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SPLIT-STEP FORWARD MILSTEIN METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider the problem of computing numerical solutions for stochastic differential equations (SDEs) of Itô form. A fully explicit method, the split-step forward Milstein (SSFM) method, is constructed for solving SDEs. It is proved that the SSFM method is convergent with strong order $\gamma = 1$ in the mean-square sense. The analysis of stability shows that the mean-square stability properties of the method proposed in this paper are an improvement on the mean-square stability properties of the Milstein method and three stage Milstein methods.

Key Words. Stochastic differential equation, Explicit method, Mean convergence, Mean square convergence, Stability, Numerical experiment.

1. Introduction

In this paper, we consider d-dimensional Itô stochastic differential equations (SDEs) of the following form

$$\begin{cases} dY(t) = f\left(Y(t)\right) dt + g\left(Y(t)\right) dW(t), t \in [t_0, T], \\ Y(t_0) = Y_0, \end{cases} \tag{1}$$

where $Y(t)$ is a random variable with value in R^d, $f : R^d \to R^d$ is called the drift function, $g : R^d \to R^d$ is called the diffusion function, and $W(t)$ is a Wiener process whose increments $\Delta W(t) = W(t + \Delta t) - W(t)$ are Gaussian random variables $N(0, \Delta t)$.

Stochastic differential equations have come to play an important role in many branches of science and industry. The importance of numerical methods for SDEs can not be overemphasized as SDEs are used in modeling of many chemical, physical, biological and economical systems [2]. SDEs arising in many applications can not be solved analytically, hence one needs to develop effective numerical methods for such systems. In recent years, many efficient numerical methods have been constructed for solving different type of SDEs with different properties, for example, Wang et al.[8], Higham [1], Platen [5], Wang [7]. These numerical schemes are now abundant and classified according to their type (strong or weak) and order of convergence [2]. In this paper, we focus our attention on schemes that converge in the strong sense. The concepts of strong convergence concern the accuracy of a numerical method over a finite interval $[t_0, T]$ for small step sizes Δt.

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2. Motivation and background

Milstein et al.[3] studied the fully implicit methods for Itô SDEs. The fully implicit methods have been constructed for stiff SDEs where some components of a stiff multidimensional system have a vanishing drift term for which semi-implicit methods can not improve the stability of the numerical solution. In this paper, we propose to solve SDEs of type (1). For such equations semi-implicit methods are applicable, however the Newton iteration is necessary for semi-implicit methods, which makes such methods expensive. Hence to avoid this issue, we need explicit methods.

In order to improve the stability properties of the explicit methods for solving SDEs, some attempts have been made to propose modified explicit Euler and Milstein methods. For example, Wang et al.[8] studied the split-step forward methods for Itô SDEs. Wang [7] studied the three-stage stochastic Runge-Kutta methods for Stratonovich SDEs. In this paper, as a fully explicit method, we discuss the split-step forward Milstein (SSFM) method which has better stability properties than the Milstein and three-stage Milstein methods. The SSFM method has unbounded stability region whereas the Milstein method has bounded stability region. In Section 5, an example is presented in order to show that the accuracy and convergence property of SSFM method are better than that of the Milstein method and three stage Milstein methods.

This paper is organized as follows. In Section 3, we introduce some notation and hypotheses of Eq. (1). In the same section we discuss the convergence of the SSFM method. The stability properties of the SSFM method are reported in Section 4. In Section 5, examples are presented in order to illustrate the applicability of our results. Conclusions are given in Section 6.

3. Numerical analysis of the method

3.1. General framework. Let there be a common underlying complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with index $t \in [t_0, T]$ on which the vector stochastic process $Y(t)$ consists of component-wise collections of random variables. Along a given sample path w, $Y(t; w)$ denotes the value taken by the random variable Y_t. We consider the numerical integration of the initial value Itô SDEs with noise in the form of

$$dY(t) = f\left(Y(t)\right) dt + g\left(Y(t)\right) dW(t) \tag{2}$$

with

$$Y(t_0) = Y_0.$$

Let $|x|$ be the Euclidean norm of vector $x \in R^d$. Let $\mathbf{E}$ denote the expectation.

