ON ERROR ESTIMATES OF THE PENALTY METHOD FOR THE UNSTEADY CONDUCTION-CONVECTION PROBLEM I: TIME DISCRETIZATION

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Abstract. In this paper, the penalty method is proposed and discussed for the unsteady conduction-convection problem in two dimensions. In addition, we analyze its time discretization which is based on the backward Euler implicit scheme. Finally, the main results of this paper that optimal error estimates are obtained for the penalty system and the time discretization under reasonable assumptions on the physical data.

Key words. Unsteady conduction-convection problem, Penalty method, Time discretization, Optimal error estimates.

1. Introduction

In this paper, let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^2$ boundary $\partial \Omega$ or a convex polygon. Now we consider the following unsteady conduction-convection problem (cf. [3, 5]).

Problem (I) : Find $u, p$ and $T$ such that for $t_N > 0$,

$$
\begin{cases}
  u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p = \lambda j T, & (x,t) \in \Omega \times (0,t_N), \\
  \text{div} u = 0, & (x,t) \in \Omega \times (0,t_N), \\
  T_t - \lambda^{-1} \Delta T + u \cdot \nabla T = 0, & (x,t) \in \Omega \times (0,t_N), \\
  u(x,t) = 0, T(x,t) = 0, & (x,t) \in \partial \Omega \times (0,t_N), \\
  u(x,0) = 0, T(x,0) = \varphi(x), & x \in \Omega,
\end{cases}
$$

(1)

where $u = (u_1(x,t), u_2(x,t))$ represents velocity vector, $p(x,t)$ the pressure, $T(x,t)$ the temperature, $\nu > 0$ the viscosity, $\lambda^{-1} > 0$ the thermal diffusivity, $j = (0,1)$ the two-dimensional unit vector, $\varphi(x,y)$ is the given function, $t_N$ is the final time.

The unsteady conduction-convection Problem (I) is an important dissipative nonlinear system in atmospheric dynamics. It is the coupled equations governing viscous incompressible flow and heat transfer process [6, 22], where the incompressible flow is the Boussinesq approximation to the unsteady Navier-Stokes equations. There are many numerical methods have been studied on the conduction-convection problem (see [2, 5]) and many literatures (see [12, 13, 14, 15, 18]) are put into the construction, analysis and implementation for conduction-convection problem. Shen [19] firstly analyzed the existence uniqueness of approximation solution for steady conduction-convection equations with the Bernadi-Raugel element. Luo and his coworkers gave an optimizing reduced PLSMFE in [14] and a least squares Galerkin/Petrov mixed finite element method in [15]. Shi provided nonconforming
mixed finite element method in [18]. An analysis of conduction natural convection conjugate heat transfer in the gap between concentric cylinders under solar irradiation was studied in [11], etc.

As we known, the velocity \( u \), the pressure \( p \) and the temperature \( T \) are coupled together by the impressibility constraint “\( \text{div} u = 0 \)” and two dissipative nonlinear equation, which make the system is difficult to solve by using the numerical methods. In order to overcome coupled problem, the penalty method as a popular pseudo-compressibility strategy which initially proposed by Courant [4] is popular used (see [7, 16, 17]). Temam [22] firstly applied it to the Navier-Stokes equations. Then, many works appeared on this subject. Shen [16] derived the optimal error estimates for the unsteady Navier-Stokes equations as follows:

\[
\tau^\frac{1}{2} \| u(t_n) - u_c(t_n) \|_{L^2} + \| u(t_n) - u_c(t_n) \|_{H^1} \leq C \varepsilon,
\]

for \( t_n \in [0, t_N] \), where \( \tau(t_n) = \text{min}\{1, t_n\} \), \( C \) is a general positive constant and \( u(t_n), u_c(t_n) \) are the solution of the Navier-Stokes equations and its penalty system, respectively. Recently, He [7] extended it to the finite element method. For the viscoelastic Oldroyd flow problem, Wang et al derived the optimal error estimates for the penalty system [23] and extended it to the fully discrete schemes [24]. This motivates our interest in solving more complicated problem by this method and we have investigated the unsteady conduction-convection problem. For the unsteady conduction-convection problem, the penalty method for Problem (I) is as follows.

Problem (II): Find \( u_c = (u_{c1}, u_{c2}), p_c \) and \( T_c \) such that for \( t_n > 0 \),

\[
\begin{align*}
&\begin{cases}
u \Delta u_c + B(u_c, u_c) + \nabla p_c = \lambda jT_c, & (x, t) \in \Omega \times (0, t_n), \\
\text{div} u_c + \frac{\partial \tau}{\partial t} = 0, & (x, t) \in \Omega \times (0, t_n), \\
T_c - \nu^{-1} \Delta T_c + B(u_c, T_c) = 0, & (x, t) \in \Omega \times (0, t_n), \\
u_c(x, 0) = 0, & (x, t) \in \partial \Omega \times (0, t_n), \\
\end{cases} \\
&u_c(x, 0) = 0, T_c(x, 0) = \varphi(x), \quad x \in \Omega,
\end{align*}
\]

where \( 0 < \varepsilon < 1 \) is a penalty parameter,

\[
\tilde{B}(u_c, v_c) = (u_c \cdot \nabla)v_c + \frac{1}{2}(\text{div} u_c)v_c \quad \text{and} \quad \tilde{B}(u_c, T_c) = u_c \cdot \nabla T_c + \frac{1}{2}(\text{div} u_c)T_c
\]

is the modified bilinear term, \( (\text{div} u_c)v_c \) and \( (\text{div} u_c)T_c \) are introduced to ensure the dissipativity of Problem (II) as \( (\text{div} u)v \) is introduced in the Navier-Stokes equations by Temam [21] to ensure the dissipativity of the Navier-Stokes equations. In this way, \( p_c \) can be eliminated to obtain a penalty system that only contains \( u_c, T_c \), which is much easier to solve than the original equations. Zhang and He have analyzed the penalty finite element for the stationary conduction convection problems [25] and the non-stationary conduction convection problems [26], they have given that, for all \( t_n \in [0, t_N] \),

\[
\| u(t_n) - u_c(t_n) \|_{L^2} + \left( \int_0^{t_n} \| u(t) - u_c(t) \|_{H^1}^2 \, dt \right)^{\frac{1}{2}} + \| T(t_n) - T_c(t_n) \|_{L^2} \leq C \sqrt{\varepsilon},
\]

under the assumptions that the exact solutions are sufficiently smooth. When we consider the discrete problem for the penalty system (2), the estimate (3) is misleading. For instance, if the backward Euler scheme is applied to the penalized
system (2), the estimate (3) would lead to
\[
\|u(t_n) - u^n\|_{L^2} + \left( k \sum_{m=1}^{n} \|u(t_m) - u^m\|^2_{H^1} \right)^{1/2} + \|T(t_n) - T^n\|_{L^2} \\
+ \left( k \sum_{m=1}^{n} \|T(t_m) - T^m\|^2_{H^1} \right)^{1/2} \leq C(k + \varepsilon),
\]
where \(0 < k < 1\) is the time step size, \(t_n = nk\) and \((u^n, T^n)\) is a penalty approximation of \((u, T)\) at the time \(t_n\). (4) suggests the choice \(\varepsilon = \Delta t^2\), which would result in a very ill conditioned system when we make a further spatial discretization (see [16]).

