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ASYMPTOTIC EXPANSION AND SUPERCONVERGENCE FOR TRIANGULAR LINEAR FINITE ELEMENT ON A CLASS OF TYPICAL MESH

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Abstract. In this paper, we present a new approach to obtain the asymptotic expansion and superconvergence for the linear element on Union Jack mesh. First, we construct a generalized interpolation function and its discrete harmonic extension by using the energy embedding method and the method of separation of variables. Then, we present elaborate estimates for the generalized interpolation function and the harmonic extension. Finally, the asymptotic expansion, superconvergence and extrapolation are obtained based on those estimates.

Key words. Finite element, Superconvergence, Asymptotic expansion, Extrapolation, Discrete harmonic extension.

1. Introduction

The research of superconvergence of triangular linear finite element can be traced to [2,3] in the late 1970's, but the work was only aimed at uniform meshes in the sense of 3-directional parallel. Later, the superconvergence results of triangular quadratic element were also obtained for uniform meshes in the sense of 3-directional parallel[15].

In the middle of 1990's the local symmetry theory developed by Schatz et.al [12] verified that on the quasi-uniform meshes, the finite element solution has the superconvergence at the mesh symmetry points far from the boundary. Babuska et.al [1] considered four typical quasi-uniform mesh patterns: Regular mesh, Criss-Cross mesh, Chevron mesh and Union Jack mesh. Applying "computer based proof", they found that the mesh symmetry points on the whole domain are derivative superconvergent points for the triangular linear finite element.

In recent years, superconvergence of finite element methods has been a subject of active research due to its strong relevance with a posteriori error estimations for the adaptive finite element method. Lin and Zhang in [10] demonstrated that under the above mentioned four mesh patterns, the mesh symmetry points are "almost" all superconvergent points for linear and high order elements. The authors of [4] and [15] proved that the mesh symmetry points are all superconvergent points for Chevron and Criss-Cross triangular linear finite elements.

In the above four meshes, the topological structure of the Union Jack mesh is comparatively complex. Furthermore, for general mesh with periodic structure, it is difficult to get the superconvergence, extrapolation and asymptotic expansion by the traditional methods, because it can't accurately characterize the influence of the Dirichlet boundary to the finite element asymptotic states. For example, numerical experiment indicates that the symmetry points near the boundary for triangular

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cubic element are not superconvergent points. Combining the numerical results of [1], they show the surprising property of the finite element superconvergence.

In this paper, we consider the Union Jack triangular linear element for the Poisson equation with Dirichlet boundary condition. Using the energy embedding method, first we construct a generalized interpolation function, whose value at the node is only high accuracy approximation to the interpolated function. Note that this interpolated function does not belong to the finite element space, because it doesn't satisfy the homogeneous boundary condition. Then, we construct a discrete harmonic extension corresponding to the interpolation function by the method of separation of variables. The difference between this discrete harmonic extension and the interpolation fuction belong to the finite element spaces. By this means, we can get asymptotic expansion, superconvergence and extrapolation. The key idea is to construct the discrete harmonic extension of the generalized interpolation function.

An outline of this paper is as follows. In section 2, we will introduce the Union Jack triangular linear finite element space and construct a generalized interpolation function with high accuracy approximation. In section 3, we introduce the discrete harmonic function and give the characteristic lemma to construct discrete harmonic function by the method of separation of variables. Then in section 4, we get the main theorem about the asymptotic expansion for the Union Jack triangular linear finite element. Also, we present superconvergence and extrapolation results for the Union Jack triangular linear finite element.

2. Union Jack Linear Element and Preliminary Lemmas

As shown in Fig.1, let $\Omega = (0,1)\times(0,1)$, Ω^h be the Union Jack partition of Ω, $2h$ and N be the mesh size and the partition number, respectively, and $N = \frac{1}{2h}$.

Linear finite element space on the Union Jack mesh is defined by

$$V_h = \{v : v \in C(\overline{\Omega}) \cap H_0^1(\Omega), v|_T \in P_1, \forall T \in \Omega^h\},$$

where P_k is the set of polynomials of degree k.

We consider the following model problem

$$\begin{cases} -\triangle u = f, & (x,y)\in\Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

The finite element solution of the above equation $u_h \in V_h$ satisfies

$$A(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h,$$

where

$$A(u,v) = \int_\Omega \nabla u \nabla v dxdy, \ (u,v) = \int_\Omega uvdxdy.$$

Denote the node index sets by

$$S = \{(k,l) : k,l = 1,2,...,2N-1\},$$

$$\overline{S} = \{(k,l) : k,l = 0,1,...,2N\},$$

$$S_p = \{(k,l) : (k,l) \in S, \frac{k+l+p}{2} \in Z\},$$

$$\overline{S}_P = \{(k,l) : (k,l) \in \overline{S}, \frac{k+l+p}{2} \in Z\},$$

where $p = 0,1$, and Z represents the set of integers.

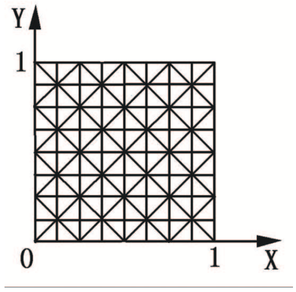

FIG 1. Union Jack mesh.

