QR-BASED METHODS FOR COMPUTING LYAPUNOV EXPONENTS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. Lyapunov exponents (LEs) play a central role in the study of stability properties and asymptotic behavior of dynamical systems. However, explicit formulas for them can be derived for very few systems, therefore numerical methods are required. Such is the case of random dynamical systems described by stochastic differential equations (SDEs), for which there have been reported just a few numerical methods. The first attempts were restricted to linear equations, which have obvious limitations from the applications point of view. A more successful approach deals with nonlinear equations defined over manifolds but is effective for the computation of only the top LE. In this paper, two numerical methods for the efficient computation of all LEs of nonlinear SDEs are introduced. They are, essentially, a generalization to the stochastic case of the well known QR-based methods developed for ordinary differential equations. Specifically, a discrete and a continuous QR method are derived by combining the basic ideas of the deterministic QR methods with the classical rules of the differential calculus for the Stratanovich representation of SDEs. Additionally, bounds for the approximation errors are given and the performance of the methods is illustrated by means of numerical simulations.

Key words. Lyapunov Exponents, Stochastic Differential Equations, QR-decomposition, numerical methods.

1. Introduction

Since A.M. Lyapunov introduced the concept of characteristic exponents [27], it has played an important role in the study of the asymptotic behavior of dynamical systems. In particular, the Lyapunov exponents (LEs) have been extensively used for analyzing stability properties of dynamical systems [1]. Some other important contributions like [4] and [5] extend the classical theory of LEs from deterministic dynamical systems to Random Dynamical Systems (RDS). This has allowed the stability analysis of a wide class of physical, mechanical and engineering processes, where the randomness becomes an essential issue for modeling their dynamics [32], [3], [36], [39], [30], [20], [34], [26], [9].

It is well known that, with the exception of some simple cases as those mentioned in [2], explicit formulas for the LEs of Stochastic Differential Equations (SDEs) are rarely known. Alternatively, some different analytic expansions have been obtained for equations driven by particular sources of noise [37]. On the other hand, asymptotic expansions in terms of noise intensity have been obtained for LEs of two-dimensional equations driven by a small noise [6], [33]. Some other asymptotic expansions for LEs have been also obtained in [3] for more general cases of large, small and slow noises. Other types of parametric expansions have been reported for certain particular systems [22], [23], [8]. In principle, the truncation of any of such expansions could be used as a numerical method for computing LEs. However this procedure lacks generality since it is only applicable for certain particular cases of SDEs.
Indeed, the development of numerical methods for computing LEs of general SDEs is a relevant issue that has not received a systematic attention. In fact, just a few papers have addressed such subject with a relative success [36], [19], [38]. The seminal works in this respect are the numerical method proposed in [36] and [38] for the class of linear stochastic equations. Afterward, this method was extended to the general case of nonlinear SDEs defined either on a compact orientable manifold or on $\mathbb{R}^d$ [19]. It was based on the discretization of the SDE by particular integrators and the corresponding approximation of the LEs for the resulting ergodic Markov chains. From a practical point of view, this method is just effective for the computation of the top LE, but it leads to numerical instabilities for the remaining ones [36].

The aim of this paper is to introduce two alternative methods for the numerical computation of the LEs of nonlinear SDEs defined on $\mathbb{R}^d$. These methods are, essentially, a generalization to the stochastic case of the well known QR-based methods for the computation of the LEs of Ordinary Differential Equations (ODEs) [10], [11], [18], [16], [17]. Specifically, a discrete and a continuous QR method are obtained by combining the basic ideas of the deterministic methods with the classical rules of the differential calculus for the Stratanovich representation of SDEs. In particular, the discrete QR method generalizes the algorithm presented in [19] for the top LE. Moreover, in contrast with that algorithm, the methods introduced here allow the efficient computation of all LEs.

The outline of the paper is as follows. Basic notations as well as essential facts about LEs of SDEs are presented in Section 2. In Section 3 the discrete and continuous QR methods are derived, whereas bounds for the approximation errors are given in Section 4. In section 5, some suggestions for the implementation of the QR methods are presented. Finally, in the last section, the performance of the methods is illustrated by means of simulations.

2. Preliminaries

2.1. Notations. For any matrix $A$, let us denote by $A^k$, $A^{kl}$, $A|_k$ and $A|_{kk}$ the $k$-th column vector, the entry $(k,l)$, the matrix of first $k$ columns and the matrix of the first $k$ rows and columns of $A$, respectively. The Frobenius norm for matrices shall be denoted by $\|\cdot\|$, and the standard scalar product in $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$. For any $k \in \mathbb{Z}^+$ and $0 \leq \delta < 1$ let $C^{k,\delta}$ be the Banach space of $C^k$ vector fields on $\mathbb{R}^d$ with growth at most linearly and bounded derivatives of order 1 up to $k$, and whose $k$-th derivatives are globally $\delta$-Holder continuous. Finally, denote by $C^l_P(\mathbb{R}^d, \mathbb{R})$ the space of $l$ time continuously differentiable functions $g : \mathbb{R}^d \to \mathbb{R}$ for which $g$ and all its partial derivatives up to order $l$ have polynomial growth.

2.2. Lyapunov Exponents of Nonlinear SDEs. Let $\{F_t, t \geq 0\}$ be an increasing right continuous family of complete sub $\sigma$-algebras of $\mathcal{F}$ and let $(\Omega, \mathcal{F}, \mathcal{P})$ be the canonical Wiener space on $\mathbb{R}^+$ (i.e., $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\}, \mathcal{F} = \mathcal{B}(\Omega)$ is the Borel $\sigma$-algebra of $\Omega$, and $\mathcal{P}$ is the Wiener measure in $\mathcal{F}$).

Consider the Stratanovich SDE on $\mathbb{R}^d$,

\begin{equation}
    dx_t = \sum_{i=0}^m f_i(x_t) \circ dw_t^i, \quad x_{t_0} = x_0, \quad t \geq t_0 \geq 0,
\end{equation}

and the variational equation on $\mathbb{R}^d \times \mathbb{R}^d$,
\( \frac{dV_t}{dt} = \sum_{i=0}^{m} Df_i(x_t) V_t \circ dw_t^i, \quad V_{t_0} = I_d, \quad t \geq t_0 \geq 0, \)

where \( w = (w^1, ..., w^m) \) is an \( m \)-dimensional \( \mathcal{F}_t \)-adapted standard Wiener process, with the convention \( dw_t^i = dt \). Here, it is assumed that the functions \( f_0, f_1, ..., f_m \) satisfy

\( f_0 \in C^{k,\delta}_b, f_1, ..., f_m \in C^{k+1,\delta}_b, \sum_{i=1}^{d} \sum_{j=1}^{d} f_i^j \frac{\partial}{\partial x_j} f_i^j \in C^{k,\delta}_b \)

for some \( k \geq 1 \) and \( \delta > 0 \); and \( f_0 \) is such that the hypothesis:

\( \exists \beta > 0 \) and \( K \subset \mathbb{R}^d \) compact, s.t. \( \langle x, f_0(x) \rangle \geq -\beta \|x\|^2, \forall x \in \mathbb{R}^d \setminus K \)

holds. Further, the infinitesimal generator of the process \( x_t \) that solves (1),

\[ L = f_0 + \frac{1}{2} \sum_{i=1}^{m} f_i^2 \]

is assumed to be strongly hypoelliptic in the sense that \( \dim \mathcal{L}(f_0, f_1, ..., f_m) = d \) for all \( x \in \mathbb{R}^d \), where \( \mathcal{L}(f_0, f_1, ..., f_m) \) denotes the Lie algebra generated by the vector fields \( f_0, ..., f_m \).

Denote by \( (\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{R}}) \) the ergodic metric dynamical system defined on \( (\Omega, \mathcal{F}, \mathcal{P}) \), where

\[ \theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t), \quad t \in \mathbb{R} \]

is the measuring preserving and ergodic shift transformation (see appendices in [5] for details). It is well known (Theorem 2.3.32 in [5]) that equation (1) generates a \( C^k \) RDS (or cocycle) \( \varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) over \( \theta \). Therefore, the system (1)-(2) generates the \( C^{k-1} \) RDS \( (\varphi, D\varphi) \) over \( \theta \), which means that \( D\varphi \) solves the variational equation (2) and is a linear matrix cocycle over the skew-product (metric dynamical system) \( \Theta = (\theta, \varphi) \).

