NUMERICAL STUDY OF TIME-PERIODIC SOLITONS IN THE DAMPED-DRIVEN NLS

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Abstract. We study localised attractors of the parametrically driven, damped nonlinear Schrödinger equation. Time-periodic solitons of this equation are obtained as solutions of the boundary-value problem on a two-dimensional domain. Stability and bifurcations of periodic solitons and their complexes is classified. We show that the bifurcation diagram can be reproduced using a few-mode approximation.

Key words. Nonlinear Schrödinger equation, temporally periodic solitons, newtonian iterative scheme, numerical continuation, stability, bifurcations

1. Introduction

We investigated the parametrically driven damped nonlinear Schrödinger equation (NLS),

\[ i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi. \]

that describes a large number of resonant phenomena in various physical media: nonlinear Faraday resonance in a vertically oscillating water trough [15], [16], [23]; the effect of phase-sensitive amplifiers on solitons in optical fibers [13], [18], [14]; magnetization waves in an easy-plane ferromagnet placed in a combination of a static and microwave field [4]; the amplitude of synchronized oscillations in vertically vibrated pendula lattices [12], [2], [11] etc. More applications of Eq.(1) are listed in [8, 9].

In Eq. (1), \( \gamma > 0 \) is the damping coefficient, \( h > 0 \) the amplitude of the parametric driver, and symbol \( * \) means the complex conjugation.

Equation (1) exhibits different classes of soliton solutions existing on the \((h, \gamma)\)-plane above the straight line \( h = \gamma \).

Two stationary solitons \( \psi_+ \) and \( \psi_- \) are available in analytic form [4]:

\[ \psi_{\pm}(x) = A_{\pm}e^{-i\theta_{\pm}}\text{sech}(A_{\pm}x), \]

where

\[ A_{\pm} = \sqrt{1 \pm \sqrt{h^2 - \gamma^2}}, \]
\[ \theta_+ = \frac{1}{2}\arcsin \frac{\gamma}{h}, \quad \theta_- = \frac{\pi}{2} - \theta_+. \]

The soliton \( \psi_{-}(x) \) is known to be unstable for all \( h \) and \( \gamma \). Stability properties of the soliton \( \psi_{+}(x) \) for various \( h \) and \( \gamma \) were examined in [4].

Other localised attractors of Eq. (1) (that have been found in numerical simulations) include: stationary multi-soliton complexes [5], uniformly travelling solitons and complexes [6, 7], time-periodic and quasi-periodic solitons [1, 10].

In this paper, we study time-periodic attractors of Eq. (1) that arise as a Hopf bifurcation of stable stationary soliton solutions. Attractors of periodic solitons on the \((h, \gamma)\)-plane were obtained in [10] on the basis of direct numerical simulation.
of Eq. (1) with initial condition in the form of stationary soliton $\psi_+$. In [22, 20] these were reobtained as solutions of a two-dimensional boundary-value problem for Eq. (1). Here, we employ the same numerical approach for our numerical analysis of time-periodic solitons. Our purpose is to clarify transformations and interconnections between coexisting periodic one- and two-soliton branches in the region of parameter $\gamma \gtrsim 0.35$.

In Section 2, we formulate the 2D boundary-value problem and describe our numerical approach. Results of numerical study are discussed in Section 3. We present the branches of time-periodic one- and two-soliton solutions for $\gamma = 0.35$. Also, we demonstrate the spatially nonsymmetric time-periodic two-soliton complex for $\gamma = 0.41$. In Section 4, a simple few-mode approximation of the 2D nonlinear boundary value problem has been suggested. Main results have been summarized in Section 5.

2. Numerical approach

2.1. Formulation of 2D boundary-value problem. We consider the time-periodic solutions Eq. (1) as solutions of the boundary value problem on the two-dimensional domain $(-\infty, \infty) \times (0, T)$. The boundary conditions are

$$\psi(x, t) = 0 \quad \text{as } x \to \pm \infty, \quad \text{and } \psi(x, t + T) = \psi(x, t).$$

The 2D boundary-value problem (1),(3) is solved numerically for the unknown time-periodic function $\psi(x, t)$, where the period $T$ is also unknown.