3.2. Assumptions. Let gg' denote a vector of length d with ith component equal to $(gg')_i = \sum_{k=1}^{d} g_k \frac{\partial g_i}{\partial y_k}$.
The following assumptions can be found in [4, 8] when considering the convergence properties of splitting schemes for Itô SDEs.
A1. The functions f, g and gg' satisfy the Lipschitz condition; that is, there exists a positive constant L_1 such that for any x_1, $x_2 \in R^d$,

$$\mid f\left(x_1\right) - f\left(x_2\right) \mid \quad \leq \quad L_1 \mid x_1 - x_2 \mid, \tag{3}$$

$$\mid g\left(x_1\right) - g\left(x_2\right) \mid \quad \leq \quad L_1 \mid x_1 - x_2 \mid, \tag{4}$$

and

$$|\ g(x_1)g'(x_1) - g(x_2)g'(x_2)\ | \quad \leq \quad L_1\ |\ x_1 - x_2\ |\ . \tag{5}$$

A2. The functions f, g and gg' satisfy a linear growth condition; that is,

$$|\ f(x_1)\ |^2 \leq C_2\left(1+ |\ x_1\ |^2\right), \tag{6}$$

$$|\ g(x_1)\ |^2 \leq C_2\left(1+ |\ x_1\ |^2\right), \tag{7}$$

$$|\ g(x_1)g'(x_1)\ |^2 \leq C_2\left(1+ |\ x_1\ |^2\right), \tag{8}$$

where C_2 is a constant.
A3. The process $Y(t)$ is adapted to filtration $\{\mathcal{F}_t, t \geq t_0\}$, where $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ for $t_1 < t_2$.

3.3. Numerical discretization.

Lemma 1. [8] *Under the assumptions* $\mathbf{A}1 - \mathbf{A}3$*, there exists a unique solution* $Y(t)$ *to Eq.*(1) *and*

$$\mathbf{E}(\sup_{t_0 \leq t \leq T} |Y(t)|^2) < K(1 + \mathbf{E}|Y_0|^2), \tag{9}$$

where K *is a constant.*

We define a mesh with a uniform step on the interval $[t_0, T], h = \frac{(T-t_0)}{N}, t_n = t_0 + nh$, where $n = 0, ..., N$.
For SDE (1), here we present explicit method based on the Milstein method in [2] and three-stage Milstein methods in [8], the split-step forward Milstein (SSFM) method:

$$\begin{aligned}
Y_{n1} &= y_n - \gamma_1 g(y_n)g'(y_n)h, \\
Y_{n2} &= Y_{n1} + h\alpha_1 f(Y_{n1}), \\
Y_{n3} &= Y_{n2} + \Delta W_n g(Y_{n2}) + \frac{1}{2}(\Delta W_n)^2 g(Y_{n2})g'(Y_{n2}), \\
Y_{n4} &= Y_{n3} + h\alpha_2 f(Y_{n3}), \\
Y_{n5} &= Y_{n4} - \gamma_2 g(Y_{n4})g'(Y_{n4})h, \\
y_{n+1} &= Y_{n5} + h\alpha_3 f(Y_{n5}),
\end{aligned} \tag{10}$$

where $\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2 \in [-1, 1]$ satisfying the conditions

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 &= 1, \\
\gamma_1 + \gamma_2 &= \frac{1}{2},
\end{aligned}$$

and the increments $\Delta W_n = W_{n+1} - W_n$ are Gaussian random variables $N(0, h)$.
Throughout the following analysis, we use $k_1, k_2, k_3, ...$, to denote generic constants that do not depend on h.

Lemma 2. *Under the assumptions* $\mathbf{A}1 - \mathbf{A}3$*, the SSFM method is consistent with order* 2 *in the mean and order* $\frac{3}{2}$ *in the mean-square sense.*

Proof. Invoke the Milstein method

$$y^M_{n+1} = y_n + hf(y_n) + g(y_n)\Delta W_n + \frac{1}{2}g(y_n)g'(y_n)\{(\Delta W_n)^2 - h\},$$

considered in [2].
We use the Lipschitz-continuity of the drift and diffusion function, properties of

multiple Itô-integrals and linear growth bounds of the drift and diffusion function.

$$
\begin{aligned}
H_1 : &= |\mathbf{E}\left(Y(t_{n+1}) - y_{n+1}\right)|\mathcal{F}_{t_n}| \\
&= |\mathbf{E}\left(Y(t_{n+1}) - y^M_{n+1} + y^M_{n+1} - y_{n+1}\right)|\mathcal{F}_{t_n}| \\
&\leq |\mathbf{E}\left(Y(t_{n+1}) - y^M_{n+1}\right)|\mathcal{F}_{t_n}| + |\mathbf{E}\left(y^M_{n+1} - y_{n+1}\right)|\mathcal{F}_{t_n}| \\
&\leq k_4(1+|y_n|^2)^{\frac{1}{2}}h^2 + H_2,
\end{aligned}
$$