In addition, let \( \mu \) be an ergodic invariant measure with respect to the cocycle \( \varphi \), i.e., a ergodic measure on \( (\Omega \times \mathbb{R}^d, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)) \) that satisfy \( \Theta(t) \mu = \mu \) for all \( t \geq 0 \) and \( \pi_\Omega \mu = \mathcal{P} \), where \( \pi_\Omega \) denotes the projection onto \( \Omega \) (see Remark below). Thus, the multiplicative ergodic theorem (MET) of Oseledets ([31]) for linear cocycles asserts the regularity and the existence of the Lyapunov spectrum (see also pages 1-26 in [4] for an excellent review). In particular, by Theorem 4.2.13 in [5], there exits an invariant set \( \tilde{\Omega} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d) \) of full \( \mu \)-measure, constants \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p \), \( p \leq d \) (the LEs of the system (1)-(2)), numbers \( d_i \in \mathbb{N} \) with \( \sum_{i=1}^{p} d_i = d \) (their multiplicities), and a random splitting

\[ \mathbb{R}^d = E_1(\omega) \oplus ... \oplus E_p(\omega) \]

of \( \mathbb{R}^d \) into measurable random subspaces \( E_i(\omega), \omega \in \tilde{\Omega} \) (the so-called Oseledets splitting) with \( \dim(E_i(\omega)) = d_i \) such that

\[ \lambda_j = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|V_{t\theta_p}\| \quad \text{if and only if} \quad p_j \in E_j(\omega) \setminus \{0\}. \]

Here, the set of vectors \( \{p_1, ..., p_d\} \) is an orthonormal basis of \( \mathbb{R}^d \), which is usually called the Lyapunov normal basis. Moreover, the Lyapunov regularity condition
can be expressed as

\[
\sum_{j=1}^{d} \lambda_j = \limsup_{t \to \infty} \frac{1}{t} \log |\det(V_t)|.
\]

**Remark** According to Theorem 2.3.45 in [5], \( \mu = P \times \rho \) is an invariant measure respecting to \( \varphi \) if and only if \( \rho \) is a stationary measure of the Markov transition probability of the one-point motion of \( \varphi \). In particular, \( \mu \) is ergodic if and only if \( \rho \) is ergodic. Thus, to fulfill the requirements of the Multiplicative Ergodic Theorem it is enough to choose an ergodic measure \( \rho \) and then define \( \mu \) as:

\[
\mu \triangleq P \times \rho.
\]

To do this just recall that the one-point motion of \( \varphi \) has a unique invariant probability measure \( \rho \) with positive density \( \nu \) respecting to the Lebesgue measure on \( \mathbb{R}^d \) if, for instance, the infinitesimal generator \( L \) is elliptic, or more generally if \( L \) is strongly hypoelliptic [21] and condition (4) holds [19].

3. **QR methods for computing LEs**

In this section the discrete and the continuous QR method for computing the LEs of (1)-(2) are derived. Basically, the same formulation of [16] and [17] for the QR methods in ODEs is followed here. Thus, the continuous QR factorization of \( V_t \) shall be the main tool for our computational purposes.

3.1. **QR formulation for LEs.** Let us rewrite the equation (2) as

\[
dV_t = \sum_{i=0}^{m} A_i(t) V_t \circ dw_i, \quad V_{t_0} = I_d,
\]

where \( A_j(t) = Df_j(x_t) \), and consider the continuous QR factorization of \( V_t \),

\[
V_t = Q_t R_t,
\]

where \( Q_t \) is orthogonal and \( R_t \) is upper triangular with positive diagonal elements \( R_{jj} \), \( j = 1, \ldots, d \). Hence, by the norm-preserving property of \( Q_t \) it is obtained

\[
\lambda_j = \lim_{t \to \infty} \frac{1}{t} \log \|V_tp_j\| = \lim_{t \to \infty} \frac{1}{t} \log \|R_tp_j\|,
\]

where \( \{p_j\}, j = 1, \ldots, d \) denotes any orthonormal basis associated to the splitting of \( \mathbb{R}^d \).

Moreover, for each \( 1 \leq k \leq d \), the solution matrix \( V_t^{(k)} \) may be considered as a diffusion process in the Grassmann bundle \( G_k(\mathbb{R}^d) \) (the manifold of the \( k \)-dimensional subspaces of \( \mathbb{R}^d \)). Then, for each \( t \), the matrix \( V_t^{(k)} \) may be seen as the direction of \( V_te_1 \wedge V_te_2 \wedge \ldots \wedge V_te_k \) in the \( k \)th exterior power \( \Lambda^k(\mathbb{R}^d) \), where \( \{e_i\}_{i=1, \ldots, d} \) is the standard basis of \( \mathbb{R}^d \). Hence, according to corollary 2.2 in [7],

\[
\lambda_1 + \lambda_2 + \ldots + \lambda_k = \lim_{t \to \infty} \frac{1}{t} \log \|V_te_1 \wedge V_te_2 \wedge \ldots \wedge V_te_k\|
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \log \sqrt{\det((<V_te_i, V_te_j>_{i,j=1,\ldots,k})}
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \log \sqrt{\det((V_t^{(k)})^T V_t^{(k)})} \quad \text{a.s.}
\]
Since $V_t^k = Q_t^k R_t^{kk}$ then,

$$
\lambda_1 + \lambda_2 + \ldots + \lambda_k = \lim_{t \to \infty} \frac{1}{t} \log \left| \det \left( (R_t^{kk})^T R_t^{kk} \right) \right| = \lim_{t \to \infty} \frac{1}{t} \log \left| \det (R_t^{kk}) \right|
$$

which implies

(6)  
$$
\lambda_j = \lim_{t \to \infty} \frac{1}{t} \log \left| R_t^{jj} \right| \quad \text{a.s.}
$$

In practice, the expression above should be approximated at a finite instant of time $T \geq t_0$. This leads to the following definition.

**Definition 3.1.** For a given time $t_0 \leq T < \infty$, the truncated LEs at $T$ are defined by

$$
\lambda_j (T) = \frac{1}{T - t_0} \log \left| R_T^{jj} \right|
$$

**3.2. Discrete QR method.** Let

$$(t)_h = \{ t_0 \leq t_1 \leq \ldots \leq t_n < \ldots < \infty \}
$$

be a time partition defined as a sequence of $\mathcal{F}_{t_n}$-measurable random times $t_n$, $n = 0, 1, \ldots$, that satisfy

$$
\sup_n (t_{n+1} - t_n) \leq h < 1, \ w.p.1,
$$

and define

$$
v_t := \max \{ n = 0, 1, 2, \ldots ; t_n \leq t \} < \infty.
$$

The main idea of the discrete QR method is to indirectly compute the fundamental solution matrix of the equation (2) by successive QR factorization of $V_t$ at discrete times $t_n \in (t)_h$, and then, expressing the LEs in terms of the factors $R_t$.

**Algorithm Description**

Set

$$
V_{t_0} = Q_{t_0} = I_d.
$$

For $s = 0, 1, \ldots$ and any initial condition $x_0$ solve the system

(7)  
$$
dx_t = \sum_{i=0}^m f_i (x_t) \circ dw_t^i, \quad x_{t_0} = x_0 \quad t_s \leq t \leq t_{s+1}
$$

(8)  
$$
dZ_t = \sum_{i=0}^m Df_i (x_t) Z_t \circ dw_t^i, \quad Z_{t_s} = Q_{t_s} \quad t_s \leq t \leq t_{s+1}
$$

and then take the QR factorization

$$
Z_{t_{s+1}} = Q_{t_{s+1}} R_{t_{s+1}}.
$$

Then one has

$$
V_{t_{s+1}} = Z_{t_{s+1}} Q_{t_s}^T V_{t_s} = Q_{t_{s+1}} R_{t_{s+1}} Q_{t_s}^T V_{t_s} = \ldots = Q_{t_{s+1}} R_{t_{s+1}} \ldots R_{t_1}
$$

Hence, the LEs and the truncated LEs are obtained by

$$
\lambda_j = \lim_{s \to \infty} \frac{1}{t_s} \sum_{n=1}^{s} \log \left| R_{t_n}^{jj} \right|
$$
and
\begin{equation}
\lambda_j(T) = \frac{1}{T-t_0} \sum_{n=1}^{n_T} \log |R_{j}^{j_n}|,
\end{equation}
respectively.

**Remark** It should be noted that the method proposed in [19] for computing the top LE is a particular case of the algorithm above. Indeed, the QR decomposition of $Z_t$ gives $R_{1}^{1} = \frac{Z_{1}^{1}}{|Z_{1}^{1}|}$, and hence,
\begin{equation}
\lambda_1 = \lim_{s \to \infty} \frac{1}{s} \sum_{n=1}^{s} \log |Z_{1}^{1_n}|,
\end{equation}
where $Z_{1}^{1}$ is the first column vector of the matrix $Z_t$, which in turn is solution of
\begin{equation}
dy_t = \sum_{i=0}^{m} Df_i(x_t) y_t \circ dw_t^i, \quad y_{t_s} = \frac{Z_{1}^{1_s}}{|Z_{1}^{1_s}|} \quad t_s \leq t \leq t_{s+1}.
\end{equation}
Then, the two previous expressions leads to the algorithm stated in [19]. Moreover, the approximation of all the LEs in [19] requires the computation of
\begin{equation}
\frac{1}{T} \log(\sqrt{\det((V_{T}^k)^{\top}V_{T}^k)})
\end{equation}
to get successively the (approximated) sums $\lambda_1 + \lambda_2 + \ldots + \lambda_k$, which is much more inefficient (and perhaps unstable) than the discrete QR algorithm presented in this section. Finally, note also that in the case of null stochastic components in the equations (1)-(2), the proposed algorithm reduces to the conventional QR discrete method for ODEs in [16] and [17].

### 3.3. Continuous QR method.

The general idea of the continuous QR method is to obtain an SDE for the factor $Q$ and then evaluating a Lebesgue integral for each of the LEs.

