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LOCALIZED POINTWISE ERROR ESTIMATES AND GLOBAL L^p ERROR ESTIMATES FOR NITSCHE'S METHOD

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Abstract. We derive localized pointwise error estimates for Nitsche's method applied to an elliptic second order problem in $\mathbf{R}^n$ $(n = 2, 3)$. Using these results, we also prove quasi-optimal global L^p error estimates as well. Numerical experiments are provided which back up the theoretical findings.

Key words. Nitsche's method, pointwise error estimates, L^p error estimates

1.1. Introduction.

We consider the following second order elliptic problem:

$$\mathcal{L}u := -\nabla \cdot (\mathbf{A}\nabla u) + \mathbf{b} \cdot \nabla u + cu = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.1b}$$

Here, $\Omega \subset \mathbf{R}^n$ $(n = 2, 3)$ is an open bounded domain with smooth boundary, $\mathbf{b} \in \mathbf{R}^n$, $c \in \mathbf{R}$, and $\mathbf{A} \in \mathbf{R}^{n\times n}$ is a symmetric positive definite matrix. More assumptions about the data and domain are given in the following subsection.

Recall that Nitsche's method [17] for (1.1) is defined as seeking a function $u_h \in V_h$ such that

$$\begin{aligned} A(u_h, v) := \int_\Omega \Big(\big(\mathbf{A}\nabla u_h\big) \cdot \nabla v + \mathbf{b} \cdot \nabla u_h v + cuv \Big)\, dx + \eta \sum_{e\in\mathcal{E}_h^b} \frac{1}{h_e} \int_e u_h v\, ds \\ - \sum_{e\in\mathcal{E}_h^b} \int_e \big(\mathbf{A}\nabla u_h\big) \cdot n_e v\, ds - \sum_{e\in\mathcal{E}_h^b} \int_e \big(\mathbf{A}\nabla v\big) \cdot n_e u_h\, ds = \int_\Omega f v\, dx \quad \forall v \in V_h, \end{aligned} \tag{1.2}$$

where V_h is the finite element space, and η denotes the penalty parameter which imposes the boundary conditions (1.1b) weakly into the variational formulation (a detailed description of the notation used above is presented in the following subsections). It is well-known that if the penalty parameter is taken sufficiently large then the method (1.2) is well-posed (cf. Lemma 2.1 below). Moreover, due to Lemma 2.2 and Céa's Lemma [3, 6], we have

$$\|u - u_h\|_{W_h^{1,2}(\Omega)} \le C \inf_{v\in V_h} \|u - v\|_{W_h^{1,2}(\Omega)}, \tag{1.3}$$

where $\|\cdot\|_{W_h^{1,2}(\Omega)}$ denotes a special energy norm defined below, and C denotes a generic positive constant. The goal of this paper is to derive localized pointwise and global L^p $(2 \le p \le \infty)$ error estimates for Nitsche's method (1.2). One motivation to derive such estimates is its use in the convergence analysis of a fully nonlinear problem [16].

Many contributions have been made to establish pointwise and L^p estimates for classical finite element methods for problems such as (1.1), and we mention the most significant results. Broadly speaking, the analysis of pointwise estimates

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can be divided into two groups. The first group, which started with the work of Natterer, Nitsche, Scott, Frehse, and Rannacher [15, 9, 23, 21, 22, 3] obtained L^p bounds with the use of certain type of *weighted L^1 estimates* for discrete Green functions. Except for the recent result found in [3, Corollary 8.2.8], all of these estimates are global, that is, the error at a certain point depends equally on the smoothness of u on all of the domain [12]. In particular, all of these estimates have the form

$$\|u - u_h\|_{L^p(\Omega)} \le Ch|\ln h|^{\bar{k}} \inf_{v\in V_h} \|u - v\|_{W^{1,p}(\Omega)}, \tag{1.4a}$$

$$\|u - u_h\|_{W^{1,p}(\Omega)} \le C \inf_{v\in V_h} \|u - v\|_{W^{1,p}(\Omega)}, \tag{1.4b}$$

where $p \in [2, \infty]$, k denotes the polynomial degree of the finite element space, and $\bar{k} = 1$ if $k = 1$ and $p = \infty$, and $\bar{k} = 0$ otherwise.

In contrast, the second group, which started with the work of Schatz [19] and later extended by various authors [11, 8, 14], used *local L^1 error estimates* of an auxiliary discrete Green function to derive weighted pointwise estimates. Using such techniques, they were able to derive estimates at an arbitrary point which depends strongly on u only near the point, namely,

$$|(u - u_h)(z)| \le Ch|\ln h|^{\bar{s}} \inf_{v\in V_h} \|u - v\|_{W^{1,\infty}(\Omega),z,s} \qquad 0 \le s \le k - 1, \tag{1.5a}$$

$$|\nabla(u - u_h)(z)| \le C|\ln h|^{\bar{\bar{s}}} \inf_{v\in V_h} \|u - v\|_{W^{1,\infty}(\Omega),z,s} \qquad 0 \le s \le k. \tag{1.5b}$$

Here, $\bar{s} = 1$ if $s = k - 1$, $\bar{s} = 0$ if $s < k - 1$, $\bar{\bar{s}} = 1$ if $s = k$, $\bar{\bar{s}} = 0$ if $s < k$, and $\|\cdot\|_{W^{1,\infty}(\Omega),z,s}$ is a weighted norm concentrated at the point z with strength s. In addition to being sharper than (1.4) in the case $p = \infty$, estimates such as (1.5) and the techniques to derive them spawned new applications such as asymptotic error expansion inequalities and new *a posteriori* residual type estimators [19, 8, 7].

In the context of discontinuous Galerkin (DG) methods, there have also been many contributions to develop L^∞ error estimates by different authors [13, 4, 5, 11]. The first by Kanschat and Rannacher [13], which generalizes earlier work of Rannacher, Frehse and Scott, used a duality argument and weighted L^1 estimates of a discrete Green function. As a result, they obtained estimates of the form (1.4) in the case $p = \infty$ and piecewise linear polynomials are used. This work was later extended and improved by Chen and Chen [4] who derived localized pointwise error estimates similar to (1.5) using techniques developed by Schatz.

Since Nitsche's method can be considered a discontinuous Galerkin method restricted to the continuous Lagrange finite element space, it seems plausible that the analysis for pointwise estimates for DG methods can be used and adapted for Nitsche's method. This is the approach we take. Since the sharpest results of pointwise estimates for DG methods were achieved by Chen and Chen, we follow their approach and derive pointwise error estimates that are similar to (1.5). As expected, most of the analysis found in [4] carries over to the case in hand. However, an added feature in our analysis is that we also derive global L^p error estimates using the analysis and techniques to derive the pointwise estimates.

1.2. Organization of Paper. In the following subsection, we set the notation that will be used throughout the paper and then state our main results, Theorems 1.1–1.3. The rest of the paper is devoted to proving these Theorems. First, in Section 2 we state some preliminary estimates that were shown in [19, 4, 17, 2, 6] which are used frequently in the main proofs. With this completed, in Section 3 we show that proving the pointwise estimates stated in Theorem 1.1 reduces to deriving weighted L^1 estimates of an auxiliary discrete Green function. We then address this issue with the help of the preliminary results in Section 2, and as a result, complete the proof of Theorem 1.1. Using similar arguments, we prove the pointwise estimates of the gradient stated in Theorem 1.2 in Section 4. In Section 5 we use some of the results in Section 4 to derive the global L^p error estimates stated in Theorem 1.3. Finally, we end the paper with some numerical experiments.

