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MIXED SPECTRAL METHOD FOR NAVIER-STOKES EQUATIONS IN AN INFINITE STRIP BY USING GENERALIZED LAGUERRE FUNCTIONS

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Abstract. In this paper, we propose a mixed spectral method for the Navier-Stokes equations in an infinite strip by using generalized Laguerre functions. We establish some results on mixed generalized Laguerre-Legendre approximation, which play important roles in spectral method for fourth order differential equations. A mixed spectral scheme is provided for stream function form of the Navier-Stokes equations. Its stability and convergence are proved. Numeric results demonstrate the efficiency of suggested algorithm.

Key words. Mixed generalized Laguerre-Legendre spectral method, stream function form of Navier-Stokes equations in an infinite strip.

1. Introduction

The Navier-Stokes equations play an important role in incompressible fluid dynamics. We often used finite difference method and finite element method for their numerical simulations, see, e.g., [6, 9, 24, 28]. As it is well known, spectral method possesses high accuracy, see [2, 3, 4, 5, 7, 10] and the references therein. Some spectral schemes were proposed for the Navier-Stokes equations, see, e.g., [2, 8, 13, 22, 25]. We usually constructed spectral schemes based on the primitive form of the Navier-Stokes equations. But, it is difficult to deal with the incompressibility and the boundary condition of the pressure. Thus, some authors provided certain spectral schemes based on the stream function form of the Navier-Stokes equations, see [11] and the references therein. However, those algorithms are only available for periodic problems and problems defined on bounded rectangular domains.

It is interesting to consider the motion of incompressible fluid flows in unbounded domains. Guo and Xu [19], and Xu and Guo [29] studied spectral and pseudospectral methods using Laguerre polynomials, for the stream function form of the Navier-Stokes equations in an infinite strip. Latter, some authors developed spectral method for the Navier-Stokes equations outside a disc or a ball, see [12, 14, 30]. Recently, Azaiez, Shen, Xu and Zhuang [1] investigated spectral method for the primitive form of the Stokes equation in an infinite strip, by using Laguerre functions as in [27]. We also refer to the work for spectral method using generalized Laguerre functions, see [15, 21]. Generally speaking, the spectral method using Laguerre or generalized Laguerre functions, gives better numerical results, if the exact solutions decay fast.

In this paper, we develop a mixed spectral method for the stream function form of the Navier-Stokes equations in an infinite strip, by using the generalized Laguerre functions. This approach has several merits:

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• It only needs to evaluate the stream function. Moreover, the incompressibility of numerical solution is fulfilled automatically.

• It does not require any approximation of boundary condition on the wall.

• Benefiting from the orthogonality of generalized Laguerre functions, we derive a sparse system with the unknown coefficients of expansion of the stream function. In addition, the numerical solution possesses the spectral accuracy in space.

• For solution decaying fast, our new algorithm oftentimes provides better numerical result than spectral method using Laguerre or generalized Laguerre polynomials.

The paper is organized as follows. In the next section, we recall and renew some results on the orthogonal approximation by using generalized Laguerre functions, which are very applicable to spectral method for fourth order problems defined on unbounded domains. In Section 3, we construct the mixed generalized Laguerre-Legendre spectral scheme for the stream function form of the Navier-Stokes equations in an infinite strip, and present the main results on its stability and convergence. In section 4, we give some numerical results demonstrating the efficiency of suggested algorithm. In section 5, we first investigate two useful mixed orthogonal projections, and then prove the stability and the spectral accuracy in space of our new method. The final section is for some concluding remarks. The techniques developed in this paper are also applicable to other fourth order problems defined on unbounded domains.

2. Preliminary

We first consider the orthogonal approximation by using generalized Laguerre functions. Let $\Lambda = \{ x \mid 0 < x < \infty\}$ and $\chi(x)$ be a certain weight function. We define the weighted space

$$L^2_\chi(\Lambda) = \{ v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{\chi,\Lambda} < \infty \},$$

with the following inner product and norm,

$$(u,v)_{\chi,\Lambda} = \int_\Lambda u(x)v(x)\chi(x)dx, \qquad \|v\|_{\chi,\Lambda} = (v,v)^{\frac{1}{2}}_{\chi,\Lambda}.$$

For any integer $r \geq 0$,

$$H^r_\chi(\Lambda) = \{ v \mid \partial^k_x v \in L^2_\chi(\Lambda), 0 \leq k \leq r \},$$

equipped with the following inner product, semi-norm and norm,

$$(u,v)_{r,\chi,\Lambda} = \sum_{0\leq k\leq r} (\partial^k_x u, \partial^k_x v)_{\chi,\Lambda}, \quad |v|_{r,\chi,\Lambda} = \|\partial^r_x v\|_{\chi,\Lambda}, \quad \|v\|_{r,\chi,\Lambda} = (v,v)^{\frac{1}{2}}_{r,\chi,\Lambda}.$$

For simplicity of statements, we omit the subscript χ in notations, whenever $\chi(x) \equiv 1$.

Let $\omega_{\alpha,\beta}(x) = x^\alpha e^{-\beta x}, \alpha > -1$ and $\beta > 0$. Especially, $\omega_\beta(x) = \omega_{0,\beta}(x) = e^{-\beta x}$. The generalized Laguerre polynomial of degree l is defined by (cf. [20])

$$\mathscr{L}^{(\alpha,\beta)}_l(x) = \frac{1}{l!}\, x^{-\alpha} e^{\beta x} \partial^l_x (x^{l+\alpha} e^{-\beta x}), \qquad l = 0, 1, 2, \cdots.$$

The generalized Laguerre functions are given by (cf. [21])

$$\tilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) = e^{-\frac{1}{2}\beta x} \mathscr{L}^{(\alpha,\beta)}_l(x), \qquad l = 0, 1, 2, \cdots. \tag{1}$$

They fulfill the following recurrence relations:

$$\tilde{\mathscr{L}}^{(\alpha,\beta)}_l(x) = \tilde{\mathscr{L}}^{(\alpha+1,\beta)}_l(x) - \tilde{\mathscr{L}}^{(\alpha+1,\beta)}_{l-1}(x), \qquad l \geq 1, \tag{2}$$

$$(3)\quad \partial_x \tilde{\mathscr{L}}_l^{(\alpha,\beta)}(x) = -\beta \tilde{\mathscr{L}}_{l-1}^{(\alpha+1,\beta)}(x) - \frac{1}{2}\beta \tilde{\mathscr{L}}_l^{(\alpha,\beta)}(x), \qquad l \geq 1,$$

$$(4)\quad (l+1)\tilde{\mathscr{L}}_{l+1}^{(\alpha,\beta)}(x) + (\beta x - 2l - \alpha - 1)\tilde{\mathscr{L}}_l^{(\alpha,\beta)}(x) + (l+\alpha)\tilde{\mathscr{L}}_{l-1}^{(\alpha,\beta)}(x) = 0, \quad l \geq 1,$$

$$(5)\quad \begin{aligned} -x\partial_x \tilde{\mathscr{L}}_l^{(\alpha,\beta)}(x) &= \frac{1}{2}(l+\alpha)\tilde{\mathscr{L}}_{l-1}^{(\alpha,\beta)}(x) + \frac{1}{2}(\alpha+1)\tilde{\mathscr{L}}_l^{(\alpha,\beta)}(x) \\ &\quad -\tfrac{1}{2}(l+1)\tilde{\mathscr{L}}_{l+1}^{(\alpha,\beta)}(x), \qquad l \geq 1. \end{aligned}$$

The relation (2) can be proved by using (2.1) and (2.3) of [20], while the relations (3)-(5) come from (3.1)-(3.3) of [21].

The set of all $\tilde{\mathscr{L}}_l^{(\alpha,\beta)}(x)$ is a complete $L^2_{x^\alpha}(\Lambda)-$orthogonal system, namely,

$$(6)\qquad (\tilde{\mathscr{L}}_l^{(\alpha,\beta)}, \tilde{\mathscr{L}}_m^{(\alpha,\beta)})_{x^\alpha,\Lambda} = \begin{cases} \gamma_l^{(\alpha,\beta)}, & l = m, \\ 0, & l \neq m, \end{cases}$$

where $\gamma_l^{(\alpha,\beta)} = \dfrac{\Gamma(l+\alpha+1)}{\beta^{\alpha+1} l!}$. Thus, for any $v \in L^2_{x^\alpha}(\Lambda)$,

$$v(x) = \sum_{l=0}^{\infty} \tilde{v}_l^{(\alpha,\beta)} \tilde{\mathscr{L}}_l^{(\alpha,\beta)}(x), \qquad \tilde{v}_l^{(\alpha,\beta)} = \frac{1}{\gamma_l^{(\alpha,\beta)}}(v, \tilde{\mathscr{L}}_l^{(\alpha,\beta)})_{x^\alpha,\Lambda}.$$

We now consider two useful orthogonal approximations. Let

$${}_0H^m_{\omega_\beta}(\Lambda) = \{\ v \in H^m_{\omega_\beta}(\Lambda) \mid \partial_x^k v(0) = 0,\ 0 \leq k \leq m-1\ \},$$

$$\mathcal{F}(\Lambda) = \{\ v \in H^1(\Lambda) \mid \text{there exists finite trace of } \partial_x v \ at\ x = 0\ \},$$

$${}_0\mathcal{F}(\Lambda) = \{ v \in \mathcal{F}(\Lambda) \mid v(0) = \partial_x v(0) = 0\ \}.$$

The space $\mathcal{F}(\Lambda)$ is meaningful. For instance, if $v \in L^2(\Lambda)$ and v is continuous near the point $x = 0$, then $v \in \mathcal{F}(\Lambda)$. Next, for any positive integer N, $\mathscr{P}_N(\Lambda)$ stands for the set of all algebraic polynomials of degree at most N, while ${}_0\mathscr{P}_N(\Lambda) = \{\ \phi \in \mathscr{P}_N(\Lambda) \mid \phi(0) = \partial_x\phi(0) = 0\ \}$. Moreover,

$$\mathcal{Q}_{N,\beta}(\Lambda) = \{\ e^{-\frac{1}{2}\beta x}\phi \mid \phi \in \mathscr{P}_N(\Lambda)\ \},$$

$${}_0\mathcal{Q}_{N,\beta}(\Lambda) = \{\ \phi \in \mathcal{Q}_{N,\beta}(\Lambda) \mid \phi(0) = \partial_x\phi(0) = 0\ \}.$$

Throughout this paper, we denote by c a generic positive constant independent of β, N and any function.