The main focus of this paper is to apply the techniques in [16] to unsteady conduction-convection problem and derive the optimal error estimates for the penalty system and its time discretization. Under some realistic assumptions of the initial value \((u_0, \varphi(x))\), we have the following error estimates
\[
\begin{align*}
\tau^2(t_n)\|u(t_n) - u(t_n)\|_{L^2} + \tau(t_n)\|u(t_n) - u(t_n)\|_{H^1} + \tau^2(t_n)\|T(t_n) - T(t_n)\|_{H^1} &\leq C \varepsilon, \\
\tau^2(t_n)\|u(t_n) - u(t_n)\|_{L^2} + \tau(t_n)\|u(t_n) - u(t_n)\|_{H^1} + \tau^2(t_n)\|T(t_n) - T(t_n)\|_{H^1} &\leq C(k + \varepsilon),
\end{align*}
\]
for sufficiently small \(\varepsilon\) and \(k\), which substantially improve the previous results (3) and (4) and lead to the proper choices of \(\varepsilon\) for time discretizations of the penalized system.

The remainder of this paper is organized as follows. We firstly introduce some notations and preliminary results for Problem (I) in the next section. Then we provide error behavior for the linearized penalty system in Section 3 and for the nonlinear penalty system in Section 4. In Section 5, we analyze the backward Euler time discretization scheme for the nonlinear penalty system. Finally, conclusions are given in Section 6.

2. Preliminaries

In this section, we describe some of the notations and results which will be frequently used in this paper. For the mathematical setting of conduction-convection Problem (I) and the penalty conduction-convection Problem (II), we introduce the following Hilbert spaces
\[
X = H^1_0(\Omega)^2, \quad W = H^1_0(\Omega), \quad M = \{ q \in L^2(\Omega); \int_\Omega qdx = 0 \}.
\]
The norm corresponding to \(H^i(\Omega)^2\) or \(H^i(\Omega)\) will be denote \(\| \cdot \|_i\) for \(i=1, 2\). In particular, we use \((\cdot, \cdot)\) and \(\| \cdot \|_0\) to denote the inner product and norm in \(L^2(\Omega)^2\) or \(L^2(\Omega)\). The spaces \(X\) and \(W\) are equipped with their usual scalar product and norm
\[
((u, v)) = (\nabla u, \nabla v), \quad \|\nabla u\|_0 = ((u, u))^{1/2}.
\]
We also introduce the Hilbert space \(H\) and \(V\) defined by
\[
H = \{ v \in L^2(\Omega)^2; \text{ div } v = 0, \ v \cdot n|_{\partial \Omega} = 0 \}, \quad V = \{ v \in X; \text{div} v = 0 \}.
\]
Define \(Au = -\Delta u\) and \(Ae u = -\Delta u - \nabla \text{div} u/\varepsilon\), which are the operators associated with the conduction-convection problem and the penalty conduction-convection
problem, respectively. They are the positive self-adjoint operators from $D(A) = H^2(\Omega)^2 \cap X$ (or $H^2(\Omega) \cap W$) onto $L^2(\Omega)^2$ (or $L^2(\Omega)$). It is valid that

$$(\tilde{A}u, v) = (A^{1/2}u, A^{1/2}v), \ \forall u \in D(A), v \in X \ (or \ W),$$

where $\tilde{A} = A$ or $A_\varepsilon$. In particular, there holds

$$(A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v), \ \forall u, v \in X \ (or \ W).$$

(5)

$$(A_\varepsilon^{1/2}u, A_\varepsilon^{1/2}v) = (A_\varepsilon^{1/2}u, A_\varepsilon^{1/2}v) + \frac{1}{\varepsilon}(\text{div}u, \text{div}v), \ \forall u, v \in X.$$ 

(6)

It is known that (see [1, 9]):

$$(7) \quad \|u\|_L^1 \leq \gamma_0\|u\|_0^{1/2}\|A^{1/2}u\|_0^{1/2}, \ \|u\|_0 \leq \gamma_0\|A^{1/2}u\|_0, \ \forall u \in X \ (or \ W),$$

$$(8) \quad \|A^{1/2}u\|_0 \leq \gamma_0\|A^{1/2}u\|_0, \ \|A^{1/2}u\|_0 \leq \gamma_0\|Au\|_0, \ \forall u \in D(A),$$

where $\gamma_0$ is a positive constant depending only on $\Omega$, which may stand for different values at its different occurrences. Furthermore, we recall the following lemma given in [16].

**Lemma 2.1.** There exists a constant $C > 0$, depending only on $\Omega$ and such that for $\varepsilon$ sufficiently small, we have

$$\|Au\|_0 \leq C\|A_\varepsilon u\|_0, \ \forall u \in D(A),$$

$$\|A_\varepsilon^{1/2}u\|_0 \leq C\|A^{1/2}u\|_0, \ \forall u \in X,$$

$$\|A_\varepsilon^{-1}u\|_0 \leq C\|u\|_{-2}, \ \forall u \in H^{-2}(\Omega)^2.$$ 

Associated with the conduction-convection Problem (I) and the penalty conduction-convection Problem (II), we define the continuous bilinear forms

$$a(u, v) = \nu(\nabla u, \nabla v), \quad a_\varepsilon(u, v) = \nu(A_\varepsilon^{1/2}u, A_\varepsilon^{1/2}v), \ \forall u, v \in X,$$

$$a(T, \varphi) = \lambda^{-1}(\nabla T, \nabla \varphi), \ \forall T, \varphi \in W,$$

$$d(v, q) = (q, \text{div}v), \ \forall v \in X, q \in M,$$

respectively. We also introduce a continuous trilinear form $\bar{b}(\cdot, \cdot, \cdot)$ on $X \times X \times X$, $\tilde{b}(\cdot, \cdot, \cdot)$ on $X \times W \times W$, respectively.

$$\bar{b}(u, v, w) = \frac{1}{2} \int_\Omega \sum_{i,k=1}^2 (u_i \frac{\partial w_k}{\partial x_i} - u_k \frac{\partial w_i}{\partial x_i})dx, \ \forall u, v, w \in X,$$

$$\tilde{b}(u, T, \psi) = \frac{1}{2} \int_\Omega \sum_{i=1}^2 (u_i \frac{\partial T}{\partial x_i} \psi - u_i \frac{\partial \psi}{\partial x_i} T)dx, \ \forall u \in X, T, \psi \in W.$$

Some estimates of the trilinear $\bar{b}(\cdot, \cdot, \cdot)$ can be found in [8, 20, 27].

(9) $$\bar{b}(u, v, w) = -\bar{b}(u, v, w), \ \forall u, v, w \in X,$$

(10) $$|\bar{b}(u, v, w)| \leq C\|u\|_1 \|v\|_1^{1/2} \|w\|_0, \ \forall u, w \in X, v \in D(A),$$

$$\|\bar{b}(u, v, w)\| + |\bar{b}(u, v, w)| + |\bar{b}(w, u, v)| \leq C\|u\|_2 \|v\|_1 \|w\|_0, \ \forall u \in D(A), v, w \in X.$$ 

Similarly, it is easy to verify that \( \hat{b}(\cdot, \cdot) \) satisfy the following important property

\[
\hat{b}(u, T, \psi) = -\hat{b}(u, \psi, T), \quad \forall u \in X, T, \psi \in W, \tag{12}
\]

\[
|\hat{b}(u, T, \psi)| \leq C\|u\|_1|T|^\frac{1}{2}|T|^\frac{1}{2}\|\psi\|_0, \quad \forall u \in X, T \in D(A), \psi \in W, \tag{13}
\]

\[
|\hat{b}(u, T, \psi)| + |\hat{b}(T, u, \psi)| + |\hat{b}(\psi, T, u)| \leq C\|u\|_1|T|\|\psi\|_0, \quad \forall T \in D(A), u \in X, \psi \in W. \tag{14}
\]

With above notations, the variational formulation of Problem (I) is written as follow.