Letting $\tilde{\psi}(x, \tilde{t}) = \psi(x, t)$, the boundary-value problem (1),(3) can be reformulated on the rectangle $(-L, L) \times (0, 1)$ (where $L$ is chosen to be sufficiently large):}

$$F \equiv i\tilde{\psi}_{\tilde{\psi}}(x, \tilde{t}) + T\Phi(\tilde{\psi}(x, \tilde{t}), h, \gamma) = 0,$$

$$(4) \quad \tilde{\psi}(\pm L, \tilde{t}) = 0,$$

Here,

$$\Phi(\tilde{\psi}(x, \tilde{t}), h, \gamma) = \tilde{\psi}_{xx} + 2|\tilde{\psi}|^2\tilde{\psi} - \tilde{\psi}^3 - h\tilde{\psi}^* + i\gamma \tilde{\psi}.$$

Eq. (4) is supplemented with an additional equation borrowed from [19]:

$$Re\Phi(\tilde{\psi}^*(x^*, \tilde{t}^*), h, \gamma) = 0, \quad x^* = t^* = 0.$$

Solutions $(T, \tilde{\psi})$ of the 2D boundary-value problem (4-6) were path-followed in $h$ for the fixed $\gamma$. The time-independent solution at the point of Hopf bifurcation is used as starting point of the continuation process. At each value of parameter $h$ we employ Newtonian iteration scheme presented in 2.2. Continuation algorithm is described in 2.3.

In what follows, we omitted tildes above $\psi$ and $t$.

For the graphical representation of solutions we are using the averaged energy defined by

$$\bar{E} = \frac{1}{T} \int_0^T dt \int_{-\infty}^{\infty} dx E(x, t),$$

where

$$E(x, t) = |\psi_x|^2 + |\psi|^2 - |\psi|^4 + hRe(\psi^2).$$

Note that the energy $\int Edx$ is not an integral of motion for $\gamma \neq 0$.

Stability of solutions is classified by examining the Floquet multipliers of the corresponding linearized equation. Details are in [22, 8].
2.2. Newtonian iteration scheme. The iteration process is based on the continuous analog of Newton's method [17]. The idea of this approach is to replace the system (4)-(6) by the set of auxiliary evolution equations with respect to a fictitious continuous parameter $\tau$:

$$
\begin{align*}
\psi_{k+1} &= \psi_k + \tau_k \psi_k ; \\
T_{k+1} &= T_k + \tau_k \mu_k ;
\end{align*}
$$

(11)

where $k$ is number of Newtonian iteration, $0 < \tau_k \leq 1$ is numerical parameter, then

$$
V_k = v_k^{(1)} + \mu_k v_k^{(2)} ;
$$

(12)

where functions $v_k^{(1)}$ and $v_k^{(2)}$ are numerical solutions of two linear two-dimensional boundary-value problems

$$
\begin{align*}
\frac{\partial}{\partial x} v_k^{(1)} + T_k \frac{\partial^2}{\partial x^2} v_k^{(1)} + \hat{A}_k v_k^{(1)} + \hat{B}_k v_k^{(1)*} &= -F_k , \\
v_k^{(1)}(\pm L, t) &= \psi(\pm L, t) , \\
v_k^{(1)}(x, 0) - v_k^{(1)}(x, 1) &= -[\psi(x, 0) - \psi(x, 1)] ;
\end{align*}
$$

(13)

and

$$
\begin{align*}
\frac{\partial}{\partial x} v_k^{(2)} + T_k \frac{\partial^2}{\partial x^2} v_k^{(2)} + \hat{A}_k v_k^{(2)} + \hat{B}_k v_k^{(2)*} &= -\Phi_k , \\
v_k^{(2)}(\pm L, t) &= 0 , \\
v_k^{(2)}(x, 0) - v_k^{(2)}(x, 1) &= 0 .
\end{align*}
$$

(14)

Here, $F_k$ and $\Phi_k$ are defined, respectively, by Eqs. (4) and (5), $A_k$ and $B_k$ are of the form:

$$
A_k = 4T_k \psi_k \psi_k^* - T_k - i\gamma T_k ;
$$

(15)

$$
B_k = 2T_k (\psi_k^2)^2 - \hbar T_k .
$$

(16)