Let $\phi_{k,l}$ be the interpolation basic function corresponding to the node (kh, lh) of V_h. Then we have

$$V_h = Span\{\phi_{k,l} : (k,l) \in S\}.$$

By simple calculation , we have

$$A(\phi_{i,j}, \phi_{k,l}) = \begin{cases} 4, & (i,j) = (k,l), \\ -1, & (i,j) = (k \pm 1, l) or (k, l \pm 1), \\ 0, & otherwise, \end{cases} \tag{1}$$

$$(1, \phi_{k,l}) = \begin{cases} \frac{4}{3}h^2, & (k,l) \in S_0, \\ \\ \frac{2}{3}h^2, & (k,l) \in S_1. \end{cases} \tag{2}$$

Lemma 2.1. *Let $F(u)$ be a linear operator, which, for any $u \in P_m$, satisfies*

$$A(F(u), \phi_{k,l}) = \begin{cases} b_0 h^m u_{k,l}^{\alpha,\beta}, & (k,l) \in S_0, \\ \\ b_1 h^m u_{k,l}^{\alpha,\beta}, & (k,l) \in S_1. \end{cases}$$

where $u_{k,l}^{\alpha,\beta} = \dfrac{\partial^{\alpha+\beta} u}{\partial x^\alpha \partial y^\beta}(kh, lh)$, and $\alpha + \beta = m$, b_0 and b_1 are constants independent of k, l, h and u. Furthermore let

$$G_m = \sum_{(i,j) \in \overline{S}} (G_m)_{i,j} \phi_{i,j},$$

where

$$(G_m)_{i,j} = \begin{cases} 0 & , \quad (i,j) \in \overline{S}_0, \\ \dfrac{2b_1 - b_0}{12} u_{i,j}^{\alpha,\beta}, & (i,j) \in \overline{S}_1. \end{cases}$$

and W_m satisfy

$$\begin{cases} -\Delta W_m = \dfrac{b_0 + b_1}{2} u^{\alpha,\beta}, & (x,y) \in \Omega, \\ \\ W_m|_{\partial\Omega} = 0. \end{cases}$$

Then for any $u \in P_m, (k,l) \in S$, we have

$$A\big(F(u) - h^m G_m - h^{m-2} W_m, \phi_{k,l}\big) = 0.$$

Proof. Using (1) and (2) noting that $u \in P_m$ implies $u^{\alpha,\beta}$ is a constant, we have

$$\begin{aligned} A(G_m, \phi_{k,l}) &= 4(G_m)_{k,l} - \sum_{p=\pm 1} \big((G_m)_{k+p,l} + (G_m)_{k,l+p}\big) \\ &= \begin{cases} -\dfrac{2b_1 - b_0}{3} u_{k,l}^{\alpha,\beta}, & (k,l) \in S_0, \\ \dfrac{2b_1 - b_0}{3} u_{k,l}^{\alpha,\beta}, & (k,l) \in S_1, \end{cases} \end{aligned}$$

and

$$A\big(F(u) - h^m G_m, \phi_{k,l}\big) = \begin{cases} \dfrac{2}{3}(b_0 + b_1) h^m u_{k,l}^{\alpha,\beta}, & (k,l) \in S_0, \\ \dfrac{1}{3}(b_0 + b_1) h^m u_{k,l}^{\alpha,\beta}, & (k,l) \in S_1. \end{cases} \tag{3}$$

For any $(k,l) \in S$, the definition of W_m yields

$$\begin{aligned} A(W_m, \phi_{k,l}) =& (-\Delta W_m, \phi_{k,l}) \\ =& (\frac{b_0 + b_1}{2} u^{\alpha,\beta}, \phi_{k,l}) \\ =& \frac{b_0 + b_1}{2} u_{k,l}^{\alpha,\beta} (1, \phi_{k,l}). \end{aligned} \tag{4}$$

And combining (2), (3) and (4), we complete the proof of Lemma 2.1. □

Remark 2.1 In Lemma 2.1, because the function G_m does not satisfy $G_m|_{\partial\Omega} = 0$, it does not belong to the space V_h.

Denote

$$u_I = \sum_{(i,j)\in S} u_{i,j}\phi_{i,j}, \ \overline{u}_I = \sum_{(i,j)\in \overline{S}} u_{i,j}\phi_{i,j},$$

i.e., u_I is the interpolation function of u in V_h. If $u|_{\partial\Omega} = 0, \overline{u}_I = u_I$.

In the following, based on Lemma 2.1, we will construct a generalized interpolation function $\bar{u}_e$ of u in three steps.

Step 1. If $u \in P_2$, by direct calculation, we deduce

$$A(u - \overline{u}_I, \phi_{k,l}) = \begin{cases} -\dfrac{1}{3} h^2 (\Delta u)_{k,l}, & (k,l) \in S_0, \\ \dfrac{1}{3} h^2 (\Delta u)_{k,l}, & (k,l) \in S_1. \end{cases} \tag{5}$$

From Lemma 2.1, we have

$$G_2 = \sum_{(i,j)\in \overline{S}} (G_2)_{i,j}\phi_{i,j}, \tag{6}$$

where

$$(G_2)_{i,j} = \begin{cases} 0 & , \quad (i,j) \in \overline{S}_0, \\ \dfrac{1}{12}(\Delta u)_{i,j} & , \quad (i,j) \in \overline{S}_1. \end{cases} \tag{7}$$

Notice that $b_0 + b_1 = 0$. Hence we have

$$W_2 \equiv 0.$$

Step 2. If $u \in P_4$, similar to the above calculation, we deduce

$$A(u - \overline{u}_I - h^2 G_2, \phi_{k,l}) = \begin{cases} h^4\big(\frac{1}{30}\triangle^2 u - \frac{1}{6}u^{2,2}\big)_{k,l}, & (k,l) \in S_0, \\ h^4\big(\frac{1}{20}\triangle^2 u - \frac{1}{6}u^{2,2}\big)_{k,l}, & (k,l) \in S_1. \end{cases} \tag{8}$$

From Lemma 2.1, we have

$$G_4 = \sum_{(i,j)\in\overline{S}} (G_4)_{i,j}\phi_{i,j},$$

where

$$(G_4)_{i,j} = \begin{cases} 0 & , \quad (i,j) \in \overline{S}_0, \\ \big(\frac{1}{180}\triangle^2 u - \frac{1}{72}u^{2,2}\big)_{i,j} & , \quad (i,j) \in \overline{S}_1. \end{cases} \tag{9}$$

Furthermore, we obtain

$$\begin{cases} -\triangle W_4 = \frac{1}{24}\triangle^2 u - \frac{1}{6}u^{2,2}, & (x,y) \in \Omega, \\ W_4|_{\partial\Omega} = 0. \end{cases} \tag{10}$$