The orthogonal projection ${}_0\Pi^1_{N,\beta,\Lambda} : {}_0\mathcal{F}(\Lambda) \to {}_0\mathcal{Q}_{N,\beta}(\Lambda)$ is defined by

$$(\partial_x(v - {}_0\Pi^1_{N,\beta,\Lambda}v), \partial_x\phi)_\Lambda = 0, \qquad \forall\ \phi \in {}_0\mathcal{Q}_{N,\beta}(\Lambda).$$

As it was shown in the proof of Lemma 2.1 of [16], if $v \in {}_0\mathcal{F}(\Lambda)$, $\partial_x^r(e^{\frac{1}{2}\beta x}v) \in L^2_{\omega_{r-2,\beta}}(\Lambda)$ and integers $2 \leq r \leq N+1$, then

$$(7)\quad \|\partial_x^\mu(v - {}_0\Pi^1_{N,\beta,\Lambda}v)\|^2_\Lambda \leq c(1 + \frac{1}{\beta^2})(\beta N)^{2-r}\|\partial_x^r(e^{\frac{1}{2}\beta x}v)\|^2_{\omega_{r-2,\beta},\Lambda}, \qquad \mu = 0,1.$$

In the forthcoming discussions, we need also another orthogonal projection. For this purpose, we introduce the auxiliary orthogonal projection ${}_0P^2_{N,\beta,\Lambda} : {}_0H^2_{\omega_\beta}(\Lambda) \to {}_0\mathscr{P}_N(\Lambda)$, defined by

$$(\partial_x^2(v - {}_0P^2_{N,\beta,\Lambda}v), \partial_x^2\phi)_{\omega_\beta,\Lambda} = 0, \qquad \forall\ \phi \in {}_0\mathscr{P}_N(\Lambda).$$

If $v \in {}_0H^2_{\omega_\beta}(\Lambda)$, $\partial_x^r v \in L^2_{\omega_{r-2,\beta}}(\Lambda)$ and integers $2 \leq r \leq N+1$, then by Lemma 2.4 of [12],

$$(8)\quad ||\partial_x^\mu({}_0P^2_{N,\beta,\Lambda}v - v)||^2_{\omega_\beta,\Lambda} \leq c(1 + \frac{1}{\beta^4})(\beta N)^{2-r}\|\partial_x^r v\|^2_{\omega_{r-2,\beta},\Lambda}, \qquad \mu = 0,1,2.$$

The orthogonal projection ${}_0\Pi^2_{N,\beta,\Lambda} : {}_0H^2(\Lambda) \to {}_0\mathcal{Q}_{N,\beta}(\Lambda)$ is defined by

$$(\partial_x^2(v - {}_0\Pi^2_{N,\beta,\Lambda}v), \partial_x^2\phi)_\Lambda + (v - {}_0\Pi^2_{N,\beta,\Lambda}v, \phi)_\Lambda = 0, \qquad \forall\ \phi \in {}_0\mathcal{Q}_{N,\beta}(\Lambda). \tag{9}$$

Lemma 2.1. If $v \in {}_0H^2(\Lambda)$, $\partial_x^r(e^{\frac{1}{2}\beta x}v) \in L^2_{\omega_{r-2,\beta}}(\Lambda)$ and integers $2 \le r \le N+1$, then

$$\|\partial_x^\mu({}_0\Pi^2_{N,\beta,\Lambda}v - v)\|^2_\Lambda \le c(\beta^4 + \frac{1}{\beta^4})(\beta N)^{2-r}\|\partial_x^r(e^{\frac{1}{2}\beta x}v)\|^2_{\omega_{r-2,\beta},\Lambda}, \quad \mu = 0, 1, 2. \tag{10}$$

Proof. Let ${}_0\hat{\Pi}^2_{N,\beta,\Lambda}v = e^{-\frac{1}{2}\beta x}{}_0P^2_{N,\beta,\Lambda}(e^{\frac{1}{2}\beta x}v) \in {}_0\mathcal{Q}_{N,\beta}(\Lambda)$. By projection theorem,

$$\begin{aligned}\|\partial_x^2({}_0\Pi^2_{N,\beta,\Lambda}v - v)\|^2_\Lambda &+ \|{}_0\Pi^2_{N,\beta,\Lambda}v - v\|^2_\Lambda = \inf_{\phi \in {}_0\mathcal{Q}_{N,\beta}(\Lambda)}(\|\partial_x^2(\phi - v)\|^2_\Lambda + \|\phi - v\|^2_\Lambda) \\ &\le \|\partial_x^2({}_0\hat{\Pi}^2_{N,\beta,\Lambda}v - v)\|^2_\Lambda + \|{}_0\hat{\Pi}^2_{N,\beta,\Lambda}v - v\|^2_\Lambda.\end{aligned} \tag{11}$$

According to the result (i) of Lemma 2.2 of [20], for any $v \in {}_0H^1_{\omega_\beta}(\Lambda)$,

$$\|v\|^2_{\omega_\beta,\Lambda} \le c\|\partial_x v\|^2_{\omega_\beta,\Lambda}. \tag{12}$$

A direct calculation, along with (8) and (12), yields

$$\begin{aligned}\|\partial_x^2({}_0\hat{\Pi}^2_{N,\beta}v - v)\|^2_\Lambda &= \int_\Lambda (\partial_x^2(e^{-\frac{1}{2}\beta x}{}_0P^2_{N,\beta,\Lambda}(e^{\frac{1}{2}\beta x}v) - e^{\frac{1}{2}\beta x}v))^2 dx \\ &= \int_\Lambda (\frac{\beta^2}{4}e^{-\frac{1}{2}\beta x}({}_0P^2_{N,\beta,\Lambda}(e^{\frac{1}{2}\beta x}v) - e^{\frac{1}{2}\beta x}v) \\ &\quad -\frac{1}{2}\beta e^{-\frac{1}{2}\beta x}\partial_x({}_0P^2_{N,\beta,\Lambda}(e^{\frac{1}{2}\beta x}v) - e^{\frac{1}{2}\beta x}v) \\ &\quad + e^{-\frac{1}{2}\beta x}\partial_x^2({}_0P^2_{N,\beta,\Lambda}(e^{\frac{1}{2}\beta x}v) - e^{\frac{1}{2}\beta x}v))^2 dx \\ &\le c(1 + \beta^4)\int_\Lambda e^{-\beta x}(\partial_x^2({}_0P^2_{N,\beta,\Lambda}(e^{\frac{1}{2}\beta x}v) - e^{\frac{1}{2}\beta x}v))^2 dx.\end{aligned}$$

We can use (8) and (12) to estimate $\|{}_0\hat{\Pi}^2_{N,\beta}v - v\|^2_\Lambda$ similarly. Then, the desired result (10) with $\mu = 0, 2$ follows from (11) and the previous statements. Finally, the result (10) with $\mu = 1$ comes from the embedding theory. $\square$

We now turn to the Legendre approximation on the interval $I = \{\ y \mid |y| < 1\}$. For any integer $r \ge 0$, we define the space $H^r_\chi(I)$ and its norm $\|v\|_{r,\chi,I}$ as usual. In particular, $H^2_{\chi,0}(I) = \{\ v \in H^2_\chi(I) \mid v(\pm 1) = \partial_y v(\pm 1) = 0\ \}$. The inner product and norm of the weighted space $L^2_\chi(I)$ are denoted by $(u, v)_{\chi,I}$ and $||v||_{\chi,I}$, respectively. We omit the subscript χ in notations, whenever $\chi(x) \equiv 1$.

Let

$$\mathcal{F}(I) = \{\ v \in H^1(I) \mid \text{there exist finite traces of } \partial_x v \ at\ x = \pm 1\ \},$$

$${}_0\mathcal{F}(I) = \{\ v \in \mathcal{F}(I) \mid v(\pm 1) = \partial_x v(\pm 1) = 0\ \}.$$

For any positive integer M, $\mathscr{P}_M(I)$ stands for the set of all polynomials of degree at most M. Especially, $\mathscr{P}^0_M(I) = \mathscr{P}_M(I) \cap H^2_0(I)$.