Formulas for the calculation of the quantity $\mu_k$ at each iteration follow from the Eq. (10):

$$
\mu_k = \frac{-G - R}{F} ,
$$

(17)

where

$$
\begin{align*}
F &= [v_R^{(2)}]_{xx}^2 + 6\psi_R^2 v_R^{(2)} + 4\psi_T^2 v_T^{(2)} + 2\psi_T^2 v_{xR}^{(2)} - v_R^{(2)} - \hbar v_R^{(2)} - \gamma v_T^{(2)} , \\
G &= [v_R^{(1)}]_{xx}^2 + 6\psi_R^2 v_R^{(1)} + 4\psi_T^2 v_T^{(1)} + 2\psi_T^2 v_{xR}^{(1)} - v_R^{(1)} - \hbar v_R^{(1)} - \gamma v_T^{(1)} .
\end{align*}
$$
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\[ R = [\psi_R]_{xx} + 2\psi_R^3 + 2\psi_R^2 \psi_R - \psi_R - h\psi_R - \gamma\psi_I \]
\[ \psi_R = \Re \psi_k(x^*, 0); \quad \psi_I = \Im \psi_k(x^*, 0); \]
\[ \psi^{(1,2)}_R(x^*, 0); \quad \psi^{(1,2)}_I(x^*, 0) \]

We iterate (11-17) until the inequality
\[ \| F \| \leq \delta \]
holds true, where \( \| \cdot \| \) is the standard C-norm and \( \delta > 0 \) is a small number chosen beforehand. (Typically, we took \( \delta = 10^{-4} \).)

Most calculations were performed on the two-dimensional interval \((-L, L) = (-50, 50)\), with the stepsizes of the fourth-order finite-difference approximation being \( \Delta x = 0.05 \) and \( \Delta t = 0.01 \).

2.3. Numerical continuation algorithm. For numerical continuation we employed an approach suggested in [21].

At each \((i + 1)\)-th step of numerical continuation the initial guess of function \( \psi \) and period \( T \) for Newtonian process was constructed using the results obtained for two previous, \( i \)-th and \((i - 1)\)-th values of parameter \( h \):

\[ \psi(h_{i+1}) = \psi(h_i) + (h_{i+1} - h_i) \cdot \frac{\psi(h_i) - \psi(h_{i-1})}{h_i - h_{i-1}} \]
\[ T(h_{i+1}) = T(h_i) + (h_{i+1} - h_i) \cdot \frac{T(h_i) - T(h_{i-1})}{h_i - h_{i-1}} \]

The \((i + 1)\)-th increment \( \Delta h_{i+1} = h_{i+1} - h_i \) is calculated as follows

\[ \Delta h_{i+1} = h_i \cdot \frac{T(h_{i-1}) - T(h_{i-2})}{T(h_i) - T(h_{i-1})} \]

Turning points of the bifurcation curve \( E(h) \) (where the direction of continuation changes as we move to the new branch) are identified with help of the following relation that is tested at each step of numerical continuation:

\[ \frac{h_i - h_{i-1}}{E(h_i) - E(h_{i-1})} < \epsilon, \]

where \( \epsilon > 0 \) is small known quantity. Note that (21) is a simple approximation of equality \( dh/dE = 0 \) that is valid at the turning points of the curve \( h(E) \).

In case we run into a turning point the sign of \( h \)-increment should be changed.

The choice of the initial guess of the form (18)-(19) and the use of the adaptive formula (20) for the increment \( \Delta h \) prevents the continuation from reversing to the previous branch of \( E(h) \) and provides a quick convergence of Newtonian iteration process at each step of numerical continuation.