Step 3. Denote

$$\overline{u}_e = \overline{u}_I + h^2 G_2 + h^4 G_4 + h^2 W_4.$$

Using Lemma 2.1 and noting the symmetry of the node base $\phi_{k,l}$, we have

$$A(u - \overline{u}_e, \phi_{k,l}) = 0, \ \forall u \in P_5, (k,l) \in S. \tag{11}$$

Using the Bramble-Hilbert lemma, we get

Lemma 2.2. *For any $u \in C^6(\overline{\Omega})$, we have*

$$A(u - \overline{u}_e, \phi_{k,l}) = O(h^6), \ (k,l) \in S.$$

3. The Discrete Harmonic Extension and Asymptotic Expansion

Denote

$$\overline{V}_h = Span\{\phi_{k,l} : (k,l) \in \overline{S}\}.$$

It is easy to see that $V_h \subset \overline{V}_h$. Hence, we can obtain the orthogonal decomposition

$$\overline{V}_h = V_h \oplus V_h^{\perp},$$

where $V_h^{\perp}$ is the orthogonal complementary subspace of V_h corresponding to $\overline{V}_h$.

The complement element is defined as follows

Definition 3.1 Let $G \in \overline{V}_h$. If g satisfies

(1) $g \in \overline{V}_h$;
(2) $A(g, \phi_{k,l}) = 0, \ \forall (k,l) \in S$;
(3) $g|_{\partial\Omega} = G$.

Then g is called the complement element or the discrete harmonic extension of G.

From the basic property of the finite dimensional inner product space, we have

Lemma 3.1. *For any $G \in \overline{V}_h$, the discrete harmonic extension of G exists and is unique.*

Lemma 3.2. *Let $G \in \overline{V}_h, g$ be the discrete harmonic extension of G. Then we have*

$$||g||_{0,\infty} \le ||G||_{0,\infty}.$$

Proof. Noting $A(g, \phi_{k,l}) = 0$ and from (1), we deduce

$$4g_{k,l} - \sum_{p=\pm 1} (g_{k+p,l} + g_{k,l+p}) = 0, \ \forall \ (k.l) \in S, \tag{12}$$

which is an irreducible diagonally dominant linear system. By the discrete maximum principle, we have

$$\max_{(kh,lh)\in\bar{\Omega}} |g_{k,l}| = \max_{(kh,lh)\in\partial\Omega} |g_{k,l}| \quad = \max_{(kh,lh)\in\partial\Omega} |G_{k,l}| \leq ||G||_{0,\infty}.$$

This is the desired result of the lemma. □

Denote

$$\lambda_r^h(t) = 2 + (-1)^r \cos\frac{t\pi}{2N} + \sqrt{\big(2 + (-1)^r \cos\frac{t\pi}{2N}\big)^2 - 1}, \tag{13}$$

$$S_{j,r}^h(t) = \frac{\lambda_r^h(t)^j - \lambda_r^h(t)^{4N-j}}{1 - \lambda_r^h(t)^{4N}}, \tag{14}$$

where $r = 1, 2$.

In the following ,we will give the following characteristic lemma to construct a discrete harmonic extension by using the method of separation of variables.

Lemma 3.3. *Let g satisfy*
(1) $g \in \overline{V}_h$;
(2) $A(g, \phi_{k,l}) = 0, \ \forall (k,l) \in S$;
(3) $g_{i,2N} = g_{0,j} = g_{2N,j} = 0, \ i,j = 0, 1, ..., 2N$ and

$$g_{i,0} = \begin{cases} 0, & i \text{ is even}, \\ 2\sin\dfrac{it\pi}{2N}, & i \text{ is odd}, \end{cases}$$

where $t \in Z^+$ and t is not integral multiple of $2N$.

Then for any $(i,j) \in \overline{S}$, we have

$$g_{i,j} = \big(S_{j,1}^h(t) + (-1)^{i-1} S_{j,2}^h(t)\big) \sin\frac{it\pi}{2N}. \tag{15}$$

Proof. Substituting (15) as the undetermined formula of $g_{i,j}$ to (12), we deduce

$$\begin{aligned} &\big[4S_{l,1}^h(t) - \big(S_{l-1,1}^h(t) + S_{l+1,1}^h(t)\big) - 2S_{l,1}^h(t)\cos\frac{t\pi}{2N}\big] \\ &+ (-1)^k\big[4S_{l,2}^h(t) - \big(S_{l-1,2}^h(t) + S_{l+1,2}^h(t)\big) + 2S_{l,2}^h(t)\cos\frac{t\pi}{2N}\big] = 0, \ \forall (k,l) \in S. \end{aligned}$$

Hence

$$S_{l-1,r}^h(t) - 2\big(2 + (-1)^r \cos\frac{t\pi}{2N}\big) S_{l,r}^h(t) + S_{l+1,r}^h(t) = 0, l = 1, 2, ..., 2N-1, r = 1, 2.$$

And combining the boundary conditions of the function g, we get the conclusion of the lemma. □

In the following ,we will use Lemma 3.3 to construct the discrete harmonic extension g_2 of G_2 (see (6) and (7)).

For any $u \in C^3(\overline{\Omega})$, based on (6), we consider the trigonometric sine expansions of the function Δu on the boundary of Ω

$$\Delta u \mid_{x=\eta} = \sum_{t=1}^{\infty} A_\eta(t) \sin t\pi y, \quad y \in (0,1);$$

$$\Delta u \mid_{y=\eta} = \sum_{t=1}^{\infty} B_\eta(t) \sin t\pi x, \quad x \in (0,1),$$

where $\eta = 0, 1$, $A_\eta(t)$ and $B_\eta(t)$ are the Fourier coefficients corresponding to the function $\Delta u \mid_{x=\eta}$ and $\Delta u \mid_{y=\eta}$, respectively, i.e.