1.3. Assumptions, Notation, and Statement of Main Results. We assume that the operator $\mathcal{L}$ defined by (1.1a) is a positive over $W_0^{1,2}(\Omega)$, that is, there exists a constant $C > 0$ such that

$$C\|w\|^2_{W^{1,2}(\Omega)} \le \int_\Omega \Big((\mathbf{A}\nabla w) \cdot \nabla w + \mathbf{b} \cdot \nabla w w + cw^2 \Big)\, dx.$$

We assume that the coefficients $\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{b} \in \mathbf{R}^n$, and $c \in \mathbf{R}$ as well as the boundary $\partial\Omega$ are smooth. Also without loss of generality, we assume that $\operatorname{diam}(\Omega) \le 1$. Due to coercivity, there exists a unique solution solution $u \in W_0^{1,2}(\Omega)$ to problem (1.1) as well as the adjoint problem. Furthermore, due to the regularity of the coefficients and boundary, if $f \in W^{m,p}(\Omega)$ for an integer $m \ge -1$ and $1 < p < \infty$, we have [10]

$$\|u\|_{W^{m+2,p}(\Omega)} \le C(p)\|f\|_{W^{m,p}(\Omega)}. \tag{1.6}$$

Similar *a priori* bounds hold for the adjoint problem as well.

Let $\mathcal{T}_h$ be a quasi-uniform, simplical, and conforming triangulation of the domain Ω where each simplex on the boundary has at most one curved side. For some $D \subset \Omega$ and fixed $z \in \overline{\Omega}$, we adopt the following notations:

- $h_T = \operatorname{diam}(T)\ \forall T \in \mathcal{T}_h$;
- $h = \max_{T\in\mathcal{T}_h} h_T$;
- $\mathcal{E}_h^b$ - the set of boundary edges/faces in $\mathcal{T}_h$;
- $h_e = \operatorname{diam}(e)\ \forall e \in \mathcal{E}_h^b$;
- n_e - the outward unit normal of $e \in \mathcal{E}_h^b$;
- $A(\cdot,\cdot)_D$ - the restriction of the bilinear form $A(\cdot,\cdot)$ to D;
- $W^{m,p}(D)$ - the set of all $L^p(D)$ functions whose distributional derivatives up to order m are in $L^p(D)$;
- $W_h^{m,p}(D) = \prod_{T\in\mathcal{T}_h,\ T\cap D\neq\emptyset} W^{m,p}(T\cap D)$;
- $|D|$ - the Lebesgue measure of D;
- $B_r(z) = \{x \in \Omega;\ |z - x| < r\}$;
- $d_j = 2^{-j}$;
- $\Omega_j = \{x \in \Omega;\ d_{j+1} < |z - x| < d_j\}$;
- $\Omega_j' = \{x \in \Omega;\ d_{j+2} < |z - x| < d_{j-1}\}$;
- $\Omega_j'' = \{x \in \Omega;\ d_{j+3} < |z - x| < d_{j-2}\}$.

For an open subset $D \subset \Omega$ and number $p \in [1,\infty)$, we define the following norm

$$\|w\|_{W_h^{1,p}(D)} = \|w\|_{W^{1,p}(D)} + \Big(\sum_{e\in\mathcal{E}_h^b} h_e^{1-p}\|w\|^p_{L^p(e\cap\overline{D})}\Big)^{1/p} + \Big(\sum_{e\in\mathcal{E}_h^b} h_e\|\nabla w\|^p_{L^p(e\cap\overline{D})}\Big)^{1/p}.$$

For the case $p = \infty$, we define

$$\|w\|_{W_h^{1,\infty}(D)} = \|w\|_{W^{1,\infty}(D)} + \max_{e\in\mathcal{E}_h^b} h_e^{-1}\|w\|_{L^\infty(e\cap\overline{D})} + \max_{e\in\mathcal{E}_h^b} \|\nabla w\|_{L^\infty(e\cap\overline{D})}.$$

Next, we define the corresponding weighted norms. To this end, for given $z \in \overline{\Omega}$, we define the weight function σ_z as

$$\sigma_z(x) = \frac{h}{|x-z|+h}. \tag{1.7}$$

Remark 1.1. *Note that $\sigma_z(x) = O(1)$ provided $|x-z| = O(h)$, while $\sigma_z(x) = O(h)$ otherwise.*

For $p \in [1,\infty)$, and $s \in \mathbf{R}$, we define the following weighted norms

$$\begin{aligned}
\|w\|_{L^p(D),z,s} &= \left(\int_D \left|\sigma_z^s(x)w(x)\right|^p dx\right)^{1/p}, \\
\|w\|_{W^{1,p}(D),z,s} &= \|w\|_{L^p(D),z,s} + \|\nabla w\|_{L^p(D),z,s}, \\
\|w\|_{W_h^{1,p}(D),z,s} &= \|w\|_{W^{1,p}(D),z,s} + \Big(\sum_{e\in\mathcal{E}_h^b} h_e^{1-p}\big\|\sigma_z^s w\big\|_{L^p(e\cap\overline{D})}^p\Big)^{1/p} \\
&\quad + \Big(\sum_{e\in\mathcal{E}_h^b} h_e\big\|\sigma_z^s \nabla w\big\|_{L^p(e\cap\overline{D})}^p\Big)^{1/p}.
\end{aligned} \tag{1.8}$$

For the case $p=\infty$, we define

$$\begin{aligned}
\|w\|_{L^\infty(D),z,s} &= \big\|\sigma_z^s w\big\|_{L^\infty(D)}, \\
\|w\|_{W^{1,\infty}(D),z,s} &= \|w\|_{L^\infty(D),z,s} + \|\nabla w\|_{L^\infty(D),z,s}, \\
\|w\|_{W_h^{1,\infty}(D),z,s} &= \|w\|_{W^{1,\infty}(D),z,s} + \max_{e\in\mathcal{E}_h^b} h_e^{-1}\big\|\sigma_z^s w\big\|_{L^\infty(e\cap\overline{D})} + \max_{e\in\mathcal{E}_h^b}\big\|\sigma_z^s \nabla w\big\|_{L^\infty(e\cap\overline{D})}.
\end{aligned}$$

Finally, for an integer $k \geq 1$, we define the Lagrange finite element space $V_h \subset W^{1,2}(\Omega)$ as follows:

- If $T \in \mathcal{T}_h$ does not have a curved edge/face, then $v\big|_T$ is a polynomial of (total) degree k in the rectilinear coordinates for T;
- If $T \in \mathcal{T}_h$ has one curved edge/face, then $v\big|_T$ is a polynomial of degree k in the curvilinear coordinates of T that are defined on the reference simplex.

Remark 1.2. *In order to avoid the proliferation of constants, we shall use the notation $A \lesssim B$ ($A \gtrsim B$) to represent the relation $A \leq$ constant $\times$ B ($A \geq$ constant $\times B$), where the constant is independent of the mesh parameter h or any distance parameters (e.g. d_j). We also use the notation $A \approx B$ to represent $A \lesssim B$ and $A \gtrsim B$.*

The main goal of this paper is to prove the following three theorems.

Theorem 1.1. *Let $u \in W^{m,\infty}(\Omega)$ with $m > 1$ satisfy* (1.1)*, and let $u_h \in V_h$ satisfy the finite element method* (1.2)*. Then for any $0 \leq s \leq k-1$ and $z \in \overline{\Omega}$, there holds*

$$|(u-u_h)(z)| \lesssim h|\ln h|^{\bar{s}} \inf_{v\in V_h} \|u-v\|_{W_h^{1,\infty}(\Omega),z,s}, \tag{1.9}$$

where $\bar{s} = 0$ if $s < k-1$ and $\bar{s} = 1$ if $s = k-1$.

Theorem 1.2. *Under the same hypotheses of Theorem 1.1, there holds*

$$|\nabla(u-u_h)(z)| \lesssim |\ln h|^{\bar{\bar{s}}} \inf_{v\in V_h} \|u-v\|_{W_h^{1,\infty}(\Omega),z,s}, \tag{1.10}$$

where $\bar{\bar{s}} = 0$ if $0 \leq s < k$ and $\bar{\bar{s}} = 1$ if $s = k$.