The orthogonal projection $\Pi^{1,0}_{M,I} : {}_0\mathcal{F}(I) \to \mathscr{P}^0_M(I)$ is defined by

$$(\partial_y(\Pi^{1,0}_{M,I}v - v), \partial_y\phi)_I = 0, \qquad \forall\phi \in \mathscr{P}^0_M(I). \tag{13}$$

Let $\chi^{(\alpha,\beta)}(y) = (1-y)^\alpha(1+y)^\beta$. We have from Lemma 2.3 of [16] that if $v \in {}_0\mathcal{F}(I)$, $\partial_y^s v \in L^2_{\chi^{(s-2,s-2)}}(I)$ and integers $2 \le s \le M + 1$, then

$$\|\partial_y^\mu(\Pi^{1,0}_{M,I}v - v)\|^2_I \le cM^{2-2s}\|\partial_y^s v\|^2_{\chi^{(s-2,s-2)},I}, \qquad \mu = 0, 1. \tag{14}$$

The orthogonal projection $\Pi^{2,0}_{M,I} : H^2_0(I) \to \mathscr{P}^0_M(I)$ is defined by

$$(\partial_y^2(\Pi^{2,0}_{M,I}v - v), \partial_y^2\phi)_I = 0, \qquad \forall\phi \in \mathscr{P}^0_M(I). \tag{15}$$

By virtue of Theorem 2.5 of [17], we assert that if $v \in H_0^2(I)$, $\partial_y^s v \in L^2_{\chi^{(s-2,s-2)}}(I)$ and integers $2 \le s \le M+1$, then

$$\|\partial_y^\mu(\Pi_{M,I}^{2,0} v - v)\|_I^2 \le cM^{2\mu-2s}\|\partial_y^s v\|^2_{\chi^{(s-2,s-2)},I}, \qquad \mu = 0,1,2. \tag{16}$$

3. Mixed Generalized Laguerre-Legendre Spectral Method

In this section, we propose the mixed spectral method for the stream function form of the Navier-Stokes equations in the strip $\Omega = \{(x,y) \mid 0 < x < \infty,\ |y| < 1\}$, with the boundary $\partial\Omega = \{\ (x,y) \mid x = 0$ or $|\mathrm{y}| = 1\}$. We define the Sobolev spaces $H^r(\Omega)$ in the usual way, with the semi-norm $|v|_{r,\Omega}$ and norm $||v||_{r,\Omega}$. Furthermore,

$$H_0^1(\Omega) = \{\ v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\ \},\ H_0^2(\Omega) = \{\ v \in H^2(\Omega) \mid \frac{\partial v}{\partial n} = v = 0 \text{ on } \partial\Omega\ \}.$$

The inner product and norm of $L^2(\Omega)$ are denoted by $(u,v)_\Omega$ and $||v||_\Omega$, respectively.

We shall use the following notations,

$$\nabla v(x,y) = (\partial_x v(x,y), \partial_y v(x,y))^T, \qquad \Delta v(x,y) = \partial_x^2 v(x,y) + \partial_y^2 v(x,y).$$

Due to (2.2) of [18], for any $v \in H_0^2(\Omega)$,

$$|v|_{2,\Omega} = ||\Delta v||_\Omega^2. \tag{17}$$

For any $v \in H^1(\Omega)$ with $v(x,-1) = 0$,

$$||v||_\Omega^2 \le 2||\partial_y v||_\Omega^2. \tag{18}$$

We also introduce the operators:

$$G(u(x,y),v(x,y)) = \partial_y u(x,y)\partial_x \Delta v(x,y) - \partial_x u(x,y)\partial_y \Delta v(x,y),$$
$$J(u,v,w) = (\Delta v, \partial_y u \partial_x w - \partial_x u \partial_y w)_\Omega.$$

Clearly,

$$J(u,v,w) + J(w,v,u) = 0, \qquad J(u,v,u) = 0. \tag{19}$$

Moreover, by integration by parts, we observe that for any $v \in H^2(\Omega)$ and $u \in H_0^2(\Omega)$,

$$J(u,u,v) = -(G(u,u),v)_\Omega. \tag{20}$$

Let $u, w \in H^2(\Omega)$ and $v \in H_0^2(\Omega)$. If, in addition, $\partial_x u(x,-1) = \partial_y u(x,-1) = \partial_x w(x,-1) = \partial_y w(x,-1) = 0$, then by the result (ii) of Lemma 2.3 and Remark 2.1 of [18],

$$|J(u,u,v)| \le 2|u|_{1,\Omega}|u|_{2,\Omega}|v|_{2,\Omega}. \tag{21}$$

$$|J(u,v,w)| \le 2\|u\|_{2,\Omega}\|v\|_{2,\Omega}\|w\|_{2,\Omega}. \tag{22}$$

Let $T > 0$. $U(x,y,t)$ and $U_0(x,y)$ are the stream function and its initial state, respectively. $f(x,y,t)$ and $\mu > 0$ are the source term and the kinetic viscosity, respectively. For simplicity, we focus on the case with fixed non-slip boundary $\partial\Omega$. Also assume that $U(x,y,t)$ and $\partial_x U(x,y,t)$ decay to zero as $x \to \infty$. Then the stream function form of the Navier-Stokes equations is of the form

$$\begin{cases} \partial_t \Delta U(x,y,t) + G(U(x,y,t),U(x,y,t)) & \\ \quad -\mu\Delta^2 U(x,y,t) = f(x,y,t), & \text{in } \Omega \times (0,T], \\ \dfrac{\partial U(x,y,t)}{\partial n} = U(x,y,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ \lim\limits_{x\to\infty} U(x,y,t) = \lim\limits_{x\to\infty} \partial_x U(x,y,t) = 0, & \text{for } y \in [-1,1],\ t \in (0,T], \\ U(x,y,0) = U_0(x,y), & \text{in } \Omega \cup \partial\Omega. \end{cases} \tag{23}$$

By virtue of (20), we derive a weak formulation of (23). It is to find $v \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H_0^2(\Omega))$ such that

$$\begin{cases} (\partial_t \nabla U(t), \nabla v)_\Omega + J(U(t),U(t),v) + \mu(\Delta U(t), \Delta v)_\Omega = -(f(t),v)_\Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\ v \in H_0^2(\Omega),\ t \in (0,T], \\ U(0) = U_0, \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in}\ \ \Omega \cup \partial\Omega. \end{cases} \tag{24}$$

According to Theorem 2.1 of [18], if $U_0 \in H_0^1(\Omega)$ and $f \in L^2(0,T;H^{-2}(\Omega))$, then problem (24) admits a unique solution.

Now, let ${}_0\mathcal{F}(\Omega) = {}_0\mathcal{F}(\Lambda) \otimes {}_0\mathcal{F}(I)$ and $V_{N,M,\beta}(\Omega) = {}_0\mathcal{Q}_{N,\beta}(\Lambda) \otimes \mathscr{P}_M^0(I)$. The orthogonal projection $\Pi_{N,M,\beta}^{1,0}:\ {}_0\mathcal{F}(\Omega) \to V_{N,M,\beta}(\Omega)$ is defined by

$$(\nabla(\Pi_{N,M,\beta}^{1,0} v - v), \nabla\phi)_\Omega = 0, \qquad\qquad \forall\ \phi \in V_{N,M,\beta}(\Omega). \tag{25}$$

The orthogonal projection $\Pi_{N,M,\beta}^{2,0}:\ H_0^2(\Omega) \to V_{N,M,\beta}(\Omega)$ is defined by

$$(\Delta(\Pi_{N,M,\beta}^{2,0} v - v), \Delta\phi)_\Omega = 0, \qquad\qquad \forall\ \phi \in V_{N,M,\beta}(\Omega). \tag{26}$$

The mixed spectral scheme for (24) is to seek $u_{N,M}(t) \in V_{N,M,\beta}(\Omega)$ for $0 \le t \le T$, such that

$$\begin{cases} (\partial_t \nabla u_{N,M}(t), \nabla\phi)_\Omega + J(u_{N,M}(t), u_{N,M}(t), \phi) + \mu(\Delta u_{N,M}(t), \Delta\phi)_\Omega \\ \quad = -(f(t),\phi)_\Omega, \qquad\qquad\qquad\qquad \forall\ \phi \in V_{N,M,\beta}(\Omega),\ t \in (0,T], \\ u_{N,M}(0) = u_{N,M,0} = \Pi_{N,M,\beta}^{1,0} U_0 \text{ or } \Pi_{N,M,\beta}^{2,0} U_0, \qquad \text{in}\ \ \Omega \cup \partial\Omega. \end{cases} \tag{27}$$

We now check the boundedness of numerical solution. Taking $\phi = 2u_{N,M}(t)$ in (27), we use (17)-(19) to deduce that

$$\begin{aligned} &\partial_t ||\nabla u_{N,M}(t))||_\Omega^2 + 2\mu||\Delta u_{N,M}(t))||_\Omega^2 = -2(f(t), u_{N,M}(t))_\Omega \\ &\quad \le \frac{\mu}{2}\|\nabla u_{N,M}(t)\|_\Omega^2 + \frac{4}{\mu}\|f(t)\|_\Omega^2 \le \mu\|\Delta u_{N,M}(t)\|_\Omega^2 + \frac{4}{\mu}\|f(t)\|_\Omega^2. \end{aligned} \tag{28}$$

Let

$$E(v,\sigma,t) = ||\nabla v(t)||_\Omega^2 + \sigma\int_0^t ||\Delta v(s)||_\Omega^2 ds. \tag{29}$$