By taking differentials in the sense of Stratanovich in the equalities $V_t = Q_t R_t$ and $Q_t^\top Q_t = I_d$ it is obtained
\begin{align}
(dV_t) &= (dQ_t) R_t + Q_t (dR_t) \\
0 &= (dQ_t^\top) Q_t + Q_t^\top (dQ_t).
\end{align}
Since
\begin{equation}
dV_t = \sum_{i=0}^{m} Df_i(x_t) Q_t R_t \circ dw_t^i
\end{equation}
holds, it is deduced from (10) that
\begin{equation}
(dR_t) R_t^{-1} = \sum_{i=0}^{m} Q_t^\top Df_i(x_t) Q_t \circ dw_t^i - dS_t,
\end{equation}
where
\begin{equation}
dS_t = Q_t^\top (dQ_t).
\end{equation}
Whereas, from the equality (11) it is concluded that $dS_t$ is a skew-symmetric matrix.
Moreover, given that \((dR_t)R_t^{-1}\) is an upper triangular matrix it is deduced from (12) that \(S_t\) satisfies the equation

\[dS_t^j = \begin{cases} \sum_{i=0}^{m} (Q_i^T Df_i (x_t) Q_t)^j_l \circ dw_t^i & j > l \\ 0 & j = l \\ -\sum_{i=0}^{m} (Q_i^T Df_i (x_t) Q_t)^j_l \circ dw_t^i & j < l \end{cases},\]

which implies the following SDE for \(Q_t\)

\[dQ_t = Q_t dS_t = \sum_{i=0}^{m} Q_t^i T_t^i (x_t, Q_t) \circ dw_t^i,\]

where the matrices \(T_t^i (x_t, Q_t), i = 0, ..., m\) are given by

\[(T_t^i(x_t, Q_t))^j_l = \begin{cases} (Q_i^T Df_i (x_t) Q_t)^j_l & j > l \\ 0 & j = l \\ -(Q_i^T Df_i (x_t) Q_t)^j_l & j < l \end{cases}, \quad j, l = 1, ..., d\]

From (12) and (13) it is obtained

\[dR_t = \sum_{i=0}^{m} (Q_i^T Df_i (x_t) Q_t - T_t^i (x_t, Q_t)) R_t \circ dw_t^i,\]

which implies the following equation for each diagonal element \(R_t^{jj}, j = 1, ..., d\)

\[dR_t^{jj} = \sum_{i=0}^{m} (Q_i^T Df_i (x_t) Q_t)^{jj} R_t^{jj} \circ dw_t^i.\]

Therefore,

\[\lambda_j = \lim_{t \to \infty} \frac{1}{t} \log R_t^{jj} = \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} (Q_u^T Df_i (x_u) Q_u)^{jj} \circ dw_u^i.\]

Note that this formula is a straightforward generalization to the stochastic case of the continuous QR method proposed in [16] and [17]. However, this expression involves the computation of \(m\) Stratanovich integrals, which could be too expensive from a practical point of view. Hence, an alternative expression is in order.

From equation (16) it is deduced that

\[d(\det(R_t^{jj})) = \sum_{i=0}^{m} tr((Q_i^T Df_i (x_t) Q_t)^{jj}) \det(R_t^{jj}) \circ dw_t^i = \sum_{i=0}^{m} tr(Df_i (x_t) Q_t^{jj} (Q_t^{jj})^\top) \det(R_t^{jj}) \circ dw_t^i,\]

for each \(1 \leq j \leq d\). Now, let \(P_t^j\) be the orthogonal projection on the \(j\)-dimensional subspace of \(\mathbb{R}^d\) spanned by the first \(j\) column vectors of \(V_t\), that is

\[P_t^j = V_t^{jj}((V_t^{jj})^\top V_t^{jj})^{-1}(V_t^{jj})^\top.\]

Using that \(V_t^{jj} = Q_t^{jj} R_t^{jj}\) such orthogonal projector can be rewritten as

\[P_t^j = Q_t^{jj}(Q_t^{jj})^\top,\]
which implies that

\[ d(\det(\mathbf{R}_{ij}^{jj})) = \sum_{i=0}^{m} \text{tr}(Df_i(x_t) \mathbf{P}_t^i) \det(\mathbf{R}_{ij}^{jj}) \circ dw_t^i. \]

Then, by using the Ito formula for \( \log(\det(\mathbf{R}_{ij}^{jj})) \) it is obtained the following equation in Ito form (see details in Theorem 3.1 in [7]):

\[ d \log(\det(\mathbf{R}_{ij}^{jj})) = \psi_j(x_t, Q_t) dt + \sum_{i=1}^{m} \text{tr}(Df_i(x_t) Q_t^j (Q_t^j)^\top) dw_t^i, \]

where

\[ \psi_j(x, Q) = \text{tr}(Df_0(x) Q_t^j (Q_t^j)^\top) + \frac{1}{2} \sum_{i=1}^{m} [\text{tr}(D(Df_i(x) f_i(x)) Q_t^j (Q_t^j)^\top) \]

\[ + \text{tr}((Df_i(x))^\top (I_d - Q_t^j (Q_t^j)^\top) Df_i(x) Q_t^j (Q_t^j)^\top)] - (\text{tr}(Df_i(x) Q_t^j (Q_t^j)^\top))^2. \]

Therefore, as it was shown in Corollary 3.1 of [7],

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t \sum_{i=1}^{m} \text{tr}(Df_i(x_t) \mathbf{P}_t^i) dw_t^i = 0 \quad \text{a.s.,} \]

which yields

\[ \lambda_1 + \lambda_2 + \ldots + \lambda_j = \lim_{t \to \infty} \frac{1}{t} \log(\det(\mathbf{R}_{ij}^{jj})) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \psi_j(x_s, Q_s) ds \quad \text{a.s.} \]

Hence,

\[ \lambda_j = \lim_{t \to \infty} \frac{1}{t} \int_0^t \Delta \psi_j(x_s, Q_s) ds \quad \text{a.s.,} \]

for all \( j = 1, \ldots, d \), where \( \Delta \psi_j = \psi_j - \psi_{j-1} \), with \( \psi_0 \equiv 0 \).

Note that the previous expression involves the computation of a single Riemann integral, which is easier to evaluate than expression (17).

**Algorithm Description**

The continuous QR algorithm is based on the integration of the system of equations

\[ dx_t = \sum_{i=0}^{m} f_i(x_t) \circ dw_t^i, \quad x_{t_0} = x_0 \]

\[ dQ_t = \sum_{i=0}^{m} Q_t T_i^j(x_t, Q_t) \circ dw_t^i, \quad Q_{t_0} = I_d \]

to obtain the factor \( Q \) at the points \( t_0 < t_1 < \ldots < t_s < \ldots \) and then computing the LEs by

\[ \lambda_j = \lim_{s \to \infty} \frac{1}{t_s} \sum_{n=0}^{s-1} \int_{t_n}^{t_{n+1}} \Delta \psi_j(x_u, Q_u) du. \]
Remark The matrix $Q_t$ may be considered as a diffusion process that induce a RDS on the projective bundle $P(\Lambda^d(\mathbb{R}^d))$(see Section 6.4.2 in [5] for more details). Hence, the existence of an unique invariant ergodic measure for such RDS and the ergodicity of $(x_t, Q_t)$ is guarantied by Theorem 6.4.5 in [5]. It is worth noting that for the derivation of (19) we basically followed the Baxendale’ results [7] (or more generally Section 6.4.2 in [5]). Indeed, the expression (19) is closely related with the Furstenberg-Khasminskii formula for the sum of LEs.

3.4. Numerical Approximations. It is well-known that, in the framework of SDEs, the use of weak integrators is the most appropriate choice for the approximation of LEs [35], [36], [19], [25]. Hence, in the sequel, it will be assumed that the solution of the systems (7-8) and (20-21) are numerically approximated by weak integrators, whose solutions shall be denoted by $(\tilde{x}_t, \tilde{Z}_t)$ and $(\tilde{x}_t, \tilde{Q}_t)$, $t \in (t)_h$, respectively. Hence, the approximated LEs $\tilde{\lambda}_j$, $j = 1, ..., d$ are given by

$$\tilde{\lambda}_j = \lim_{s \to \infty} \frac{1}{t_s} \sum_{n=1}^{s} \log \left| R_n^{ij} \right|,$$

in the discrete case and by

$$\tilde{\lambda}_j = \lim_{s \to \infty} \frac{1}{t_s} \sum_{n=0}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \Delta \psi_j(\tilde{x}_u, \tilde{Q}_u) du.$$

in the continuous case. Correspondingly, the respective truncated approximations are given by

$$\tilde{\lambda}_j(T) = \frac{1}{T-t_0} \sum_{n=1}^{n_T} \log \left| R_n^{ij} \right|$$

and

$$\tilde{\lambda}_j(T) = \frac{1}{T-t_0} \sum_{n=0}^{n_T-1} \int_{t_n}^{t_{n+1}} \Delta \psi_j(\tilde{x}_u, \tilde{Q}_u) du,$$

where, without loss of generality it is assumed that $t_{n_T} = T$.