$$A_\eta(t) = 2\int_0^1 \Delta u \mid_{x=\eta} \sin t\pi y dy, \tag{16}$$

$$B_\eta(t) = 2\int_0^1 \Delta u \mid_{y=\eta} \sin t\pi x dx. \tag{17}$$

From Lemma 3.3 and the linear superposition principle, we have

$$g_2 = \sum_{(i,j)\in\overline{S}} (g_2)_{i,j}\phi_{i,j}, \tag{18}$$

where

$$\begin{aligned}(g_2)_{i,j} = \frac{1}{24}\sum_{t=1}^{\infty} \Big[& A_0(t)\sin\frac{jt\pi}{2N}\big(S^h_{i,1}(t) + (-1)^{j-1}S^h_{i,2}(t)\big) \\ &+ A_1(t)\sin\frac{jt\pi}{2N}\big(S^h_{2N-i,1}(t) + (-1)^{j-1}S^h_{2N-i,2}(t)\big) \\ &+ B_0(t)\sin\frac{it\pi}{2N}\big(S^h_{j,1}(t) + (-1)^{i-1}S^h_{j,2}(t)\big) \\ &+ B_1(t)\sin\frac{it\pi}{2N}\big(S^h_{2N-j,1}(t) + (-1)^{i-1}S^h_{2N-j,2}(t)\big)\Big].\end{aligned} \tag{19}$$

In the following, we will give the convergent result of the series (19).

Lemma 3.4. *Let $\phi \in C^3[0,1]$, a_t be the Fourier sine series's coefficient of the function ϕ. Then we have*

$$a_t = \frac{2}{\pi t}\big(\phi(0) - (-1)^t\phi(1)\big) + O\big(\frac{1}{t^3}\big), \ \forall\ t \in Z^+.$$

Proof. Using integration by parts,we have

$$\begin{aligned} a_t &= 2\int_0^1 \phi(x)\sin t\pi x dx \\ &= \frac{2}{\pi t}\big(\phi(0) - (-1)^t\phi(1)\big) + \frac{2}{\pi t}\int_0^1 \phi^{'}(x)\cos t\pi x dx \\ &= \frac{2}{\pi t}\big(\phi(0) - (-1)^t\phi(1)\big) - \frac{2}{(\pi t)^2}\int_0^1 \phi^{''}(x)\sin t\pi x dx \\ &= \frac{2}{\pi t}\big(\phi(0) - (-1)^t\phi(1)\big) + \frac{2}{(\pi t)^3}\Big[\big(\phi^{''}(x)\cos t\pi x\big)\Big|_0^1 - \int_0^1 \phi^{'''}(x)\cos t\pi x dx\Big] \\ &= \frac{2}{\pi t}\big(\phi(0) - (-1)^t\phi(1)\big) + O\big(\frac{1}{t^3}\big). \end{aligned}$$

□

Lemma 3.5. *Let $0 \le i \le 2N, r \in \{1,2\}, t \in Z^+$. Then*

(1) $S^h_{i,r}(4kN \pm t) = S^h_{i,r}(t),\ \forall\ k \in Z$;

(2) $|S^h_{i,r}(t)| \le 4$.

Proof. (1) From (13) and (14), the conclusion is obviously valid.
(2) Noting that

$$S^h_{i,\,2}(t) = S^h_{i,\,1}(|t - 2N|).$$

Hence we only need to prove the boundness of $S^h_{i,\,1}(t)$.
If t is the integral multiple of $4N$, $\lambda^h_1(t) = 1$. Thus we have

$$|S^h_{i,\,1}(t)| = 1 - \frac{i}{2N} \leq 4.$$

If $t \in Z^+$and t is not integral multiple of $4N$, from(13), we have

$$\begin{aligned}\lambda^h_1(t) &= 2 - \cos\frac{t\pi}{2N} + \sqrt{(2 - \cos\frac{t\pi}{2N})^2 - 1}\\ &\geq 1 + \sqrt{(2 - \cos\frac{t\pi}{2N})^2 - 1}\\ &\geq 1 + 2\sin\frac{\pi}{4N} \geq 1 + \frac{1}{N},\end{aligned}$$

$$\lambda^h_1(t)^{4N} - 1 \geq \frac{1}{2}\lambda^h_1(t)^{4N}. \tag{20}$$

Thus the bound

$$\begin{aligned}|S^h_{i,1}(t)| &= \left|\frac{\lambda^h_1(t)^i - \lambda^h_1(t)^{4N-i}}{1 - \lambda^h_1(t)^{4N}}\right|\\ &\leq 2(\lambda^h_1(t)^{i-4N} + \lambda^h_1(t)^{-i}) \leq 4,\end{aligned}$$

is valid and the proof is completed. □

Lemma 3.6. *Let* $(i,j) \in \overline{S}$, $r \in \{1,2\}$. *Then the series* $\sum_{t=1}^{\infty} \frac{S^h_{j,\,r}(t)}{t}\sin\frac{it\pi}{2N}$ *and* $\sum_{t=1}^{\infty} \frac{(-1)^t S^h_{j,\,r}(t)}{t}\sin\frac{it\pi}{2N}$ *are convergent.*

Proof. Let

$$a_t = \frac{S^h_{j,\,r}(t)}{t}\sin\frac{it\pi}{2N}.$$

Then for any $k \in Z^+$, by Lemma 3.5, we have

$$\begin{aligned}\left|\sum_{t=1}^{2N-1}(a_{4kN-t} + a_{4kN+t})\right| &= \left|\sum_{t=1}^{2N-1}\left(\frac{1}{4kN+t} - \frac{1}{4kN-t}\right) S^h_{j,\,r}(t)\sin\frac{it\pi}{2N}\right|\\ &\leq \sum_{t=1}^{2N-1}\frac{8t}{(4kN)^2 - t^2} \leq \frac{4}{4k^2-1}.\end{aligned}$$

Hence the series $\sum\limits_{k=1}^{\infty}\sum\limits_{t=1}^{2N-1}(a_{4kN-t} + a_{4kN+t})$ is convergent.
Using Lemma 3.5 again, we have

$$\begin{aligned}\sum_{t=1}^{2N-1}(|a_{4kN-t}| + |a_{4kN+t}|) &\leq 4\sum_{t=1}^{2N-1}\left(\frac{1}{4kN+t} + \frac{1}{4kN-t}\right)\\ &\leq \frac{16k}{4k^2-1} \to 0 \quad (k \to \infty).\end{aligned}$$

So, we deduce that the series $\sum_{t=1}^{\infty} a_t$ is convergent. Similarly, the series $\sum_{t=1}^{\infty}(-1)^t a_t$ is convergent. □

Combining Lemma 3.4~Lemma 3.6, we have

Lemma 3.7. *For any $u \in C^6(\overline{\Omega})$, the convergence of the series (19) holds true for any $(i,j) \in \overline{S}$.*

Now we will prove the main theorem of this section.