where

$$
\begin{aligned}
H_2 &= |\mathbf{E}\left(y^M_{n+1} - y_{n+1}\right)|\mathcal{F}_{t_n}| \\
&= |\mathbf{E}(y^M_{n+1} - Y_{n5} - h\alpha_3 f(Y_{n5}))|\mathcal{F}_{t_n}| \\
&= |\mathbf{E}(y^M_{n+1} - Y_{n3} - h\alpha_2 f(Y_{n3}) + \gamma_2 hg(Y_{n4})g'(Y_{n4}) \\
&\quad -h\alpha_3 f(Y_{n5}))|\mathcal{F}_{t_n}| \\
&\leq |\mathbf{E}(\Delta W_n(g(Y_{n2}) - g(y_n)))|\mathcal{F}_{t_n}| \\
&\quad +|\mathbf{E}(\frac{1}{2}(\Delta W_n)^2(g(Y_{n2})g'(Y_{n2}) - g(y_n)g'(y_n)))|\mathcal{F}_{t_n}| \\
&\quad +|\mathbf{E}(h\alpha_2(f(Y_{n3}) - f(y_n)))|\mathcal{F}_{t_n}| \\
&\quad +|\mathbf{E}(h\alpha_3(f(Y_{n5}) - f(y_n)))|\mathcal{F}_{t_n}| \\
&\quad +|\mathbf{E}(h\alpha_1(f(y_n - \gamma_1 hg(y_n)g'(y_n)) - f(y_n)))|\mathcal{F}_{t_n}| \\
&\quad +|\mathbf{E}(h\gamma_2(g(Y_{n4})g'(Y_{n4}) - g(y_n)g'(y_n)))|\mathcal{F}_{t_n}|,
\end{aligned}
$$

from Eqs. (3), (4) and (5), we have

$$
\begin{aligned}
H_2 \leq\ & k_5 h(|\mathbf{E}(Y_{n2} - y_n)|\mathcal{F}_{t_n}| + |\mathbf{E}(Y_{n3} - y_n)|\mathcal{F}_{t_n}| + |\mathbf{E}(Y_{n5} - y_n)|\mathcal{F}_{t_n}| \\
& +|\mathbf{E}(Y_{n4} - y_n)|\mathcal{F}_{t_n}| + |\mathbf{E}(y_n - \gamma_1 hg(y_n)g'(y_n) - y_n)|\mathcal{F}_{t_n}|),
\end{aligned}
$$

from Eqs. (6), (7) , (8) and inequality $|a| < (1+|a|^2)^{\frac{1}{2}}$, we have

$$
H_2 \leq k_6(1+|y_n|^2)^{\frac{1}{2}}h^2,
$$

where k_4, k_5 and k_6 are constants. Similarly by standard arguments, we can prove the following

$$
\begin{aligned}
H_3 &= \left(\mathbf{E}\left(|Y(t_{n+1}) - y_{n+1}|^2\right)|\mathcal{F}_{t_n}\right)^{\frac{1}{2}} \\
&\leq \left(\mathbf{E}\left(|Y(t_{n+1}) - y^M_{n+1}|^2\right)|\mathcal{F}_{t_n}\right)^{\frac{1}{2}} + (\mathbf{E}|\Delta W_n(g(Y_{n2}) - g(y_n))|^2|\mathcal{F}_{t_n})^{\frac{1}{2}} \\
&\quad +(\mathbf{E}|\frac{1}{2}(\Delta W_n)^2(g(Y_{n2})g'(Y_{n2}) - g(y_n)g'(y_n))|^2|\mathcal{F}_{t_n})^{\frac{1}{2}} \\
&\quad +(\mathbf{E}|h\alpha_2(f(Y_{n3}) - f(y_n))|^2|\mathcal{F}_{t_n})^{\frac{1}{2}} \\
&\quad +(\mathbf{E}|h\alpha_3(f(Y_{n5}) - f(y_n))|^2|\mathcal{F}_{t_n})^{\frac{1}{2}} \\
&\quad +(\mathbf{E}|h\alpha_1(f(y_n - \gamma_1 hg(y_n)g'(y_n)) - f(y_n))|^2|\mathcal{F}_{t_n})^{\frac{1}{2}} \\
&\quad +(\mathbf{E}|h\gamma_2(g(Y_{n4})g'(Y_{n4}) - g(y_n)g'(y_n))|^2|\mathcal{F}_{t_n})^{\frac{1}{2}} \\
&\leq k_7(1+|y_n|^2)^{\frac{1}{2}}h^{\frac{3}{2}},
\end{aligned}
$$