Problem (III): Find \((u, p, T) \in (X, M, W)\), for all \(t \in [0, t_N]\), such that for all \((v, \phi, \psi) \in (X, M, W)\):

\[
\begin{cases}
(u_t, v) + a(u, v) + \hat{b}(u, u, v) - d(v, p) = \lambda (jT, v), \\
\phi_t + a(T, \phi) + \hat{b}(T, \phi) = 0,
\end{cases} \tag{15}
\]

and the variational formulation of the penalty system Problem (II) reads as.

Problem (IV): Find \((u_\varepsilon, p_\varepsilon, T_\varepsilon) \in (X, M, W)\), for all \(t \in [0, t_N]\), such that for all \((v, \phi, \psi) \in (X, M, W)\):

\[
\begin{cases}
(u_\varepsilon_t, v) + a(u_\varepsilon, v) + \hat{b}(u_\varepsilon, u_\varepsilon, v) - d(v, p_\varepsilon) = \lambda (jT_\varepsilon, v), \\
\phi_t + a(T_\varepsilon, \phi) + \hat{b}(T_\varepsilon, \phi) = 0, \\
\end{cases} \tag{16}
\]

with \(u(x, 0) = u_\varepsilon(x, 0) = 0, T(x, 0) = T_\varepsilon(x, 0) = \varphi(x)\), respectively.

Theorem 2.2. [13] If \(\varphi(x) \in H^2(\Omega)\), Problem (III) has a unique solution \((u, p, T) \in [L^2(0, t_N; X) \cap H^1(0, t_N; V)] \times L^2(0, t_N; M) \times H^1(0, t_N; W)\), satisfies

\[
\|\nabla T^{(i)}\|_0 + \|T^{(i)}\|_0 + \|\nabla u^{(i)}\|_0 + \|u^{(i)}\|_0 \leq \theta(t), \quad 0 \leq i \leq 3,
\]

where \(\theta(t)\) is continuous general positive function about \(t\) only depends on the data \(\varphi(x)\).

Theorem 2.3. If \(\varphi(x) \in H^2(\Omega)\), Problem (IV) has a unique solution \((u_\varepsilon, p_\varepsilon, T_\varepsilon) \in [L^\infty(0, t_N; H^2(\Omega)) \cap L^2(0, t_N; X)] \times L^2(0, t_N; M) \times H^1(0, t_N; W)\).

The proof of Theorem 2.3 and the solution \((u_\varepsilon, p_\varepsilon, T_\varepsilon)\) of Problem (IV) converges to the solution \((u, p, T)\) of Problem (III) uniformly as \(\varepsilon \to 0\) are similar to the proof of Theorem 3.1 and Theorem 3.2 of [26] and much easier than that, here we omit it.

By using a similar argument to [21], we have the following properties.

(A1). Assume that the initial velocity \(u_0 \in V\), the initial temperature \(\varphi(x) \in W, T \in L^\infty(0, T; L^2(\Omega))\), there exists a finite time \(t_{M1} \leq t_N\) such that

\[
u \in C([0, t_{M1}); V) \cap L^2([0, t_{M1}]; H^2(\Omega)^2),
\]

\[
T \in L^2([0, t_{M1}]; H^2(\Omega)), \quad p \in L^2([0, t_{M1}]; H^1(\Omega)/R). \tag{17}
\]

(A2). Assume that \(\varphi(x) \in H^2(\Omega)\), then \(tT_t \in L^2([0, t_N]; L^2(\Omega))\).

By using the smoothing property of the conduction-convection problem, that

\[
tp_\varepsilon \in L^2([0, t_N]; H^1(\Omega)). \tag{18}
\]

Similar to the penalty Navier-Stokes equation [16], one can show that assumption (A1), there exists a finite time \(t_{M2} \leq t_N\) and a constant \(C\) independent of \(\varepsilon\) such that

\[
\|u_\varepsilon(t)\|^2 + \int_0^t \|u_\varepsilon(t)\|^2 dt \leq C, \quad \forall t \in [0, t_{M2}].
\]
In the sequel, we restrict ourselves to the interval \([0, t_M]\) with \(t_M = \min\{t_{M1}, t_{M2}\}\).

We also recall two lemmas of Gronwall type which will be frequently used [21].

**Lemma 2.4.** (Gronwall Lemma). Let \(y(t)\), \(g(t)\), \(h(t)\), \(f(t)\) be nonnegative functions such that
\[
\int_0^t g(s)ds \leq y(0) + \int_0^t (g(s)y(s) + f(s))ds, \forall 0 \leq t \leq t_N,
\]
then
\[
y(t) + \int_0^t h(s)ds \leq \exp(M)\left(y(0) + \int_0^t f(s)ds\right), \forall 0 \leq t \leq t_N.
\]

**Lemma 2.5.** (Discrete Gronwall Lemma). Let \(C\) and \(a_n\), \(b_n\), \(d_n\), for integer \(1 \leq m \leq \frac{t_N}{k}\), be nonnegative numbers, such that
\[
a_m + k \sum_{n=0}^m b_n \leq k \sum_{n=0}^m a_n d_n + C, \forall 1 \leq m \leq \frac{t_N}{k}.
\]
Assume that \(kd_n < \frac{1}{2}\) and for all \(1 \leq n \leq \frac{t_N}{k}\) then
\[
a_m + k \sum_{n=1}^m b_n \leq C \exp(2k \sum_{n=1}^{m-1} a_n), \forall 1 \leq m \leq \frac{t_N}{k}.
\]

### 3. Error estimates for the linearized problem

In this section, we will consider the following linear problem. The results in this section will be used in the next section as an intermediate step for analyzing the nonlinear conduction-convection problem.

\[
\begin{align*}
\begin{cases}
u \Delta u + \nabla p &= \lambda jT, \div u = 0, & (x, t) \in \Omega \times (0, t_N), \\
\lambda^{-1} \Delta T &= 0, & (x, t) \in \Omega \times (0, t_N), \\
u u_t + \nabla u_x &= 0, T_x(x, t) = 0, & (x, t) \in \partial\Omega \times (0, t_N).
\end{cases}
\end{align*}
\]

The penalty method of (21) is as follows:
\[
\begin{align*}
\begin{cases}
u \Delta u_x + \nabla p_x &= \lambda jT_x, \div u_x + \frac{\xi}{\nu}p_x = 0, & (x, t) \in \Omega \times (0, t_N), \\
\lambda^{-1} \Delta T_x &= 0, & (x, t) \in \Omega \times (0, t_N), \\
u u_t(x, t) = 0, T_x(x, t) = 0, & (x, t) \in \partial\Omega \times (0, t_N),
\end{cases}
\end{align*}
\]

with \(u(x, 0) = u_x(x, 0) = 0, T(x, 0) = T_x(x, 0) = \varphi(x)\), respectively.

Similarly, the variational formulation of the problem (21) is defined as follows: find \((u, p, T) \in (X, M, W)\), for all \(t \in [0, t_N]\), such that for all \((v, \phi, \psi) \in (X, M, W)\),
\[
\begin{align*}
\begin{cases}
u u_t + a(u, v) - d(v, p) + d(u, \phi) &= \lambda jT, \\
T_t(\psi) &= \hat{a}(T, \psi) = 0.
\end{cases}
\end{align*}
\]

The penalty method applied to (23) is that: find \((u_x, p_x, T_x) \in (X, M, W)\), for all \(t \in [0, t_N]\), such that for all \((v, \phi, \psi) \in (X, M, W)\),
\[
\begin{align*}
\begin{cases}
u u_t + \alpha(u, v) - d(v, p_x) + d(u, \phi) + \xi(p_x, \phi) &= \lambda jT_x, \\
T_t(\psi) + \hat{a}(T, \psi) &= 0,
\end{cases}
\end{align*}
\]

with \(u(x, 0) = u_x(x, 0) = 0, T(x, 0) = T_x(x, 0) = \varphi(x)\), respectively.