Theorem 3.1 Let $u \in C^6(\overline{\Omega}) \cap H_0^1(\Omega)$ be the solution of the model problem. Then the finite element solution u_h of u satisfies

$$u_h = u_I + h^2(G_2 - g_2) + h^2 W_4 + O(h^{4-\varepsilon}), \tag{21}$$

where u_I is the interpolation function of u in V_h, G_2 is defined by (6) and (7), W_4 by (10), g_2 by (18) and (19), respectively, and ε is an arbitrary positive constant independent of h.

Proof. Let $(W_4)_h$ be the finite element projection of W_4 in V_h. By the definition (10) of W_4 and [5], we know that $W_4 \in C^{2-\varepsilon}(\overline{\Omega})$ and

$$(W_4)_h = W_4 + O(h^{2-\varepsilon}). \tag{22}$$

Let g_m be the discrete harmonic extension of $G_m (m = 2, 4)$. From Lemma 3.2, we have

$$\|g_m\|_{0,\infty} \leq \|G_m\|_{0,\infty} \lesssim \|u\|_{m,\infty}, \qquad m = 2, 4. \tag{23}$$

Set

$$u_e = u_I + h^2(G_2 - g_2) + h^4(G_4 - g_4) + h^2(W_4)_h. \tag{24}$$

Obviously, $u_e \in V_h$. Using Lemma 2.2, we have

$$A(u_h - u_e, \phi_{k,l}) = A(u - \overline{u}_e, \phi_{k,l}) = O(h^6), \ \forall (k,l) \in S.$$

From [7], the estimate

$$A(u_h - u_e, v) = O(h^4)\|v\|_1, \tag{25}$$

holds true for any $v \in V_h$.

Combining (22)~(25), we complete the proof of Theorem 3.1 as follows

$$\begin{aligned} u_h &= u_e + O(h^4|\ln h|^{\frac{1}{2}}) \\ &= u_I + h^2(G_2 - g_2) + h^2 W_4 + O(h^{4-\varepsilon}). \end{aligned}$$

□

4. Superconvergence of the Union Jack Linear Element

In this section, based on Theorem 3.1, we will give the superconvergence of the Union Jack linear element. In the asymptotic expansion (21) of the Union Jack linear element, it is clear that u_I and W_4 are global smooth function, G_2 is a piecewise smooth function corresponding to the difference types of nodes, but g_2 is a singular function. Thus the asymptotic characteristics of g_2 will be discussed in this section and it is the key to get the superconvergence of the Union Jack linear element.

4.1. Superconvergence of Finite Element Derivative.

Denote

$$(Q_x)_{i,j} = \sum_{p=0}^{1}((g_2)_{i+1,j-p} - (g_2)_{i-1,j-p} + (g_2)_{i+1,j-p+1} - (g_2)_{i-1,j-p+1}), \tag{26}$$

$$(Q_y)_{i,j} = \sum_{p=0}^{1}((g_2)_{i-p,j+1} - (g_2)_{i-p,j-1} + (g_2)_{i-p+1,j+1} - (g_2)_{i-p+1,j-1}). \tag{27}$$

Assume that the model problem solution $u \in C^6(\overline{\Omega}) \bigcap H_0^1(\Omega)$. From the honogeneous boundary and regularity of u, we know that $\Delta u(A_i) = 0$, where $A_i (i = 1, ..., 4)$ are the four corners of Ω. Thus for the Fourier coefficients which are defined by (16) and (17), form Lemma 3.4, we have

$$A_\eta(t) = O(\frac{1}{t^3}), \qquad B_\eta(t) = O(\frac{1}{t^3}), \tag{28}$$

where $t \in Z^+, \eta = 0, 1$.
Combining (26), (19) and Lemma 3.5, for any $(i, j) \in S$, we have

$$\begin{aligned} \big|(Q_x)_{i,j}\big| \lesssim \sum_{t=1}^{\infty} \frac{1}{t^3}\Big[&\big|S^h_{i+1,1}(t) - S^h_{i-1,1}(t)\big| + \big|S^h_{2N-i-1,1}(t) - S^h_{2N-i+1,1}(t)\big| \\ &+ \big|\sin\frac{(j-1)t\pi}{2N} - 2\sin\frac{jt\pi}{2N} + \sin\frac{(j+1)t\pi}{2N}\big| \\ &+ \big|\sin\frac{(i+1)t\pi}{2N} - \sin\frac{(i-1)t\pi}{2N}\big|\Big]. \end{aligned} \tag{29}$$

If $1 \le k \le 2N-1, 1 \le t \le N$, from (13), (14) and (20), we have

$$\begin{aligned} \Big|S^h_{k+1,1}(t) - S^h_{k-1,1}(t)\Big| &= 2\sqrt{(2-\cos\frac{t\pi}{2N})^2 - 1}\Big|\frac{\lambda_1^h(t)^k + \lambda_1^h(t)^{4N-k}}{1-\lambda_1^h(t)^{4N}}\Big| \\ &\lesssim \sqrt{(2-\cos\frac{t\pi}{2N})^2 - 1}\big(\lambda_1^h(t)^{k-4N} + \lambda_1^h(t)^{-k}\big) \\ &\lesssim \frac{t}{N}. \end{aligned} \tag{30}$$

Combining (29), (30) and Lemma 3.5, we have

$$\big|(Q_x)_{i,j}\big| \lesssim \frac{1}{N}\sum_{t=1}^{N}\frac{1}{t^2} + \sum_{t=N+1}^{\infty}\frac{1}{t^3} \lesssim h. \tag{31}$$