Theorem 1.3. *Suppose that* $u \in W^{m,p}(\Omega)$ *for some* $p \in [2,\infty)$ *and* $m > 1 + 1/p$. *Then there holds*

$$\|\nabla(u-u_h)\|_{L^p(\Omega)} \lesssim \inf_{v\in V_h} \|u-v\|_{W_h^{1,p}(\Omega)}, \tag{1.11}$$

$$\|u-u_h\|_{L^p(\Omega)} \lesssim h \inf_{v\in V_h} \|u-v\|_{W_h^{1,p}(\Omega)}. \tag{1.12}$$

Remark 1.3. *The restriction* $m > 1 + 1/p$ *in Theorem 1.3 ensures that the right-hand side of* (1.11)–(1.12) *is well-defined* [1].

2. Preliminary Results

In order to prove Theorems 1.1–1.3, we must first state some well-known results. The first two Lemmas, whose proofs can be found in [17] and [4], concern the coercivity and continuity of the bilinear form $A(\cdot,\cdot)$.

Lemma 2.1. (**Coercivity**) *There exists a* $\eta_0 > 0$ *independent of* h *such that for* $\eta \ge \eta_0$ *there holds*

$$\|v\|^2_{W_h^{1,2}(\Omega)} \lesssim A(v,v) \qquad \forall v \in V_h. \tag{2.1}$$

Lemma 2.2. (**Continuity**) *Let* $D \subset \Omega$ *be an open set,* $p, p' \in [1,\infty]$ *with* $1/p + 1/p' = 1$, $s \in \mathbf{R}$, *and* $z \in \overline{\Omega}$. *Then for any* $w \in W_h^{m,p}(D)$ *and* $q \in W_h^{r,p'}(D)$ *with* $m > 1 + 1/p$ *and* $r > 1 + 1/p'$ *there holds*

$$A(w,q)_D \lesssim \|w\|_{W_h^{1,p}(D),z,s} \|q\|_{W_h^{1,p'}(D),z,-s}. \tag{2.2}$$

The next three results concern the approximation properties of the finite element space as well as interior error estimates of Nitsche's method. Their proofs can found be found in [6, 18, 4, 2, 19].

Lemma 2.3. (**Approximation Properties**) *There exists a constant* $\kappa > 0$ *independent of* h *such that if* $D_0 \subset D_1 \subset \Omega$ *satisfies* $\operatorname{dist}(D_0, \partial D_1 \backslash \partial\Omega) \ge \kappa h$ *and* $\varphi \in W^{m,p}(D_1)$ *with* $1 \le p \le \infty$ *and* $1 + 1/p < m \le k+1$, *then there exists a* $v \in V_h$ *such that*

$$\|\varphi - v\|_{L^p(D_0)} + h\|\varphi - v\|_{W_h^{1,p}(D_0)} \lesssim h^m \|\varphi\|_{W^{m,p}(D_1)}. \tag{2.3}$$

Lemma 2.4. (**Inverse Estimates**) *If* $D_0 \subset D_1 \subset \Omega$ *with* $\operatorname{dist}(D_0, \partial D_1 \backslash \partial\Omega) \ge \kappa h$, *then for all* $v \in V_h$

$$\|v\|_{L^p(D_0)} \lesssim h^{n(1/p-1/2)} \|v\|_{L^2(D_1)} \qquad \forall p \in [2,\infty], \tag{2.4}$$

$$\|v\|_{W_h^{1,2}(D_0)} \lesssim h^{-1} \|v\|_{L^2(D_1)}, \tag{2.5}$$

$$\|v\|_{L^2(D_0)} \lesssim h^{-1} \|v\|_{W^{-1,2}(D_1)}, \tag{2.6}$$

where

$$\|v\|_{W^{-1,2}(D)} = \sup_{\substack{\lambda \in C_0^\infty(D) \\ \|\lambda\|_{W^{1,2}(D)} = 1}} \int_D v\lambda \, dx. \tag{2.7}$$

Lemma 2.5. (**Interior Error Estimates**) *Let* $D_0 \subset D_1 \subset \Omega$ *with* $d = \operatorname{dist}(D_0, \partial D_1 \backslash \partial\Omega) \ge 4\kappa h$. *Suppose that* $\varphi \in W^{m,2}(\Omega) \cap W^{k+1,2}(D_1)$ *with* $m > 3/2$ *and* $\varphi_h \in V_h$ *satisfy*

$$A(v, \varphi - \varphi_h) = 0 \qquad \forall v \in V_h.$$

Then there holds

$$\|\varphi - \varphi_h\|_{W_h^{1,2}(D_0)} \lesssim h^k \|\varphi\|_{W^{k+1,2}(D_1)} + d^{-1} \|\varphi - \varphi_h\|_{L^2(D_1)}. \tag{2.8}$$

Next, we state and prove some estimates for the discrete weighted norms that will be used in Sections 3–4.

Lemma 2.6. (**Estimates of Discrete Weighted Norms**) *For any* $M > 1,\ s \geq 0$, *and* $p' \in [1,2]$, *there holds*

$$\|w\|_{W_h^{1,p'}(\Omega_j),z,-s} \lesssim d_j^{n/q+s} h^{-s} \|w\|_{W_h^{1,2}(\Omega_j)}, \tag{2.9}$$

$$\|w\|_{W_h^{1,p'}(B_{Mh}(z)),z,-s} \lesssim M^{n/q+s} h^{n/q} \|w\|_{W_h^{1,2}(B_{Mh}(z))}, \tag{2.10}$$

where $q \in [2,\infty]$ *satisfies* $1/q + 1/2 = 1/p'$.

Proof. First, we note that by (1.7)

$$\|\sigma_z^{-s}\|_{L^\infty(\Omega_j)} = \max_{x\in\overline{\Omega}_j} \left(\frac{|x-z|+h}{h}\right)^s \leq \left(\frac{d_j+h}{h}\right)^s \lesssim d_j^s h^{-s}, \tag{2.11}$$

and

$$\|\sigma_z^{-s}\|_{L^\infty(B_{Mh}(z))} = \max_{x\in\overline{B_{Mh}(z)}} \left(\frac{|x-z|+h}{h}\right)^s \lesssim \left(\frac{Mh+h}{h}\right)^s \lesssim M^s. \tag{2.12}$$

Therefore by (1.8), Hölder's inequality, the quasi-uniformity of the mesh and (2.11), we have

$$\begin{aligned}
\|w\|_{W_h^{1,p'}(\Omega_j),z,-s} &\lesssim \left\|\sigma_z^{-s}\right\|_{L^\infty(\Omega_j)} \Bigg(|\Omega_j|^{1/q} \|w\|_{W^{1,2}(\Omega_j)} \\
&\quad + \Big(\sum_{e\in\mathcal{E}_h^b} h_e^{1-p'+p'(n-1)/q} \|w\|_{L^2(e\cap\overline{\Omega}_j)}^{p'}\Big)^{1/p'} \\
&\quad + \Big(\sum_{e\in\mathcal{E}_h^b} h_e^{1+p'(n-1)/q} \|\nabla w\|_{L^2(e\cap\overline{\Omega}_j)}^{p'}\Big)^{1/p'} \Bigg) \\
&\lesssim d_j^s h^{-s} \Bigg(d_j^{n/q} \|w\|_{W^{1,2}(\Omega_j)} + h^{n/q} \Big(\sum_{e\in\mathcal{E}_h^b} h_e^{-p'/2} \|w\|_{L^2(e\cap\overline{\Omega}_j)}^{p'}\Big)^{1/p'} \\
&\quad + h^{n/q} \Big(\sum_{e\in\mathcal{E}_h^b} h_e^{p'/2} \|\nabla w\|_{L^2(e\cap\overline{\Omega}_j)}^{2}\Big)^{1/p'} \Bigg) \\
&\lesssim d_j^{s+n/q} h^{-s} \|w\|_{W_h^{1,2}(\Omega_j)}.
\end{aligned}$$

Similarly, by (2.12) we have

$$\begin{aligned}
\|w\|_{W_h^{1,p'}(B_{Mh}(z)),z,-s} &\lesssim \left\|\sigma_z^{-s}\right\|_{L^\infty(B_{Mh}(z))} \Bigg(|B_{Mh}(z)|^{1/q} \|w\|_{W^{1,2}(B_{Mh}(z))} \\
&\quad + h^{n/q} \Big(\sum_{e\in\mathcal{E}_h^b} h_e^{-p'/2} \|w\|_{L^2(e\cap\overline{B_{Mh}(z)})}^{p'}\Big)^{1/p'} \\
&\quad + h^{n/q} \Big(\sum_{e\in\mathcal{E}_h^b} h_e^{p'/2} \|\nabla w\|_{L^2(e\cap\overline{B_{Mh}(z)})}^{p'}\Big)^{1/p'} \Bigg) \\
&\lesssim M^{n/q+s} h^{n/q} \|w\|_{W_h^{1,2}(\Omega_j)}.
\end{aligned}$$

□

Finally, we end this section with some finite element approximation properties for a solution to an auxiliary problem, which relies on some estimates of the corresponding Green function to problem (1.1).