Setting \(e = u - u_x\), \(q = p - p_x\), \(\xi = T - T_x\), it follows that \(e(0) = \xi(0) = 0\). We shall derive a sequence of estimates for the penalty error \(e, q\) and \(\xi\).
Subtracting (24) from (23), we obtain
\[
\begin{aligned}
(e_t, v) + a(e, v) - d(v, q) + d(e, \phi) + \tilde{g}(q, \phi) &= \lambda(j\xi, v) + \tilde{g}(p, \phi), \\
(\xi_t, \psi) + \tilde{a}(\xi, \psi) &= 0.
\end{aligned}
\] (25)

**Lemma 3.1.** Suppose (A1) is valid, \(\varepsilon\) sufficiently small, for all \(t \in [0, t_M]\), we have
\[
\|e\|_0^2 + \int_0^t \|\nabla e\|_0^2 ds + \varepsilon \int_0^t \|q\|_0^2 ds \leq C\varepsilon,
\] (26)
\[
\int_0^t \|e\|_0^2 ds \leq C\varepsilon^2,
\] (27)
\[
\|\xi\|_0^2 + \int_0^t \|\nabla \xi\|_0^2 ds = 0.
\] (28)

**Proof.** Taking \((v, \phi, \psi) = (e, q, \xi)\) in (25) and using (7), we have
\[
\frac{1}{2} \frac{d}{dt} \|e\|_0^2 + \nu \|\nabla e\|_0^2 + \varepsilon \nu \|q\|_0^2 \leq \varepsilon \nu \|q\|_0^2 + \nu \|\nabla e\|_0^2 + C \|\nabla \xi\|_0^2,
\] (29)
and
\[
\frac{1}{2} \frac{d}{dt} \|\xi\|_0^2 + \lambda^{-1} \|\nabla \xi\|_0^2 = 0.
\] (30)

Integrating the above two inequalities from 0 to \(t \leq t_M\), thanks to \(e(0) = 0\), \(\xi(0) = 0\) and (17), we have
\[
\|e\|_0^2 + \int_0^t \|\nabla e\|_0^2 ds + \varepsilon \int_0^t \|q\|_0^2 ds \leq C\varepsilon,
\] (31)
\[
\|\xi\|_0^2 + \int_0^t \|\nabla \xi\|_0^2 ds = 0.
\] (32)

Now we use the standard parabolic duality argument. For any \(0 < t \leq t_M\), we define \((w, \varphi)\) by
\[
\begin{aligned}
w_s + \nu \Delta w + \nabla \varphi &= e(s), &0 < s \leq t, \\
div w &= 0, &w(t) = 0.
\end{aligned}
\] (33)

There are the following inequality (see [16]),
\[
\nu \int_0^t \|w\|_0^2 ds + \int_0^t \|\nabla \varphi\|_0^2 ds \leq C \int_0^t \|e\|_0^2 ds.
\] (34)

Taking the inner product of (33) with \(e(s)\), because of (25) and \(\text{div} w = 0\), we derive
\[
\|e\|_0^2 = (w_s, e) + \nu (\Delta w, e) + (\nabla \varphi, e)
\]
\[
= \frac{d}{ds} (w, e) - \lambda(j\xi, w) - \varepsilon \frac{\varepsilon}{\nu} (\varphi, p_e).
\]

Integrating from 0 to \(t\), using Schwarz inequality, (7), (32) and (34), since \(w(t) = e(0) = 0\), we have
\[
\int_0^t \|e\|_0^2 ds \leq C\varepsilon^2 \int_0^t \|p_e\|_0^2 ds \leq C\varepsilon^2, \quad \forall t \in [0, t_M].
\]

**Remark:** Note that equation (27) implies that \(\xi\) is equal to zero, this indicates that equation (25) is the same as equation (3.3)-(3.4) in Shen [16]. Now, we recall some results from the reference [16].
Lemma 3.2. Suppose (A1) and (A2) are valid, $\varepsilon$ sufficiently small, for all $t \in [0, t_M]$, we have
\[ \int_0^t s^2 \| p_\varepsilon \|_0^2 \, ds \leq C. \]

Theorem 3.3. Suppose (A1) and (A2) are valid, $\varepsilon$ sufficiently small, for all $t \in [0, t_M]$, we have
\[ t \| e \|_0^2 + \int_0^t s \| \nabla e \|_0^2 \, ds + \varepsilon \int_0^t s \| q \|_0^2 \, ds \leq C \varepsilon^2, \]
\[ t^2 \| \nabla e \|_0^2 + \int_0^t s^2 \| q \|_0^2 \, ds \leq C \varepsilon^2. \]

4. Error estimates for the nonlinear problem

We consider the following intermediate linear problem:
\[ \begin{cases}
 v_t - \nu \Delta v + \nabla \gamma = \lambda_j T - \tilde{B}(u, u), \\
 \text{div } v + \frac{\varepsilon}{\nu} \gamma = 0, \\
 T_t - \lambda^{-1} \Delta T = -\tilde{B}(u, T),
\end{cases} \]
with $v(x, 0) = 0$, $T(x, 0) = \varphi(x)$, where $(u, T)$ is the solution of Problem (I).

Letting $\rho = v - u$, $\sigma = \gamma - p$ and subtracting Problem (I) from (37), we obtain
\[ \begin{align*}
 \rho_t - \nu \Delta \rho + \nabla \sigma &= 0, \\
 \text{div } \rho + \frac{\varepsilon}{\nu} \sigma &= -\frac{\varepsilon}{\nu} p,
\end{align*} \]
with $\rho(x, 0) = 0$.

Lemma 4.1. Suppose (A1) and (A2) are valid, $\varepsilon$ sufficiently small, for all $t \in [0, t_M]$, we have
\[ t \| \rho \|_0^2 + t^2 \| \nabla \rho \|_0^2 + \int_0^t \| \rho \|_0^2 \, ds + \int_0^t \| \nabla \rho \|_0^2 \, ds + \int_0^t \| \sigma \|_0^2 \, ds \leq C \varepsilon^2. \]

Proof. From Section 3, we note that the assumption (A1) for a linear problem can be replaced by the weaker condition $\lambda_j T \in L^2([0, t_M]; L^2(\Omega))$. Thanks to (17), we have $\lambda_j T - \tilde{B}(u, u) \in L^2([0, t_M]; L^2(\Omega))$. On the other hand, it can be easily shown (see for instance [9] that $tu_t \in L^2([0, t_M]; L^2(\Omega))$). Hence
\[ t \partial_t (\lambda_j T - \tilde{B}(u, u)) = t(\lambda_j T_t - \tilde{B}(u_t, u) - \tilde{B}(u, u_t)) \in L^2([0, t_M]; L^2(\Omega)). \]

Lemma 4.1 is then a direct consequence of Lemma 3.1 and Theorem 3.3 applied to (38)-(39).