where k_7 is a constant. □

Theorem 1. [3] *Assume for a one-step discrete time approximation y that the local mean error and mean-square error for all $N = 1, 2, ...,$ and $n = 0, 1, ..., N-1$ satisfy the estimates*

$$
|\mathbf{E}(Y(t_n) - y_{n+1})|\mathcal{F}_{t_n}| \leq k_1(1+|y_n|^2)^{\frac{1}{2}}h^{p_1}
$$

and

$$(\mathbf{E}|Y(t_n) - y_{n+1}|^2|\mathcal{F}_{t_n})^{\frac{1}{2}} \leq k_2(1 + |y_n|^2)^{\frac{1}{2}} h^{p_2},$$

with $p_2 \geq \frac{1}{2}$ *and* $p_1 \geq p_2 + \frac{1}{2}$. *Then,*

$$(\mathbf{E}|Y(t_n) - y_{n+1}|^2|\mathcal{F}_{t_0})^{\frac{1}{2}} \leq k_3(1 + |y_0|^2)^{\frac{1}{2}} h^{p_2 - \frac{1}{2}}.$$

Theorem 2. *Under the assumptions* $\mathbf{A}1 - \mathbf{A}3$*, the numerical solution produced by the SSFM method* (10) *converges to the exact solution of Eq.* (1) *in the mean-square sense with strong order of convergence* 1.

Proof. According to Eq. (1), SSFM method (10), Lemma 1 and Lemma 2, we can easily see that all conditions of Theorem 1 are satisfied. Thus, this conclusion can be considered as a corollary of Theorem 1. □

4. Stability properties of the method

Following the established practice [6, 7, 8] for analyzing the stochastic stability of a numerical integrator in the mean-square sense, we take a one-dimensional linear test SDE with a single channel of noise:

$$dY(t) = aY(t)dt + bY(t)dW(t), \tag{11}$$

with known solution $Y(t) = Y_0 e^{(a-b^2/2)t+bW(t)}$ which is represented by

$$y_{n+1} = R(a, b, h, J)y_n,$$

where J is the standard Gaussian random variable $J \sim N(0, 1)$ and we assume that $Y_0 \neq 0$ with probability 1. Saito and Mitsui [6] introduced the following definition of the mean-square (MS) stability.

Definition 1. *The numerical method is said to be MS-stable , if*

$$r(a, b, h) = \mathbf{E}(R^2(a, b, h, J)) < 1.$$

$r(a, b, h)$ *is called MS-stability function of the numerical method.*

The MS-stability function of the SSFM method is given by

$$r_1(p, q) = (1 + \alpha_1 p)^2(1 + \alpha_2 p)^2(1 + \alpha_3 p)^2(1 - \gamma_1 q^2)^2(1 - \gamma_2 q^2)^2(1 + 2q^2 + \frac{3}{4}q^4),$$

where $p = ah$ and $q = b\sqrt{h}$.

Figs. 1, 2, 3, 4 and 5 give the MS-stable regions of the SSFM method. Fig.

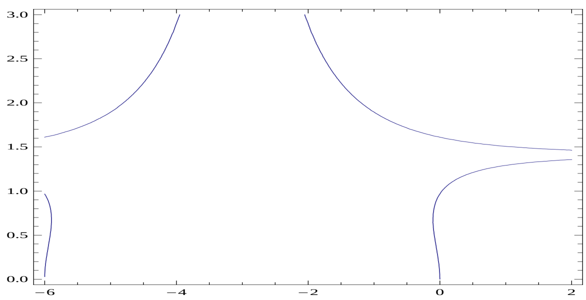

FIGURE 1. MS-stable region of the SSFM method ($\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}, \gamma_1 = 0, \gamma_2 = \frac{1}{2}$) for linear SDE.

6 gives the MS-stable region of the Milstein method. Fig. 7 gives the MS-stable region of the three-stage Milstein methods. The MS-stable regions of the Milstein

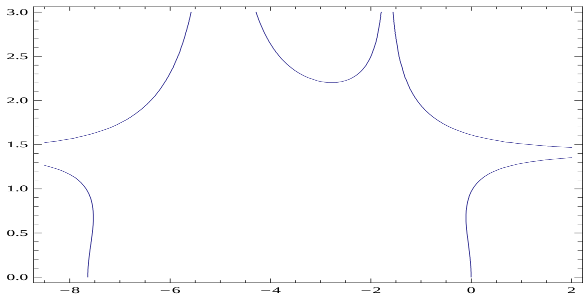

FIGURE 2. MS-stable region of the SSFM method ($\alpha_1 = \alpha_2 = \frac{1}{5}, \alpha_3 = \frac{3}{5}, \gamma_1 = 0, \gamma_2 = \frac{1}{2}$) for linear SDE.