Lemma 2.7. (**Finite Element Approximations of Auxiliary Problem**) *For* $\lambda \in C_0^\infty(\Omega_j)$ *with* $\|\lambda\|_{L^2(\Omega)} = 1$*, let* $w \in W^{1,2}(\Omega)$ *satisfy*

$$\mathcal{L}w = \lambda \qquad \text{in } \Omega, \tag{2.13a}$$

$$w = 0 \qquad \text{on } \partial\Omega. \tag{2.13b}$$

Then there exists $v \in V_h$ *such that*

$$\|w - v\|_{W_h^{1,2}(\Omega_j'')} \lesssim h, \tag{2.14}$$

$$\|w - v\|_{W_h^{1,\infty}(\Omega\backslash\Omega_j'')} \lesssim h^k d_j^{1-k-n/2}. \tag{2.15}$$

Proof. By (2.3), we can choose $v \in V_h$ such that

$$\|w - v\|_{W_h^{1,2}(\Omega_j'')} \lesssim h\|w\|_{W^{2,2}(\Omega)}.$$

Therefore, by (1.6) we have

$$\|w - v\|_{W_h^{1,2}(\Omega_j'')} \lesssim h\|\lambda\|_{L^2(\Omega)} = h,$$

which proves (2.14).

To prove (2.15), we again use (2.3) to conclude

$$\|w - v\|_{W_h^{1,\infty}(\Omega\backslash\Omega_j'')} \lesssim h^k\|w\|_{W^{k+1,\infty}(\Omega\backslash\Omega_j')}. \tag{2.16}$$

To estimate $\|w\|_{W^{k+1,\infty}(\Omega\backslash\Omega_j')}$, we introduce the Green function G_x associated with problem (2.13) with singularity $x \in \Omega\backslash\Omega_j'$. We then have [24, 4]

$$w(x) = \int_{\Omega_j} G_x(y)\lambda(y)\, dy, \tag{2.17}$$

and

$$\left|\frac{\partial^{\alpha+\beta}G_x(y)}{\partial x^\alpha \partial y^\beta}\right| \lesssim |x-y|^{2-n-|\alpha|-|\beta|} \qquad \text{for } |\alpha| + |\beta| > 0. \tag{2.18}$$

Differentiating (2.17) with resect to x, we have for $|\alpha| \le k+1$

$$\left|\frac{\partial^\alpha w(x)}{\partial x^\alpha}\right| = \left|\int_{\Omega_j} \frac{\partial^\alpha G_x(y)}{\partial x^\alpha}\lambda(y)\, dy\right|.$$

Noting $y \in \Omega_j$ and $x \in \Omega\backslash\Omega_j'$ imply $|x-y| \ge d_j$, we have by (2.18)

$$\left|\frac{\partial^\alpha w(x)}{\partial x^\alpha}\right| \lesssim d_j^{2-n-(k+1)}|\Omega_j|^{1/2}\|\lambda\|_{L^2(\Omega)} \lesssim d_j^{1-k-n/2}. \tag{2.19}$$

Combining (2.16) with (2.19), we obtain (2.15). □

3. Proof of Theorem 1.1

With the preliminary results established, we now go on to prove Theorem 1.1. To this end, we set

$$\rho(x) = \begin{cases} \dfrac{h^{-n/2}(u-u_h)(x)}{\|u-u_h\|_{L^2(B_{2\kappa h}(z))}} & x \in B_{2\kappa h}(z), \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where κ is defined in Lemma 2.3. Let g_z be the solution to the following auxiliary problem

$$\mathcal{L}^* g_z = \rho \qquad \text{in } \Omega, \tag{3.2a}$$

$$g_z = 0 \qquad \text{on } \partial\Omega, \tag{3.2b}$$

where $\mathcal{L}^*$ denotes the adjoint operator of $\mathcal{L}$. Let $g_{z,h} \in V_h$ denote the corresponding finite element approximation, i.e.,

$$A(v, g_{z,h}) = 0 \qquad \forall v \in V_h. \tag{3.3}$$

Setting $e_z = g_z - g_{z,h}$, we have the following result.

Lemma 3.1. *There holds the following estimate*

$$|(u-u_h)(z)| \lesssim \Big(\|e_z\|_{W_h^{1,1}(\Omega),z,-s} + h\Big) \inf_{v\in V_h} \|u-v\|_{W_h^{1,\infty}(\Omega),z,s}. \tag{3.4}$$

Proof. By the inverse estimate (2.4), we have for any $v \in V_h$

$$\begin{aligned}
|(u-u_h)(z)| &\le |(u-v)(z)| + \|v-u_h\|_{L^\infty(B_{\kappa h}(z))} \\
&\le |(u-v)(z)| + h^{-n/2}\|v-u_h\|_{L^2(B_{2\kappa h}(z))} \\
&\lesssim |(u-v)(z)| + h^{-n/2}\Big(\|u-v\|_{L^2(B_{2\kappa h}(z))} + \|u-u_h\|_{L^2(B_{2\kappa h}(z))}\Big) \\
&\lesssim \|u-v\|_{L^\infty(B_{2\kappa h}(z))} + h^{-n/2}\|u-u_h\|_{L^2(B_{2\kappa h}(z))}.
\end{aligned}$$

Therefore by (2.3), we have

$$\begin{aligned}
|(u-u_h)(z)| &\lesssim h\|u\|_{W^{1,\infty}(B_{3\kappa h}(z))} + h^{-n/2}\|u-u_h\|_{L^2(B_{2\kappa h}(z))} \\
&\lesssim h\|u\|_{W_h^{1,\infty}(\Omega),z,s} + h^{-n/2}\|u-u_h\|_{L^2(B_{2\kappa h}(z))}.
\end{aligned}$$

Replacing u with $u-v$ and u_h by $u_h - v$ in the argument above, we conclude

$$|(u-u_h)(z)| \lesssim h\|u-v\|_{W_h^{1,\infty}(\Omega),z,s} + h^{-n/2}\|u-u_h\|_{L^2(B_{2\kappa h}(z))}. \tag{3.5}$$

Next by (3.1)–(3.2) and (2.2) we have for any $v \in V_h$

$$\begin{aligned}
h^{-n/2}\|u-u_h\|_{L^2(B_{2\kappa h}(z))} &= \int_\Omega (u-u_h)\rho\, dx \\
&= A(u-u_h, g_z) \\
&= A(u-v, e_z) \\
&\lesssim \|u-v\|_{W_h^{1,\infty}(\Omega),s,z}\|e_z\|_{W_h^{1,1}(\Omega),-s,z}.
\end{aligned} \tag{3.6}$$

Combining the estimate (3.6) with (3.5), we obtain (3.4). $\square$

In light of Lemma 3.1, the proof of Theorem 1.1 reduces to the proof of the following lemma.

Lemma 3.2. *For any* $0 \le s \le k-1$, *there holds*

$$\|e_z\|_{W_h^{1,1}(\Omega),-s,z} \lesssim h|\ln h|^{\bar{s}}, \tag{3.7}$$

where $\bar{s}$ *is defined in Theorem 1.1.*

The proof of Lemma 3.2 is based on a sequence of highly technical lemmas which rely on a dyadic decomposition of Ω as well as the estimates stated in Lemmas 2.3–2.7. As a first step, we state and prove a preliminary estimate.