Now, letting $\eta = u_x - v$, $\delta = p_x - \gamma$, $\varsigma = T_x - T$ and subtracting (37) from Problem (II), we obtain
\[ \begin{cases}
 \eta_t - \nu \Delta \eta + \tilde{B}(u_x, \eta + \rho) + \tilde{B}(\eta + \rho, u) + \nabla \delta = \lambda_j \varsigma, \\
 \text{div } \eta + \frac{\varepsilon}{\nu} \delta = 0, \\
 \varsigma_t - \lambda^{-1} \Delta \varsigma + \tilde{B}(u_x, \varsigma) + \tilde{B}(\eta + \rho, T) = 0,
\end{cases} \]
with $\eta(x, 0) = 0$, $\varsigma(x, 0) = 0$, respectively. The variational formulation of the problem (40) is defined as follows.
Find \((\eta, \delta, \varsigma) \in (X, M, W)\), for all \(t \in [0, t_M]\), such that for all \((v, \phi, \psi) \in (X, M, W)\),

\[
\begin{cases}
(\eta, v) + a(\eta, v) + \tilde{b}(u_x, \eta + \rho, v) + \tilde{b}(\eta + \rho, u, v) - d(v, \delta) = \lambda j\varsigma, v),
\end{cases}
\]

Taking \(\psi = A^{-1}\varsigma\) in (42), thanks to Lemma 2.1, (11) and Schwarz inequality, we get

\[
\frac{1}{2} \frac{d}{dt}||A^{-\frac{2}{2}}\eta||_2^2 + \lambda^{-1}||\varsigma||_2^2 = -\tilde{b}(u_x, \varsigma, A^{-1}\varsigma) - \tilde{b}(\eta + \rho, T, A^{-1}\varsigma)
\]

Taking \(\psi = A^{-1}\varsigma\) in (42), thanks to (14) and Schwarz inequality, we get

\[
\frac{1}{2} \frac{d}{dt}||A^{-\frac{2}{2}}\varsigma||_2^2 + \lambda^{-1}||\varsigma||_2^2 = -\tilde{b}(u_x, \varsigma, A^{-1}\varsigma) - \tilde{b}(\eta + \rho, T, A^{-1}\varsigma)
\]

Since \(\int_0^{t_M} (||T||_2^2 + ||u_x||_2^2 + ||u||_2^2) dt \leq C\), we can apply Lemma 2.4 to above two inequalities, use Lemma 4.1, we obtain

\[
(43) \quad ||A^{-\frac{2}{2}}\eta||_2^2 + ||A^{-\frac{2}{2}}\varsigma||_2^2 + \nu \int_0^t ||\eta||_2^2 dt + \lambda^{-1} \int_0^t ||\varsigma||_2^2 dt \leq Ce^2.
\]

Now, taking \((v, \phi) = t(\eta, \delta)\) in (41), summing up the two relations, using (11) and Schwarz inequality, we derive

\[
\frac{1}{2} \frac{d}{dt} ||\eta||_2^2 + \nu ||\nabla\eta||_2^2 + \frac{\nu}{\lambda} \int_0^t ||\eta||_2^2 dt
\]

Taking \(\psi = \varsigma\) in (41), using (9), (14) and Schwarz inequality, we derive

\[
\nu \int_0^t ||\nabla\eta||_2^2 dt + \lambda t ||\nabla\varsigma||_2^2 = \frac{1}{2} ||\varsigma||_2^2 - \tilde{b}(u_x, \varsigma, \varsigma) - \tilde{b}(\eta + \rho, T, \varsigma)
\]

Integrating over \([0, t]\), we can apply (43), Lemma 4.1 and Lemma 2.4 to above two inequalities, such that

\[
(44) \quad t||\eta||_2^2 + t||\varsigma||_2^2 + \nu \int_0^t s||\nabla\eta||_2^2 ds + \lambda^{-1} \int_0^t s||\nabla\varsigma||_2^2 ds + \frac{\nu}{\lambda} \int_0^t s||\eta||_2^2 ds \leq Ce^2.
\]
Next, we take the partial derivative with respect to $t$ of the second term of (41) to obtain
\begin{equation}
(t^2 \eta_t + \frac{\nu}{\nu} \frac{d}{dt} t^2 \nabla \eta) + \frac{\nu}{\nu} \frac{d}{dt} t^2 \| \delta \|_0^2 = 0.
\end{equation}

Now, taking $v = t^2 \eta$ in (41) and $\phi = t^2 \delta$ in (45). Then adding them up, using (11) and Schwarz inequality, we get
\begin{align*}
n &= t^2 \eta_t + \frac{\nu}{\nu} \frac{d}{dt} t^2 \nabla \eta + \frac{\nu}{\nu} \frac{d}{dt} t^2 \| \delta \|_0^2 \\
&= t^2 \eta_t + \frac{\nu}{\nu} \frac{d}{dt} t^2 \| \delta \|_0^2 - t^2 \tilde{b}(u_e, \eta + \rho, \eta_t) - t^2 \tilde{b}(\eta + \rho, u, \eta) + \lambda t^2 (j, \eta_t) \\
&\leq t^2 \eta_t + \frac{\nu}{\nu} \frac{d}{dt} t^2 \| \delta \|_0^2 + \frac{\nu}{\nu} \frac{d}{dt} t^2 \| \eta \|_0^2 + C (\| u \|_0^2 + \| u_e \|_0^2) (\epsilon^2 + t^2 \| \nabla \eta \|_0^2) + \frac{\lambda^2 t^2 \epsilon^2}{2} \| \delta \|_0^2.
\end{align*}

Taking $\psi = t^2 \xi$ in (41), using (9), (14) and Schwarz inequality, we derive
\begin{align*}
1 &\frac{d}{dt} \frac{1}{2} t^2 \xi_t + \lambda^{-1} t^2 \| \nabla \xi \|_0^2 \\
&= t^2 \xi_t + \lambda^{-1} t^2 d^2 (\| \nabla \eta \|_0^2 + \| \nabla \rho \|_0^2). &
\end{align*}

Integrating over $[0, t]$, using (44), Lemma 2.4 and Lemma 4.1 to above two inequalities and taking $\delta_3$ sufficiently small, we derive
\begin{equation}
\begin{aligned}
\nu t^2 \| \nabla \eta \|_0^2 + t^2 \| \xi \|_0^2 + \frac{\nu}{\nu} \frac{d}{dt} t^2 \| \delta \|_0^2 + \int_0^t s^2 \| \eta \|_0^2 ds + \lambda^{-1} \int_0^t s^2 \| \nabla \xi \|_0^2 ds & \leq C \epsilon^2. \\
\nu t^2 \| \nabla \eta \|_0^2 + t^2 \| \xi \|_0^2 + \frac{\nu}{\nu} \frac{d}{dt} t^2 \| \delta \|_0^2 + \int_0^t s^2 \| \eta \|_0^2 ds + \lambda^{-1} \int_0^t s^2 \| \nabla \xi \|_0^2 ds & \leq C \epsilon^2.
\end{aligned}
\end{equation}

Taking $\psi = t^2 \xi_t$ in (41), using (11), (14) and Schwarz inequality, we have
\begin{align*}
t^2 \| \xi_t \|_0^2 + \frac{\lambda^{-1}}{2} \frac{d}{dt} t^2 \| \nabla \xi \|_0^2 \\
&= \lambda^{-1} t^2 \| \nabla \xi \|_0^2 - t^2 \tilde{b}(u_e, \xi, \xi_t) - t^2 \tilde{b}(\eta + \rho, T, \xi_t) \\
&\leq \lambda^{-1} t^2 \| \nabla \xi \|_0^2 + \frac{t^2}{2} \| \xi \|_0^2 + C t^2 \| u_e \|_0^2 \| \nabla \xi \|_0^2 + C t^2 \| T \|_0^2 \| \nabla (\eta + \rho) \|_0^2.
\end{align*}

Integrating over $[0, t]$, using (44), (46), Lemma 2.4 and Lemma 4.1, we derive
\begin{align*}
\lambda^{-1} t^2 \| \nabla \xi \|_0^2 + \int_0^t s^2 \| \xi_t \|_0^2 ds & \leq C \epsilon^2.
\end{align*}

From (40), there holds
\begin{align*}
\nabla \delta = -\nabla t + \nu \Delta t - \tilde{B}(u_e, \eta + \rho) - \tilde{B}(\eta + \rho, \xi) + \lambda f - \xi_t.
\end{align*}

Therefore by using previous estimates on the above equation, we derive
\begin{align*}
\int_0^{t_M} s^2 \| \delta \|_0^2 ds & \leq C \epsilon^2.
\end{align*}

By combining Lemma 4.1 with Lemma 4.2, we obtain the following error estimate result.