It should be noted that, in general, $\tilde{\lambda}_j$ are not the LEs of the discrete Markov chains $(\tilde{x}_t, \tilde{Z}_t)$ and $(\tilde{x}_t, \tilde{Q}_t)$, $t \in (t)_h$ that approximate the continuous ergodic process $(x_t, Z_t)$ and $(x_t, Q_t)$, respectively. In [36] and [19] the authors demonstrate the applicability of a discrete Multivariate Ergodic Theorem and as a consequence the existence of LEs for the Markov chains resulting from particular numerical integrators, namely, the weak Euler and Milstein schemes. As one can see there, proving the existence of LEs for these particular schemes requires a tedious and long algebraic manipulations, mainly addressed to showing that the ergodicity property of the process $(x_t, Z_t)$ and $(x_t, Q_t)$ is inherited by the Markov chains $(\tilde{x}_t, \tilde{Z}_t)$ and $(\tilde{x}_t, \tilde{Q}_t)$, $t \in (t)_h$. Our approach here differs from that in [36] and [19] in the sense that we are not assuming that $\tilde{\lambda}_j$ are LEs but they are just numerical approximations to the true LEs $\lambda_j$. However, our numerical simulations carried out in the last section were implemented with both weak Milstein and weak Euler schemes, for which the ergodicity property has been already proved.
4. Convergence Analysis

In this section an analysis of the approximation errors in the computation of the LEs is carried out. The results presented here are somehow quite similar to those given in [17] and [28] for the case of the QR methods in ODEs. However, a difference appears in the stochastic case, namely, the quantities $\tilde{\lambda}_j(T)$ are random variables. Since the LEs are in fact functionals of the solution process of (1)-(2), then the weak convergence criterion is the more natural choice to deal with [36], [19]. However, the approach followed in this section is somewhat different from that of [36], [19]. There, the authors obtained convergence results for the particular case of the Euler and Milstein schemes when $T$ goes to infinity, provided the approximations $\tilde{\lambda}_j$ were in fact LEs of discrete Markov chains (resulting form the numerical schemes). Our results here corresponds to the standard convergence analysis carried out when dealing with weak approximations, computed as usual, up to a finite time instant. Specifically, our approach consists on giving bounds for the errors $|\lambda_j - E(\tilde{\lambda}_j(T))|$, $i = 1, \ldots, d$, which can be decomposed as

$$|\lambda_j - E(\tilde{\lambda}_j(T))| \leq |\lambda_j - E(\lambda_j(T))| + \left|E(\tilde{\lambda}_j(T)) - E(\lambda_j(T))\right|.$$  

From this perspective, a convergence analysis for both terms $|\lambda_j - E(\lambda_j(T))|$ and $\left|E(\tilde{\lambda}_j(T)) - E(\lambda_j(T))\right|$ will be given in the sequel. For the first term we have the following theorem, which is a stochastic version of Lemma 4.2 in [17].

**Theorem 4.1.** Let $T > 0$ be a large enough fixed number. Then, for each $\epsilon > 0$ there exists a positive constant $K_\epsilon$ independent of $T$ such that

$$|\lambda_j - E(\lambda_j(T))| \leq \frac{K_\epsilon}{T - t_0} + \epsilon$$

**Proof.** Denote by $R_\epsilon : \Omega \to [1, \infty)$ an $\epsilon-$slowly random variable defined as in [5]. That is,

$$-\epsilon + \frac{1}{t} \log (R_\epsilon) \leq \frac{1}{t} \log (R_\epsilon(\theta_t)) \leq \frac{1}{t} \log (R_\epsilon) + \epsilon,$$

where $\theta_t$ is the ergodic shift transformation of the ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t \in \mathbb{R}})$ defined in Section 2.

According to Theorem 4.3.4 in [5], for each $\epsilon > 0$ there exists an $\epsilon-$slowly random variable $R_\epsilon$ such that

$$\frac{1}{R_\epsilon}e^{\lambda_j t - \epsilon |t|} \|x\| \leq \|V_t x\| \leq R_\epsilon e^{\lambda_j t + \epsilon |t|} \|x\|,$$

where $x$ is any vector in the subspace of $\mathbb{R}^d$ corresponding to the LE $\lambda_j$ according to the Lyapunov spectrum of (1)-(2). Taking $\|x\| = 1$ and $t = T - t_0$ it is obtained

$$-\epsilon - \frac{1}{T - t_0} \log (R_\epsilon) \leq \lambda_j(T) - \lambda_j \leq \frac{1}{T - t_0} \log (R_\epsilon) + \epsilon,$$

which implies

$$|\lambda_j - E(\lambda_j(T))| \leq \frac{1}{T - t_0} E(|\log (R_\epsilon)|) + \epsilon.$$  

On the other hand, (26) implies that

$$-\epsilon \leq \liminf_{t \to \infty} \frac{1}{t} \log (R_\epsilon(\theta_t)) \leq \limsup_{t \to \infty} \frac{1}{t} \log (R_\epsilon(\theta_t)) \leq \epsilon,$$
and so
\[ E(\limsup_{t \to \infty} \frac{1}{t} \log (R_t(\theta_i))) \leq \epsilon. \]
Thus, by the Fatou’s Lemma,
\[ \limsup_{t \to \infty} E(\frac{1}{t} \log (R_t(\theta_i))) \leq \epsilon. \]
Therefore, for all \( \epsilon' > 0 \) there exists a \( T' \) such that
\[ E(\frac{1}{t} \log (R_t(\theta_i))) \leq \epsilon' \]
for all \( t > T' \), where \( \epsilon'' = \epsilon + \epsilon' \). Additionally, (26) also implies that
\[-\epsilon + \frac{1}{t} \log (R_t(\theta_i)) \leq \frac{1}{t} \log (R_t) \leq \frac{1}{t} \log (R_t(\theta_i)) + \epsilon \]
for all \( t \). Thus,
\[ \frac{1}{t} \log (R_t) \leq \frac{1}{t} \log (R_t(\theta_i)) + \epsilon. \]
and so
\[ \frac{1}{t} E(\log (R_t)) \leq \frac{1}{t} E(\log (R_t(\theta_i))) + \epsilon. \]
This and (28) imply that
\[ K_e = E(\log (R_t)) < \infty. \]
With this bound and inequality (27), the prove is concluded. \( \square \)

Bounds for the second error term \( |E(\tilde{\lambda}_j(T)) - E(\lambda_j(T))| \) will be given in the following two subsections.

4.1. Weak convergence to the truncated LEs: continuous QR method.

Let
\[ y_t = (x_t^1, ..., x_t^d, Q_t^{11}, ..., Q_t^{dd}) \in \mathbb{R}^{d+d^2}, \quad t \in [t_0, T] \]
be the solution of the system
\[ dx_t = \sum_{i=0}^{m} f_i(x_t) \circ dw_t^i, \quad x_{t_0} = x_0, \]
\[ dQ_t = \sum_{i=0}^{m} Q_t^{ii} (x_t, Q_t) \circ dw_t^i, \quad Q_{t_0} = I_d, \]
and let \( \tilde{y}_t = (\tilde{x}_t^1, ..., \tilde{x}_t^d, \tilde{Q}_t^{11}, ..., \tilde{Q}_t^{dd}), \quad t \in (t)_b \) be any numerical approximation to \( y_t \). Denote by \( y_t(s, \bar{r}) \) the solution at \( t \) of the system above with initial condition \( y_s = \bar{r} \), and let \( \tilde{y}_t(s, \bar{r}) \) be an approximation to \( y_t(s, z) \) that satisfy \( \tilde{y}_s = \bar{r} \). Hence, with this notation, it is obvious that \( y_t(t_0, y_{t_0}) = y_t \) and \( \tilde{y}_t(t_0, y_{t_0}) = \tilde{y}_t \). For \( l = 1, 2, ..., \) define \( P_l = \{ p \in \{1, ..., d + d^2\} \} \) and for each \( p = (p_1, ..., p_l) \in P_l \) define the function \( F_p : \mathbb{R}^{d+d^2} \to \mathbb{R} \) as
\[ F_p(y) = \prod_{i=1}^{l} Q_{p_i}^{y_{p_i}}, \]
for all \( y = (y^1, ..., y^{d+d^2}) \). The following theorem provides condition under which a weak order \( \beta \) numerical approximation \( \tilde{y}_t \) yields to a weak order \( \beta \) approximation of \( \tilde{\lambda}_j(T) \) to \( \lambda_j(T) \) as well. It is obtained as a direct application of the method for
evaluating functional integrals by weak approximations (see [35] and Chapter 17 in [25]).

**Theorem 4.2.** Let \( \beta \in \{1, 2, \ldots \} \) be given and suppose that \( f_0, \ldots, f_m, k = 1, \ldots, d \) are such that

\[
f_0 \in C_p^{2\beta+3}(\mathbb{R}^d, \mathbb{R}), \quad f_1, \ldots, f_m \in C_p^{2\beta+4}(\mathbb{R}^d, \mathbb{R}), \quad \sum_{i=1}^{m} \sum_{j=1}^{d} f_i^j \frac{\partial}{\partial x_j} f_i \in C_p^{2\beta+3}(\mathbb{R}^d, \mathbb{R}).
\]

In addition suppose that for each \( q = 1, 2, \ldots \), there exist constants \( C < \infty \) and \( r \in \mathbb{N}^+ \), which do not depend on \( h \), such that

\[
E(\max_{0 \leq n \leq \tau} ||\tilde{y}_{tn}||^{2q}/F_{tn}) \leq C(1 + ||y_0||^{2r})^q,
\]

and

\[
E(||\tilde{y}_{tn+1}(t_n, \tilde{y}_{tn}) - \tilde{y}_{tn}||^{2q}/F_{tn}) \leq C(1 + ||\tilde{y}_{tn}||^{2r})(t_{n+1} - t_n)^q,
\]

\[
E(||F_p(\tilde{y}_{tn+1}(t_n, \tilde{y}_{tn}) - \tilde{y}_{tn}) - F_p(\tilde{y}_{tn+1}(t_n, \tilde{y}_{tn}) - \tilde{y}_{tn})/F_{tn})|| \leq C(1 + ||\tilde{y}_{tn}||^{2r})(t_{n+1} - t_n)^{h\beta},
\]

for all \( n = 0, \ldots, n_T - 1 \), \( p \in P_k \) and \( l = 1, \ldots, 2\beta + 1 \). Then, there exist positive constants \( K_j(T) \), which do not depend on \( h \) such that

\[
\left| E(\lambda_j(T)) - E(\tilde{\lambda}_j(T)) \right| \leq K_j(T)h^\beta,
\]

for all \( j = 1, \ldots, d \), where \( \tilde{\lambda}_j(T) \) is the corresponding approximation to the truncated LE (25).

