n+1}_h \times u_h^{n+1}, \chi_h)
\end{equation}

Let $\chi_h = A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h$ and simplify to get
\begin{equation}
(3.12) \quad \frac{1}{\Delta t}(\overline{w}^{n+1}_h - \overline{w}^n_h, A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h) + \gamma \nu^{-1}\|\text{curl} A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h \times u_h^{n+\frac{1}{2}}\|^2 + \nu\|\nabla \overline{w}^{n+\frac{1}{2}}_h\|^2 = (\text{curl} f^{n+\frac{1}{2}}, A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h) - 2(D(\overline{w}^{n+\frac{1}{2}}_h)u_h^{n+\frac{1}{2}}, A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h) - \gamma \nu^{-1}(\text{curl} A_h^{-1}\overline{w}^{n+1}_h \times u_h^{n+1}, A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h)
\end{equation}

It is straightforward to show that $A_h^{-1}$ is a symmetric operator on $V_{h0}$ and thus
\begin{equation}
(3.13) \quad \frac{1}{\Delta t}(\overline{w}^{n+1}_h - \overline{w}^n_h, A_h^{-1}\overline{w}^{n+\frac{1}{2}}_h) = \frac{1}{2\Delta t}[(\overline{w}^{n+1}_h, A_h^{-1}\overline{w}^{n+T}_h) + (\overline{w}^{n+T}_h, A_h^{-1}\overline{w}^{n+1}_h) - (\overline{w}^n_h, A_h^{-1}\overline{w}^{n+T}_h)]
\end{equation}

This completes the proof.
Using (2.5) on the first RHS term of (3.12) yields

\[
\begin{align*}
(\text{curl } f^{n+\frac{1}{2}}, A_h^{-1} w_h^{n+\frac{1}{2}}) &= (f^{n+\frac{1}{2}}, \text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \\
&\leq C \left( \left\| f^{n+\frac{1}{2}} \right\|_{V^*} + h \left\| f^{n+\frac{1}{2}} \right\| \right) \left\| w_h^{n+\frac{1}{2}} \right\| \\
&\leq C(\epsilon) \nu^{-1} (\left\| f^{n+\frac{1}{2}} \right\|_{V^*}^2 + h^2 \left\| f^{n+\frac{1}{2}} \right\|^2) + \nu \left\| w_h^{n+\frac{1}{2}} \right\|^2.
\end{align*}
\]

(3.14)

Using vector identities, integration by parts, and that \( A_h^{-1} w_h^{n+\frac{1}{2}} \) is divergence free, on the first trilinear term in (3.12) gives

\[
\begin{align*}
2(D(w_h^{n+\frac{1}{2}}) u_h^{n+\frac{1}{2}}, A_h^{-1} w_h^{n+\frac{1}{2}}) &= |(\text{curl } w_h^{n+\frac{1}{2}} \times u_h^{n+\frac{1}{2}}, A_h^{-1} w_h^{n+\frac{1}{2}})| \\
&= |(\text{curl } w_h^{n+\frac{1}{2}} \times u_h^{n+\frac{1}{2}}, A_h^{-1} w_h^{n+\frac{1}{2}}), A_h^{-1} w_h^{n+\frac{1}{2}})| \\
&= |(\text{curl } w_h^{n+\frac{1}{2}} \times u_h^{n+\frac{1}{2}}, A_h^{-1} w_h^{n+\frac{1}{2}}, \nabla \cdot A_h^{-1} w_h^{n+\frac{1}{2}})| \\
&= |(\text{curl } w_h^{n+\frac{1}{2}} \times u_h^{n+\frac{1}{2}}, \text{curl } A_h^{-1} w_h^{n+\frac{1}{2}})| \\
&\leq \frac{\gamma^{-1} \nu}{2} \left\| w_h^{n+\frac{1}{2}} \right\|^2 + \frac{\gamma \nu^{-1}}{2} \left\| (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}} \right\|^2.
\end{align*}
\]

(3.15)