Integrating the inequality (28) with respect to t, we obtain

$$E(u_{N,M},\mu,t) \le \|\nabla u_{N,M,0}\|_\Omega^2 + \frac{4}{\mu}\int_0^t \|f(\eta)\|_\Omega^2 d\eta. \tag{30}$$

Remark 3.1. For the existence of solution of (24), we only require $U_0 \in H_0^1(\Omega)$. It means that $\frac{\partial U}{\partial n}$ could be discontinuous at $t = 0$. In this case, we may approximate the initial value in another way. To do this, let $V_{N,M,\beta}^*(\Omega) = \mathcal{Q}_{N,\beta}^*(\Lambda) \otimes \mathscr{P}_M^*(I)$ where

$$\mathcal{Q}_{N,\beta}^*(\Lambda) = \{\ \phi \in \mathcal{Q}_{N,\beta}(\Lambda) \mid \phi(0) = 0\ \}, \qquad \mathscr{P}_M^*(I) = \{\ \phi \in \mathscr{P}_M(I) \mid \phi(\pm 1) = 0\ \}.$$

The orthogonal projection $\Pi_{N,M,\beta}^{1,*}:\ H_0^1(\Omega) \to V_{N,M,\beta}^*(\Omega)$ is defined by

$$(\nabla(\Pi_{N,M,\beta}^{1,*} v - v), \nabla\phi)_\Omega = 0, \qquad\qquad \forall\phi \in V_{N,M,\beta}^*(\Omega). \tag{31}$$

Accordingly, we take $u_{N,M,0} = \Pi_{N,M,\beta}^{1,*} U_0$ in (27). We can derive a priori estimate similar to (30).

We next consider the stability of scheme (27). Because of the nonlinearity, it does not possess the usual stability. But it might be of the generalized stability, as described in [9, 10]. We assume that f and $u_{N,M,0}$ have the errors $\tilde{f}$ and $\tilde{u}_{N,M,0}$ respectively, which induce the error of numerical solution $u_{N,M}$, denoted by $\tilde{u}_{N,M}$.

Theorem 3.1. Let $\tilde{u}_{N,M}(x,y,t)$ be the error of solution of (27), induced by the errors $\tilde{f}$ and $\tilde{u}_{N,M,0}$. We have

$$
\begin{aligned}
E(\tilde{u}_{N,M},\mu,t) \leq\ & e^{\frac{8}{\mu}\int_0^t \|\Delta u_{N,M}(\eta)\|_\Omega^2 d\eta}(||\nabla \tilde{u}_{N,M,0}||_\Omega^2 \\
& +\frac{8}{\mu}\int_0^t e^{-\frac{8}{\mu}\int_0^\eta \|\Delta u_{N,M}(\xi)\|_\Omega^2 d\xi}\|\tilde{f}(\eta)\|_\Omega^2 d\eta).
\end{aligned} \tag{32}
$$

Remark 3.2. According to (30), the integral $\int_0^t \|\Delta u_{N,M}(\eta)\|_\Omega^2 d\eta$ is finite. Therefore, the scheme (27) is of the generalized stability.

Finally, we deal with the convergence of scheme (27). For description of numerical errors, we introduce the following quantities with non-negative integers r and s:

$$
\begin{aligned}
A_{r,s,\beta}(v) =\ & \iint_\Omega (1-y^2)^{s-2}e^{-\beta x}(\partial_x^2\partial_y^s(e^{\frac{1}{2}\beta x}v))^2 dxdy \\
& +\iint_\Omega (1-y^2)^{s-2}((\partial_x\partial_y^s v)^2+(\partial_y^s v)^2)dxdy \\
& +\iint_\Omega x^{r-2}e^{-\beta x}((\partial_x^r\partial_y(e^{\frac{1}{2}\beta x}v))^2+(\partial_x^r(e^{\frac{1}{2}\beta x}v))^2)dxdy, \\
B_{r,s,\beta}(v) =\ & \iint_\Omega (1-y^2)^{s-2}e^{-\beta x}(\partial_x^2\partial_y^s(e^{\frac{1}{2}\beta x}v))^2 dxdy \\
& +\iint_\Omega (1-y^2)^{s-2}((\partial_y^s v)^2+(\partial_x^2\partial_y^s v)^2)dxdy \\
& +\iint_\Omega x^{r-2}e^{-\beta x}((\partial_x^r(e^{\frac{1}{2}\beta x}v))^2+(\partial_x^r\partial_y^2(e^{\frac{1}{2}\beta x}v))^2)dxdy, \\
R_{r,s,\beta}(v,\sigma,t) =\ & \frac{1}{\sigma}(B_{r,s,\beta}(\partial_t v(t))+(\beta^4+\frac{1}{\beta^4})B_{2,2,\beta}(v(t))+\|\Delta v(t)\|_\Omega^2), \\
V_\beta(v,\sigma,t) =\ & \frac{c}{\sigma}((\beta^4+\frac{1}{\beta^4})B_{2,2,\beta}(v(t))+\|\Delta v(t)\|_\Omega^2).
\end{aligned}
$$

Besides, if $u_{N,M,0}=\Pi_{N,M,\beta}^{2,0}U_0$, we take $D_{r,s,\beta}(U_0)=0$. If $u_{N,M,0}=\Pi_{N,M,\beta}^{1,0}U_0$, then $D_{r,s,\beta}(U_0)=A_{r,s,\beta}(U_0)+B_{r,s,\beta}(U_0)$.

Theorem 3.2. Let $U(x,y,t)$ and $u_{N,M}(x,y,t)$ be the solutions of (24) and (27), respectively. If for integers $2\leq r\leq N+1$ and $2\leq s\leq M+1$, the quantities $R_{r,s,\beta}(U,\mu,t)$ and $D_{r,s,\beta}(U_0)$ are finite, then

$$
\begin{aligned}
E(U-u_{N,M},\mu,t) \ & \leq c\big((\beta^4+\frac{1}{\beta^4})(\beta N)^{2-r}+M^{4-2s}\big)\big(e^{\int_0^t V_\beta(U,\mu,\eta)d\eta} \\
& \times\big(\int_0^t e^{-\int_0^\eta V_\beta(U,\mu,\xi)d\xi}R_{r,s,\beta}(U,\mu,\eta)d\eta+D_{r,s,\beta}(U_0)\big) \\
& +\mu\int_0^t B_{r,s,\beta}(U(\eta))d\eta+B_{r,s,\beta}(U(t))\big).
\end{aligned} \tag{33}
$$

Remark 3.3. The result (33) implies

$$
E(U-u_{N,M},\mu,t)=\mathcal{O}((\beta^4+\frac{1}{\beta^4})(\beta N)^{2-r}+M^{4-2s}). \tag{34}
$$

Therefore, the smoother the exact solution, the more accurate the numerical result.

Remark 3.4. We may also take $u_{N,M,0}=\Pi_{N,M,\beta}^{1,*}U_0$ in (27), and derive the error estimation of numerical solution, which is similar to (33).

Remark 3.5. The spectral method (27) is the same as the method given in the reference [29] mathematically, except that we now put a parameter β for the flexibility of computation. But in actual computation, our new algorithm is much easier to be carried out. In fact, Xu and Guo [29] introduced the variable transformation $U = e^{-\frac{1}{2}x}W$, and then used the Laguerre polynomials to solve the reformed problem

$$
\begin{cases}
\partial_t \triangle(e^{-\frac{x}{2}}W) + G(e^{-\frac{x}{2}}W, e^{-\frac{x}{2}}W) & \\
\qquad -\mu\triangle^2(e^{-\frac{x}{2}}W) = f(x,y,t), & \text{in } \Omega\times(0,T], \\
\dfrac{\partial W}{\partial n} = W = 0, & \text{on } \partial\Omega\times[0,T], \\
\lim\limits_{x\to\infty} e^{-\frac{x}{2}}W = \lim\limits_{x\to\infty} e^{-\frac{x}{2}}\partial_x W = 0, & \text{for } y\in[-1,1],\ t\in(0,T], \\
W(x,y,0) = e^{\frac{x}{2}}U_0(x,y) = W_0(x,y), & \text{in } \Omega\cup\partial\Omega.
\end{cases}
\tag{35}
$$

Clearly, this is a twisted way. Therefore, it is quite complicated to derive the algorithm and estimate the error of numerical solution. In opposite, we now use the Laguerre functions directly to solve problem (24), based on the new approximation results of [20]. Accordingly, it is much simpler to do calculation and the error estimate. In particular, we could use the recurrence relations (2)-(5) to derive the corresponding discrete system easily and use the new approximation results (40) and (46) of this paper to estimate the numerical error more precisely.