Lemma 3.3. (**A Preliminary Estimate**) *Let $M > 1$ be a real number and let J be an integer such that $Mh = 2^{-J}$. Then there holds*

$$\|e_z\|_{W_h^{1,1}(\Omega),z,-s} \lesssim h + F_h, \tag{3.8}$$

where

$$F_h = h^{-s} \sum_{j=0}^{J} d_j^{n/2+s} \|e_z\|_{W_h^{1,2}(\Omega_j)}. \tag{3.9}$$

Proof. By standard finite element techniques, (1.6), and noting $\|\rho\|_{L^2(\Omega)} = h^{-n/2}$, we have

$$\begin{aligned} \|e_z\|_{W_h^{1,2}(B_{Mh}(z))} &\le \|e_z\|_{W_h^{1,2}(\Omega)} \lesssim h\|g_z\|_{W^{2,2}(\Omega)} \\ &\lesssim h\|\rho\|_{L^2(\Omega)} \lesssim h^{1-n/2}. \end{aligned} \tag{3.10}$$

Thus, by (2.9)–(2.10), (3.10) and the assumption $\operatorname{diam}(\Omega) = 1$, we have

$$\begin{aligned} \|e_z\|_{W_h^{1,1}(\Omega),z,-s} &\lesssim \|e_z\|_{W_h^{1,1}(B_{Mh}(z)),z,-s} + \sum_{j=0}^{J} \|e_z\|_{W_h^{1,1}(\Omega_j),z,-s} \\ &\lesssim h^{n/2}\|e_z\|_{W_h^{1,2}(B_{Mh}(z))} + F_h \lesssim h + F_h. \end{aligned}$$

□

In light of the estimate (3.8), in order to prove Lemma 3.2 we must derive estimates of F_h. This is achieved in the following two lemmas. First, using the interior estimates stated in Lemma 2.5 we are able to replace the terms $\|e_z\|_{W_h^{1,2}(\Omega_j)}$ appearing in (3.9) with $d_j^{-1}\|e_z\|_{L^2(\Omega_j')}$.

Lemma 3.4. (**Estimate of F_h I**) *There holds the following estimate*

$$F_h \lesssim h|\ln h|^{\bar{s}} + h^{-s} \sum_{j=0}^{J} d_j^{s+n/2-1} \|e_z\|_{L^2(\Omega_j')}. \tag{3.11}$$

Proof. First we note that by (2.8), we have for any $j = 0, ..., J$

$$\|e_z\|_{W_h^{1,2}(\Omega_j)} \lesssim h^k \|g_z\|_{W^{k+1,2}(\Omega_j')} + d_j^{-1}\|e_z\|_{L^2(\Omega_j')}. \tag{3.12}$$

To estimate the first term in the expression above, we use the identity

$$g_z(x) = \int_{B_{2\kappa h}(z)} G_x(y)\rho(y)\, dy, \tag{3.13}$$

where G_x is the Green function with singularity at $x \in \Omega_j'$.

Differentiating (3.13) with respect to x, we have for $|\alpha| \le k+1$

$$\left|\frac{\partial^\alpha g_z(x)}{\partial x^\alpha}\right| = \left|\int_{B_{2\kappa h}(z)} \frac{\partial^\alpha G_x(y)}{\partial x^\alpha}\rho(y)\, dy\right|.$$

Noting $y \in B_{2\kappa h}(z)$ and $x \in \Omega_j'$ implies $|x - y| \gtrsim d_j$, we have by (2.18), and the fact that $\|\rho\|_{L^2(\Omega)} = h^{-n/2}$

$$\left|\frac{\partial^\alpha g_z(x)}{\partial x^\alpha}\right| \lesssim |B_{\kappa h}(z)|^{1/2} d_j^{2-n-(k+1)} \|\rho\|_{L^2(\Omega)} \lesssim d_j^{1-k-n}.$$

It then follows that

$$\|g_z\|_{W^{k+1,2}(\Omega_j')} \lesssim |\Omega_j'|^{1/2} d_j^{1-k-n} \lesssim d_j^{1-k-n/2}. \tag{3.14}$$

Combining (3.14) with (3.12), we obtain

$$\|e_z\|_{W_h^{1,2}(\Omega_j)} \lesssim d_j^{1-k-n/2} h^k + d_j^{-1} \|e_z\|_{L^2(\Omega_j')}. \tag{3.15}$$

Next, combining (3.9) and (3.15), we have

$$\begin{aligned} F_h &= h^{-s} \sum_{j=0}^{J} d_j^{n/2+s} \|e_z\|_{W_h^{1,2}(\Omega_j)} \\ &\lesssim \sum_{j=0}^{J} \Big(d_j^{1+s-k} h^{k-s} + d_j^{n/2+s-1} h^{-s} \|e_z\|_{L^2(\Omega_j')} \Big) \\ &= h\Theta(k-1-s) + h^{-s} \sum_{j=0}^{J} d_j^{n/2+s-1} \|e_z\|_{L^2(\Omega_j')}, \end{aligned} \tag{3.16}$$

where

$$\Theta(\gamma) = \sum_{j=0}^{J} (h/d_j)^{\gamma}. \tag{3.17}$$

Noting that [4]

$$\Theta(\gamma) \lesssim \begin{cases} |\ln h| & \gamma = 0, \\ \dfrac{1}{M^{\gamma}(1-2^{-\gamma})} & \gamma > 0, \end{cases} \tag{3.18}$$

and $s \leq k-1$, $M > 1$, and $k \geq 1$, we have $\Theta(k-1-s) \lesssim |\ln h|^{\bar{s}}$. Combining this last bound with (3.16), we obtain (3.11).

□

Next, using a duality argument and the results presented in Lemma 2.7, we are able to estimate the summation involving $\|e_z\|_{L^2(\Omega_j')}$ in (3.11) in terms of $\|e_z\|_{W_h^{1,1}(\Omega)}$ and $\|e_z\|_{W_h^{1,2}(\Omega_j'')}$. As a result, using the definition of F_h (3.9) we are able to 'kick-back' the terms involving $\|e_z\|_{W_h^{1,2}(\Omega_j'')}$ provided M is sufficiently large.

Lemma 3.5. (**Estimate of F_h II**) *For M sufficiently large, there holds*

$$F_h \lesssim h|\ln h|^{\bar{s}} + \Theta(k-s)\|e_z\|_{W_h^{1,1}(\Omega)}, \tag{3.19}$$

where $\Theta(\cdot)$ is defined by (3.17).

Proof. The proof is based on the bound (3.11) and estimating $\|e_z\|_{L^2(\Omega)}$ using a duality argument, noting that

$$\|e_z\|_{L^2(\Omega_j)} = \sup_{\substack{\lambda \in C_0^{\infty}(\Omega_j) \\ \|\lambda\|_{L^2(\Omega)} = 1}} \int_{\Omega_j} e_z \lambda \, dx. \tag{3.20}$$

To this end, for arbitrary $\lambda \in C_0^{\infty}(\Omega_j)$ with $\|\lambda\|_{L^2(\Omega)} = 1$, let w satisfy (2.13). We then have for any $v \in V_h$

$$\int_{\Omega} e_z \lambda \, dx = A(w, e_z) = A(w - v, e_z).$$

Therefore by (2.2) and (2.14)–(2.15), for appropriately chosen $v \in V_h$ we have

$$\int_{\Omega} e_z \lambda \, dx \lesssim \|e_z\|_{W_h^{1,1}(\Omega \setminus \Omega_j'')} \|w - v\|_{W_h^{1,\infty}(\Omega \setminus \Omega_j'')} + \|e_z\|_{W_h^{1,2}(\Omega_j'')} \|w - v\|_{W_h^{1,2}(\Omega_j'')}$$

$$\lesssim h^k d_j^{1-k-n/2} \|e_z\|_{W_h^{1,1}(\Omega)} + h\|e_z\|_{W_h^{1,2}(\Omega_j'')}.$$

It then follows from (3.20) that

$$\|e_z\|_{L^2(\Omega_j)} \lesssim h^k d_j^{1-k-n/2} \|e_z\|_{W_h^{1,1}(\Omega)} + h\|e_z\|_{W_h^{1,2}(\Omega_j'')}.$$