The part of the penalty term on the right-hand side of (3.12) is majorized as

\[
\begin{align*}
-\gamma \nu^{-1} ((\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}}, (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}}) \\
&\leq \gamma \nu^{-1} \left\| (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}} \right\| \left\| (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}} \right\| \\
&\leq \gamma \nu^{-1} \left\| (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}} \right\|^2 + \frac{\gamma \nu^{-1}}{4} \left\| (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}} \right\|^2.
\end{align*}
\]

(3.16)

The first term on the RHS of (3.16) can be bounded using Holder’s inequality, (3.10) and Lemma 2.1

\[
\begin{align*}
\left\| (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}} \right\|^2 \leq C \left\| \text{curl } A_h^{-1} w_h^{n+\frac{1}{2}} \right\|^2_{L^2} \left\| u_h^{n+\frac{1}{2}} \right\|^2_{L^2} \\
&\leq C C_\nu^2 \left\| \nabla u_h^{n+\frac{1}{2}} \right\|^2 \\
&\leq C C_\nu^2 C_\nu \left\| \nabla u_h^{n+\frac{1}{2}} \right\|^2.
\end{align*}
\]

(3.17)

(3.18)

Substituting back into (3.16) we now have

\[
\begin{align*}
-\gamma \nu^{-1} ((\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}}, (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}}) \\
&\leq \gamma \nu^{-1} C C_\nu^2 \left\| \nabla u_h^{n+\frac{1}{2}} \right\|^2 + \frac{\gamma \nu^{-1}}{4} \left\| (\text{curl } A_h^{-1} w_h^{n+\frac{1}{2}}) \times u_h^{n+\frac{1}{2}} \right\|^2.
\end{align*}
\]

(3.19)
Substitute into (3.12) using (3.13)-(3.23) to get

\[
\nu(3.21) = \frac{1}{\Delta t} (w_h^{n+\frac{1}{2}} - w_h^{n,\ast}, A_h^{-1} w_h^{n+\frac{1}{2}}) \leq C \nu(1) \frac{1}{\Delta t} \| I_h(\nabla \cdot (u^{n+1} - u^n)) \|_{L^\infty} \leq C_\nu
\]

(3.20)

\[
\leq C(\nu) C_\nu^{-1} \| \nabla w_h^{n+\frac{1}{2}} \|_{L^2}^2 + \nu \| w_h^{n+\frac{1}{2}} \|_{L^2}^2
\]

(3.21)

\[
\leq C(\nu) C_\nu^{-1} + \nu \| w_h^{n+\frac{1}{2}} \|_{L^2}^2.
\]

Using (2.7) gives

\[
\nu(\nabla w_h^{n+\frac{1}{2}}, \nabla A_h^{-1} w_h^{n+\frac{1}{2}}) \leq C \nu(\| w_h^{n+\frac{1}{2}} \| + \| w_h^{n+\frac{1}{2}} \| + \| w_h^{n+\frac{1}{2}} \| - \frac{1}{\Delta t, \Omega}) \| w_h^{n+\frac{1}{2}} \|_{L^2}^2
\]

(3.22)

\[
\leq C(\nu) \nu(\| w_h^{n+\frac{1}{2}} \| + \| w_h^{n+\frac{1}{2}} \| + \| w_h^{n+\frac{1}{2}} \| - \frac{1}{\Delta t, \Omega}) \| w_h^{n+\frac{1}{2}} \|_{L^2}^2 + \nu \| w_h^{n+\frac{1}{2}} \|_{L^2}^2
\]

Finally, Cauchy Schwarz, Young’s inequality, and the definition of \( w_h^{n,\ast} \) yield

\[
\frac{1}{\Delta t} (w_h^{n+1,\ast} - w_h^{n,\ast}, A_h^{-1} w_h^{n+\frac{1}{2}}) \leq \| I_h(\nabla \cdot (u^{n+1} - u^n)) \|_{L^\infty} \leq C_0 \| I_h(\nabla \cdot (u^{n+1} + u(t^*))\|_{L^\infty} \| w_h^{n+\frac{1}{2}} \|_{L^2} + \nu(1 - \frac{1}{2\gamma} - 4\epsilon) \| w_h^{n+\frac{1}{2}} \|_{L^2}^2
\]

\[
\leq C_0 C(\nu) \| I_h(\nabla \cdot (u^{n+1} + u(t^*))\|_{L^2}^2 + \nu \| w_h^{n+\frac{1}{2}} \|_{L^2}^2.
\]

