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FINITE ELEMENT APPROXIMATIONS OF OPTIMAL CONTROLS FOR THE HEAT EQUATION WITH END-POINT STATE CONSTRAINTS

GENGSHENG WANG AND LIJUAN WANG

Abstract. This study presents a new finite element approximation for an optimal control problem (P) governed by the heat equation and with end-point state constraints. The state constraint set S is assumed to have an empty interior in the state space. We begin with building a new penalty functional where the penalty parameter is an algebraic combination of the mesh size and the time step. Based on it, we establish a discrete optimal control problem $(P_{h\tau})$ without state constraints. With the help of Pontryagin's maximum principle and by suitably choosing the above-mentioned combination, we successfully derive error estimate between optimal controls of problems (P) and $(P_{h\tau})$, in terms of the mesh size and time step.

Key words. Error estimate, optimal control problem, the heat equation, end-point state constraint, discrete.

1. Introduction

Let Ω be a bounded convex domain (with a smooth boundary $\partial\Omega$) in $\mathbb{R}^d$, $d = 1, 2, 3$. Let ω be an open subset of Ω and T be a positive number. We write Q for the product set $\Omega \times (0, T)$ and χ_ω for the characteristic function of the subset ω. Let $\langle \cdot, \cdot \rangle$ denote the inner product of the space $L^2(\Omega)$. Consider the following optimal control problem:

$$(P) \qquad \mathrm{Min} J(y, u)$$

over all such pairs $(y, u) \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$ that

$$\begin{cases} \partial_t y - \triangle y = \chi_\omega u & \text{in} \quad \Omega \times (0, T), \\ y = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ y(0) = y_0 & \text{in} \quad \Omega \end{cases} \tag{1.1}$$

and

$$y(T) \in S.$$

Here, the initial data y_0 is a given function in $H_0^1(\Omega) \cap H^2(\Omega)$, the cost functional J is defined by

$$J(y, u) = \frac{1}{2} \int_0^T \int_\Omega (y - y_d)^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_\Omega u^2 \, dx \, dt,$$

the reference function y_d is taken from the space $H^1(0, T; L^2(\Omega))$, and the constraint set S satisfies the following conditions:

$(A1)$ $S \subset H_1^\perp$ is a convex and closed subset with a nonempty interior in $H_1^\perp$. Here, $H_1^\perp$ denotes the orthogonal subspace of H_1 in $L^2(\Omega)$, while H_1 is a subspace

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spanned by $f_1, f_2, \cdots, f_{n_0}$ with f_i, $i = 1, 2, \cdots, n_0$, being functions in the space $H_0^1(\Omega)$ and n_0 being a positive integer.

$(A2)$ The boundary of S, denoted by ∂S, is a C^1-manifold with one codimension in $H_1^{\perp}$. Furthermore, $\partial S = \{y \in H_1^{\perp} : F(y) = 0\}$, where $F \in C^1(H_1^{\perp})$ holds the property that $F'(\xi) \in H_0^1(\Omega)$ whenever $\xi \in H_0^1(\Omega) \cap H_1^{\perp}$.

The purpose of this paper is to build a discrete approximating optimal control problem $(P_{h\tau})$ (where h and τ are the mesh size and time step, respectively), and then present an error estimate between optimal controls for those two problems. The main steps to reach the goals are as follows: We first set up a new penalty functional, where the penalty parameter is a suitable algebraic combination of the mesh size and the time step, then establish, with the aid of the penalty functional, a discrete approximating optimal control problem $(P_{h\tau})$ without state constraint, and finally, derive, with the help of the Pontryagin's maximum principle, an error estimate of optimal controls for those two problems. The main result of the paper can be approximately stated as: *the order of the L^2-error between optimal controls of the problems (P) and $(P_{h\tau})$ is $h^{\frac{1}{2}}$ whenever $\tau \approx O(h^2)$.*

In general, for parabolic equations, the study of optimal control problems with state constraints is much more difficult than the study of those without state constraints. This can be seen from the following points of view: (1) It is harder to show the existence of optimal controls for the problems with state constraints than those without state constraints. It may happen that a problem without state constraints has optimal controls while the same problem with a state constraint has no solution. (2) Some optimal control problems without state constraints hold the Pontryagin maximum principle, while the same problems with some state constraints do not have the Pontryagin maximum principle (see [5]). Therefore, to guarantee the problem (P) having optimal controls and holding the Pontryagin maximum principle, it is necessary to impose some conditions on S. It will be proved that when S satisfies the above-mentioned conditions $(A1)$ and $(A2)$, the problem (P) has a unique optimal control and holds the Pontryagin maximum principle. These two conditions are quite close to the finite codimensionality condition provided in [5].