**Proof.** Since

\[
\lambda_j(T) = \frac{1}{T - t_0} \int_{t_0}^{T} \Delta \psi_j(x_t, Q_t) \, dt,
\]

which implies that

\[
\left| E(\lambda_j(T)) - E(\tilde{\lambda}_j(T)) \right| \leq \sup_{t \in [t_0, T]} |E(b_j(y_t)) - E(b_j(\tilde{y}_t))|,
\]

where

\[
b_j(y) := \Delta \psi_j(x, Q) \in C_p^{2\beta+2}(\mathbb{R}^{d+d^2}, \mathbb{R})
\]

From this and conditions (30), (31) and (32) it can be obtained from Theorem 9.1 in [29] that

\[
\sup_{t \in [t_0, T]} |E(b_j(y_t)) - E(b_j(\tilde{y}_t))| \leq C_j(1 + ||y_0||^{2r})h^\beta,
\]

for some positive constants \( C_j \) and \( r \in \mathbb{N}^+ \), which concludes the proof.

\[\square\]

**4.2. Weak convergence to the truncated LEs: discrete QR method.** Similarly to the previous subsection, let

\[
y_t = (x_t^1, \ldots, x_t^d, Z_t^{11}, \ldots, Z_t^{dd}) \in \mathbb{R}^{d+d^2}
\]

be the solution of the system (7)-(8) and let \( \tilde{y}_t = (\tilde{x}_t^1, \ldots, \tilde{x}_t^d, \tilde{Z}_t^{11}, \ldots, \tilde{Z}_t^{dd}) \) be its approximation. Then the following theorem holds.
Theorem 4.3. Let $\beta \in \{1, 2, \ldots\}$ be given and suppose that the components $f^k_0, \ldots, f^k_m$, $k = 1, \ldots, d$ are such that
\begin{equation}
\label{eq:33}
f^k_0 \in C_p^{2\beta+3}(\mathbb{R}^d, \mathbb{R}), \quad f^k_1, \ldots, f^k_m \in C_p^{2\beta+4}(\mathbb{R}^d, \mathbb{R}), \quad \sum_{j=1}^{d} \sum_{i=1}^{m} f^k_j \partial_{x_j} f^k_i \in C_p^{2\beta+3}(\mathbb{R}^d, \mathbb{R}).
\end{equation}

In addition suppose that for each $q = 1, 2, \ldots$, there exists constants $C < \infty$ and $r \in \mathbb{N}^+$, which do not depend on $h$, such that
\begin{equation}
\label{eq:34}
E\left(\max_{0 \leq n \leq n_T} \|\tilde{y}_n\|^2q / F_{t_0}\right) \leq C(1 + \|y_0\|^{2r}),
\end{equation}
\begin{equation}
\label{eq:35}
E\left(\max_{0 \leq n \leq n_T} \left\|\tilde{Z}_n^{-1}\right\|^q / F_{t_0}\right) \leq C(1 + \|Z_0\|^{2r}),
\end{equation}
and
\begin{align*}
E\left(\|\tilde{y}_{n+1}(t_n, \tilde{y}_n) - \tilde{y}_n\|^2q / F_{t_n}\right) & \leq C(1 + \|\tilde{y}_n\|^{2r})(t_{n+1} - t_n)^q, \\
|E(F_p(\tilde{y}_{n+1}(t_n, \tilde{y}_n) - \tilde{y}_n) - F_p(\tilde{y}_{n+1}(t_n, \tilde{y}_n) - \tilde{y}_n)) / F_{t_n}| & \leq C(1 + \|\tilde{y}_n\|^{2r})(t_{n+1} - t_n)^{\beta},
\end{align*}
for all $n = 0, \ldots, n_T - 1$, $p \in P_1$ and $l = 1, \ldots, 2\beta + 1$. Then for each $g \in C_p^{2\beta+2}$ there exists positive constants $K_j(T)$, which do not depend on $h$ such that
\begin{equation*}
\left|E(\lambda_j(T)) - E(\tilde{\lambda}_j(T))\right| \leq K_j(T) h^\beta,
\end{equation*}
for all $j = 1, \ldots, d$, where $\tilde{\lambda}_j(T)$ is the corresponding approximation to truncated LE (24).

Proof. According to the expressions (9) and (24) it is obtained
\begin{equation*}
|E(\lambda_j(T)) - E(\tilde{\lambda}_j(T))| = \frac{1}{T - t_0}|E(\log |R_{jT}^j|) - E(\log |\tilde{R}_{jT}^j|)|.
\end{equation*}

The QR decomposition of the matrices $Z_T$ and $\tilde{Z}_T$ allows one to rewrite this expression as
\begin{equation*}
|E(\lambda_j(T)) - E(\tilde{\lambda}_j(T))| = |E(b_j(y_T)) - E(b_j(\tilde{y}_T))|,
\end{equation*}
where for any
\begin{equation*}
y = (x^1, \ldots, x^d, Z^{11}, \ldots, Z^{dd}) \in \mathbb{R}^{d+d^2},
\end{equation*}
the functions $b_j, j = 1, \ldots, d$ are defined by
\begin{align*}
b_1(y) & = \log \|Z^1\|, \\
b_j(y) & = \log \left\|Z^j - \sum_{i=1}^{j-1} \langle Z^i, Q^i \rangle Q^i \right\|,
\end{align*}
with
\begin{equation*}
Q^1 = \frac{Z^1}{R^1}, \quad R^{11} = \|Z^1\|, \\
Q^j = \frac{Z^j - \sum_{i=1}^{j-1} \langle Z^i, Q^i \rangle Q^i}{R^{jj}}, \quad R^{jj} = \left\|Z^j - \sum_{i=1}^{j-1} \langle Z^i, Q^i \rangle Q^i \right\|, \quad j = 2, \ldots, d.
\end{equation*}

It is easy to check from (33) that, for all $j = 1, \ldots, d$,
\begin{equation}
\label{eq:36}
b_j \in C_p^{2\beta+2}(\mathbb{R}^{d+d^2}, \mathbb{R}).
\end{equation}
On the other hand, for each $k = 1, ..., d + d^2$,
\[
\left| \frac{\partial b_1(y)}{\partial y^k} \right| = \|Z^1\|^{-1} \left| \frac{\partial \|Z^1\|}{\partial y^k} \right| \\
= \|Z^1\|^{-3/2} < Z^1, \frac{\partial Z^1}{\partial y^k} > \\
\leq |R^{11}|^{-3/2} \|Z^1\| \left| \frac{\partial Z^1}{\partial y^k} \right| \\
\leq \|R^{11}\|^{-3} + \|y\|^2 \\
\leq \|Z^{-1}\|^3 + \|y\|^2
\]

In a similar way, it can be shown that for each $p \in P_1, l = 1, ..., 2\beta + 2$ and $j = 1, ..., d$ there exits constants $C > 0$ and $r_{pk}, s_{pk} = 1, 2, ..., s_{pk}$ such that
\[
(37) \quad \left| \frac{\partial^p b_j(y)}{\partial y^p} \right| \leq C(\sum_{p=1}^{|p|} \sum_{k=1}^{|p|} |R^{lj}|^{-r_{pk}} + \|y\|^s_{pk}) \\
\leq C(\sum_{p=1}^{|p|} \sum_{k=1}^{|p|} \|Z^{-1}\|^{|r_{pk}} + \|y\|^s_{pk}),
\]

If the condition $b_j \in C^{2\beta+2}_p(\mathbb{R}^{d+d^2}, \mathbb{R})$ were true for the functions $b_j$ then it were enough to apply Theorem 9.1 in [29] to conclude our proof. Although this is not the case, it can be seen that, for the purpose of Theorem 9.1 in [29], conditions $b_j \in C^{2\beta+2}_p(\mathbb{R}^{d+d^2}, \mathbb{R})$ and (34) are equivalent to conditions (36), (37), (35) and (34). Thus, a slightly variation in the proof of such theorem yields
\[
|E(b_j(y_T)) - E(b_j(y_T))| \leq K_j(T)h^\beta,
\]
for certain positive constants $K_j(T), j = 1, ..., d$. \hfill \Box

It should be noted here that due to inequality (37), even with accurate approximations to $Z_t$, some difficulties can be expected for the approximation of negative LEs of large magnitude. In fact, this has been reported in the literature regarding the computation of LEs of ODEs by the discrete QR methods.

**Remark** As we already mentioned, the analysis carried out in this section does not assume any ergodicity property on the discrete Markov chains resulting from the numerical approximations. This means that, in principle, our analysis can be applied to any numerical integrator. However, for those integrators where such ergodicity property had already been proved (e.g. Euler, Milstein and the second order schemes given in [35]), one is able to obtain more accurate results. Indeed, ergodicity property would imply the uniform bounds (in $T$) on the constants $K_j(T)$, $j = 1, ..., d$. On the other hand, under ergodicity assumptions one may obtains bounds for the almost sure convergence of $\lambda_j$ (or $\lambda_j(T)$) to $\lambda_j$ (as in [36] and [19]). Therefore, as a general rule, we recommend the use of ergodic numerical schemes for the approximations of LEs by QR-based methods.