Substitute into (3.12) using (3.13)-(3.23) to get

\[
(3.23) \quad \frac{1}{\Delta t} (\| \nabla A_h^{-1} w_h^{n+1} \|_{L^2}^2 - \| \nabla A_h^{-1} w_h^n \|_{L^2}^2) + \nu(1 - \frac{1}{2\gamma} - 4\epsilon) \| w_h^{n+\frac{1}{2}} \|_{L^2}^2
\]

\[
\leq C \nu^{-1} \| f^{n+\frac{1}{2}} \|_{V^\ast}^2 + h^2 \| f^{n+\frac{1}{2}} \|_{V^\ast}^2 + \gamma \nu^{-1} C_0 C_u^2 \| \nabla u_h^{n+\frac{1}{2}} \|_{L^2}^2 + C(\nu) C_\nu^{-1} C_u^2 + C(\nu) C_u^2 + C_0 C(\nu) \nu^{-1}.
\]
Choosing an arbitrary small \( \epsilon \), the penalty parameter \( \gamma \) satisfying \( \left( 1 - \frac{1}{2\gamma} - 4\epsilon \right) > 0 \), multiplying by \( 2\Delta t \), and summing from 0 to \( M - 1 \) yield

\[
\left\| \nabla A_h^{-1} \overline{w_h^n} \right\|^2 + \Delta t \sum_{n=0}^{M-1} \frac{1}{4} \nu \left\| \overline{w_h^{n+\frac{1}{2}}} \right\|^2 \leq \nu^{-1} \Delta t C \sum_{n=0}^{M-1} \left( \left\| f_{n+\frac{1}{2}} \right\|_{V^*}^2 + h^2 \left\| r_{n+\frac{1}{2}} \right\|^2 \right) + C(\nu^{-1}, C_4, C_u, \nu) + \nu^{-2} C C_0 C_u \Delta t \sum_{n=0}^{M-1} \nu \left\| \nabla u_h^{n+\frac{1}{2}} \right\|^2.
\]

Using the result for the velocity stability bound on the last sum of (3.24) finishes the proof. \( \square \)

4. Numerical Results

We now present two numerical experiments to test the VVH method studied in this paper. For all tests, we use \((P_3, P_2^{disc})\) Scott-Vogelius elements, on barycenter-refined tetrahedral meshes. To solve the linear systems, we use the robust and efficient method proposed in [17] for this element choice. This is the lowest order element pair that is LBB stable on this mesh. The first experiment confirms expected convergence rates, and the second tests the method on 3D channel flow over a step.

All computations use \( \gamma = 0 \). In our computations, vorticity appears to be stable with this choice, and so it was not necessary to add this (costly) stabilization term. However, proving discrete stability of vorticity does not seem possible in this case, and so its use is believed by the authors to cover a gap in the analysis only.

4.1. Convergence Rates. Our first experiment is used to test convergence rates for the problem \( \Omega = (0,1)^3 \), where the true solution is given by

\[
u = 1, \quad \text{and initial condition } u_0 = P_V(u(0)), w_0 = P_V(\text{curl } u(0)). \quad \text{We compute with end time } T = 1, \quad \text{and monitor error while decreasing the values of } \Delta t \text{ with } h. \quad \text{Uniform meshes are used in the sense that each mesh divides } \Omega \text{ into equal size cubes, then divides each cube into six tetrahedra, and then performs a barycenter refinement of each tetrahedra. In the tables, } h \text{ denotes the length of a side of a cube. For the velocity boundary condition, we use the nodal interpolant of the true solution on the boundary. For the vorticity boundary condition, we compute three different ways, all using a Dirichlet condition for discrete vorticity: using the nodal interpolant of the true vorticity, using the nodal interpolant of the } L^2 \text{ projection of the curl of the discrete velocity into } V_h, \text{ and also using a simple local averaging of the curl of the discrete velocity.}