**Theorem 4.3.** Suppose (A1) and (A2) are valid, $\epsilon$ sufficiently small, for all $t \in [0, t_M]$, the following error estimates holds.
\begin{align*}
t || u(t) - u_e(t) ||_0^2 + t^2 || \nabla (u(t) - u_e(t)) ||_0^2 + t || T(t) - T_e(t) ||_0^2 \\
+ t^2 || \nabla (T(t) - T_e(t)) ||_0^2 + \int_0^t s^2 || p(t) - p_e(t) ||_0^2 ds & \leq C \epsilon^2.
\end{align*}
5. Time discretizations of the penalized system

In this section, we will analyze the backward Euler time discretization scheme for the nonlinear penalty system. Let $0 < k < 1$ be the time-step size and $t_n = nk$.

**Lemma 5.1.** In addition to (A1) and (A2), we assume $u_0 \in H^2(\Omega)^2$, then the solution $(u_{\varepsilon}, T_{\varepsilon})$ of Problem (II) satisfies

\begin{align}
(47) \quad u_{\varepsilon} &\in L^2([0, T_M]; X), \quad A_{\varepsilon} \frac{1}{2} u_{\varepsilon t t} + \nabla T_{\varepsilon t t} \in L^2([0, T_M]; L^2(\Omega)^2), \\
(48) \quad T_{\varepsilon} &\in L^2([0, T_M]; W), \quad A_{\varepsilon} \frac{1}{2} T_{\varepsilon t t} + \nabla T_{\varepsilon t t} \in L^2([0, T_M]; L^2(\Omega)).
\end{align}

**Proof.** By using a similar argument which used by He in [7], we known that $\|u_{\varepsilon t}(0)\|_0$ and $\|T_{\varepsilon t}(0)\|_0$ is bounded. We take the partial derivative with respect to $t$ of problem (IV) to obtain

\begin{align}
(49) \quad \begin{cases}
(u_{\varepsilon t}, v) + a_{\varepsilon}(u_{\varepsilon t}, v) + b(u_{\varepsilon t}, u_{\varepsilon t}, v) = \lambda(jT_{\varepsilon t}, v), \\
(T_{\varepsilon t}, \psi) + \bar{a}(T_{\varepsilon t}, \psi) + b(u_{\varepsilon t}, T_{\varepsilon t}, \psi) + \bar{b}(u_{\varepsilon t}, T_{\varepsilon t}, \psi) = 0,
\end{cases}
\end{align}

Now, taking $v = u_{\varepsilon t}$ in (49), using Lemma 2.1, (14) and Schwarz inequality, we obtain

\begin{align}
\frac{1}{2} \frac{d}{dt}\|u_{\varepsilon t}\|_0^2 + \nu\|A_{\varepsilon} \frac{1}{2} u_{\varepsilon t}\|_0^2 = \lambda(jT_{\varepsilon t}, u_{\varepsilon t}) - b(u_{\varepsilon t}, u_{\varepsilon t}, u_{\varepsilon t}) \\
\leq \frac{\nu}{2}\|A_{\varepsilon} \frac{1}{2} u_{\varepsilon t}\|_0^2 + C\|T_{\varepsilon t}\|_1^2 + C\|u_{\varepsilon t}\|_0^2 \|u_{\varepsilon t}\|_0^2.
\end{align}

Taking $\psi = T_{\varepsilon t}$ in (49), using (14) and Schwarz inequality, we obtain

\begin{align}
\|T_{\varepsilon t}\|_0^2 + \frac{\lambda}{2} \frac{d}{dt}\|\nabla T_{\varepsilon t}\|_0^2 = -\bar{b}(u_{\varepsilon t}, T_{\varepsilon t}, T_{\varepsilon t}) \leq \frac{1}{2}\|T_{\varepsilon t}\|_0^2 + C\|u_{\varepsilon t}\|_0^2 \|\nabla T_{\varepsilon t}\|_0^2.
\end{align}

Integrating over $[0, t]$, using Lemma 2.4 to above two inequalities, we derive

\begin{align}
(50) \quad \|u_{\varepsilon t}\|_0^2 + \|T_{\varepsilon t}\|_0^2 + \int_0^t \|A_{\varepsilon} \frac{1}{2} u_{\varepsilon t}\|_0^2 ds + \int_0^t \|T_{\varepsilon t}\|_0^2 ds \leq C.
\end{align}

Taking $\psi = T_{\varepsilon t}$ in (49), using Lemma 2.4, (14) and Schwarz inequality, we obtain

\begin{align}
\frac{1}{2} \frac{d}{dt}\|T_{\varepsilon t}\|_0^2 + \lambda^{-1} \frac{d}{dt}\|A_{\varepsilon} \frac{1}{2} T_{\varepsilon t}\|_0^2 = -\bar{b}(u_{\varepsilon t}, T_{\varepsilon t}, T_{\varepsilon t}) \\
\leq \frac{1}{2}\|A_{\varepsilon} \frac{1}{2} u_{\varepsilon t}\|_0^2 + C\|T_{\varepsilon t}\|_0^2 \|T_{\varepsilon t}\|_0^2.
\end{align}

Integrating over $[0, t]$ and using Lemma 2.4, we derive

\begin{align}
(51) \quad \|T_{\varepsilon t}\|_0^2 + \int_0^t \|A_{\varepsilon} \frac{1}{2} T_{\varepsilon t}\|_0^2 ds \leq C.
\end{align}

From Problem (II), we known that

\begin{align}
\lambda^{-1} \Delta T_{\varepsilon} = T_{\varepsilon t} + \bar{B}(u_{\varepsilon t}, T_{\varepsilon t}).
\end{align}

Taking the inner products of the last relation with $\Delta T_{\varepsilon}$, using Theorem 2.3, and (51), we derive readily that

\begin{align}
\|T_{\varepsilon}\|_2 \leq C.
\end{align}

By using Lemma 2.1 and (50), we have $\int_0^t \|A_{\varepsilon} \frac{1}{2} u_{\varepsilon t}\|_0^2 ds \leq C$. Then, we get

\begin{align}
\|A_{\varepsilon} \frac{1}{2} \bar{B}(u_{\varepsilon t}, u_{\varepsilon t})\|_0 \leq \|\bar{B}(u_{\varepsilon t}, u_{\varepsilon t})\|_1 \leq \sup_{v \in X} \frac{\bar{b}(u_{\varepsilon t}, u_{\varepsilon t}, v)}{\|v\|_1} \leq C\|u_{\varepsilon t}\|_0 \|u_{\varepsilon t}\|_2.
\end{align}
The same is true for $\tilde{B}(u, u)$. Thus

$$A^{-\frac{\lambda}{2}} u_{\text{ext}} = A^{\frac{\lambda}{2}} \{ \lambda j T - A u_{\text{ext}} - \tilde{B}(u_{\text{ext}}, u) - \hat{B}(u_{\text{ext}}, u) \} \in L^2([0, t_M], L^2(\Omega))$$

$$A^{-\frac{\lambda}{2}} T_{\text{ext}} = A^{\frac{\lambda}{2}} \{ \lambda^{-1} A T_{\text{ext}} - \tilde{B}(u_{\text{ext}}, T) - \hat{B}(u_{\text{ext}}, T) \} \in L^2([0, t_M], L^2(\Omega)).$$