The end-point state constraint is a very important kind of state constraints in the field of optimal controls for parabolic equations. To our surprise, the studies on error estimates for numerical approximations to optimal control problems for parabolic differential equations with end-point state constraint are very limited. Here we quote two related papers [11] and [12]. In [11], the authors studied numerical approximations of optimal controls for linear parabolic equations. The state constraint set in that paper was assumed to have interior points in the state space. In [12], the authors studied such a problem where the constraint set is a non-degenerate closed unit ball centered at the origin of the state space. An error estimate was established in [12]. Moreover, that estimate is better than what we have in this paper. However, the problem studied in the current paper properly covers the case in [12]. This will be seen from the following example:

Write $\{e_k\}_{k=1}^{\infty} \subset H_0^1(\Omega)$ for an orthonormal basis of $L^2(\Omega)$. Set $H_1^{\perp} = \mathrm{span}\{e_{n_0+1}, e_{n_0+2}, \cdots\}$, where n_0 is a positive integer. Let $S \equiv \{y \in H_1^{\perp} : \|y\|_{L^2(\Omega)} \leq 1\}$. It is easy to check that S satisfies $(A1)$. Moreover, if we define $F : H_1^{\perp} \to (-\infty, +\infty)$ by $F(y) = \|y\|_{L^2(\Omega)}^2 - 1, \forall\, y \in H_1^{\perp}$, then $\partial S = \{y \in H_1^{\perp} : \|y\|_{L^2(\Omega)} = 1\} = \{y \in H_1^{\perp} : F(y) = 0\}$ and $F'(y) = 2y$, which imply that S satisfies $(A2)$.

Obviously, the above-mentioned S is a degenerate closed unit ball centered at the origin of the state space. Therefore, the framework of this paper properly covers the cases studied in [12]. Moreover, the above-mentioned example on the state constraint fits in our setting but not in one of [11]. The essential difference between S and the state constraint sets in [11] and [12] is that S can have no interior point while the constraint sets in those papers have interior points. From both perspective of infinite dimensional optimal control theory and numerical approximations for optimal controls, the case where the state constraint set has no interior points is much more complicated than the case that the state constraint set has an interior point.

Because of the constraint set S, the discrete problem cannot be constructed by directly projecting the problem (P) via the classical space-time discretization scheme (The authors of [11] and [12] did in this way). The reason is that if we did it in such a way, then the constraint set S would be projected into the set $S \cap V_h$, where V_h is a finite element space with the mesh size h. Thus, we cannot guarantee the existence of admissible controls for this discrete problem (and it is very hard to prove otherwise). As a result, we are not able to guarantee the existence of solutions for the above-mentioned discrete problem. To overcome this difficulty, we create a penalty functional where the penalized parameter is chosen to be the combination $2(h + \tau^{\frac{1}{2}})$ of the mesh size h and the time step τ. This penalty functional leads us to a right way to set up our discrete problem $(P_{h\tau})$, which is an optimal control problem without state constraints. Furthermore, the problem $(P_{h\tau})$ has a unique optimal control.

When we apply Pontryagin's maximum principle to study error estimates between optimal controls to the problems $(P_{h\tau})$ and (P), another barrier appears. Namely, the multipliers, which are the initial data of the adjoint state equations in Pontryagin's maximum principle of problems (P) and $(P_{h\tau})$, respectively, lack quantitative information. Therefore, one cannot expect to estimate the difference between optimal controls for problems (P) and $(P_{h\tau})$ by directly estimating the errors between solutions for the state equations or adjoint state equations. Fortunately, in the paper [12], the authors observed that this barrier can be passed if the multipliers belong to the space $H_0^1(\Omega)$. Because of the specific construction of our discrete problem $(P_{h\tau})$, the multiplier for the problem $(P_{h\tau})$ stays in this space. Thus, we only need to have the $H_0^1(\Omega)-$regularity for the multiplier, denoted by $-\mu^*$, corresponding to the problem (P). In general, this is not the case. However, we can prove that it is true whenever the set S holds the above-mentioned properties $(A1)$ and $(A2)$.