The results are shown in Tables 1-3, respectively, and all three boundary conditions provide optimal convergence of velocity in the \( L^2(0,T;H^1(\Omega)) \) norm. With our choice of elements and a trapezoidal time discretization, optimal error is \( O(\Delta t^2 + h^3) \), and since we tie together the spatial and temporal refinements by cutting \( \Delta t \) in half when \( h \) is cut in half, \( O(h^3) \) is optimal; this rate is observed. An optimal rate is also observed for the velocity in the \( L^2(0,T;L^2(\Omega)) \) norm. For the vorticity, it appears that optimal rates are recovered by the finest mesh in the \( L^2(0,T;H^1(\Omega)) \) norm.
norm. Here, while the errors observed using the (more practical) non-exact boundary conditions are expectably larger, the rates of convergence observed do not seem to decrease.

We note it is somewhat unexpected that the rates for the non-exact vorticity boundary conditions are nearly identical to those from the exact case. Although for the non-exact case, we see analogous results to what was found in [16] (although now for higher order elements), for the exact case one might expect fourth order convergence of vorticity in the $L^\infty(0,T;L^2(\Omega))$. A complete convergence theory for the method currently appears impenetrable without several assumptions not needed for usual NSE analysis, but progress on this front will likely lead to answers about boundary-dependence of convergence rates.

<table>
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<th>Rate</th>
<th>$|u - u_h|_{L^2(0,T,H^1(\Omega))}$</th>
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<th>Rate</th>
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**Table 1.** Velocity and Vorticity errors and convergence rates using the nodal interpolant of the true vorticity for the vorticity boundary condition.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\Delta t$</th>
<th>$|u - u_h|_{L^2(0,T,L^2(\Omega))}$</th>
<th>Rate</th>
<th>$|u - u_h|_{L^2(0,T,H^1(\Omega))}$</th>
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<td>8.6168e-2</td>
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<th>Rate</th>
<th>$|w - w_h|_{L^2(0,T,H^1(\Omega))}$</th>
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<td>2.9715</td>
</tr>
</tbody>
</table>

**Table 2.** Velocity and Vorticity errors and convergence rates using the nodal interpolant of the $L^2$ projection of the curl of the discrete velocity into $V_h$, for the vorticity boundary condition.

### 4.2. 3D Channel Flow Over a Forward-Backward Facing Step.

The next experiment tests the scheme on 3D flow over a forward-backward facing step, studied in [10, 5]. In the problem the channel is modeled by a $[0, 10] \times [0, 40] \times [0, 10]$ rectangular box, with a $10 \times 1 \times 1$ step on the bottom of the channel, beginning 5 units into the channel. A diagram of the flow domain is shown in Figure 1.

We compute to end-time $T = 10$, $\nu = \frac{1}{200}$, and $\Delta t = .025$. No-slip boundary conditions are used on the top, bottom, and sides of the channel, as well as on
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\[ \| \mathbf{u} - \mathbf{u}_h \|_{L^2(0,T; L^2(\Omega))} \quad \text{Rate} \quad \| \mathbf{u} - \mathbf{u}_h \|_{L^2(0,T; H^1(\Omega))} \quad \text{Rate} \]

\begin{array}{cccccc}
\hline
h & \Delta t & \| \mathbf{u} - \mathbf{u}_h \|_{L^2(0,T; L^2(\Omega))} & \text{Rate} & \| \mathbf{u} - \mathbf{u}_h \|_{L^2(0,T; H^1(\Omega))} & \text{Rate} \\
\hline
1/2 & 1/1 & 3.8609e-2 & - & 8.6168e-2 & - \\
1/4 & 1/3 & 2.3317e-3 & 4.0495 & 9.5342e-3 & 3.1760 \\
1/6 & 1/6 & 4.873e-4 & 3.9568 & 2.7361e-3 & 3.0788 \\
1/8 & 1/9 & 1.5074e-4 & 3.9435 & 1.1361e-3 & 3.0552 \\
1/10 & 1/18 & 5.9864e-5 & 4.1385 & 5.4720e-4 & 3.2739 \\
\hline
\end{array}