Therefore, noting (3.17), (3.9), and $\max_{0\le j\le J} d_j^{-1} = 2^J = 1/(hM)$, we have

$$\begin{aligned}\sum_{j=0}^J d_j^{s+n/2-1} h^{-s} \|e_z\|_{L^2(\Omega_j')} &\lesssim \sum_{j=0}^J \Big(h^{k-s} d_j^{s-k} \|e_z\|_{W_h^{1,1}(\Omega)} + d_j^{s+n/2-1} h^{1-s} \|e_z\|_{W_h^{1,2}(\Omega_j'')} \Big) \\ &= \Theta(k-s)\|e_z\|_{W_h^{1,1}(\Omega)} + h^{1-s} \sum_{j=0}^J d_j^{s+n/2-1} \|e_z\|_{W_h^{1,2}(\Omega_j'')} \\ &\lesssim \Theta(k-s)\|e_z\|_{W_h^{1,1}(\Omega)} + \frac{h^{-s}}{M} \sum_{j=0}^J d_j^{s+n/2} \|e_z\|_{W_h^{1,2}(\Omega_j'')} \\ &\lesssim \Theta(k-s)\|e_z\|_{W_h^{1,1}(\Omega)} + \frac{F_h}{M}.\end{aligned}$$

Combining this last estimate with (3.11), we find

$$F_h \lesssim h|\ln h|^{\bar{s}} + \Theta(k-s)\|e_z\|_{W_h^{1,1}(\Omega)} + \frac{F_h}{M}.$$

Therefore for M sufficiently large, we can absorb the term F_h/M into the left-hand side to get

$$F_h \lesssim h|\ln h|^{\bar{s}} + \Theta(k-s)\|e_z\|_{W_h^{1,1}(\Omega)}.$$

□

Proof. of Lemma 3.2 Combining (3.19) with (3.8), we have

$$\|e_z\|_{W_h^{1,1}(\Omega),z,-s} \lesssim h|\ln h|^{\bar{s}} + \Theta(k-s)\|e_z\|_{W_h^{1,1}(\Omega)}. \tag{3.21}$$

In particular, the case $s = 0$ gives us

$$\|e_z\|_{W_h^{1,1}(\Omega)} \lesssim h|\ln h|^{\bar{s}} + \Theta(k)\|e_z\|_{W_h^{1,1}(\Omega)}.$$

Therefore, we may choose M large enough such that

$$\|e_z\|_{W_h^{1,1}(\Omega)} \lesssim h|\ln h|^{\bar{s}}.$$

Using this last estimate in (3.21), we conclude

$$\|e_z\|_{W_h^{1,1}(\Omega),z,-s} \lesssim h|\ln h|^{\bar{s}} + \Theta(k-s)h|\ln h|^{\bar{s}} \lesssim h|\ln h|^{\bar{s}}.$$

□

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1 so we omit some details and focus only on the main points. As a first step, we state and prove a preliminary result.

Lemma 4.1. *For any $D \subset \Omega$ and $w \in L^p(D)$ with $p \in [2,\infty]$, there holds*

$$\|w\|_{W^{-1,2}(D)} \lesssim \operatorname{diam}(D)|D|^{(p-2)/2p} \|w\|_{L^p(D)}. \tag{4.1}$$

In particular, if $\operatorname{diam}(D) \approx h,\ |D| \approx h^n$, *then*

$$\|w\|_{W^{-1,2}(D)} \lesssim h^{1+n/2-n/p} \|w\|_{L^p(D)}.$$

Proof. By (2.7), Hölder's inequality, and Poincare's inequality, we have for arbitrary $\lambda \in C_0^\infty(D)$ with $\|\lambda\|_{W^{1,2}(D)} = 1$

$$\begin{aligned}\int_D w\lambda \, dx &\le |D|^{(p-2)/2p}\|w\|_{L^p(D)}\|\lambda\|_{L^2(D)} \\ &\lesssim \operatorname{diam}(D)|D|^{(p-2)/2p}\|w\|_{L^p(D)}\|\nabla\lambda\|_{L^2(D)} \\ &\le \operatorname{diam}(D)|D|^{(p-2)/2p}\|w\|_{L^p(D)}.\end{aligned}$$

□

We now prove Theorem 1.2. For any $i = 1, 2, ..., n$ and $v \in V_h$, we have by (2.4), (2.6), (4.1) and by integration by parts

$$\begin{aligned}&\big|\partial(u-u_h)(z)/\partial x_i\big| \\ &\quad\lesssim \big\|\partial(u-v)/\partial x_i\big\|_{L^\infty(B_{\kappa h}(z))} + h^{-n/2-1}\big\|\partial(u_h-v)/\partial x_i\big\|_{W^{-1,2}(B_{3\kappa h}(z))} \\ &\quad\lesssim \big\|\partial(u-v)/\partial x_i\big\|_{L^\infty(B_{\kappa h}(z))} \\ &\qquad + h^{-n/2-1}\Big(\big\|\partial(u-v)/\partial x_i\big\|_{W_h^{-1,2}(B_{3\kappa h}(z))} + \big\|\partial(u-u_h)/\partial x_i\big\|_{W^{-1,2}(B_{3\kappa h}(z))}\Big) \\ &\quad\lesssim \|u-v\|_{W_h^{1,\infty}(\Omega),z,s} + h^{-n/2-1}\big\|\partial(u-u_h)/\partial x_i\big\|_{W^{-1,2}(B_{3\kappa h}(z))} \\ &\quad= \|u-v\|_{W_h^{1,\infty}(\Omega),z,s} \\ &\qquad + h^{-n/2-1} \sup_{\substack{\varphi \in C_0^\infty(B_{3\kappa h}(z)) \\ \|\varphi\|_{W^{1,2}(B_{3\kappa h}(z))}=1}} \int_{B_{3\kappa h}(z)} (u-u_h)\big(\partial\varphi/\partial x_i\big)\, dx.\end{aligned}$$

For arbitrary $\varphi \in C_0^\infty(B_{3\kappa h}(z))$, with $\|\varphi\|_{W^{1,2}(B_{3\kappa h}(z))} = 1$, we extend φ to Ω by zero, and let $\hat{g}_z$ be the solution to

$$\begin{aligned}\mathcal{L}^*\hat{g}_z &= h^{-n/2-1}\partial\varphi/\partial x_i &&\text{in } \Omega, \\ \hat{g}_z &= 0 &&\text{on } \partial\Omega.\end{aligned}$$

Let $\hat{g}_{z,h} \in V_h$ satisfy

$$A(v, \hat{g}_{z,h}) = h^{-n/2-1}\int_\Omega \big(\partial\varphi/\partial x_i\big) v \, dx \qquad \forall v \in V_h.$$

Setting $\hat{e}_z = \hat{g}_z - \hat{g}_{z,h}$, we have for arbitrary $v \in V_h$

$$\begin{aligned}h^{-n/2-1}\int_{B_{3\kappa h}(z)} (u-u_h)\partial\varphi/\partial x_i \, dx &= A(u-v, \hat{e}_z) \\ &\lesssim \|u-v\|_{W_h^{1,\infty}(\Omega),z,s}\|\hat{e}_z\|_{W_h^{1,1}(\Omega),z,-s},\end{aligned}$$

and therefore

$$\big|\partial(u-u_h)(z)/\partial x_i\big| \lesssim \|u-v\|_{W_h^{1,\infty}(\Omega),z,s}\Big(\|\hat{e}_z\|_{W_h^{1,1}(\Omega),z,-s} + 1\Big). \tag{4.3}$$

In light of (4.3), we need to derive estimates of $\|\hat{e}_z\|_{W_h^{1,1}(\Omega),z,-s}$ to complete the proof of Theorem 1.2. In the following lemma, we drive estimates of $\|\hat{e}_z\|_{W^{1,p'}(\Omega),z,-s}$ for all $p' \in [1, 2]$. This generalization will be useful when deriving global L^p error estimates in the following section.