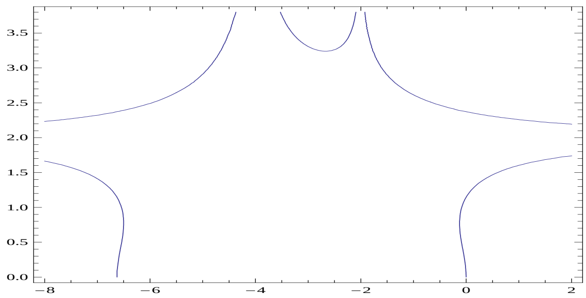

FIGURE 3. MS-stable region of the SSFM method ($\alpha_1 = \alpha_2 = \frac{1}{4}, \alpha_3 = \frac{1}{2}, \gamma_1 = \gamma_2 = \frac{1}{4}$) for linear SDE.

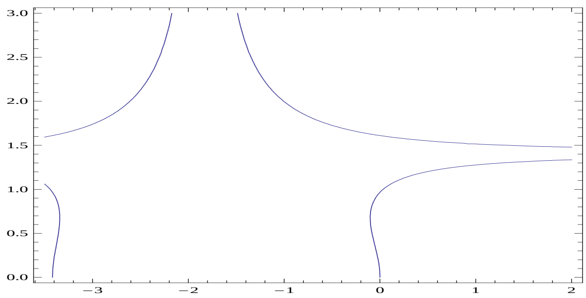

FIGURE 4. MS-stable region of the SSFM method ($\alpha_1 = 0.5, \alpha_2 = 0.6, \alpha_3 = -0.1, \gamma_1 = 0, \gamma_2 = \frac{1}{2}$) for linear SDE.

method, three stage Milstein methods and SSFM method are the areas between the plotted curves and are symmetric about the X-axis. From these figures, we can see that the MS-stability properties of the split-step forward Milstein method are better than that of the Milstein method and three stage Milstein methods. In particular, the MS-stable regions of the split-step forward Milstein method are unbounded.

5. Examples

In this section, we shall discuss the examples to illustrate our theory.

Example 1. Denoting y_{iN} the numerical approximation to $Y_i(t_N)$ at step point t_N in the ith simulation of all 5000 simulations, we use mean of absolute errors M,

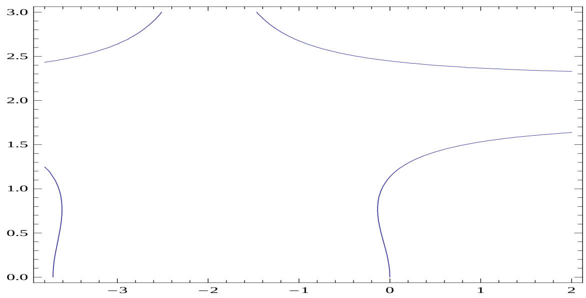

FIGURE 5. MS-stable region of the SSFM method ($\alpha_1 = 0.5, \alpha_2 = 0.6, \alpha_3 = -0.1, \gamma_1 = 0.2, \gamma_2 = 0.3$) for linear SDE.

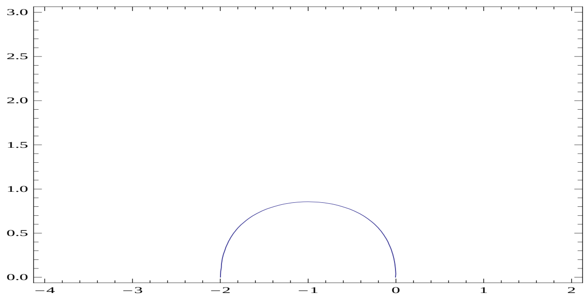

FIGURE 6. MS-stable region of the Milstein method for linear SDE.