Remark 3.6. According to Theorem 3.2 of [29], the error of numerical solution $w_{N,M}$ of problem (35), is bounded above by

$$
\begin{aligned}
&\int\int_\Omega e^{-\frac{1}{2}x}|\nabla(W(t)-w_{N,M}(t))|^2dxdy \\
&\quad +\mu\int_0^t\int\int_\Omega e^{-\frac{1}{2}x}|\Delta(W(s)-w_{N,M}(s))|^2dxdyds \le cd_1(N^{2-r}+M^{4-s})
\end{aligned}
$$

where d_1 is a positive constant depending on the norms of certain derivatives of W, with the weight $e^{-qx}, q<1$. Since $W = e^{\frac{1}{2}x}U$, the above result implies

$$
\begin{aligned}
&\int\int_\Omega |\nabla(U(t)-u_{N,M}(t))|^2dxdy \\
&\quad +\mu\int_0^t\int\int_\Omega |\Delta(U(s)-u_{N,M}(s))|^2dxdyds \le cd_2(N^{2-r}+M^{4-s}), \qquad q<1
\end{aligned}
$$

where d_2 is a positive constant depending on the norms of certain derivatives of U, with the weight $e^{(1-q)x}$. In other words, the error estimate is valid only for such solutions whose certain derivatives decay exponeintially. This is a very strong restriction. However, by (33),

$$
\begin{aligned}
&\int\int_\Omega |\nabla(U(t)-u_{N,M}(t))|^2dxdy \\
&\quad +\mu\int_0^t\int\int_\Omega |\Delta(U(s)-u_{N,M}(s))|^2dxdyds \le cd_3(N^{2-r}+M^{4-s}),
\end{aligned}
$$

where d_3 is a positive constant depending on the weighted norms of certain derivatives of U, without the weight $e^{(1-q)x}$. In other words, this error estimate is valid for all solutions whose certain derivatives are in $L^2(\Omega)$. Thereby, our new result is an essential improvement of the existing results.

4. Numerical Results

We now present some numerical results. We first describe the implementation for scheme (27). Let $L_l(y)$ be the Legendre polynomial of degree l, and

$$
\phi_k(x) = \tilde{\mathscr{L}}_k^{(0,\beta)}(x) - 2\tilde{\mathscr{L}}_{k+1}^{(0,\beta)}(x) + \tilde{\mathscr{L}}_{k+2}^{(0,\beta)}(x), \qquad 0\le k\le N-2.
$$

$$\psi_l(y) = \frac{1}{\sqrt{2(2l+3)^2(2l+5)}}(L_l(y) - \frac{2(2l+5)}{2l+7}L_{l+2}(y) + \frac{2l+3}{2l+7}L_{l+4}(y)), \quad 0 \le l \le M-4.$$

Obviously, $\phi_k(0) = \partial_x\phi_k(0) = \psi_l(\pm 1) = \partial_y\psi_l(\pm 1) = 0$, cf. [23, 26]. Take $\Phi_{k,l}(x,y) = \phi_k(x)\psi_l(y)$. The set of all $\Phi_{k,l}(x,y)$ conforms a basis of $V_{N,M,\beta}(\Omega)$.

We expand the numerical solution $u_{N,M}$ as

$$u_{N,M}(x,y,t) = \sum_{i=0}^{N-2}\sum_{j=0}^{M-4} u_{i,j}(t)\Phi_{i,j}(x,y).$$

For notational convenience, we also set

$$q_{k,l}(t) = -J\big(u_{N,M}(t), u_{N,M}(t), \Phi_{k,l}\big), \qquad f_{k,l}(t) = -(f(t), \Phi_{k,l})_\Omega.$$

Taking $\phi(x,y) = \Phi_{k,l}(x,y)$ in (27), we obtain a linear system of ordinary differential equations, as

(36)
$$\begin{aligned}
&\sum_{i=0}^{N-2}\sum_{j=0}^{M-4} \partial_t u_{i,j}(t) \iint_\Omega \big(\partial_x\Phi_{i,j}(x,y)\partial_x\Phi_{k,l}(x,y) + \partial_y\Phi_{i,j}(x,y)\partial_y\Phi_{k,l}(x,y)\big)dxdy \\
&\quad +\mu\sum_{i=0}^{N-2}\sum_{j=0}^{M-4} u_{i,j}(t) \iint_\Omega \big(\partial_x^2\Phi_{i,j}(x,y)\partial_x^2\Phi_{k,l}(x,y) + \partial_x^2\Phi_{i,j}(x,y)\partial_y^2\Phi_{k,l}(x,y) \\
&\qquad +\partial_y^2\Phi_{i,j}(x,y)\partial_x^2\Phi_{k,l}(x,y) + \partial_y^2\Phi_{i,j}(x,y)\partial_y^2\Phi_{k,l}(x,y)\big)dxdy \\
&= q_{k,l}(t) + f_{k,l}(t), \qquad 0 \le k \le N-2,\ 0 \le l \le M-4.
\end{aligned}$$

We could rewrite the above system in a compact matrix form. To do this, we set

$$X(t) = (u_{0,0}(t), \cdots, u_{0,M-4}(t), u_{1,0}(t), \cdots, u_{1,M-4}(t), \cdots, u_{N-2,0}(t), \cdots, u_{N-2,M-4}(t))^T,$$
$$Q(t) = (q_{0,0}(t), \cdots, q_{0,M-4}(t), q_{1,0}(t), \cdots, q_{1,M-4}(t), \cdots, q_{N-2,0}(t), \cdots, q_{N-2,M-4}(t))^T,$$
$$F(t) = (f_{0,0}(t), \cdots, f_{0,M-4}(t), f_{1,0}(t), \cdots, f_{1,M-4}(t), \cdots, f_{N-2,0}(t), \cdots, f_{N-2,M-4}(t))^T.$$

We also introduce the matrices $A_{\sigma,\lambda} = (a_{k,i}^{(\sigma,\lambda)})$ and $B_{\sigma,\lambda} = (b_{l,j}^{(\sigma,\lambda)})$, with the following entries,

$$a_{k,i}^{(1,1)} = \int_\Lambda \partial_x\phi_k(x)\partial_x\phi_i(x)dx, \qquad 0 \le k, i \le N-2,$$

$$a_{k,i}^{(1,2)} = a_{k,i}^{(2,4)} = \int_\Lambda \phi_k(x)\phi_i(x)dx, \qquad 0 \le k, i \le N-2,$$

$$a_{k,i}^{(2,1)} = \int_\Lambda \partial_x^2\phi_k(x)\partial_x^2\phi_i(x)dx, \qquad 0 \le k, i \le N-2,$$

$$a_{k,i}^{(2,2)} = a_{i,k}^{(2,3)} = \int_\Lambda \partial_x^2\phi_k(x)\phi_i(x)dx, \qquad 0 \le k, i \le N-2,$$

and

$$b_{l,j}^{(1,1)} = b_{l,j}^{(2,1)} = \int_I \psi_l(y)\psi_j(y)dy, \qquad 0 \le l, j \le M-4,$$

$$b_{l,j}^{(1,2)} = \int_I \partial_y\psi_l(y)\partial_y\psi_j(y)dy, \qquad 0 \le l, j \le M-4,$$

$$b_{l,j}^{(2,2)} = b_{j,l}^{(2,3)} = \int_I \psi_l(y)\partial_y^2\psi_j(y)dy, \qquad 0 \le l, j \le M-4,$$

$$b_{l,j}^{(2,4)} = \int_I \partial_y^2\psi_l(y)\partial_y^2\psi_j(y)dy, \qquad 0 \le l, j \le M-4.$$

Furthermore, let

$$A = A_{11} \otimes B_{11} + A_{12} \otimes B_{12}, \qquad B = A_{21} \otimes B_{21} + A_{22} \otimes B_{22} + A_{23} \otimes B_{23} + A_{24} \otimes B_{24}.$$

Then, the system (36) can be rewritten as the following compact matrix form,

$$A\partial_t X(t) + \mu B X(t) = Q(t) + F(t). \tag{37}$$

In actual computation, we use the fourth order Runge-kutta approximation with the step size τ in time.

For description of numerical errors, we denote the nodes and weights of Laguerre-Gauss-Radau quadrature using generalized Laguerre functions (cf. [21]), by $\xi_{\beta,i}$ and $\omega_{\beta,i}$, respectively. Meanwhile, we denote the nodes and weights of Legendre-Gauss-Labatto quadrature by ζ_j and κ_j, respectively. The numerical errors are measured by the quantity

$$E_{N,M}(t) = \Big(\sum_{i=0}^{N-2}\sum_{j=0}^{M-4}(U(\xi_{\beta,i},\zeta_j,t) - u_{N,M}(\xi_{\beta,i},\zeta_j,t))^2\omega_{\beta,i}\kappa_j\Big)^{\frac{1}{2}} \approx \|U(t) - u_{N,M}(t)\|_\Omega.$$

We use scheme (27) to solve problem (24). We first take the test function

$$U(x,y,t) = x^2(1-y^2)^2 e^{-x} sin(kxt + kyt),$$

which oscillates and decays exponentially as x increases. In Figure 1, we plot the value of $\log_{10} E_{N,M}(1)$ with $k = 0.2,\ \beta = 1,\ N = 4M$ and $\tau = 0.005,\ 0.001,\ 0.0001$, vs. the mode M. Clearly, the numerical error decays very fast when M increases and τ decreases. This confirms the prediction by Theorem 3.2. In Figure 2, we plot the values of $\log_{10} E_{N,M}(t)$ for $0 \le t \le 5$, with $k = 0.1,\ \beta = 1,\ N = 4M = 48$ and $\tau = 0.001$. They demonstrate the stability of scheme (27), as predicted by Theorem 3.1.

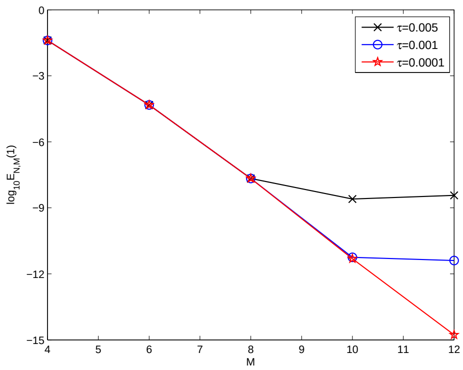

FIGURE 1. Convergence rate.