3. Results of numerical study

3.1. Hopf bifurcation points. In [20, 22], we applied the above numerical approach to periodic single-solitons solutions with \( \gamma = 0.3, 0.265 \), and to periodic two-soliton solutions with the strong damping \( \gamma = 0.565 \) as representative sections of various parts of the attractor chart in [10]. Here, we focus on the region around \( \gamma \gtrsim 0.35 \) for which the periodic one- and two-soliton solutions coexist. Indeed, \( \psi_+ \) solutions undergo Hopf bifurcation for \( \gamma < 0.356 \) [4]. On the other hand, stable stationary two-soliton complexes \( \psi_{(+)} \) (and corresponding Hopf bifurcation points) exist only for \( \gamma > 0.345 \).
Figure 1. (a) The energy of the periodic solutions with $\gamma = 0.30$ and $\gamma = 0.35$. The solid curves show the stable and the dashed ones unstable branches. The absolute value of the periodic solution with $\gamma = 0.35$: (b) $h = 0.95$, $T = 2.329$, stable branch; (c) $h = 0.85$, $T = 2.337$, unstable branch. In each case several periods of oscillation are shown.
First, we consider the case $\gamma = 0.35$. The Hopf bifurcation point for the single soliton is known to be $h_{H0} = 0.75$ [4]. Soliton $\psi_+$ is stable in case $\gamma < h < h_{H0}$. At the point $h_{H0}$ the stationary soliton loses stability and the branch of temporally periodic soliton detaches [8].

Stability properties and bifurcations of stationary two-soliton solutions $\psi_{(+)}$ were analyzed, numerically, on the basis of solving the corresponding linearized eigenvalue problem [5]. It was established [9] that the stationary two-soliton complex undergoes two Hopf bifurcations for $0.345 < \gamma < 0.39$. Each Hopf bifurcation gives a rise to a coexisting branches of temporally periodic two-soliton solutions.

The stability domain of the $\psi_{++}$ soliton with $\gamma = 0.35$ is bounded by two Hopf bifurcations: $h_{H1} \leq h < h_{H2}$ where $h_{H1} = 0.806$ and $h_{H2} = 0.832$. So, in the case $\gamma = 0.35$ the periodic single-soliton solution coexists with the periodic two-soliton complexes; however our calculations show that the corresponding branches are not connected.

Second, Hopf bifurcation of the $\psi_+$ soliton does not occur for $\gamma > 0.356$ while two-soliton complex undergoes four Hopf bifurcations for $\gamma$ between 0.39 and 0.413. We take $\gamma = 0.41$ to exemplify this case and consider one of these Hopf bifurcations in Section 3.4.

3.2. Time-periodic single-soliton solutions; $\gamma=0.35$. We start the numerical continuation from the point $h_H = 0.75$. Our investigation shows that in the case $\gamma = 0.35$, the transformation of the solution is similar to the one in the case $\gamma = 0.30$; see Fig. 1.

The left endpoint of each of the two curves in Fig. 1a corresponds to the stationary single-soliton solution $\psi_+$. The corresponding value of $h$ equals $h_{H0} = 0.385$ for $\gamma = 0.30$ and $h_{H0} = 0.750$ for $\gamma = 0.35$. At this value of $h$ a stable time-periodic soliton is born. Near the leftmost point of the curve $\bar{E}(h)$ in Fig. 1a the periodic solution looks like a single soliton with periodically oscillating amplitude and width; see Fig. 1b.

The periodic solution loses its stability at the turning point $h_{sn}$; $h_{sn} = 0.8761$ for $\gamma = 0.30$ and $h_{sn} = 1.0186$ for $\gamma = 0.35$. As we continue back along the unstable branch of the curve $\bar{E}(h)$, the oscillating solution evolves into a three-hump structure which may be interpreted as a triplet of solitons, see Fig. 1c. The end point of the unstable branch ($h = 0.61$ for $\gamma = 0.30$ and $h = 0.760$ for $\gamma = 0.35$) corresponds to a stationary unstable three-soliton complex $\psi_{(---)}$.

3.3. Time-periodic two-soliton complexes; $\gamma=0.35$ and $\gamma=0.38$. The first, “lower” Hopf bifurcation occurs at $h_{H1} = 0.806$ for $\gamma = 0.35$. This bifurcation is supercritical; for $h < h_{H1}$, the unstable stationary two-soliton solution is replaced by a stable periodic two-soliton complex. Results of numerical continuation are presented in Fig. 2a.

As we continue the periodic complex in the direction of smaller $h$, the periodic solution is stable while $0.79 < h < 0.806$; representative stable solution is shown in Fig. 2b for $h = 0.795$. At $h = 0.79$ it loses its stability to a double-periodic complex of two solitons. As we continue, the unstable branch $\bar{E}(h)$ makes a number of turns (Fig. 2a) and the spatiotemporal complexity of the solution increases (see Fig. 2c) but it never regains its stability.