Similarly, we have

$$\big|(Q_y)_{i,j}\big| \lesssim h. \tag{32}$$

Denote

$$\begin{aligned} (\overline{\partial}_x u_h)_{i,j} =& \frac{1}{8h}\Big[(u_h)_{i+1,j-1} - (u_h)_{i-1,j-1} + 2\big((u_h)_{i+1,j} - (u_h)_{i-1,j}\big) \\ &+ (u_h)_{i+1,j+1} - (u_h)_{i-1,j+1}\Big], \\ (\overline{\partial}_y u_h)_{i,j} =& \frac{1}{8h}\Big[(u_h)_{i-1,j+1} - (u_h)_{i-1,j-1} + 2\big((u_h)_{i,j+1} - (u_h)_{i,j-1}\big) \\ &+ (u_h)_{i+1,j+1} - (u_h)_{i+1,j-1}\Big]. \end{aligned}$$

According to Theorem 3.1, (31) and (32), we have

Theorem 4.1 Assume that the model problem solution $u \in C^6(\overline{\Omega}) \bigcap H_0^1(\Omega)$. Then the conclusions

$$(\overline{\partial}_x u_h)_{i,j} = u_{i,j}^{1,0} + O(h^2),$$
$$(\overline{\partial}_y u_h)_{i,j} = u_{i,j}^{0,1} + O(h^2),$$

are valid for any $(i,j) \in S$.

4.2. Extrapolation of the Union Jack Linear Element.

Corresponding to (19), denote

$$\begin{aligned}(g_2^{h_k})_{i,j} = \frac{1}{24}\sum_{t=1}^{\infty}\Big[&A_0(t)\sin\frac{jt\pi}{2N_k}\big(S_{i,1}^{h_k}(t) + (-1)^{j-1}S_{i,2}^{h_k}(t)\big)\\ &+ A_1(t)\sin\frac{jt\pi}{2N_k}\big(S_{2N_k-i,1}^{h_k}(t) + (-1)^{j-1}S_{2N_k-i,2}^{h_k}(t)\big)\\ &+ B_0(t)\sin\frac{it\pi}{2N_k}\big(S_{j,1}^{h_k}(t) + (-1)^{i-1}S_{j,2}^{h_k}(t)\big)\\ &+ B_1(t)\sin\frac{it\pi}{2N_k}\big(S_{2N_k-j,1}^{h_k}(t) + (-1)^{i-1}S_{2N_k-j,2}^{h_k}(t)\big)\Big],\end{aligned} \tag{33}$$

where $N_k = 2^k N, h_k = h/2^k, k = 0, 1$.

Now, we consider the estimate of $(g_2^{h_1})_{2i,2j} - (g_2^{h_0})_{i,j}$. Combining (18), (28), (33) and Lemma 3.5, we have

$$\begin{aligned}\big|(g_2^{h_1})_{2i,2j} - (g_2^{h_0})_{i,j}\big| \lesssim \sum_{t=1}^{N}\frac{1}{t^3}\sum_{k\in E_{i,j}}&\Big[\big|S_{2k,1}^{h_1}(t) - S_{k,1}^{h}(t)\big|\\ &+ \big|S_{2k,2}^{h_1}(t)\big| + \big|S_{k,2}^{h}(t)\big|\Big] + \sum_{t=N+1}^{\infty}\frac{1}{t^3},\end{aligned} \tag{34}$$

where $E_{i,j} = \{i, j, 2N-i, 2N-j\}$.

Lemma 4.1. *The estimate*

$$|\lambda_1^{h_1}(t)^2 - \lambda_1^{h}(t)| \lesssim \big(\frac{t}{N}\big)^3,$$

holds true for $1 \le t \le N$.

Proof. Denote

$$\phi(z) = \big(1 + 2\sin^2 z + 2\sin z\sqrt{1+\sin^2 z}\big)^2 - \big(1 + 2\sin^2 2z + 2\sin 2z\sqrt{1+\sin^2 2z}\big).$$

Because the limit $\lim\limits_{z\to+0}\frac{\phi(z)}{z^3}$ exists, $\frac{\phi(z)}{z^3}$ is a continuous function on $[0,1]$. Hence, we have

$$|\phi(z)| \lesssim z^3,\ \forall\ z \in [0,1]. \tag{35}$$

From the definition (13) of $\lambda_1^h(t)$, we have

$$\lambda_1^{h_1}(t)^2 - \lambda_1^{h}(t) = \phi\big(\frac{t\pi}{8N}\big). \tag{36}$$

Combining (35) and (36), we complete the proof of Lemma 4.1. □

Let

$$d_{i,j} = dist\big((ih, jh), \partial\Omega\big),$$

be the distance of the node (ih, jh) to the boundary of Ω.

Lemma 4.2. *Let* $(i,j)\in S, k\in E_{i,j}$. *Then the following estimate holds true*

$$\left|S^{h_1}_{2k,1}(t)-S^{h}_{k,1}(t)\right|\lesssim \frac{t^3}{N^2}e^{-d_{i,j}t},\ \forall\ 0\le t\le N.$$

Proof. Denote

$$F(z)=\frac{z^k-z^{4N-k}}{1-z^{4N}}.$$

Combining (14) and Lagrange's mean value theorem , we have

$$\begin{aligned} S^{h_1}_{2k,1}(t)-S^{h}_{k,1}(t) &= F\big(\lambda_1^{h_1}(t)^2\big)-F\big(\lambda_1^{h}(t)\big)\\ &= \big(\lambda_1^{h_1}(t)^2-\lambda_1^{h}(t)\big)F'(\xi), \end{aligned}\tag{37}$$

where ξ is located between $\lambda_1^h(t))$ and $\lambda_1^{h_1}(t)^2$.
When $1\le t\le N$, from (13), the estimates

$$\lambda_1^h(t)=1+2sin^2\frac{t\pi}{4N}+2sin\frac{t\pi}{4N}\sqrt{1+sin^2\frac{t\pi}{4N}}\ge 1+\frac{t}{N},$$
$$\lambda_1^{h_1}(t)^2=\Big(1+2sin^2\frac{t\pi}{4N_1}+2sin\frac{t\pi}{4N_1}\sqrt{1+sin^2\frac{t\pi}{4N_1}}\ \Big)^2\ge 1+\frac{t}{N},$$

are valid.
Thus we obtain

$$\xi^{4N}-1\ge\frac{1}{2}\xi^{4N},\tag{38}$$

$$\xi^{-2N}\le\ \big((1+\frac{t}{N})^{\frac{N}{t}+1}\big)^{-\frac{2Nt}{N+t}}\le e^{-t}.\tag{39}$$