Lemma 4.2. *Let $p \in [2, \infty]$, $p' \in [1, 2]$ such that $1/p + 1/p' = 1$. Then for any $0 \le s \le k + n/p$ there holds*

$$\|\hat{e}_z\|_{W_h^{1,p'}(\Omega),z,-s} \lesssim |\ln h|^{\bar{\bar{s}}(p)}h^{-n/p},$$

where $\bar{\bar{s}}(p) = 1$ if $s = k + n/p$ and $\bar{\bar{s}}(p) = 0$ otherwise.

Remark 4.1. *The case $p'=1,\ p=\infty$ completes the proof of Theorem 1.2.*

Proof. By Lemma 2.6 and using similar arguments in the proof of Lemma 3.3, we get

$$\|\hat{e}_z\|_{W_h^{1,p'}(\Omega),z,-s} \lesssim h^{n/q}\|\hat{e}_z\|_{W^{1,2}(B_{Mh}(z))} + h^{-s}\sum_{j=0}^{J} d_j^{s+n/q}\|\hat{e}_z\|_{W_h^{1,2}(\Omega_j)} \lesssim h^{-n/p} + \hat{F}_h, \tag{4.4}$$

where

$$\hat{F}_h = h^{-s}\sum_{j=0}^{J} d_j^{n/q+s}\|\hat{e}_z\|_{W_h^{1,2}(\Omega_j)}, \tag{4.5}$$

and $q\in[2,\infty]$ satisfies $1/q+1/2=1/p'$.

Next, by (2.8) we have

$$\|\hat{e}_z\|_{W_h^{1,2}(\Omega_j)} \lesssim h^k\|\hat{g}_z\|_{W^{k+1,2}(\Omega_j')} + d_j^{-1}\|\hat{e}_z\|_{L^2(\Omega_j')}. \tag{4.6}$$

To estimate $\|\hat{g}_z\|_{W^{k+1,2}(\Omega_j')}$, we use (2.18) to get for any $|\alpha|=k+1$

$$\begin{aligned}
\left|\frac{\partial^\alpha \hat{g}_z(x)}{\partial x^\alpha}\right| &= h^{-n/2-1}\left|\int_{B_{3\kappa h}(z)} \frac{\partial^\alpha G_x(y)}{\partial x^\alpha}\frac{\partial \varphi(y)}{\partial y_i}\,dy\right| \\
&= h^{-n/2-1}\left|\int_{B_{3\kappa h}(z)} \frac{\partial}{\partial y_i}\frac{\partial^\alpha G_x(y)}{\partial x^\alpha}\varphi(y)\,dy\right| \\
&\lesssim h^{-n/2-1}|B_{\kappa h}(z)|^{1/2} d_j^{2-n-(k+2)}\|\varphi\|_{L^2(B_{Mh}(z))} \\
&\lesssim d_j^{-n-k},
\end{aligned}$$

where we used Poincare's inequality to derive the last estimate. It then follows that

$$\|\hat{g}_z\|_{W^{k+1,2}(\Omega_j')} \lesssim |\Omega_j'|^{1/2} d_j^{-n-k} \lesssim d_j^{-n/2-k},$$

and therefore by (4.6)

$$\|\hat{e}_z\|_{W_h^{1,2}(\Omega_j)} \lesssim d_j^{-k-n/2}h^k + d_j^{-1}\|\hat{e}_z\|_{L^2(\Omega_j')}. \tag{4.7}$$

Hence, by (4.7), (4.5), and (3.17), we have

$$\begin{aligned}
\hat{F}_h &\lesssim \sum_{j=0}^{J}\Big(d_j^{s-k-n/p}h^{k-s} + h^{-s}d_j^{n/q+s-1}\|\hat{e}_z\|_{L^2(\Omega_j')}\Big) \\
&= h^{-n/p}\Theta(k-s+n/p) + h^{-s}\sum_{j=0}^{J} d_j^{n/q+s-1}\|\hat{e}_z\|_{L^2(\Omega_j')} \\
&\lesssim h^{-n/p}|\ln h|^{\bar{\bar{s}}(p)} + h^{-s}\sum_{j=0}^{J} d_j^{n/q+s-1}\|\hat{e}_z\|_{L^2(\Omega_j')}.
\end{aligned} \tag{4.8}$$

Next, using a duality argument similar to that in Lemma 3.5, we get

$$\|\hat{e}_z\|_{L^2(\Omega_j')} \lesssim h^k d_j^{1-k-n/2}\|\hat{e}_z\|_{W_h^{1,1}(\Omega)} + h\|\hat{e}_z\|_{W_h^{1,2}(\Omega_j'')}, \tag{4.9}$$

and therefore by (4.8)–(4.9) and (4.5), we obtain

$$\hat{F}_h \lesssim h^{-n/p}|\ln h|^{\bar{\bar{s}}(p)} + h^{1-s}\sum_{j=0}^{J} d_j^{n/q+s-1}\|\hat{e}_z\|_{W^{1,2}(\Omega_j'')} \tag{4.10}$$

$$+ h^{k-s}\sum_{j=0}^{J} d_j^{s-k-n/p}\|\hat{e}_z\|_{W_h^{1,1}(\Omega)}$$

$$\lesssim h^{-n/p}|\ln h|^{\bar{\bar{s}}(p)} + \frac{\hat{F}_h}{M} + h^{k-s}\sum_{j=0}^{J} d_j^{s-k-n/p}\|\hat{e}_z\|_{W_h^{1,1}(\Omega)}$$

$$= h^{-n/p}|\ln h|^{\bar{\bar{s}}(p)} + \frac{\hat{F}_h}{M} + h^{-n/p}\Theta(k-s+n/p)\|\hat{e}_z\|_{W_h^{1,1}(\Omega)}.$$

Thus by (4.4) and (4.10), for M sufficiently large we have

$$\|\hat{e}_z\|_{W_h^{1,p'}(\Omega),z,-s} \lesssim h^{-n/p}|\ln h|^{\bar{\bar{s}}(p)} + h^{-n/p}\Theta(k-s+n/p)\|\hat{e}_z\|_{W_h^{1,1}(\Omega)}. \tag{4.11}$$

In particular, the case $p'=1,\ p=\infty,\ s=0$ gives us

$$\|\hat{e}_z\|_{W_h^{1,1}(\Omega)} \lesssim 1 + \Theta(k)\|\hat{e}_z\|_{W_h^{1,1}(\Omega)}.$$

Thus, for M sufficiently large we conclude

$$\|\hat{e}_z\|_{W_h^{1,1}(\Omega)} \lesssim 1.$$

Completing the proof, we use this last estimate in (4.11) to get

$$\|\hat{e}_z\|_{W^{1,p'}(\Omega),z,-s} \lesssim h^{-n/p}|\ln h|^{\bar{\bar{s}}(p)} + h^{-n/p}\Theta(k-s+n/p) \lesssim h^{-n/p}|\ln h|^{\bar{\bar{s}}(p)}.$$

□

5. Proof of Theorem 1.3

In this section we prove Theorem 1.3 and derive global L^p estimates with the aid of Lemma 4.2 and using similar arguments to those found in [22]. As a first step, we need the following Lemma which follows from several applications of Fubini's Theorem and (1.7)–(1.8).