The second, “upper” Hopf bifurcation occurs at $h_{H2} = 0.832$ for $\gamma = 0.35$. This bifurcation is subcritical: the emerging periodic branch is unstable and coexists with
Figure 2. (a) The first ("lower") $\bar{E}(h)$ branch of the two-soliton periodic solution for $\gamma = 0.35$. The solid curve marks the stable and the dashed one unstable branch. The circle indicates the starting point $h_{H1} = 0.806$ of the continuation. (b) A stable two-soliton periodic solution with $h = 0.795$, $T = 8.788$. (c) A two-soliton periodic solution with complex temporal behavior arising at the end point of the curve presented for $\gamma = 0.35$, $h = 0.741$, $T = 15.9$. Both figures (b) and (c) show the solution over one period.
Figure 3. The second (“upper”) $\bar{E}(h)$ (a) and $T(h)$ (b) branches of the two-soliton periodic solution for $\gamma = 0.35$ and $\gamma = 0.38$. The solid curve marks the stable and the dashed one unstable branch. The circles indicate the starting point of the continuation (the point where the stationary two-soliton complex undergoes the Hopf bifurcation). (c) The two-soliton periodic solution on the stable branch. Here $\gamma = 0.38$, $\bar{h} = 0.95$, $T = 2.476$. Several periods of oscillation are shown.
the stable stationary branch (i.e. the periodic branch continues in the direction of lower $h$, see Fig. 3a,b). The entire branch is unstable in the case $\gamma = 0.35$.

However, the increase of $\gamma$ results in the stabilization of the periodic two-soliton solution. This is exemplified by $\gamma = 0.38$; the corresponding “upper” Hopf bifurcation point is $h_{H2} = 0.89$. The second branch in Fig. 3a,b features a stable interval $h_1 < h < h_2$, with $h_1 = 0.9415$ and $h_2 = 1.015$. At the bifurcation points $h_{1,2}$ the periodic two-soliton solution loses stability to a quasi-periodic two-soliton complex. The Fig. 3b demonstrates the stable periodic solution for $h = 0.95$.

3.4. Nonsymmetric time-periodic two-soliton complex; $\gamma=0.41$. We have already noted that for $\gamma = 0.41$ the stationary complex $\psi(\pm)$ undergoes four Hopf bifurcation [9]. Here, we only consider the case of “upper” Hopf bifurcation $h_{H4} = 1.037$. In contrast with all other periodic branches, the time-periodic solutions emanating from $h = h_{H4}$ are not symmetric in space.

The Hopf bifurcation is supercritical: the emerging periodic branch is stable and extends up in $h > h_{H4}$. Corresponding $T(h)$ curve is shown at the Fig. 4a. Solution is stable between $h = h_{H4}$ and the point $h = 1.049$ where the periodic solution loses its stability and the $T(h)$ branch turns back. After several turns, the periodic branch rejoins the branch of (unstable) stationary complexes $\psi(\pm)$ at the endpoint $h = 1.082$.

Representative stable and unstable solutions are shown on Fig. 4b,c. It can be seen that the complex consists of two identical solitons that oscillate out-of-phase.

4. Few-mode approximation

With all its advantages, the approach to periodic solitons as solutions of a 2D boundary-value problem has a serious shortcoming. The computational capacity is quickly saturated as any of the two sides of the 2D domain is enlarged. As a result, one cannot access solutions with long periods or slow spatial decay. In this section, we explore a possibility of using a truncated Fourier expansion to reduce the computationally expensive two-dimensional boundary-value problem to a problem on a one-dimensional interval.

We start by decomposing $\psi$ as

$$\psi = A_+ [U(\bar{x}, \bar{t}) + iV(\bar{x}, \bar{t})] e^{-i\theta_+},$$

where $A_+$ and $\theta_+$ are as in Eq. (2), and $\bar{t} = A_+^2 t$, $\bar{x} = A_+ x$. This casts Eq. (1) in the following form:

$$-V_t - 2TV = -U_{xx} + U - 2(U^2 + V^2)U,$$
$$+U_t + 2HV = -V_{xx} + V - 2(U^2 + V^2)V$$

where we have omitted bars above $x$ and $t$. Here we have introduced $\Gamma = \gamma/A_+^2$ and $H = \sqrt{h^2 - \gamma^2/A_+^4}$.