5. **Computational Aspects**

In this section some computational aspects to take into account when applying both the discrete and the continuous QR algorithms are given. It is also briefly presented a practical criterion for the choice of the truncation time $T$. 
According to Section 3.2, the discrete $QR$ algorithm requires the numerical integration of the equations (7)-(8), which can be carried out by any of the standard weak numerical integrators for SDE [25]. However, the continuous $QR$ algorithm requires the integration of the system (20)-(21) in such a way of preserving the orthogonality of the factor $Q_t$ for each $t \geq t_0$. This can be achieved by two different classes of numerical schemes. The first class are usually called projected orthogonal schemes. They are based on generating approximations by any standard numerical scheme and then projecting the solution into the set of orthogonal matrices. The schemes belonging to the second class are those that preserve the orthogonality during the numerical integration process. For that reason they are called automatic orthogonal integrators. In a recent paper [14], a class of such automatic orthogonal schemes for SDEs was introduced. These schemes have order of convergence 1 and constitute a stochastic version of the automatic orthogonal integrators for ODEs [15]. So, the continuous $QR$ presented here can be implemented by either such automatic orthogonal schemes or by a projected method, which shall be called Continuous Automatic and Continuous Projected $QR$ methods, respectively.

In general, as in any numerical integration problem, there is a number of practical aspects that could improve the efficiency and accuracy of the numerical solutions of (1)-(2). Those aspects focus, for instance, on an adequate selection of the numerical integrator and efficient adaptive step size control of the numerical solutions. It would become very important for avoiding undesirable explosive realizations in those situations where the SDE (1) has unstable solutions.

Another aspect to take into account in the implementation of the continuous $QR$ algorithm is the evaluation of the integrals in (25). Firstly it should be noted that the expression (18) can be rewritten as

$$
\psi_j(x, Q) = \text{tr} (Q^\top Df_i(x) Q)_{jj} + \frac{1}{2} \sum_{i=1}^m \left[ \text{tr} \left( Q^\top D (Df_i(x) f_i(x)) Q \right)_{jj} \right] + \text{tr}(Q^\top (Df_i(x))^\top (I_d - Q_{jj}Q_{jj}^\top) Df_i(x) Q_{jj}) \\
- \text{tr} \left( Q^\top Df_i(x) Q \right)_{jj}^2,
$$

which is easier to implement numerically and allows the computation of all $\Delta \psi_j, j = 1, \ldots, d$ at once. Additionally, the trivial case $\psi_d(x, Q)$ does not depend on $Q$ and, for linear SDEs, it is a constant function that only depends on the Jacobian $Df_i, i = 0, \ldots, m$ evaluated at $x = 0$. In other words, the continuous $QR$ method produces exact results while computing the sum of all LEs corresponding to linear SDEs. Here, we used the composite trapezoidal rule for approximating the integrals

$$
\int_{t_n}^{t_{n+1}} \Delta \psi_j(\tilde{x}_u, \tilde{Q}_u) du, \ n \geq 0, \ j = 1, \ldots, d - 1.
$$

However, an alternative approach can be obtained by the numerical integration of the extended system

$$
dx_t = \sum_{i=0}^m f_i(x_t) \circ dw_t^i, \ x_{t_0} = x_0,
$$

$$
dQ_t = \sum_{i=0}^m Q_t T_i^x (x_t, Q_t) \circ dw_t^i, \ Q_{t_0} = I_d,
$$

$$
dz_t^j = \Delta \psi_j (x_t, Q_t) \circ dw_t^j, \ z_{t_0} = \Delta \psi_j (x_0, I_d), \ j = 1, \ldots, d,
$$
for which one has
\[ \tilde{\lambda}_j(T) = \sum_{n=0}^{n_T-1} \int_{t_n}^{t_{n+1}} \Delta \psi_j(\tilde{x}_u, \tilde{Q}_u) \, du \]
\[ = \frac{z_j^T}{T-t_0}. \]

Finally, some considerations about the choice of the truncation time \( T \) are in order. In [36] it was enunciated a Central-Limit theorem (Theorem 5.2) as a possible solution to this problem. However, as it was noted there, this approach is extremely difficult to use in practice because the variance of the limit laws depends on the unknown LEs. Alternatively, it was proposed the heuristic criterion of taking \( T = t_0 + Nh \) such that \( \tilde{\lambda}_j(t_0 + nh) \) be almost constant for \( n \) around \( N \). However, usually one does not expect large changes in \( \tilde{\lambda}_j(t_0 + nh) \) from \( n \) to \( n+1 \). A different criterion introduced in [13] could be followed. The basic idea is to take \( T = T_0 \) such that, for given \( \Delta T > 0 \) large enough, the approximations \( \tilde{\lambda}_j(T_0 + k\Delta T) \), \( k = 0, 1, \ldots \), becomes almost constant with respect to \( k \).

6. Numerical Examples

In this section, the performance of the QR methods is illustrated through two examples. For the Discrete and Continuous Projected QR-methods, the classical Euler and Milstein weak integrators [25] for SDEs were used, which will be denoted by "D – Euler", "D – Milstein", "CP – Euler", and "CP – Milstein", respectively. These integrators were also used in combination with the 2-stage automatic orthogonal Runge-Kutta scheme [14] to integrate, respectively, the equations (20) and (21) for the Continuous QR method. These Continuous Automatic methods will be called "CA – Euler – RK2" and "CA – Milstein – RK2".

Example 1. Let us consider the nonlinear 1-dimensional Itô SDE given by
\[ dx_t = (-ax_t + F(x_t)) \, dt + G(x_t) \, dw_t, \]
where \( F(x) = \arctan(x) \) and \( G(x) = \sqrt{1+x^2} \). For this equation the LE can be explicitly computed by the expression
\[ \lambda = -a + \int_{\mathbb{R}} \left( F'(x) - \frac{1}{2} G'(x)^2 \right) p(x) \, dx, \]
for any value of the parameter \( a \), where \( p(x) \) is the invariant probability law of \( x_t \) [19]. For example, for \( a = 2 \) the "true" LE is given by \( \lambda = -1.3385 \).

The Table I summarizes the values of the approximate LE \( \tilde{\lambda}(T) \) computed by the discrete and the continuous QR methods at \( T = 500 \) for different step sizes. For each step size we also included the relative CPU, which is obtained as the ratio of the actual CPU time for each numerical scheme to the minimum of all CPU times. Note that the discrete QR methods are the less computationally expensive, in contrast to the continuous automatic variants, which are up to 3.5 times slower. Since we have not proved almost sure convergence to the true LEs, we included the values of \( E(\tilde{\lambda}(T)) \), estimated on the base of 100 independent realizations. As expected, the values of \( E(\tilde{\lambda}(T)) \) are closer to the true \( \lambda \) than the approximation obtained with a single realization of \( \tilde{\lambda}(T) \). However, despite no almost sure convergence was proven, we can easily observe that single realizations of \( \tilde{\lambda}(T) \) also provide a very accurate approximation to \( \lambda \). Therefore, we conjecture that our
approximations $\tilde{\lambda}(T)$ are in fact LEs of the discrete numerical schemes, which converge almost surely to the true $\lambda$. However, proving such kind of results is going to be subject of another work in the near future.

<table>
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<tr>
<th>Method</th>
<th>Step Size $h$</th>
<th>$\lambda(T)$</th>
<th>$E(\lambda(T))$</th>
<th>Rel. CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D - Euler$</td>
<td>$2^{-6}$</td>
<td>$-1.3638$</td>
<td>$-1.3635$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$2^{-9}$</td>
<td>$-1.3491$</td>
<td>$-1.3433$</td>
<td>1</td>
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<tr>
<td>$D - Milstein$</td>
<td>$2^{-6}$</td>
<td>$-1.3576$</td>
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<tr>
<td></td>
<td>$2^{-9}$</td>
<td>$-1.3409$</td>
<td>$-1.3407$</td>
<td>1.0115</td>
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<tr>
<td>$CP - Euler$</td>
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<td>$-1.3411$</td>
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<tr>
<td></td>
<td>$2^{-9}$</td>
<td>$-1.3446$</td>
<td>$-1.3389$</td>
<td>1.3392</td>
</tr>
<tr>
<td>$CP - Milstein$</td>
<td>$2^{-6}$</td>
<td>$-1.3455$</td>
<td>$-1.3379$</td>
<td>1.3327</td>
</tr>
<tr>
<td></td>
<td>$2^{-9}$</td>
<td>$-1.3303$</td>
<td>$-1.3395$</td>
<td>1.3409</td>
</tr>
<tr>
<td>$CA - Euler - RK2$</td>
<td>$2^{-6}$</td>
<td>$-1.3281$</td>
<td>$-1.3383$</td>
<td>1.6815</td>
</tr>
<tr>
<td></td>
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<td>$-1.3359$</td>
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<tr>
<td>$CA - Milstein - RK2$</td>
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<td>$-1.3297$</td>
<td>$-1.3390$</td>
<td>3.5655</td>
</tr>
<tr>
<td></td>
<td>$2^{-9}$</td>
<td>$-1.3404$</td>
<td>$-1.3387$</td>
<td>3.5722</td>
</tr>
</tbody>
</table>

Table I. QR-based approximations to $\lambda = -1.3385$ of Example 1, computed for two different step sizes with $T = 500$. For each method, relative CPU times (ratio of the actual CPU time to the minimum of all CPU times) are also reported.