Taking $v = tu_{\text{ext}}$ in (49), thanks to Lemma 2.1, (11) and Schwarz inequality, we obtain

$$t \| u_{\text{ext}} \|^2_0 + \frac{\nu}{2} \frac{d}{dt} t \| A^{\frac{\lambda}{2}} u_{\text{ext}} \|^2_0$$

$$= \frac{\nu}{2} \| A^{\frac{\lambda}{2}} u_{\text{ext}} \|^2_0 + t \lambda (j T, u_{\text{ext}}) - \tilde{b}(u_{\text{ext}}, u, u_{\text{ext}}) - T \hat{b}(u_{\text{ext}}, u, u_{\text{ext}})$$

$$\leq \frac{\nu}{2} \| A^{\frac{\lambda}{2}} u_{\text{ext}} \|^2_0 + C t \| T \|^2_0 + \frac{t}{2} \| u_{\text{ext}} \|^2_0 + C \| u \|_2^2.\| A^{\frac{\lambda}{2}} u_{\text{ext}} \|^2_0.$$
Problem (IV) at $t = t_{n+1}$, we get

\[
\begin{aligned}
    (\frac{1}{2}(e^{n+1} - e^n, v) + a(u^{n+1}, v) + b(u^{n+1}, e^{n+1}, v) + b(e^{n+1}, u(t_{n+1}), v) \\
    = (R_{uz}^n, v) + \lambda(j\delta^{n+1}, v), \\
    (\frac{1}{2}(\delta^{n+1} - \delta^n, \psi) + \lambda(\delta^{n+1}, \psi) + b(u^{n+1}, \delta^{n+1}, \psi) + b(e^{n+1}, T(z(t_{n+1})), \psi) \\
    = (R_{T}^n, \psi),
\end{aligned}
\]

where

\[
\begin{aligned}
    R_{uz}^n &= u_{zt}(t_{n+1}) - \frac{1}{k}(u_z(t_{n+1}) - u_z(t_n)) = \frac{1}{k} \int_{t_n}^{t_{n+1}} (t - t_n) u_{ztt}(t) dt, \\
    R_{T}^n &= T_z(t_{n+1}) - \frac{1}{k}(T_z(t_{n+1}) - T_z(t_n)) = \frac{1}{k} \int_{t_n}^{t_{n+1}} (t - t_n) T_{ztt}(t) dt.
\end{aligned}
\]

Taking $(v, \psi) = 2k(e^{n+1}, \delta^{n+1})$ in (53), using (11), (14) and Schwarz inequality, we obtain

\[
\begin{aligned}
    ||e^{n+1}||_0^2 - ||e^n||_0^2 + ||e^{n+1} - e^n||_0^2 + 2\nu k ||A^\frac{1}{2} e^{n+1}||_0^2 \\
    = 2k(R_{uz}^n, e^{n+1}) - 2k\tilde{b}(e^{n+1}, u_z(t_{n+1}), e^{n+1}) + 2\lambda k(j\delta^{n+1}, e^{n+1}) \\
    \leq \nu k ||A^\frac{1}{2} e^{n+1}||_0^2 + Ck^2 \int_{t_n}^{t_{n+1}} ||A^{-\frac{1}{2}} u_{ztt}||_0^2 dt \\
    + Ck(||u_z(t_{n+1})||_2^2 + 1)||e^{n+1}||_0^2 + k||\delta^{n+1}||_0^2.
\end{aligned}
\]

and

\[
\begin{aligned}
    ||\delta^{n+1}||_0^2 - ||\delta^n||_0^2 + ||\delta^{n+1} - \delta^n||_0^2 + 2\lambda^{-1} k ||A^\frac{1}{2}\delta^{n+1}||_0^2 \\
    = 2k(R_{T}^n, \delta^{n+1}) - 2k\tilde{b}(e^{n+1}, T_z(t_{n+1}), \delta^{n+1}) \\
    \leq \lambda^{-1} k ||A^\frac{1}{2}\delta^{n+1}||_0^2 + Ck^2 \int_{t_n}^{t_{n+1}} ||A^{-\frac{1}{2}}T_{ztt}||_0^2 dt + Ck||T_z(t_{n+1})||_2^2 ||e^{n+1}||_0^2.
\end{aligned}
\]

Taking the summation of (56), (57) for $n$ from 0 to $m$, using Lemma 2.5 and Lemma 5.1, for all $0 \leq m \leq t_M/k - 1$, we have

\[
\begin{aligned}
    ||e^{m+1}||_0^2 + ||\delta^{m+1}||_0^2 + \nu k \sum_{n=0}^{m} ||A^\frac{1}{2} e^{n+1}||_0^2 + \lambda^{-1} k \sum_{n=0}^{m} ||A^\frac{1}{2}\delta^{n+1}||_0^2 \leq Ck^2.
\end{aligned}
\]

Thus

\[
\begin{aligned}
    ||\nabla u^{m+1}||_0 \leq C, \quad ||\nabla T^{m+1}||_0 \leq C, \quad \forall m \leq t_M/k - 1.
\end{aligned}
\]

Taking $v = 2kt_{n+1}e^{n+1}$ in (53), using Lemma 2.1, (11) and Schwarz inequality, we obtain

\[
\begin{aligned}
    t_{n+1}(||e^{n+1}||_0^2 - ||e^n||_0^2 + ||e^{n+1} - e^n||_0^2) + 2\nu kt_{n+1}||A^\frac{1}{2} e^{n+1}||_0^2 \\
    = 2kt_{n+1}(R_{uz}^n, e^{n+1}) + 2kt_{n+1}\lambda(j\delta^{n+1}, e^{n+1}) \\
    - 2kt_{n+1}\tilde{b}(e^{n+1}, u_z(t_{n+1}), e^{n+1}) \\
    \leq \nu kt_{n+1}||A^\frac{1}{2} e^{n+1}||_0^2 + Ck^2 \int_{t_n}^{t_{n+1}} ||u_{ztt}||_0^2 dt + kt_{n+1}||\delta^{n+1}||_0^2 \\
    + Ckt_{n+1}(||u_z(t_{n+1})||_2^2 + 1)||e^{n+1}||_0^2.
\end{aligned}
\]
Taking \( \psi = 2kt_{n+1}\delta^{n+1} \) in (53), using (14) and Schwarz inequality, we get

\[
t_{n+1}(\|\delta^{n+1}\|_0^2 - \|u^n\|_0^2) + \|\delta^{n+1} - \Delta t \delta^n\|_0^2 + 2\lambda^{-1}kt_{n+1}\|A^\frac{1}{2}\delta^{n+1}\|_0^2 \\
= 2kt_{n+1}(R_{\epsilon T}^n, \delta^{n+1}) - 2kt_{n+1}\hat{b}(e^{n+1}, T_e(t_{n+1}), \delta^{n+1})
\]

\[
\leq \lambda^{-1}kt_{n+1}\|A^\frac{1}{2}\delta^{n+1}\|_0^2 + Ck^2 \int_{t_n}^{t_{n+1}} t\|T_{zt}\|_0^2 dt
\]

(61)

We known that

\[
t_{n+1}(\|\delta^{n+1}\|_0^2 - \|u^n\|_0^2) = t_{n+1}\|v^{n+1}\|_0^2 - t_n\|v^n\|_0^2 - k\|v^n\|_0^2.
\]