Hence, we have

$$\mid F'(\xi)\mid\lesssim N\xi^{-2d_{i,j}N}\lesssim Ne^{-d_{i,j}t}.\tag{40}$$

Combining (37), (40) and Lemma 4.1, we complete the proof of Lemma 4.2. □

Lemma 4.3. *The estimates*

$$\mid S^h_{k,1}(t)\mid\lesssim\begin{cases} e^{-kht}, & 0<t\le N,\\ 3^{-k}, & N<t\le 3N,\\ e^{-kh(4N-t)}, & 3N<t<4N,\end{cases}$$

and

$$\mid S^h_{k,2}(t)\mid\lesssim\begin{cases} 3^{-k}, & 0<t\le N,\\ e^{-kh|2N-t|}, & N<t\le 3N,\\ 3^{-k}, & 3N<t<4N,\end{cases}$$

are valid for $1\le k<2N$.

Proof. If $0<t\le N$, combining (13), (14)and (20), we have

$$\lambda_1^h(t)\ge 1+\frac{t}{N},$$

and

$$\begin{aligned}\mid S^h_{k,2}(t)\mid &\lesssim \lambda_1^h(t)^{k-4N}+\lambda_1^h(t)^{-k}\\ &\lesssim (1+\frac{t}{N})^{-k}\lesssim e^{-kht}.\end{aligned}$$

Similarly, if $N<t\le 3N$, we have

$$\lambda_1^h(t)\ge 3,$$

and

$$\mid S^h_{k,1}(t)\mid\lesssim 3^{-k}.$$

The proof of other cases is similar. □

Lemma 4.4. *For any $u \in C^6(\overline{\Omega}) \bigcap H_0^1(\Omega)$, the following the estimate holds true*

$$| (g_2^{h_1})_{2i,2j} - (g_2^{h_0})_{i,j} | \lesssim (\frac{h}{d_{i,j}})^2, \ \forall \ (i,j) \in S.$$

Proof. Combining (34), Lemma 4.2 and Lemma 4.3 , the estimate

$$\begin{aligned} | (g_2^{h_1})_{2i,2j} - (g_2^{h_0})_{i,j} | &\lesssim \frac{1}{N^2} \sum_{t=1}^{\infty} e^{-d_{i,j}t} + \sum_{k \in E_{i,j}} 3^{-k} \\ &\lesssim (\frac{h}{d_{i,j}})^2, \end{aligned}$$

holds true, which completes the proof of Lemma 4.4. □

Using Theorem 3.1 and Lemma 4.4 , we have

Theorem 4.2 Aussme that the model problem solution $u \in C^6(\overline{\Omega}) \bigcap H_0^1(\Omega)$. Then for any $(i,j) \in S_0$, we have

$$\frac{4}{3}(u_{h_1} - \frac{1}{4}u_h)(ih, jh) = u(ih, jh) + O\big(h^4(d_{i,j}^{-2} + h^{-\varepsilon})\big),$$

where $h_1 = \frac{h}{2}$ and ε is an arbitrary positive constant independent of h.

To get the extrapolation for the derivative of the Union Jack finite element solution, we denote

$$(R_x^{h_k})_{i,j} = (g_2^{h_k})_{i+1,j} - (g_2^{h_k})_{i-1,j},$$
$$(R_y^{h_k})_{i,j} = (g_2^{h_k})_{i,j+1} - (g_2^{h_k})_{i,j-1},$$

where $k = 0, 1$.

In the following, we will estimate $2(R_x^{h_1})_{2i,2j} - (R_x^{h_0})_{i,j}$ and $2(R_y^{h_1})_{2i,2j} - (R_y^{h_0})_{i,j}$.

Assume that $u \in C^6(\overline{\Omega}) \bigcap H_0^1(\Omega), (i,j) \in S$. Combining (19), (28) and (33), Lemma 3.5 and Lemma 4.3, we have

$$\begin{aligned} &\big|2(R_x^{h_1})_{2i,2j} - (R_x^{h_0})_{i,j}\big| \\ \lesssim &\sum_{t=1}^{N} \frac{1}{t^3}\Big[\sum_{k \in E_i} \big|2\big(S_{2k+1,1}^{h_1}(t) - S_{2k-1,1}^{h_1}(t)\big) - \big(S_{k+1,1}^{h}(t) - S_{k-1,1}^{h}(t)\big)\big| \\ &+ \sum_{k \in E_j} \big|2\big(sin\frac{(2i+1)t\pi}{2N_1} - sin\frac{(2i-1)t\pi}{2N_1}\big)S_{2k,1}^{h_1}(t) \\ &- \big(sin\frac{(i+1)t\pi}{2N} - sin\frac{(i-1)t\pi}{2N}\big)S_{k,1}^{h}(t)\big|\Big] + 3^{-2d_{i,j}N} + \sum_{t=N+1}^{\infty} \frac{1}{t^3}, \end{aligned} \tag{41}$$

where $E_i = \{i, 2N-i\}$, $E_j = \{j, 2N-j\}$.