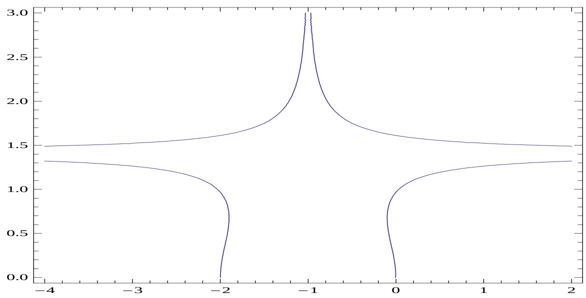

FIGURE 7. MS-stable region of the three stage Milstein methods for linear SDE.

convergence rates R_h defined by [8]

$$M = \frac{1}{5000}\sum_{i=1}^{5000} |y_{iN} - Y_i(t_N)|, \qquad R_h = \frac{M}{h},$$

to measure the accuracy and convergence property of the SSFM method.
The test equation is a nonlinear SDE, whose Itô form is given by

$$(12)\, dY(t) = Y(t)(1 - Y^2(t))dt - (1 - Y^2(t))dW(t), \quad Y(0) = 2, \quad t \in [0, 3].$$

The exact solution of Eq. (12) is $Y(t) = coth(W(t) + arccoth(Y_0))$.
For Eq. (12), the errors of Milstein method, three-stage Milstein (TSM 1b) method and SSFM method are shown in Table 1. For Eq. (12), the convergence rates of Milstein method, three-stage Milstein (TSM 1b) method and SSFM method are

shown in Table 2. The accuracy of SSFM method is better than that of the Milstein and three-stage Milstein(TSM 1b) methods.

TABLE 1. Mean (M) of absolute errors $\times 10^4$

method\\h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
$\alpha_1 = \alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{2}$, $\gamma_1 = 0, \gamma_2 = \frac{1}{2}$	4.01	3.57	1.57	1.63	1.29	1.03	0.976	0.574
$\alpha_1 = 0.5, \alpha_2 = 0.6$, $\alpha_3 = -0.1$, $\gamma_1 = 0, \gamma_2 = \frac{1}{2}$	3.93	3.05	1.60	1.39	1.24	1.21	0.892	0.487
$\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{3}$, $\gamma_1 = 0, \gamma_2 = \frac{1}{2}$	3.81	3.25	1.81	1.54	1.31	1.17	0.883	0.539
Milstein method	4.52	3.93	2.61	2.05	1.79	1.52	0.991	0.696
TSM 1b	4.23	3.77	2.03	1.69	1.41	1.24	0.985	0.658

TABLE 2. Convergence rates (R_h) $\times 10^2$

method\\R_h	$R_{2^{-3}}$	$R_{2^{-4}}$	$R_{2^{-5}}$	$R_{2^{-6}}$	$R_{2^{-7}}$	$R_{2^{-8}}$	$R_{2^{-9}}$	$R_{2^{-10}}$
$\alpha_1 = \alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{2}$, $\gamma_1 = 0, \gamma_2 = \frac{1}{2}$	0.320	0.571	0.502	1.04	1.65	2.63	4.99	5.87
$\alpha_1 = 0.5, \alpha_2 = 0.6$, $\alpha_3 = -0.1$, $\gamma_1 = 0, \gamma_2 = \frac{1}{2}$	0.314	0.488	0.512	0.889	1.58	3.09	4.56	4.98
$\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{3}$, $\gamma_1 = 0, \gamma_2 = \frac{1}{2}$	0.304	0.520	0.579	0.985	1.67	2.99	4.52	5.51
Milstein method	0.361	0.628	0.835	1.31	2.29	3.89	5.07	7.12
TSM 1b	0.338	0.603	0.649	1.08	1.80	3.17	5.04	6.73

Example 2. We consider the stochastic rotating problem,

$$(13) \begin{cases} dY_1(t) = 5Y_2(t) + 2(Y_1(t) + Y_2(t))dW(t), \\ dY_2(t) = -5Y_1(t) + 2(Y_1(t) + Y_2(t))dW(t), \quad Y_1(0) = 1, Y_2(0) = 0. \end{cases}$$

For this equation, a version with two Wiener processes can be found in [3] and a version with single Wiener process can be found in [8]. Stochastic stiffness is a generalization of the deterministic notion of stiffness, so a stiff ordinary differential equation is also stiff in the stochastic sense. The deterministic version of the Eq.(13) is a stiff system which implies that the Eq.(13) is a stiff system . In [8], we can see that the Milstein method can not give the stable solution for Eq. (13) when $h = 0.02$. For Eq. (13), Figs. 8, 9, 10, 11 and 12 illustrate the numerical simulation of the split-step forward Milstein method when $h = 0.02$. We observe in Figs. 8, 9, 10, 11 and 12 that the approximate trajectory of the split-step forward Milstein method stays close to the origin, which replicates the behavior of the exact solution.