Taking the summation of (60), (61) for \( n \) from 0 to \( m \), using (58), (62), Lemma 2.5 and Lemma 5.1, for all \( 0 \leq m \leq T_M/k - 1 \), we derive

\[
t_{m+1}(\|\epsilon^{n+1}\|_0^2 + \|\delta^{m+1}\|_0^2) + k\sum_{n=0}^{m} t_{n+1}(\nu\|A^\frac{1}{2}\epsilon^{n+1}\|_0^2 + \lambda^{-1}\|A^\frac{1}{2}\delta^{n+1}\|_0^2) \leq Ck^2.
\]

Taking \( v = 2kt_{n+1}\epsilon^{n+1} \) in (53), using (10), (11), (58), (59), Young inequality and Lemma 2.1, we obtain

\[
t_{n+1}(\|A^\frac{1}{2}\epsilon^{n+1}\|_0^2 - \|A^\frac{1}{2}\epsilon^n\|_0^2 + \|A^\frac{1}{2}(\epsilon^{n+1} - \epsilon^n)\|_0^2) + 2\nu kt_{n+1}\|A\epsilon^{n+1}\|_0^2
\]

\[
= 2kt_{n+1}(R_{\epsilon T}^n, A\epsilon^{n+1}) - 2kt_{n+1}\hat{b}(u^{n+1}, e^{n+1}, A\epsilon^{n+1})
\]

\[
- 2kt_{n+1}\hat{b}(e^{n+1}, u_c(t_{n+1}), A\epsilon^{n+1}) + 2kt_{n+1}\lambda(j\delta^{n+1}, A\epsilon^{n+1})
\]

\[
\leq \nu kt_{n+1}\|A\epsilon^{n+1}\|_0^2 + Ck^2 \int_{t_n}^{t_{n+1}} t\|\epsilon_{zt}\|_0^2 dt + Ckt_{n+1}\|\delta^{n+1}\|_0^2
\]

(63)

Then, taking \( \psi = 2kt_{n+1}A\epsilon^{n+1} \) in (53), using (13), (14), (59), Young inequality and Lemma 2.1, we obtain

\[
t_{n+1}(\|A^\frac{1}{2}\delta^{n+1}\|_0^2 - \|A^\frac{1}{2}\delta^n\|_0^2 + \|A^\frac{1}{2}(\delta^{n+1} - \delta^n)\|_0^2) + 2\lambda^{-1}kt_{n+1}\|A\delta^{n+1}\|_0^2
\]

\[
= 2kt_{n+1}(R_{\epsilon T}^n, A\delta^{n+1}) - 2kt_{n+1}\hat{b}(u^{n+1}, \delta^{n+1}, A\delta^{n+1})
\]

\[
- 2kt_{n+1}\hat{b}(e^{n+1}, T_e(t_{n+1}), A\delta^{n+1})
\]

\[
\leq \lambda^{-1}kt_{n+1}\|A\delta^{n+1}\|_0^2 + Ck^2 \int_{t_n}^{t_{n+1}} t\|\delta_{zt}\|_0^2 dt
\]

(64)

\[
+ Ckt_{n+1}\|T_e(t_{n+1})\|_0^2 \|A^\frac{1}{2}\epsilon^{n+1}\|_0^2 + Ckt_{n+1}\|A^\frac{1}{2}\delta^{n+1}\|_0^2.
\]

We known that

\[
t_{n+1}(\|\epsilon^{n+1}\|_0^2 - \|\epsilon^n\|_0^2) = t_{n+1}\|v^{n+1}\|_0^2 - t_n\|v^n\|_0^2 - (2kt_n + k^2)\|v^n\|_0^2.
\]

Taking the summation of (63), (64) for \( n \) from 0 to \( m \), using (65), Lemma 2.5 and Lemma 5.1, for all \( 0 \leq m \leq T_M/k - 1 \), we derive

\[
t_{m+1}(\|A^\frac{1}{2}\epsilon^{m+1}\|_0^2 + \|A^\frac{1}{2}\delta^{m+1}\|_0^2) + k\sum_{n=0}^{m} t_{n+1}(\nu\|A\epsilon^{n+1}\|_0^2 + \lambda^{-1}\|A\delta^{n+1}\|_0^2) \leq Ck^2.
\]

**Lemma 5.3.** Under the assumptions of (A1) and (A2), then, for any \( 0 \leq m \leq T_M/k \), it is valid that

\[
k\sum_{n=0}^{m} t_{n}^2\|p_T(t_n) - p^n\|_0^2 \leq Ck^2.
\]
Proof. Taking \( v = t_{n+1}^2 (e^{n+1} - e^n) \) in (53), using Lemma 2.1, (11) and Schwarz inequality, we obtain

\[
kt_{n+1}^2 \frac{\|e^{n+1} - e^n\|_k}{k} + v t_{n+1}^2 \{ \|A^\frac{1}{2} e^{n+1}\|_0^2 - \|A^\frac{1}{2} e^n\|_0^2 + \|A^\frac{1}{2} (e^{n+1} - e^n)\|_0^2 \} = t_{n+1}^2 (R_{e^n, e^{n+1}} - e^n) - t_{n+1}^2 \tilde{b}(u^{n+1}, e^{n+1}, e^{n+1} - e^n)
\]

\[
+ t_{n+1}^2 \tilde{b}(e^{n+1}, u_{\epsilon}(t_{n+1}), e^{n+1} - e^n) + \lambda t_{n+1}^2 (\eta^{n+1}, e^{n+1} - e^n)
\]

\[
\leq \frac{kt_{n+1}^2}{2} \frac{\|e^{n+1} - e^n\|_0^2}{k} + Ckt_{n+1}^2 \int_{t_n}^{t_{n+1}} t\|u_{\epsilon t}\|_0^2 dt + Ckt_{n+1}^2 \|A^\frac{1}{2} \delta^{n+1}\|_0^2
\]

(66)

Taking the summation of (66) for \( n \) from 0 to \( m \), using Lemma 2.5, (59) and Lemma 5.2, we derive

\[
k \sum_{n=1}^{m} t_{n+1}^2 \frac{\|e^{n+1} - e^n\|_0^2}{k} \leq Ck^2, \quad \forall m \leq t_M/k - 1.
\]

Then using the equation (53) and the available estimates for \( e^n \) and \( \delta^n \), we can prove

\[
k \sum_{n=1}^{m} t_{n+1}^2 \|p_{\epsilon}(t_n) - p^n\|_0^2 \leq Ck^2, \quad \forall m \leq t_M/k.
\]

Finally, by combining Theorem 4.3 with Lemma 5.2, Lemma 5.3, we obtain the following theorem.

Theorem 5.4. Under the assumptions of (A1) and (A2), \( \varepsilon \) sufficiently small, then, for any \( 0 \leq m \leq t_M/k \), it is valid that

\[
t_m(\|u(t_m) - u^m\|_0^2 + \|T(t_m) - T^m\|_0^2) + t_m^2 (\|\nabla (u(t_m) - u^m)\|_0^2 + \|\nabla (T(t_m) - T^m)\|_0^2)
\]

\[
+ k \sum_{n=1}^{m} t_{n+1}^2 \|p(t_n) - p^n\|_0^2 \leq C(k^2 + \varepsilon^2).
\]

6. Conclusions

In this paper, we studied the penalty method for the two-dimensional unsteady conduction-convection problem under some realistically assumptions. By using the penalty method, we overcome the coupled problem and can efficiently separate the computation of the velocity from that of the pressure. Optimal error estimate of the numerical velocity, pressure and temperature for the penalized system and the backward Euler scheme are derived, we will extend the present analysis to a fully discrete scheme by combining it with the finite element approximation results in our future work.

References


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