We next take the test function

$$U(x,y,t) = \frac{x^2(1-y^2)^2 sin(kxt + kyt)}{(2+x+y)^h}, \qquad h > 0,$$

which oscillates and decays algebraically as x tends to infinity. In Table 1, we list the values of $\log_{10} E_{N,M}(1)$ with $k = 0.2,\ h = 4,6,\ \beta = 1,\ N = 4M$ and $\tau = 0.001$, vs. the mode M. It shows again the convergence of scheme (27). We also observe that our method provides more accurate numerical results for solutions decaying faster. This coincides with theoretical analysis.

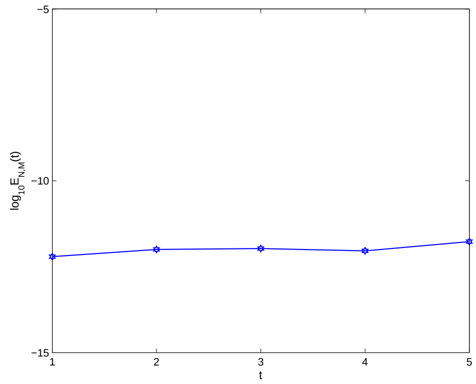

FIGURE 2. Stability of computation.

TABLE 1. Convergence rate with $h = 3,\ 4$.

	$M=4$	$M=6$	$M=8$	$M=10$	$M=12$	$M=14$
$h=3$	1.77E-2	3.58E-4	2.28E-4	1.60E-4	1.16E-4	8.99E-5
$h=4$	2.81E-3	8.51E-5	1.33E-5	2.57E-6	9.30E-7	5.13E-7

Finally, we take the test function

$$U(x,y,t) = \frac{x^2(1-y^2)^2 sin(\dfrac{kt}{x+d} + kyt)}{(2+x+y)^h}, \qquad d > 0.$$

This function with small $d > 0$, oscillates very seriously near $x = 0$, and decays algebraically as x tends to infinity. In Table 2, we list the values of $\log_{10} E_{N,M}(1)$ with $d = 0.1,\ k = 1,\ h = 4,\ N = 4M,\ \tau = 0.001$ and $\beta = 1,\ 1.5$, vs. the mode M. We find that the suitable parameter β leads to better numerical results.

TABLE 2. Convergence rate with $\beta = 1,\ 1.5$.

	$M=4$	$M=6$	$M=8$	$M=10$	$M=12$
$\beta=1$	2.06E-3	8.24E-4	4.48E-4	2.41E-4	1.25E-4
$\beta=1.5$	1.81E-3	3.53E-4	1.26E-4	4.46E-5	2.92E-5

5. Error Analysis

In this section, we prove Theorems 3.1 and 3.2.

5.1. Proof of Theorem 3.1. We have from (27) that

$$\begin{aligned}(\partial_t \nabla \tilde{u}_{N,M}(t), \nabla\phi)_\Omega + \mu(\Delta\tilde{u}_{N,M}(t), \Delta\phi)_\Omega + J(u_{N,M}(t), \tilde{u}_{N,M}(t), \phi)\\ +J(\tilde{u}_{N,M}(t), u_{N,M}(t) + \tilde{u}_{N,M}(t), \phi) = -(\tilde{f}(t), \phi)_\Omega.\end{aligned} \tag{38}$$

Taking $\phi = 2\tilde{u}_{N,M}$ in (38), we use the two results of (19) to obtain

$$\begin{aligned}\partial_t\|\nabla\tilde{u}_{N,M}(t)\|_\Omega^2 + 2\mu\|\Delta\tilde{u}_{N,M}(t)\|_\Omega^2 - 2J(\tilde{u}_{N,M}(t), \tilde{u}_{N,M}(t), u_{N,M}(t))\\ = -2(\tilde{f}(t), \tilde{u}_{N,M}(t))_\Omega.\end{aligned} \tag{39}$$

Thanks to (17), (18), (21) and the Cauchy inequality, we derive that

$$\begin{aligned}|J(\tilde{u}_{N,M}(t),\tilde{u}_{N,M}(t),u_{N,M})| &\le 2\|\Delta\tilde{u}_{N,M}(t)\|_\Omega\|\nabla\tilde{u}_{N,M}(t)\|_\Omega\|\Delta u_{N,M}(t)\|_\Omega\\ &\le \frac{\mu}{4}\|\Delta\tilde{u}_{N,M}(t)\|^2_\Omega+\frac{4}{\mu}\|\Delta u_{N,M}(t)\|^2_\Omega\|\nabla\tilde{u}_{N,M}(t)\|^2_\Omega,\end{aligned}$$

$$|(\tilde{f}(t),\tilde{u}_{N,M}(t))_\Omega| \le \frac{\mu}{8}\|\nabla\tilde{u}_{N,M}(t)\|^2_\Omega+\frac{4}{\mu}\|\tilde{f}(t)\|^2_\Omega \le \frac{\mu}{4}\|\Delta\tilde{u}_{N,M}(t)\|^2_\Omega+\frac{4}{\mu}\|\tilde{f}(t)\|^2_\Omega.$$

Let $E(v,\sigma,t)$ be the same as in (29). Substituting the above two inequalities into (39), yields

$$\partial_t E(\tilde{u}_{N,M},\mu,t) \le \frac{8}{\mu}(\|\Delta u_{N,M}(t)\|^2_\Omega E(\tilde{u}_{N,M},\mu,t)+\|\tilde{f}(t)\|^2_\Omega).$$

This implies

$$\partial_t(e^{-\frac{8}{\mu}\int_0^t\|\Delta u_{N,M}(\eta)\|^2_\Omega d\eta}E(\tilde{u}_{N,M},\mu,t)) \le \frac{8}{\mu}e^{-\frac{8}{\mu}\int_0^t\|\Delta u_{N,M}(\eta)\|^2_\Omega d\eta}\|\tilde{f}(t)\|^2_\Omega.$$

Integrating the above inequality with respect to t, we obtain the desired result (32).

5.2. Some approximation results. In order to prove Theorem 3.2, we need some results on the mixed generalized Laguerre-Legendre approximation. In the sequel, the meanings of $A_{r,s,\beta}(v)$ and $B_{r,s,\beta}(v)$ are the same as in Theorem 3.2.

Lemma 5.1. If $v\in {}_0\mathcal{F}(\Omega)$ and $A_{r,s,\beta}(v)$ are finite for integers $2\le r\le N+1, 2\le s\le M+1$, then

$$\|\nabla(\Pi^{1,0}_{N,M,\beta}v-v)\|^2_\Omega \le c((1+\frac{1}{\beta^2})(\beta N)^{2-r}+M^{4-2s})A_{r,s,\beta}(v). \tag{40}$$

Proof. Clearly, $\Pi^{1,0}_{M,I}({}_0\Pi^1_{N,\beta,\Lambda}v)\in V_{N,M,\beta}(\Omega)$. By projection theorem,

$$\begin{aligned}\|\nabla(\Pi^{1,0}_{N,M}v-v)\|^2_\Omega &= \inf_{\phi\in V_{N,M,\beta}(\Omega)}\|\nabla(\phi-v)\|^2_\Omega \le \|\nabla(\Pi^{1,0}_{M,I}({}_0\Pi^1_{N,\beta,\Lambda}v)-v)\|^2_\Omega\\ &\le \|\partial_x(\Pi^{1,0}_{M,I}({}_0\Pi^1_{N,\beta,\Lambda}v)-{}_0\Pi^1_{N,\beta,\Lambda}v)\|^2_\Omega\\ &\quad+\|\partial_y(\Pi^{1,0}_{M,I}({}_0\Pi^1_{N,\beta,\Lambda}v)-{}_0\Pi^1_{N,\beta,\Lambda}v)\|^2_\Omega\\ &\quad+\|\partial_x({}_0\Pi^1_{N,\beta,\Lambda}v-v)\|^2_\Omega+\|\partial_y({}_0\Pi^1_{N,\beta,\Lambda}v-v)\|^2_\Omega.\end{aligned} \tag{41}$$

By using (14) with $\mu=0$ and (7) with $\mu=1,r=2$ successively, we deduce that for $s\ge2$,

$$\begin{aligned}&\|\partial_x(\Pi^{1,0}_{M,I}({}_0\Pi^1_{N,\beta,\Lambda}v)-{}_0\Pi^1_{N,\beta,\Lambda}v)\|^2_\Omega\\ &\quad\le cM^{2-2s}\iint_\Omega(1-y^2)^{s-2}(\partial_x\partial_y^s({}_0\Pi^1_{N,\beta,\Lambda}v))^2dxdy\\ &\quad\le c(1+\frac{1}{\beta^2})M^{2-2s}\iint_\Omega(1-y^2)^{s-2}e^{-\beta x}(\partial_x^2\partial_y^s(e^{\frac12\beta x}v))^2dxdy\\ &\qquad+cM^{2-2s}\iint_\Omega(1-y^2)^{s-2}(\partial_x\partial_y^s v)^2dxdy.\end{aligned} \tag{42}$$