Figure 1 shows the value of the approximate LE $\tilde{\lambda}(T)$, $0 \leq T \leq 500$, obtained by the QR-methods with step-size $h = 2^{-9}$. The left panel corresponds to the methods $D - Euler$, $CP - Euler$ and $CA - Euler - RK2$. Here, the $CP - Euler$ and the $CA - Euler - RK2$ methods seems to be those of the better performance. Whereas, the right panel corresponds to the methods $D - Milstein$, $CP - Milstein$ and $CA - Milstein - RK2$, which also suggest that better results are obtained by the continuous $QR$ methods.
This is better illustrated in Figure 2, which shows the mean value $\hat{\lambda}(T)$ = $\frac{1}{100} \sum_{i=1}^{100} \tilde{\lambda}(T)$ of the approximated LEs $\tilde{\lambda}(T)$, $i = 1, \ldots, 100$, computed by the methods $D –$ Milstein, $CP –$ Milstein and $CA –$ Milstein $-$ RK2 in 100 trials, with step-size $h = 2^{-4}$, $2^{-6}$, $2^{-8}$ and truncation times $T = 100, 200, 300, 400, 500$. Each subplot in the figure shows a box and whisker plot (boxplot) with one box for each of the methods. The boxes have lines at the mean value $\hat{\lambda}(T)$ and at the 95% confidence interval $[\hat{\lambda}(T) - 1.96\sigma_\lambda(T), \hat{\lambda}(T) + 1.96\sigma_\lambda(T)]$, where $\sigma_\lambda(T)$ denotes the standard deviation of the 100 values $\tilde{\lambda}(T)$. The whiskers extending from each end of the boxes show the minimum and the maximum value of $\tilde{\lambda}(T)$, $i = 1, \ldots, 100$. In this example, for $h$ relatively large ($h = 2^{-4}$) the method $D –$ Milstein presents the worse performance; even more, this situation does not improves with the increasing of $T$. Note also that for this step size, the continuous QR method $CP –$ Milstein achieves the best results. It can be also seen that for the other two step sizes, the estimation of $\hat{\lambda}(T)$ remains almost with the same value for the three methods. On the other hand, as expected, the length of the confidence interval (determined by the standard deviation $\sigma_\lambda(T)$) decreases with the increasing of $T$. At glance, $\hat{\lambda}(T)$ seems to be a constant for all values of $T$. However, it gets closer to the true $\lambda$ when $T$ increases, which is shown in the next figure.

Indeed, Figure 3 shows the errors $e_\lambda(T) = |\lambda - \hat{\lambda}(T)|$, $T = 200, 300, \ldots, 600$, for the $CP –$ Milstein method with step size $h = 2^{-6}$, where $\hat{\lambda}(T)$ was estimated from 500 values $\tilde{\lambda}(T)$. For these values of $e_\lambda(T)$, a nonlinear regression model of the form

$$e_\lambda(T) = a + b(1/T)^c$$
Figure 3. Plot of the errors $e_\lambda(T) = \left| \lambda - \hat{\lambda}(T) \right|$, $T = 200, 300, \ldots, 600$, for the method $PC - Milstein$ with step size $h = 2^{-6}$ and the estimated curve $e_\lambda(T) = 2.54 \times 10^{-4} + 0.43(1/T)^{1.04}$.

Figure 4. Plot of the errors $\log_2(e_\lambda(h)) = \log_2(\left| \lambda - \hat{\lambda}(500) \right|)$ for the method $PC - Milstein$ method with step sizes $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$ and the estimated curve $\log_2(e_\lambda(h)) = -4.89 + 1.03 \log_2(h)$.

was fitted. This yields the estimates $\hat{a} = 2.54 \times 10^{-4}$, $\hat{b} = 0.43$ and $\hat{c} = 1.04$. The figure also shows the curve $\hat{a} + \hat{b}(1/T)^{\hat{c}}$ for $T \in [200, 600]$. Notice that the estimated value of $c$ agrees with the theoretical value $c = 1$ provided by Theorem 4.1.

On the other hand, the errors $e_\lambda(h) = \left| \lambda - \hat{\lambda}(500) \right|$ were also computed for the $CP - Milstein$ method with step sizes $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$ and $T = 500$, where
\( \hat{\lambda}(T) \) was estimated from 500 values \( \hat{\lambda}(T) \). A regression model of the form

\[
\log_2(\epsilon_\lambda(h)) = \alpha + \gamma \log_2(h)
\]

was fitted, which yields to the estimations \( \hat{\alpha} = -4.89 \) and \( \hat{\gamma} = 1.03 \). Figure 4 shows these error values and the curve \( \hat{\alpha} + \hat{\gamma} \log_2(h) \). As expected, the estimated value of \( \gamma \) is very close to the theoretical value \( \gamma = 1 \), which is the weak order of convergence of the weak Milstein scheme.

Example 2. The second example corresponds to the damped harmonic oscillator that is defined by the 2-dimensional linear SDE

\[
(39) \quad dx_t = \begin{pmatrix} 0 & 1 \\ -\alpha & 2\beta \end{pmatrix} x_t dt + \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix} x_t \circ dw_t,
\]

where \( \beta \) is the damping constant and the parameter \( \alpha \) controls the strength of the restoring force. It is known from [22] and [23] that, for \( |\beta| = \sqrt{\alpha} \) and \( \gamma = \frac{\alpha}{\beta} \), the Lyapunov exponents are given by

\[
\lambda_1 = \beta + 12^{1/3} \frac{\Gamma(1/2)}{\Gamma(1/6)} \frac{\gamma^{1/3}}{2} \quad \text{and} \quad \lambda_2 = \beta - 12^{1/3} \frac{\Gamma(1/2)}{\Gamma(1/6)} \frac{\gamma^{1/3}}{2}.
\]

In this example, two different sets of values for \( \beta \) and \( \sigma \) were selected. The first one is \( \sigma = 1 \) and \( \beta = -12^{1/3} \frac{\Gamma(1/2)}{\Gamma(1/6)} \frac{\gamma^{1/3}}{2} \), which yield the LEs \( \lambda_1 = 0 \) and \( \lambda_2 = -0.5786 \). The second one corresponds to \( \sigma = 1 \) and \( \beta = 12^{1/3} \frac{\Gamma(1/2)}{\Gamma(1/6)} \frac{\gamma^{1/3}}{2} \), resulting in the LEs \( \lambda_1 = 0.5786 \) and \( \lambda_2 = 0 \). Note that, for this linear equation, the fixed point \( x_0 = 0 \) generates a stationary orbit for the cocycle \( \varphi \) and the Dirac measure \( \mu = \delta_{x_0} \) is \( \varphi \)-invariant. Then, the linear matrix cocycle \( D\varphi \) can be computed directly from (2) in such a way of avoiding the numerical integration of the equation (39) (i.e. setting \( x_t \equiv 0 \) in (2)). Nevertheless, in order to illustrate the performance of the proposed numerical methods on general SDEs with unknown invariant measure, the equation (39) was solved numerically as well, where the point \( x_0 = (0,0,1) \) was used as initial condition.

For the first set of parameters, Table II shows the performance of the QR methods for two different step sizes.

<table>
<thead>
<tr>
<th>Method</th>
<th>Step Size h</th>
<th>( \lambda_1(T) )</th>
<th>( \lambda_2(T) )</th>
<th>( \Sigma \lambda(T) )</th>
<th>Rel. CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-Euler</td>
<td>2\textsuperscript{-b}</td>
<td>0.0573</td>
<td>-0.6374</td>
<td>-0.5801</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2\textsuperscript{-9}</td>
<td>0.0103</td>
<td>-0.6014</td>
<td>-0.5911</td>
<td>1</td>
</tr>
<tr>
<td>D-Milstein</td>
<td>2\textsuperscript{-b}</td>
<td>-0.0141</td>
<td>-0.5523</td>
<td>-0.5664</td>
<td>1.0158</td>
</tr>
<tr>
<td></td>
<td>2\textsuperscript{-9}</td>
<td>0.0104</td>
<td>-0.5901</td>
<td>-0.5797</td>
<td>1.0030</td>
</tr>
<tr>
<td>CP-Euler</td>
<td>2\textsuperscript{-b}</td>
<td>0.0237</td>
<td>-0.6023</td>
<td>-0.5786</td>
<td>1.3438</td>
</tr>
<tr>
<td></td>
<td>2\textsuperscript{-9}</td>
<td>0.0132</td>
<td>-0.5918</td>
<td>-0.5786</td>
<td>1.1977</td>
</tr>
<tr>
<td>CP-Milstein</td>
<td>2\textsuperscript{-b}</td>
<td>-0.0079</td>
<td>-0.5707</td>
<td>-0.5786</td>
<td>1.3601</td>
</tr>
<tr>
<td></td>
<td>2\textsuperscript{-9}</td>
<td>-0.0018</td>
<td>-0.5767</td>
<td>-0.5786</td>
<td>1.2007</td>
</tr>
<tr>
<td>CA-Euler-RK2</td>
<td>2\textsuperscript{-b}</td>
<td>-0.0181</td>
<td>-0.5604</td>
<td>-0.5786</td>
<td>3.2036</td>
</tr>
<tr>
<td></td>
<td>2\textsuperscript{-9}</td>
<td>-0.0012</td>
<td>-0.5773</td>
<td>-0.5786</td>
<td>2.1503</td>
</tr>
<tr>
<td>CA-Milstein-RK2</td>
<td>2\textsuperscript{-b}</td>
<td>-0.0109</td>
<td>-0.5676</td>
<td>-0.5786</td>
<td>3.2223</td>
</tr>
<tr>
<td></td>
<td>2\textsuperscript{-9}</td>
<td>-0.0011</td>
<td>-0.5775</td>
<td>-0.5786</td>
<td>2.1540</td>
</tr>
</tbody>
</table>

Table II. QR-based approximations to the LEs \( \lambda_1 = 0 \) and \( \lambda_2 = -0.5786 \) of Example 2, computed for two different step sizes with \( T = 500 \). For each method, relative CPU times.
COMPUTING LYAPUNOV EXPONENTS OF SDES

Figure 5. QR-based approximations to the LEs $\tilde{\lambda}_1(T)$ and $\tilde{\lambda}_2(T)$ of the Example 2, implemented with a) the Euler integrator, and b) the Milstein integrator, with step size $h = 2^{-9}$, $T = 500$ and parameters $\sigma = 1$, $\beta = -6^{1/3} \frac{1}{1(1/2)} \frac{2}{2(1/6)}$.