The periodic functions $U(x, t)$ and $V(x, t)$ can be expanded in the Fourier series

$$U = \sum_{n=-N}^{N} U_n e^{in\Omega t} + c.c., \quad V = \sum_{n=-N}^{N} V_n e^{in\Omega t} + c.c.$$

Here $N = \infty$, $\Omega = 2\pi/(A_+^2 T)$ where $T$ is the period of solution to Eq. (1).

Near the Hopf bifurcation point, the Fourier amplitudes $U_n, V_n$ scale as $e^{n|}$ [4]. Computer analysis indicates that this scaling law remains valid over large regions.
Figure 4. (a) The $T(h)$ branch of periodic two-soliton complex for $\gamma = 0.41$. The solid curve marks the stable and the dashed one unstable branch. The empty circle marks the starting point of the continuation $h_{H4} = 1.037$. The full circle marks the end point. The periodic two-soliton complex oscillating out of phase with each other (several periods of oscillation are shown): (b) the stable solution with $h = 1.0493$ and $T = 1.991$; (c) The unstable solution with $h = 0.9$ and $T = 3.07$. 
in the parameter space. Hence when \( \epsilon \) is small, higher harmonics have negligible amplitudes and the series can be truncated at some finite \( N \).

The crudest approximation results by retaining just the first and zeroth harmonics (i.e. taking \( N = 1 \) in (23)). To simplify the notation, we re-denote the amplitudes of the two modes as follows:

\[
\begin{align*}
U(x,t) &= u(x) + A(x)e^{i\Omega t} + A^*(x)e^{-i\Omega t}, \\
V(x,t) &= v(x) + B(x)e^{i\Omega t} + B^*(x)e^{-i\Omega t},
\end{align*}
\]

where \( u \) and \( v \) are real and \( A \) and \( B \) complex coefficient functions, decaying to zero as \( |x| \to \infty \). Substituting in the equations (22) and equating coefficients of like
harmonics, we get the three-mode reduced system

\[
\begin{align*}
\begin{cases}
    u_{xx} - u + 2(u^2 + v^2)u + 4(3|A|^2 + |B|^2)u + 4(A^*B + A^*B)v - 2\Gamma v = 0 \\
v_{xx} - v + 2(u^2 + v^2)v + 4(|A|^2 + 3|B|^2)v + 4(A^*B + A^*B)u + 2\Gamma v = 0, \\
A_{xx} - A + 2(3u^2 + v^2)A + 2|B|^2 + 2(2uv + A^*B)B - 2\Gamma B - i\Omega B = 0, \\
B_{xx} - B + 2(u^2 + 3v^2)B + 2(2uv + B^*A)A + 2HB + i\Omega A = 0.
\end{cases}
\end{align*}
\]

One homoclinic solution of the system (25) exists for all $H$ and $\Gamma$; it is $u = \text{sech} x$, $v = A = B = 0$. This solution corresponds to the stationary soliton $\psi_+(x)$ of equation (1). A nontrivial homoclinic solution (corresponding to a periodically oscillating soliton of Eq. (1)) bifurcates from it as $H$ is increased for the fixed $\Gamma$. The bifurcation point corresponds to the point of the Hopf bifurcation of the stationary soliton within Eq. (1).

Next, consider the reduced system including the first, zeroth and second harmonics, i.e. put $N = 2$. Letting

\[
\begin{align*}
U(x, t) &= u(x) + P(x)e^{i\Omega t} + c.c. + A^*(x)e^{2i\Omega t} + c.c., \\
V(x, t) &= v(x) + Q(x)e^{i\Omega t} + c.c. + B^*(x)e^{2i\Omega t} + c.c.,
\end{align*}
\]

one can obtain the five-mode reduced system with respect to real functions $u$ and $v$ and complex coefficient functions $P, Q, A$ and $B$, decaying to zero as $|x| \to \infty$:

\[
\begin{align*}
\begin{cases}
u_{xx} - u + 2u(u^2 + v^2 + 2(|P|^2 + |Q|^2) + 2(|Q|^2 + |B|^2)) \\
v_{xx} - v + 2v(u^2 + v^2 + 2(|P|^2 + |A|^2) + 6(|Q|^2 + |B|^2)) \\
P_{xx} - P + 2P(3u^2 + v^2 + 3|P|^2 + 2|Q|^2 + 6|A|^2 + 2|B|^2) \\
Q_{xx} - Q + 2Q(u^2 + 3u^2 + 2|P|^2 + 3|Q|^2 + 2|A|^2 + 6|B|^2) \\
A_{xx} - A + 2A(3u^2 + v^2 + 6|P|^2 + 2|Q|^2 + 3|A|^2 + 2|B|^2) \\
B_{xx} - B + 2B(u^2 + 3u^2 + 2|P|^2 + 6|Q|^2 + 2|A|^2 + 3|B|^2)
\end{cases}
\end{align*}
\]

The few-mode approximation was tested by continuing the periodic solution arising from the $\psi_+$ soliton with $\gamma = 0.3$ and 0.35.

The three-mode system (25) and the five-mode system (27) were solved, numerically, for the cases $\gamma = 0.3$ and $\gamma = 0.35$. We continued in $h$ starting from the Hopf bifurcation point. At each step of continuation the Newtonian iteration with the Numerov’s fourth-order approximation was employed. The curves $T(h)$ for the two-dimensional boundary-value problem (1),(3) and the corresponding curves for three- and five-mode systems are compared in Figs. 5 and 6.

Fig. 5 compares results obtained using the $N = 1$ and $N = 2$ reduced systems to the solution of the full PDE (1) for $\gamma = 0.3$. The solid curve was obtained by solving Eq. (1),(3) on the $(x, t)$-domain. The dash-pointed curve results from the solution
of the three-mode system Eq. (25). The dashed curve corresponds the five-mode approximation Eq. (27). It is seen that the few-mode approximation can reproduce, qualitatively, the transformation of periodic solitons as \( h \) is varied. Indeed, taking \( N = 1 \) was found to be sufficient to explain the transformation of a one-soliton periodic solution to a three-soliton complex. However the first and zeroth harmonics alone were not enough to reproduce the shape of the corresponding bifurcation curve. In order to make the approximation better, we have to increase number of modes.

The next simplest approximation involves the \( N = 2 \) truncation. Corresponding curves are shown by dashed lines in Fig. 5 for \( \gamma = 0.3 \) and Fig. 6 for \( \gamma = 0.35 \). It is seen that the three-soliton branch is reproduced more accurately than the one-soliton branch. The explanation is that the amplitude of the first harmonic (and hence all higher harmonics) is smaller in the three-soliton case (see Fig. 1b,c). The \( \psi_- \) solitons bound in the complex tend to damp the oscillations of the \( \psi_+ \) soliton.

Another observation is that the few-harmonic approximation is accurate for relatively large values of \( \gamma \). As \( \gamma \) is decreased, the amplitude of oscillations tends to grow. Accordingly, the amplitudes of higher harmonics (neglected in the approximation) become nonnegligible. On the other hand, when \( \gamma \) is growing, the agreement of five-mode approximation with solution of the boundary problem (1),(3) is improved. It is seen from comparison of results for \( \gamma = 0.3 \) and \( \gamma = 0.35 \) (see Fig. 5 and Fig. 6).

The reduced system allows one not only to determine the domain of existence of the breathers, but also to study their stability and bifurcations [3]. The shortcoming of the method is that if the amplitude of the temporal oscillation is not small enough, the reduced system with low \( N \) (such as \( N = 2 \)) may provide a poor approximation to the solution of the PDE. However, this drawback can be overcome simply by increasing \( N \), without risk of running out of computational resources.

5. Conclusions

The temporally periodic solitons of parametrically driven damped NLS are obtained numerically as solutions of the 2D boundary-value problem (1),(3). Our numerical approach allows to investigate transformations of time-periodic solitons and interconnection between coexisting branches of stable and unstable oscillating single solitons and complexes. New temporally periodic two-solitons solutions have been found.

We propose a simple few-mode approximation that reduces the study of periodic solitons to the solution of a system of ordinary differential equations for the (space-dependent) Fourier coefficients. This few-mode approximation is shown to correctly reproduce the one-soliton bifurcation diagram.

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