Similarly, we use (14) with $\mu=1$ and (7) with $\mu=0,r=2$ successively, to obtain that for $s\ge2$,

$$\begin{aligned}&\|\partial_y(\Pi^{1,0}_{M,I}({}_0\Pi^1_{N,\beta,\Lambda}v)-{}_0\Pi^1_{N,\beta,\Lambda}v)\|^2_\Omega\\ &\quad\le cM^{2-2s}\iint_\Omega(1-y^2)^{s-2}(\partial_y^s({}_0\Pi^1_{N,\beta,\Lambda}v))^2dxdy\\ &\quad\le c(1+\frac{1}{\beta^2})M^{2-2s}\iint_\Omega(1-y^2)^{s-2}e^{-\beta x}(\partial_x^2\partial_y^s(e^{\frac12\beta x}v))^2dxdy\\ &\qquad+cM^{2-2s}\iint_\Omega(1-y^2)^{s-2}(\partial_y^s v)^2dxdy.\end{aligned} \tag{43}$$

It is easy to verify that for $r \geq 2$,

$$\|\partial_x({}_0\Pi^1_{N,\beta,\Lambda}v - v)\|^2_\Omega \leq c(1+\frac{1}{\beta^2})(\beta N)^{2-r}\iint_\Omega x^{r-2}e^{-\beta x}(\partial_x^r(e^{\frac{1}{2}\beta x}v))^2dxdy, \tag{44}$$

$$\|\partial_y({}_0\Pi^1_{N,\beta,\Lambda}v - v)\|^2_\Omega \leq c(1+\frac{1}{\beta^2})(\beta N)^{2-r}\iint_\Omega x^{r-2}e^{-\beta x}(\partial_x^r\partial_y(e^{\frac{1}{2}\beta x}v))^2dxdy. \tag{45}$$

Then, a combination of (41)-(45) leads to the desired result. □

We next deal with the error of projection $\Pi^{2,0}_{N,M,\beta}v$, which plays an essential role in the error estimate of numerical solution.

Lemma 5.2. If $v \in H^2_0(\Omega)$ and $B_{r,s,\beta}(v)$ are finite for integers $2 \leq r \leq N+1, 2 \leq s \leq M+1$, then

$$\|\Delta(\Pi^{2,0}_{N,M}v - v)\|^2_\Omega \leq c((\beta^4+\frac{1}{\beta^4})(\beta N)^{2-r} + M^{4-2s})B_{r,s,\beta}(v). \tag{46}$$

Proof. Obviously, $\Pi^{2,0}_{M,I}({}_0\Pi^2_{N,\beta,\Lambda}v) \in V_{N,M,\beta}(\Omega)$. By projection theorem,

$$\begin{aligned}\|\Delta(\Pi^{2,0}_{N,M}v - v)\|^2_\Omega &= \inf_{\phi\in V_{N,M,\beta}(\Omega)}\|\Delta(\phi - v)\|^2_\Omega \leq \|\Delta(\Pi^{2,0}_{M,I}({}_0\Pi^2_{N,\beta,\Lambda}v) - v)\|^2_\Omega\\ &\leq 2\|\partial_x^2(\Pi^{2,0}_{M,I}({}_0\Pi^2_{N,\beta,\Lambda}v) - {}_0\Pi^2_{N,\beta,\Lambda}v)\|^2_\Omega\\ &\quad+2\|\partial_y^2(\Pi^{2,0}_{M,I}({}_0\Pi^2_{N,\beta,\Lambda}v) - {}_0\Pi^2_{N,\beta,\Lambda}v)\|^2_\Omega\\ &\quad+2\|\partial_x^2({}_0\Pi^2_{N,\beta,\Lambda}v - v)\|^2_\Omega + \|\partial_y^2({}_0\Pi^2_{N,\beta,\Lambda}v - v)\|^2_\Omega.\end{aligned} \tag{47}$$

We use (16) with $\mu = 0$ and (10) with $\mu = r = 2$ successively, to obtain that for $s \geq 2$,

$$\begin{aligned}&\|\partial_x^2(\Pi^{2,0}_{M,I}({}_0\Pi^2_{N,\beta,\Lambda}v) - {}_0\Pi^2_{N,\beta,\Lambda}v)\|^2_\Omega\\ &\quad\leq cM^{-2s}\iint_\Omega(1-y^2)^{s-2}(\partial_x^2\partial_y^s({}_0\Pi^2_{N,\beta,\Lambda}v))^2dxdy\\ &\quad\leq c(\beta^4+\frac{1}{\beta^4})M^{-2s}\iint_\Omega(1-y^2)^{s-2}e^{-\beta x}(\partial_x^2\partial_y^s(e^{\frac{1}{2}\beta x}v))^2dxdy\\ &\qquad+cM^{-2s}\iint_\Omega(1-y^2)^{s-2}(\partial_x^2\partial_y^s v)^2dxdy.\end{aligned} \tag{48}$$

In the same manner, we use (16) with $\mu = 2$ and (10) with $\mu = 0, r = 2$ successively, to verify that for $s \geq 2$,

$$\begin{aligned}&\|\partial_y^2(\Pi^{2,0}_{M,I}({}_0\Pi^2_{N,\beta,\Lambda}v) - {}_0\Pi^2_{N,\beta,\Lambda}v)\|^2_\Omega\\ &\quad\leq cM^{4-2s}\iint_\Omega(1-y^2)^{s-2}(\partial_y^s({}_0\Pi^2_{N,\beta,\Lambda}v))^2dxdy\\ &\quad\leq c(\beta^4+\frac{1}{\beta^4})M^{4-2s}\iint_\Omega(1-y^2)^{s-2}e^{-\beta x}(\partial_x^2\partial_y^s(e^{\frac{1}{2}\beta x}v))^2dxdy\\ &\qquad+cM^{4-2s}\iint_\Omega(1-y^2)^{s-2}(\partial_y^s v)^2dxdy.\end{aligned} \tag{49}$$

Similarly, for $r \geq 2$,

$$\|\partial_x^2({}_0\Pi^2_{N,\beta,\Lambda}v - v)\|^2_\Omega \leq c(\beta^4+\frac{1}{\beta^4})(\beta N)^{2-r}\iint_\Omega x^{r-2}e^{-\beta x}(\partial_x^r(e^{\frac{1}{2}\beta x}v))^2dxdy, \tag{50}$$

$$\|\partial_y^2({}_0\Pi^2_{N,\beta,\Lambda}v - v)\|^2_\Omega \leq c(\beta^4+\frac{1}{\beta^4})(\beta N)^{2-r}\iint_\Omega x^{r-2}e^{-\beta x}(\partial_x^r\partial_y^2(e^{\frac{1}{2}\beta x}v))^2dxdy. \tag{51}$$

Then, the desired result frows from a combination of (47)-(51) . □

5.3. Proof of Theorem 3.2. We are now in position to estimate the error of numerical solution $u_{N,M}$.

Let $U^*_{N,M} = \Pi^{2,0}_{N,M,\beta} U$. Then, by virtue of (24) and (26), we have

$$\begin{aligned}(\partial_t \nabla U^*_{N,M}(t), \nabla\phi)_\Omega + \mu(\Delta U^*_{N,M}(t), \Delta\phi)_\Omega + J(U,U,\phi)\\ = G_1(\phi,t) + (f(t),\phi)_\Omega, \qquad \forall \phi \in V_{N,M,\beta}(\Omega),\end{aligned}$$

with

$$G_1(\phi,t) = (\partial_t \nabla(U^*_{N,M} - U), \nabla\phi)_\Omega.$$

Further, we set $\tilde{U}_{N,M} = u_{N,M} - U^*_{N,M}$. Subtracting the above equation from (27), we obtain

$$\left\{\begin{array}{ll}(\partial_t \nabla \tilde{U}_{N,M}(t), \nabla\phi)_\Omega + \mu(\Delta\tilde{U}_{N,M}(t), \Delta\phi)_\Omega + G_1(\phi,t) & \\ \quad +J(u_{N,M}, u_{N,M}, \phi) - J(U,U,\phi) = 0, & \forall\phi \in V_{N,M}(\Omega),\\ \tilde{U}_{N,M,0} = \Pi^{1,0}_{N,M,\beta}U_0 - \Pi^{2,0}_{N,M,\beta}U_0 \text{ or } 0. & \end{array}\right. \tag{52}$$

With the aid of (19), it can be checked that if $u,\ v,\ v^* \in H^2(\Omega)$ and $u - v^* \in H^2_0(\Omega)$, then

$$\begin{aligned}J(u,u,u-v^*) - J(v,v,u-v^*) = \ & -J(u-v^*, u-v^*, v^*) + J(v^*-v, v^*, u-v^*)\\ & +J(v, v^*-v, u-v^*).\end{aligned}$$

As a result,

$$J(u_{N,M}, u_{N,M}, \tilde{U}_{N,M}) - J(U,U,\tilde{U}_{N,M}) = \sum_{j=2}^{4} G_j(\tilde{U}_{N,M}, t)$$

where

$$\begin{aligned}G_2(\tilde{U}_{N,M},t) &= -J(\tilde{U}_{N,M}(t), \tilde{U}_{N,M}(t), U^*_{N,M}(t)),\\ G_3(\tilde{U}_{N,M},t) &= J(U^*_{N,M}(t) - U(t), U^*_{N,M}(t), \tilde{U}_{N,M}(t)),\\ G_4(\tilde{U}_{N,M},t) &= J(U(t), U^*_{N,M}(t) - U(t), \tilde{U}_{N,M}(t)).\end{aligned}$$