If $1 \leqslant t \leqslant N, k \in E_j$, using Lemma 4.2 and Lemma 3.5, we have

(42)

$$\begin{aligned} &\big|2\big(sin\frac{(2i+1)t\pi}{2N_1} - sin\frac{(2i-1)t\pi}{2N_1}\big)S_{2k,1}^{h_1}(t) - \big(sin\frac{(i+1)t\pi}{2N} - sin\frac{(i-1)t\pi}{2N}\big)S_{k,1}^{h}(t)\big| \\ \lesssim &sin\frac{t\pi}{4N}\big|S_{2k,1}^{h_1}(t) - cos\frac{t\pi}{4N}S_{k,1}^{h}(t)\big| \\ \lesssim &sin\frac{t\pi}{4N}\big(\big|S_{2k,1}^{h_1}(t) - S_{k,1}^{h}(t)\big| + (1 - cos\frac{t\pi}{4N})\big|S_{k,1}^{h}(t)\big|\big) \\ \lesssim &\frac{t^4}{N^3}e^{-d_{i,j}t} + (\frac{t}{N})^3. \end{aligned}$$

Lemma 4.5. *Let $(i,j)\in S, k\in E_i$. For any $1\le t\le N$, we have the estimate*

$$\big|2\big(S^{h_1}_{2k+1,1}(t)-S^{h_1}_{2k-1,1}(t)\big)-\big(S^{h}_{k+1,1}(t)-S^{h}_{k-1,1}(t)\big)\big|\lesssim \frac{t^4}{N^3}e^{-d_{i,j}t}+(\frac{t}{N})^3.$$

Proof. Denote

$$G(z)=\frac{z^k+z^{4N-k}}{1-z^{4N}}.$$

From (13) and (14), we have

$$S^{h}_{k+1,1}(t)-S^{h}_{k-1,1}(t)=2\sqrt{(2-cos\frac{t\pi}{2N})^2-1}G(\lambda^h_1(t)), \tag{43}$$

and

$$S^{h_1}_{2k+1,1}(t)-S^{h_1}_{2k-1,1}(t)=2\sqrt{(2-cos\frac{t\pi}{2N_1})^2-1}G(\lambda^{h_1}_1(t)^2). \tag{44}$$

From (13)and(20), we have

$$\big|G\big(\lambda^h_1(t)\big)\big|\lesssim \lambda^h_1(t)^{k-4N}+\lambda_1(t)^{-k}\lesssim 1. \tag{45}$$

Using the *Lagrange's* mean value theorem and Lemma 4.1, and combining the similar estimate as (40), we deduce

$$\big|G\big(\lambda^{h_1}_1(t)^2\big)-G\big(\lambda^h_1(t)\big)\big|=\big|\big(\lambda^{h_1}_1(t)^2-\lambda^h_1(t)\big)G'(\xi)\big|\quad\lesssim\frac{t^3}{N^2}e^{-d_{i,j}t}. \tag{46}$$

Combing (43)∼(46), we obtain

$$\begin{aligned}&\big|2\big(S^{h_1}_{2k+1,1}(t)-S^{h_1}_{2k-1,1}(t)\big)-\big(S^{h}_{k+1,1}(t)-S^{h}_{k-1,1}(t)\big)\big|\\ =&\big|4\sqrt{(2-cos\frac{t\pi}{4N})^2-1}\Big(G\big(\lambda^{h_1}_1(t)^2\big)-G\big(\lambda^h_1(t)\big)\Big)\\ &+2\Big(2\sqrt{(2-cos\frac{t\pi}{4N})^2-1}-\sqrt{(2-cos\frac{t\pi}{2N})^2-1}\ \Big)G(\lambda^h_1(t))\big|\\ \lesssim&\frac{t^4}{N^3}e^{-d_{i,j}t}+(\frac{t}{N})^3,\end{aligned}$$

which completes the proof of Lemma 4.5. □

Lemma 4.6. *For any $u\in C^6(\overline{\Omega})\bigcap H^1_0(\Omega)$ and $z\in\{x,y\}$, we have the estimate*

$$\mid 2(R^{h_1}_z)_{2i,2j}-(R^{h_0}_z)_{i,j}\mid\lesssim(\frac{h}{d_{i,j}})^2.$$

Proof. Using the symmetry of x and y, combining (41),(42) and Lemma 4.5, we have

$$\begin{aligned}\mid 2(R^{h_1}_z)_{2i,2j}-(R^{h_0}_z)_{i,j}\mid\lesssim&\quad\frac{1}{N^3}\sum_{t=1}^{N}te^{-d_{i,j}t}+\frac{1}{N^2}+3^{-2d_{i,j}N}\\ \lesssim&\quad h^2(1+\frac{h}{d^2_{i,j}})+3^{-2d_{i,j}N}\lesssim(\frac{h}{d_{i,j}})^2.\end{aligned}$$

This completes the proof of Lemma 4.6. □

Denote

$$\begin{aligned}(\widetilde{\partial}_x u_h)(ih,jh)=&\frac{4}{3}\Big[\frac{u_{h_1}\big((2i+1)h_1,2jh_1\big)-u_{h_1}\big((2i-1)h_1,2jh_1\big)}{2h_1}\\ &-\frac{1}{4}\frac{u_h\big((i+1)h,jh\big)-u_h\big((i-1)h,jh\big)}{2h}\Big],\end{aligned}$$

and

$$(\widetilde{\partial}_y u_h)(ih, jh) = \frac{4}{3}\Big[\frac{u_{h_1}\big(2ih_1, (2j+1)h_1\big) - u_{h_1}\big(2ih_1, (2j-1)h_1\big)}{2h_1} - \frac{1}{4}\frac{u_h\big(ih, (j+1)h\big) - u_h\big(ih, (j-1)h\big)}{2h}\Big].$$

Using Theorem 3.1 and Lemma 4.6, we have

Theorem 4.3 Assume that the model problem solution $u \in C^6(\overline{\Omega}) \bigcap H_0^1(\Omega)$, and u_h is the Union jack linear element solution. For any positive constant ε independent of h, we have

$$(\widetilde{\partial}_x u_h)(ih, jh) = u^{1,0}(ih, jh) + O\big(h^3(d_{i,j}^{-2} + h^{-\varepsilon})\big),$$
$$(\widetilde{\partial}_y u_h)(ih, jh) = u^{0,1}(ih, jh) + O\big(h^3(d_{i,j}^{-2} + h^{-\varepsilon})\big).$$

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