Example 3. We consider the following SDE,

$$(14) \quad \begin{cases} dY_1(t) = -12Y_1(t) + 4Y_1(t)dW(t), \\ dY_2(t) = -12Y_2(t) + 4Y_1(t)dW(t), \qquad Y_1(0) = 0.3, Y_2(0) = 0.1. \end{cases}$$

For Eq. (14), Fig. 13 illustrates the numerical simulation of the split-step forward Milstein method when $h = \frac{1}{5}$ and Fig. 14 illustrates the numerical simulation of

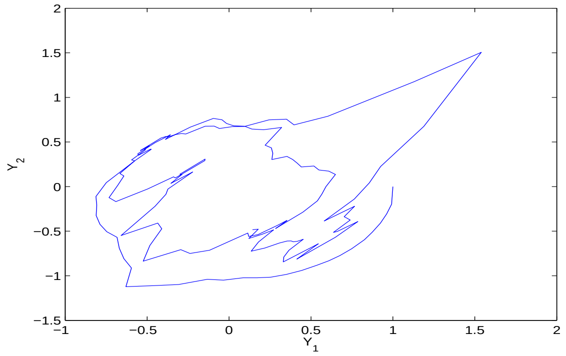

FIGURE 8. Numerical simulation of the SDE (13) by SSFM method ($\alpha_1 = 0.2, \alpha_2 = 0.2, \alpha_3 = \frac{3}{5}, \gamma_1 = 0, \gamma_2 = \frac{1}{2}$) .

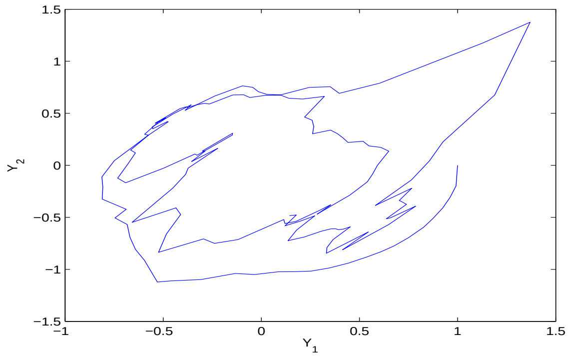

FIGURE 9. Numerical simulation of the SDE (13) by SSFM method ($\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{4}, \alpha_3 = \frac{1}{2}, \gamma_1 = 0, \gamma_2 = \frac{1}{2}$).

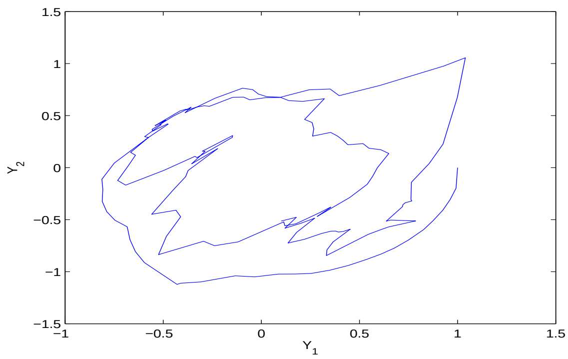

FIGURE 10. Numerical simulation of the SDE (13) by SSFM method ($\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}, \gamma_1 = 0, \gamma_2 = \frac{1}{2}$) .

the three-stage Milstein methods when $h = \frac{1}{5}$. Fig. 15 illustrates the deterministic

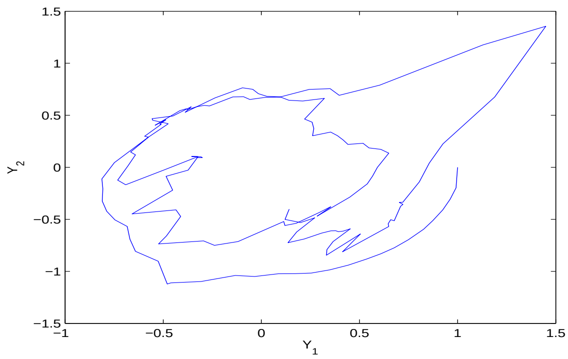

FIGURE 11. Numerical simulation of the SDE (13) by SSFM method ($\alpha_1 = 0.5, \alpha_2 = 0.6, \alpha_3 = -0.1, \gamma_1 = 0, \gamma_2 = \frac{1}{2}$).

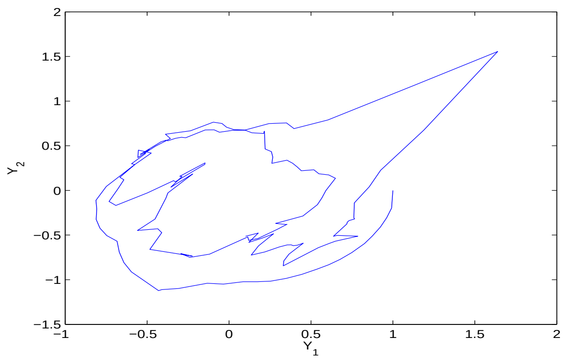

FIGURE 12. Numerical simulation of the SDE (13) by SSFM method ($\alpha_1 = 0.5, \alpha_2 = 0.6, \alpha_3 = -0.1, \gamma_1 = 0.2, \gamma_2 = 0.3$).

solution of Eq.(14). We observe in Figs. 13 and 14 that the split-step forward Milstein method gives stable solution for Eq. (14), while three-stage Milstein methods give unstable solution for Eq. (14).