We take $\phi = 2\tilde{U}_{N,M}$ in (52). Then

$$\partial_t \|\nabla\tilde{U}_{N,M}(t)\|^2_\Omega + 2\mu\|\Delta\tilde{U}_{N,M}(t)\|^2_\Omega + 2\sum_{j=1}^{4} G_j(\tilde{U}_{N,M},t) = 0. \tag{53}$$

We now estimate $|G_i(\tilde{U}_{N,M}(t),t)|, 1 \le j \le 4$. Let $R_{r,s,\beta}(v,\sigma,t)$ and $D_{r,s,\beta}(v)$ be the same as in Theorem 3.2. Since $\partial_x U^*_{N,M}(x,y,t), \partial_y U^*_{N,M}(x,y,t), \partial_x U_{N,M}(x,y,t)$ and $\partial_y U_{N,M}(x,y,t)$ vanish at $y=-1$, we can use (18), (17) and (46) successively to deduce that

(54)

$$\begin{aligned}|G_1(\tilde{U}_{N,M}(t),t)| &\le \frac{4}{\mu}\|(\nabla(\partial_t U^*_{N,M}(t) - \partial_t U(t))\|^2_\Omega + \frac{\mu}{16}\|\nabla\tilde{u}_{N,M}(t)\|^2_\Omega\\ &\le \frac{8}{\mu}\|(\Delta(\partial_t U^*_{N,M}(t) - \partial_t U(t))\|^2_\Omega + \frac{\mu}{8}\|\Delta\tilde{u}_{N,M}(t)\|^2_\Omega\\ &\le \frac{\mu}{8}\|\Delta\tilde{u}_{N,M}\|^2_\Omega + \frac{c}{\mu}((\beta^4 + \frac{1}{\beta^8})(\beta N)^{2-r} + M^{4-2s})B_{r,s,\beta}(\partial_t U(t)).\end{aligned}$$

Next, by virtue of (46) with $r=s=2$, we have

$$\begin{aligned}\|\Delta U^*_{N,M}(t)\|^2_\Omega &\le \|\Delta(U^*_{N,M}(t) - U(t))\|^2_\Omega + \|\Delta U(t)\|^2_\Omega\\ &\le c(\beta^4 + \frac{1}{\beta^4})B_{2,2,\beta}(U(t)) + \|\Delta U(t)\|^2_\Omega.\end{aligned} \tag{55}$$

Accordingly, we use (21), (17) and (55) successively to verify that

$$
\begin{aligned}
|G_2(\tilde{U}_{N,M}(t),t)| &\leq 2|\tilde{U}_{N,M}(t)|_{1,\Omega}|\tilde{U}_{N,M}(t)|_{2,\Omega}|U^*_{N,M}(t)|_{2,\Omega} \\
&= 2\|\nabla\tilde{U}_{N,M}(t)\|_\Omega\|\Delta\tilde{U}_{N,M}(t)\|_\Omega\|\Delta U^*_{N,M}(t)\|_\Omega \\
&\leq \frac{\mu}{8}\|\Delta\tilde{U}_{N,M}(t)\|^2_\Omega + \frac{8}{\mu}\|\nabla\tilde{U}_{N,M}(t)\|^2_\Omega\|\Delta U^*_{N,M}\|^2_\Omega \\
&\leq \frac{\mu}{8}\|\Delta\tilde{U}_{N,M}(t)\|^2_\Omega + \frac{c}{\mu}((\beta^4+\frac{1}{\beta^4})B_{2,2,\beta}(U(t)) \\
&\quad +\|\Delta U(t)\|^2_\Omega)\|\nabla\tilde{U}_{N,M}(t)\|^2_\Omega.
\end{aligned}
\tag{56}
$$

Furthermore, by virtue of (17) and (18), for any $v \in H^2_0(\Omega)$,

$$
||v||^2_{2,\Omega} \leq |v|^2_{2,\Omega} + 2||\partial_x\partial_y v||^2_\Omega + 2||\partial^2_y v||^2_\Omega + 4||\partial^2_y v||^2_\Omega \leq 7|v|^2_{2,\Omega} = 7||\Delta v||^2_\Omega.
$$

Therefore, with the aid of (22), (46) and (55), we derive that

$$
\begin{aligned}
|G_3(\tilde{U}_{N,M}(t),t)| &\leq 2\|U^*_{N,M}(t)-U(t)\|_{2,\Omega}\|\tilde{U}_{N,M}(t)\|_{2,\Omega}\|U^*_{N,M}(t)\|_{2,\Omega} \\
&\leq 2\sqrt{343}\|\Delta(U^*_{N,M}(t)-U(t))\|_\Omega\|\Delta\tilde{U}_{N,M}(t)\|_\Omega\|\Delta U^*_{N,M}(t)\|_\Omega \\
&\leq \frac{\mu}{8}\|\Delta\tilde{U}_{N,M}(t)\|^2_\Omega + \frac{2744}{\mu}\|\Delta(U^*_{N,M}(t)-U(t))\|^2_\Omega\|\Delta U^*_{N,M}(t)\|^2_\Omega \\
&\leq \frac{\mu}{8}\|\Delta\tilde{U}_{N,M}(t)\|^2_\Omega + \frac{c}{\mu}((\beta^4+\frac{1}{\beta^4})(\beta N)^{2-r}+M^{4-2s}) \\
&\quad \times((\beta^4+\frac{1}{\beta^4})B_{2,2,\beta}(U(t))+\|\Delta U(t)\|^2_\Omega)B_{r,s,\beta}(U(t)).
\end{aligned}
\tag{57}
$$

Similarly,

$$
\begin{aligned}
|G_4(\tilde{U}_{N,M}(t),t)| &\leq \frac{\mu}{8}\|\Delta\tilde{U}_{N,M}(t)\|^2_\Omega + \frac{2744}{\mu}\|\Delta(U^*_{N,M}(t)-U(t))\|^2_\Omega\|\Delta U(t)\|^2_\Omega \\
&\leq \frac{\mu}{8}\|\Delta\tilde{U}_{N,M}(t)\|^2_\Omega \\
&\quad +\frac{c}{\mu}((\beta^4+\frac{1}{\beta^8})(\beta N)^{2-r}+M^{4-2s})\|\Delta U(t)\|^2_\Omega B_{r,s,\beta}(U(t)).
\end{aligned}
\tag{58}
$$

If we take $u_{N,M,0} = \Pi^{2,0}_{N,M,\beta}U_0$, then $\tilde{U}_{N,M,0} = 0$. If we take $u_{N,M,0} = \Pi^{1,0}_{N,M,\beta}U_0$, then we use (17), (18), (40) and (46) to deduce that

$$
\begin{aligned}
||\tilde{U}_{N,M,0}||^2_\Omega &\leq \|\nabla(\Pi^{1,0}_{N,M}U_0-U_0)\|^2_\Omega + \|\nabla(U_0-\Pi^{2,0}_{N,M}U_0)\|^2_\Omega \\
&\leq \|\nabla(\Pi^{1,0}_{N,M}U_0-U_0)\|^2_\Omega + 2\|\Delta(U_0-\Pi^{2,0}_{N,M}U_0)\|^2_\Omega \\
&\leq c((1+\frac{1}{\beta^2})(\beta N)^{2-r}+M^{4-2s})A_{r,s,\beta}(U_0) \\
&\quad +c((\beta^4+\frac{1}{\beta^4})(\beta N)^{2-r}+M^{4-2s})B_{r,s,\beta}(U_0).
\end{aligned}
\tag{59}
$$

By substituting (54) and (56)-(58) into (53), we obtain

$$
\begin{aligned}
\partial_t E(\tilde{U}_{N,M},\mu,t) \leq\ & V_\beta(U,\mu,t)E(\tilde{U}_{N,M},\mu,t) \\
& +c((\beta^4+\tfrac{1}{\beta^4})(\beta N)^{2-r}+M^{4-2s})R_{r,s,\beta}(U,\mu,t).
\end{aligned}
$$

Then, by (59) and an argument like the last part of Subsection 5.1, we obtain

$$
\begin{aligned}
E(\tilde{U}_{N,M},\mu,t) &\leq c((\beta^4+\frac{1}{\beta^4})(\beta N)^{2-r}+M^{4-2s})e^{\int_0^t V_\beta(U,\mu,\eta)d\eta} \\
&\quad \times\Big(\int_0^t e^{-\int_0^\eta V_\beta(U,\mu,\xi)d\xi}R_{r,s,\beta}(U,\mu,\eta)d\eta + D_{r,s,\beta}(U_0)\Big).
\end{aligned}
$$

Finally, we use (46) again to reach the desired result (33).

6. Concluding Discussion

In this work, we proposed the mixed spectral method for the stream function form of the Navier-Stokes equations in an infinite strip, by using generalized Laguerre functions. We proved its generalized stability and convergence. Numerical results demonstrated its efficiency and confirmed the analysis well. This approach has several fascinating merits as discussed in the first section of this paper.

We introduced a new mixed generalized Laguerre-Legendre orthogonal approximation and established the basic approximation results in certain non-uniformly weighted Sobolev spaces, which play important roles in spectral method for fourth order problems defined on unbounded domains.

Although we only considered the Navier-Stokes equations in an infinite strip, the suggested method and the approximation results are also applicable to many other problems, such as certain problems arising in fluid dynamics, quantum mechanics, and other fields.

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