\text{Table 3. Velocity and Vorticity errors and convergence rates using nodal averages of the curl of the discrete velocity for the vorticity boundary condition.}

\[ \| \mathbf{w} - \mathbf{w}_h \|_{L^2(0,T; L^2(\Omega))} \quad \text{Rate} \quad \| \mathbf{w} - \mathbf{w}_h \|_{L^2(0,T; H^1(\Omega))} \quad \text{Rate} \]

\begin{array}{cccccc}
\hline
h & \Delta t & \| \mathbf{w} - \mathbf{w}_h \|_{L^2(0,T; L^2(\Omega))} & \text{Rate} & \| \mathbf{w} - \mathbf{w}_h \|_{L^2(0,T; H^1(\Omega))} & \text{Rate} \\
\hline
1/2 & 1/1 & 6.991e-1 & - & 2.6394 & - \\
1/4 & 1/3 & 8.0245e-2 & 3.0666 & 5.0373e-1 & 2.3895 \\
1/6 & 1/6 & 2.0989e-2 & 3.3075 & 1.9434e-1 & 2.3490 \\
1/8 & 1/9 & 8.2120e-3 & 3.2619 & 1.0115e-1 & 2.2699 \\
1/10 & 1/18 & 3.8818e-3 & 3.3579 & 5.2427e-2 & 2.9451 \\
\hline
\end{array}

\text{Figure 1. Shown above is the flow domain for the 3D step test problem.}

the step, and an inflow=outflow condition is employed for both . For the initial condition, we use the \( Re = 20 \) steady solution. Note this is consistent with [5] but in contrast to [10], where a constant inflow profile \( (\mathbf{u}(x, 0, z) = <0, 1, 0>) \) is used; such a boundary condition is non-physical, but also not usable in a method that solves for vorticity (since it will blow up as \( h \to 0 \) at the inflow edges). We compute the solution on a barycenter-refined tetrahedral mesh, which provides 1,282,920 total degrees of freedom. For the vorticity boundary condition on the walls and sides, we tried Dirichlet conditions that it be a nodal interpolant of the local average of the curl of the velocity, simply zero, and the projection of the curl of the velocity into \( V_h \). Only for the case of nodal averaging did we see the expected results, shown in Figure 2 as a speed contour plot of the sliceplane \( x=5 \) with overlaying streamlines, where eddies form behind the step and shed. Plots of vorticity magnitude and helical density are also provided. For the case of zero vorticity boundary condition latter, the simulation did not capture eddy detachment, and for the projection boundary condition, we saw instabilities occur early and a bad solution resulted.

5. Conclusions and Future Directions

We have studied a finite element method for the time-dependent NSE based on the recently developed VVH formulation. We have shown that the velocity is unconditionally stable, as is the vorticity if we penalize the discrete solution to be ‘close’, in some sense, to the curl of the discrete velocity. Numerical experiments show mixed results: for an idealized problem, results look good. However, on
Figure 2. Shown above are (top) speed contours and streamlines, (middle) vorticity magnitude, and (bottom) helical density, from the fine mesh computation at time $t = 10$ at the $x = 5$ mid-slice-plane for the 3D step problem with nodal averaging vorticity boundary condition.
channel flow problems over a step, it is clear that improved vorticity boundary conditions need developed in order for this method to be competitive when higher order elements are used.

In addition to proper vorticity boundary conditions, developing appropriate stabilization/subgrid models for the formulation is necessary to handle higher Re flows. This also should provide natural framework in which stability estimates for vorticity can be proven without extra artificial terms.

References


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