Lemma 5.1. *Let $p \in [2,\infty)$ and $w \in L^p(\Omega)$. Then there holds*

$$\int_\Omega \int_{B_{3\kappa h}(z)} |w(x)|^p\,dx\,dz \lesssim h^n \|w\|^p_{L^p(\Omega)}. \tag{5.1}$$

Moreover for any $s > n/p$ and $w \in W^{m,p}(\Omega)$ with $m > 1 + 1/p$, there holds

$$\int_\Omega \|w\|^p_{W_h^{1,p}(\Omega),z,s}\,dz \lesssim \frac{h^n}{ps-n}\|w\|^p_{W_h^{1,p}(\Omega)}. \tag{5.2}$$

To derive global L^p error estimates, we first use (2.4), (2.6), the triangle inequality, and (4.1) to obtain for any $z \in \Omega$

$$\begin{aligned}
&|\partial u_h(z)/\partial x_i| \\
&\quad\lesssim h^{-n/2-1}\|\partial u_h/\partial x_i\|_{W^{-1,2}(B_{3\kappa h}(z))} \\
&\quad\lesssim h^{-n/2-1}\Big(\|\partial(u-u_h)/\partial x_i\|_{W^{-1,2}(B_{3\kappa h}(z))} + \|\partial u/\partial x_i\|_{W^{-1,2}(B_{3\kappa h}(z))}\Big) \\
&\quad\lesssim h^{-n/p}\|\partial u/\partial x_i\|_{L^p(B_{3\kappa h}(z))} + h^{-n/2-1}\|\partial(u-u_h)/\partial x_i\|_{W^{-1,2}(B_{3\kappa h}(z))}.
\end{aligned} \tag{5.3}$$

Replacing u by $u-v$ and u_h by $u_h - v$ for some $v \in V_h$ in the argument above, we conclude

$$\begin{aligned}
|\partial(u_h - v)(z)/\partial x_i| \lesssim{}& h^{-n/p}\|\partial(u-v)/\partial x_i\|_{L^p(B_{3\kappa h}(z))} \\
&+ h^{-n/2-1}\|(u-u_h)/\partial x_i\|_{W^{-1,2}(B_{3\kappa h}(z))}.
\end{aligned}$$

Therefore, using a duality argument similar to the one found in the proof of Theorem 1.2 and using Lemma 4.2, we have

$$|\nabla(u_h - v)(z)| \lesssim h^{-n/p}\|\nabla(u-v)\|_{L^p(B_{3\kappa h}(z))} + h^{-n/p}|\ln h|^{\bar{\bar{s}}(p)}\|u-v\|_{W_h^{1,p}(\Omega),z,s}, \tag{5.4}$$

where $\bar{\bar{s}}(p)$ is defined in Lemma 4.2. Raising (5.4) by the power p and integrating over Ω with respect to z, we conclude

$$\|\nabla(u_h - v)\|_{L^p(\Omega)} \lesssim \left(h^{-n}\int_\Omega \|\nabla(u-v)\|^p_{L^p(B_{3\kappa h}(z))}\,dz\right)^{1/p} + \left(h^{-n}|\ln h|^{\bar{\bar{s}}(p)p}\int_\Omega \|u-v\|^p_{W_h^{1,p}(\Omega),z,s}\,dz\right)^{1/p}.$$

Next, we choose s such that $n/p < s < k + n/p$. Then $|\ln h|^{\bar{\bar{s}}(p)} = 0$, and by (5.1)–(5.2)

$$\|\nabla(u_h - v)\|_{L^p(\Omega)} \lesssim \|u - v\|_{W_h^{1,p}(\Omega)}.$$

The estimate (1.11) then follows from the triangle inequality.

To obtain the L^p estimate (1.12), we use a standard duality argument. To this end, let $\varphi \in W_0^{1,2}(\Omega)$ satisfy

$$\begin{aligned} \mathcal{L}^*\varphi &= \operatorname{sgn}(u-u_h)|u-u_h|^{p-1} && \text{in } \Omega,\\ \varphi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

By (1.6), $\varphi \in W^{2,p'}(\Omega)$ and there holds

$$\|\varphi\|_{W^{2,p'}(\Omega)} \lesssim \||u-u_h|^{p-1}\|_{L^{p'}(\Omega)} = \|u-u_h\|^{p-1}_{L^p(\Omega)}. \tag{5.5}$$

It then follows that for any $v \in V_h$

$$\begin{aligned} \|u-u_h\|^p_{L^p(\Omega)} &= A(u-u_h,\varphi)\\ &= A(u-u_h,\varphi - v)\\ &\lesssim \|u-u_h\|_{W_h^{1,p}(\Omega)}\|\varphi - v\|_{W_h^{1,p'}(\Omega)}. \end{aligned}$$

Therefore by Lemma 2.3 and (5.5) we have

$$\begin{aligned} \|u-u_h\|^p_{L^p(\Omega)} &\lesssim h\|u-u_h\|_{W_h^{1,p}(\Omega)}\|\varphi\|_{W^{2,p'}(\Omega)}\\ &\lesssim h\|u-u_h\|_{W_h^{1,p}(\Omega)}\|u-u_h\|^{p-1}_{L^p(\Omega)}. \end{aligned}$$

Dividing by $\|u-u_h\|^{p-1}_{L^p(\Omega)}$ in the last expression completes the proof.

6. Numerical Experiments

In this section, we perform some numerical experiments in two dimensions which back up some of the theoretical findings. In all tests below we take $\mathbf{A} = I_{2\times 2}$, $\mathbf{b} = \mathbf{0}$, and $c = 0$.

6.1. Numerical Test 1. In this test, we compute (1.2) with $\eta = 15$ and $\Omega = (0,1)^2$ for varying h and k-values and choose the data such that that the exact solution is

$$u = \sin(10\pi x_1)\sin(10\pi x_2).$$

We record the errors in Table 1 along with with the corresponding rates of convergence. The numerical experiments indicate that $\|u-u_h\|_{L^\infty(\Omega)} = O(h^{k+1})$ and $\|\nabla(u-u_h)\|_{L^\infty(\Omega)} = O(h^k)$, which is in accordance with Theorems 1.1–1.2.

TABLE 1. Numerical Test 1: L^∞ errors and rates of convergence with respect to h. $\Omega = (0,1)^2$ and $\eta = 15$.

	h	$\|u - u_h\|_{L^\infty(\Omega)}$	rate	$\|\nabla(u - u_h)\|_{L^\infty(\Omega)}$	rate
$k = 1$	1/8	8.58E+00		1.01E+02	
	1/16	9.79E-01	3.13	8.54E+01	0.24
	1/32	3.28E-01	1.58	5.47E+01	0.64
	1/64	8.24E-02	1.99	3.10E+01	0.82
	1/128	2.19E-02	1.91	1.60E+01	0.96
	1/256	5.34E-03	2.03	8.24E+00	0.96
$k = 2$	1/8	5.99E-01		1.14E+02	
	1/16	1.11E-01	2.43	6.68E+01	0.77
	1/32	1.68E-02	2.72	2.22E+01	1.59
	1/64	1.89E-03	3.15	5.58E+00	1.99
	1/128	2.48E-04	2.93	1.31E+00	2.10
	1/256	2.80E-05	3.15	3.30E-01	1.98

6.2. Numerical Test 2. In this test, we compute (1.2) with $\eta = 50$ and $\Omega = B(0,1) \subset \mathbf{R}^2$ for varying h and k-values and choose the data such that that the exact solution is

$$u = \sin(5\pi x_1)\sin(5\pi x_2)(x_1^2 + x_2^2 - 1).$$

Again, we record the errors in Table 2 along with with the corresponding rates of convergence. Similar to the previous test, we observe $\|u - u_h\|_{L^\infty(\Omega)} = O(h^{k+1})$ and $\|\nabla(u - u_h)\|_{L^\infty(\Omega)} = O(h^k)$.

TABLE 2. Numerical Test 2: L^∞ errors and rates of convergence with respect to h. $\Omega = B(0,1)$ and $\eta = 50$.

	h	$\|u - u_h\|_{L^\infty(\Omega)}$	rate	$\|\nabla(u - u_h)\|_{L^\infty(\Omega)}$	rate
$k = 1$	1/8	4.36E-01		2.07E+01	
	1/16	1.35E-01	1.69	1.21E+01	0.78
	1/32	3.49E-02	1.96	6.15E+00	0.97
	1/64	8.59E-03	2.02	3.28E+00	0.91
	1/128	2.19E-03	1.97	1.70E+00	0.95
$k = 2$	1/8	9.16E-02		8.72E+00	
	1/16	1.07E-02	3.09	3.37E+00	1.37
	1/32	1.58E-03	2.77	8.53E-01	1.98
	1/64	2.13E-04	2.89	2.25E-01	1.92
	1/128	3.04E-05	2.81	6.04E-02	1.90

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