Similarly to Figure 1, Figure 5 shows the approximate LEs $\tilde{\lambda}_1(T)$, $\tilde{\lambda}_2(T)$, $0 \leq T \leq 500$, for different numerical schemes. As in the previous example above, the continuous methods seem to be those with best performance.

For the second set of parameters, Table III shows the results obtained by the QR methods with two different step sizes.

<table>
<thead>
<tr>
<th>Method</th>
<th>Step Size</th>
<th>$\lambda_1(T)$</th>
<th>$\lambda_2(T)$</th>
<th>$\Sigma\lambda(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-Euler</td>
<td>$2^{-6}$</td>
<td>0.5582</td>
<td>0.0203</td>
<td>0.5785</td>
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<tr>
<td></td>
<td>$2^{-9}$</td>
<td>0.5898</td>
<td>-0.0130</td>
<td>0.5768</td>
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<tr>
<td>D-Milstein</td>
<td>$2^{-6}$</td>
<td>0.5869</td>
<td>-0.0094</td>
<td>0.5775</td>
</tr>
<tr>
<td></td>
<td>$2^{-9}$</td>
<td>0.5844</td>
<td>-0.0064</td>
<td>0.5780</td>
</tr>
<tr>
<td>CP-Euler</td>
<td>$2^{-6}$</td>
<td>0.5922</td>
<td>-0.0136</td>
<td>0.5786</td>
</tr>
<tr>
<td></td>
<td>$2^{-9}$</td>
<td>0.5603</td>
<td>0.0183</td>
<td>0.5786</td>
</tr>
<tr>
<td>CP-Milstein</td>
<td>$2^{-6}$</td>
<td>0.5911</td>
<td>-0.0125</td>
<td>0.5786</td>
</tr>
<tr>
<td></td>
<td>$2^{-9}$</td>
<td>0.5895</td>
<td>-0.0109</td>
<td>0.5786</td>
</tr>
<tr>
<td>CA-Euler-RK2</td>
<td>$2^{-6}$</td>
<td>0.5628</td>
<td>0.0158</td>
<td>0.5786</td>
</tr>
<tr>
<td></td>
<td>$2^{-9}$</td>
<td>0.5695</td>
<td>0.0091</td>
<td>0.5786</td>
</tr>
<tr>
<td>CA-Milstein-RK2</td>
<td>$2^{-6}$</td>
<td>0.5894</td>
<td>-0.0108</td>
<td>0.5786</td>
</tr>
<tr>
<td></td>
<td>$2^{-9}$</td>
<td>0.5823</td>
<td>-0.0037</td>
<td>0.5786</td>
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</table>

Table III. QR-based approximations to the LEs $\lambda_1 = 0.5786$ and $\lambda_2 = 0$ of Example 2, computed for two different step sizes with $T = 500$. 
Figure 6. QR-based approximations to the LEs $\tilde{\lambda}_1(T)$ and $\tilde{\lambda}_2(T)$ for the Example 2, implemented with a) the Euler integrator, and b) the Milstein integrator, with step size $h = 2^{-9}$, $T = 500$ and parameters $\sigma = 1, \beta = 6^{1/3} \left(2\Gamma(1/2)/2\Gamma(1/6)\right)$.

Figure 6 shows the approximate LEs $\tilde{\lambda}_1(T), \tilde{\lambda}_2(T), 0 \leq T \leq 500$, in this case. Note that it is very similar to Figures 1 and 5. None of the numerical realizations corresponding to $h = 2^{-6}, 2^{-9}$ produced explosive behavior in any numerical integrator.

It is easy to check from Tables II and III that all continuous QR methods automatically preserve the Lyapunov regularity condition. On the other hand, although not exacts, discrete QR methods provide good approximations to the sum of all LEs.

Example 3. The final example corresponds to the stochastic Lorenz system [24], which is given by the SDE
\[ d\mathbf{x}_t = (\mathbf{f} + \mathbf{B}\mathbf{x}_t + \mathbf{F}(\mathbf{x}_t)) dt + \sigma \mathbf{x}_t \circ dw_t, \]
where
\[ \mathbf{f} = \begin{pmatrix} 0 & 0 \\ -b(r+s) & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -s & s & 0 \\ -s & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \quad \mathbf{F}(\mathbf{x}) = \begin{pmatrix} 0 \\ -\mathbf{x}_1 \mathbf{x}_3 \\ \mathbf{x}_1 \mathbf{x}_2 \end{pmatrix}, \]
s, r, b, $\sigma$ are positive constants and $\mathbf{f}$ is an external force. As in [24], we concentrate on the parameter values $s = 10, b = 5/3$ and study changes on the Lyapunov spectrum for different values of $r > 0$.

As it was shown in [24], two different dynamical behaviors can be obtained by varying $r$. Namely, for small values of $r$ all LEs are negative, which corresponds to a one point random attractor. While increasing $r$, the top LE reaches zero (indicating a stochastic bifurcation) to finally become positive for large values of $r$.

It is worth mentioning that, despite being defined by a nonlinear SDE, the Lorenz system is a special case for which $\psi_d(\mathbf{x}, \mathbf{Q})$ is a constant function. Indeed,
\[
\psi_d(\mathbf{x}, \mathbf{Q}) = tr(\mathbf{B} + D\mathbf{F}(\mathbf{x})) + \frac{1}{2}(tr(\sigma^2\mathbf{I}_3) - tr(\sigma\mathbf{L}))^2
\]
\[
= -s - 1 - b - 3\sigma^2 = -13.9367.
\]
Thus, it is another example for which the continuous QR method is exact while approximating the sum of all LEs.

Table IV shows the computed LEs \((T = 500)\) for two extreme regimes \((r = 5\) and \(r = 25\)) corresponding to \(\sigma = 0.3\) and step size \(h = 2^{-8}\).

<table>
<thead>
<tr>
<th>Method</th>
<th>(\lambda(T), r = 5)</th>
<th>(\lambda(T), r = 25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D-Euler</td>
<td>-0.688 -1.073 -12.172</td>
<td>0.644 -0.106 -14.239</td>
</tr>
<tr>
<td>D-Milstein</td>
<td>-0.618 -1.151 -12.200</td>
<td>0.637 -0.096 -14.304</td>
</tr>
<tr>
<td>CP-Euler</td>
<td>-0.689 -1.186 -12.062</td>
<td>0.487 -0.364 -14.060</td>
</tr>
<tr>
<td>CP-Milstein</td>
<td>-0.648 -1.209 -12.079</td>
<td>0.425 -0.366 -13.996</td>
</tr>
<tr>
<td>CA-Euler-RK2</td>
<td>-0.659 -1.194 -12.084</td>
<td>0.450 -0.371 -14.016</td>
</tr>
<tr>
<td>CA-Milstein-RK2</td>
<td>-0.708 -1.145 -12.083</td>
<td>0.442 -0.372 -14.007</td>
</tr>
</tbody>
</table>

Table IV. QR-based approximations to the LEs of Example 3 for two different values of \(r\), with \(h = 2^{-8}\) and \(T = 500\).

As seen on the table, all QR-based methods reproduce well the two different dynamical behaviors corresponding to \(r = 5\) and \(r = 25\), and the Lyapunov regularity property.

7. Conclusions

In this work, two numerical methods for approximating the Lyapunov Exponents of stochastic differential equations were introduced. To the authors knowledge, this work is the first attempt for a systematic study of the QR-based methods in the stochastic case. Such methods constitutes a stochastic version of the well-known QR-based methods that have been long used for ordinary differential equations. In contrast with previous works reported in the literature, the QR-based methods presented here perform well for the approximation of the all Lyapunov exponents.

The numerical examples show that the continuous QR methods seem to be the ones with best performance. This fact can be also corroborated from the errors analysis carried out in section 4. There, it was shown that some difficulties may occur in the computation of negative LEs of large magnitude with discrete QR methods. This limitation can be avoided through the use of continuous QR methods, but at the expense of increasing the computational complexity.

Finally, it is worth to remark that although the present work follows essentially the same ideas exposed in [17], it can be also extended for adapting some promising results about continuous QR methods in ODEs [12] to the stochastic case. Namely, the computation of just a few largest LEs of an SDE by means of the numerical integration of a weak skew-symmetric stochastic SDE. This issue and some other extensions of the theory of the QR-based methods for SDEs will be subject of a future work.

References


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