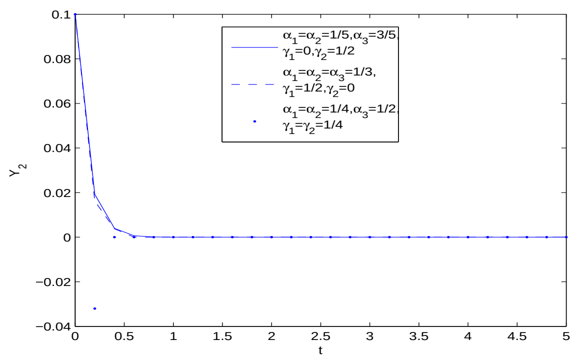

FIGURE 13. Numerical simulation of the SDE (14) by SSFM method

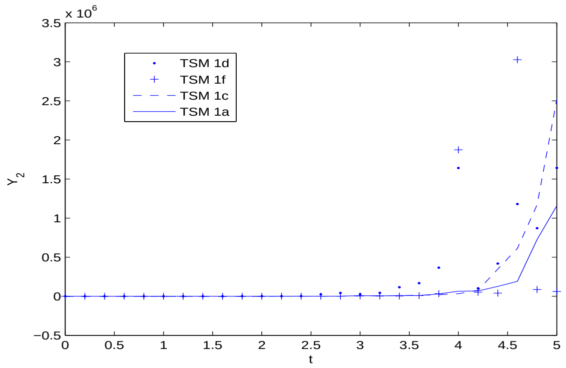

FIGURE 14. Numerical simulation of the SDE (14) by three-stage Milstein methods

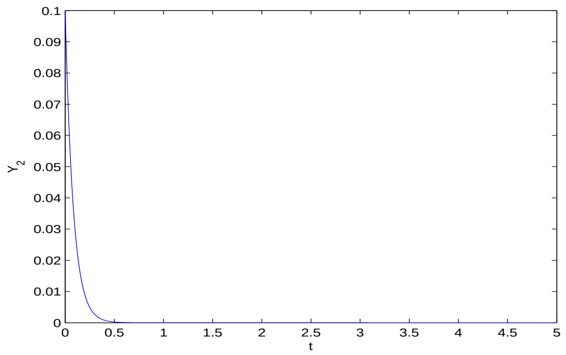

FIGURE 15. Deterministic solution of Eq. (14)

6. Conclusion

In this paper, we have constructed a new explicit method (split-step forward Milstein method), a significant feature of our method is its better stability and error properties for stochastic differential equations as compared to the Milstein method and three stage Milstein methods. The ability to use larger step sizes justifies the additional computational work required to implement the SSFM method. From the numerical results, it is also clear that the SSFM method is suitable for solving stiff stochastic differential equations. We will consider constructing a method with better stability properties and higher convergence order (under global and local Lipschitz condition) in a future work.

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References

[1] D.J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM Rev., 43 (2001) 525-546.

[2] P.E. Kloeden and E. Platen, Numerical solution of Stochastic Differential Equations, Springer-Verlag, Berlin, 1999.
[3] G.N. Milstein, E. Platen and H. Schurz, Balanced Implicit methods for stiff stochastic systems, SIAM J. Numer. Anal., 35 (1998) 1010-1019.
[4] W.P. Petersen, A general implicit splitting for stabilizing numerical simulations of Itô stochastic differential equations, SIAM. J. Numer. Anal., 35 (1998) 1439-1451.
[5] E. Platen, An introduction to numerical methods for stochastic differential equation, Acta Numer., 8 (1999) 197-246.
[6] Y. Saito and T. Mitsui, Stability analysis of numerical schemes for stochastic differential equations, SIAM J. Numer. Anal., 33 (1996) 2254-2267.
[7] P. Wang, Three-stage stochastic Runge-Kutta methods for stochastic differential equations, J. Comput. Appl. Math., 222 (2008) 324-332.
[8] P. Wang and Y. Li , Split-step forward methods for stochastic differential equations, J. Comput. Appl. Math., 233 (2010) 2641-2651.

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