Next, we would like to explain that the assumption $(A1)$ and the assumption $(A2)$ on the constraint set S are fairly reasonable from the perspectives of Pontryagin's maximum principle and the numerical approximation to the problem (P). On one hand, because of the difference between finite and infinite dimensional spaces, for optimal control problems of the infinite dimensional spaces and with end-point state constraint, Pontryagin's maximum principle doesn't necessarily hold for only closed and convex constraint set ([5]). It is known ([5]) that if the constraint set S is finite codimensional when it is convex and closed in the state space, Pontryagin's maximum principle holds for problem (P). *The most important characteristic of the sets of finite codimension in* $L^2(\Omega)$ *can be roughly stated as: if* S *is a set of finite codimension in* $L^2(\Omega)$*, then it may have an empty interior in* $L^2(\Omega)$*, but inevitably have a non-empty interior in a finite codimensional subspace of* $L^2(\Omega)$. There are indeed other conditions on the constraint set S, under which, Pontryagin's maximum principle of the corresponding optimal control problem holds ([2]). However, in our specific case, these conditions are related to the attainable set of the internally controlled heat equation. And we have quite limited knowledge about this sophisticated attainable set. Hence, if we only assume that the constraint set S satisfies the above-mentioned conditions, then it would be very hard to study the numerical approximation to the corresponding optimal control problems by applying Pontryagin's maximum principle. Thus, it appears that in order to apply Pontryagin's maximum principle to get our error estimates, we should assume that the constraint set is of finite codimension when it is a convex and closed subset. On the other hand, the following argument may not be correct: *if the constraint set S is of finite codimension in $L^2(\Omega)$ when it is convex and closed, then the above-mentioned multiplier $-\mu^*$ is in the space $H_0^1(\Omega)$* (and it is very hard to prove otherwise). From this point of view, it is quite reasonable to study such constraint sets that have realistically stronger properties than the finite codimension. Clearly, when the set S satisfies the property $(A1)$ and the property $(A2)$, it is a set of finite codimension in $L^2(\Omega)$. Moreover, we can prove that the corresponding multiplier $-\mu^*$ has $H_0^1(\Omega)$-regularity (Proposition 2.5) . To conclude, the assumption $(A1)$ and the assumption $(A2)$ are rational assumptions for our study.

The rest of this paper is organized as the following: In section 2, we first prove that the problem (P) has a unique optimal control. Then we state the Pontryagin's maximum principle for the problem (P). Finally, we show the regularity of the above-mentioned multiplier $-\mu^*$. In section 3, we introduce some notations and existing results that will be used in the rest of the paper. In section 4, we first set up a discrete problem $(P_{h\tau})$ for the problem (P), and then show that the problem $(P_{h\tau})$ has a unique solution. Finally, we establish Pontryagin's maximum principle for the problem $(P_{h\tau})$. Section 5 presents the main result of this paper, namely, an error estimate between optimal controls to the problems (P) and $(P_{h\tau})$.

2. Some properties of the optimal control for the problem (P)

First of all, we derive the existence and uniqueness of the optimal control for the problem (P). The proof is based on the following existing result.

Lemma 2.1. *([9]) Let E be a subspace of $L^2(\Omega)$ of finite dimension and Π_E be the orthogonal projection over E. Given z_0 and z_1 in $L^2(\Omega)$ and $\varepsilon > 0$, then there exists a control $f \in L^2(0,T;L^2(\Omega))$ such that the solution of*

$$\begin{cases} \partial_t z - \triangle z = \chi_\omega f & \text{in} \quad \Omega \times (0,T), \\ z = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ z(0) = z_0(x) & \text{in} \quad \Omega \end{cases}$$

satisfies simultaneously $\Pi_E(z(T)) = \Pi_E(z_1)$ and $\|z(T) - z_1\|_{L^2(\Omega)} \leq \varepsilon$.

Theorem 2.2. *If the constraint set S has the finite codimension in $L^2(\Omega)$ when it is closed and convex, then the problem (P) has a unique optimal control.*

Proof. Let $\mathcal{U}_{ad} \triangleq \{u \in L^2(0,T;L^2(\Omega)) \ : \ y(u)(T) \in S\}$, where $y(u)(\cdot)$ denotes the solution of the equation (1.1) corresponding to the control u. Each u in $\mathcal{U}_{ad}$ is called an admissible control for the problem (P). Two observations are given in order. First, if we can show that $\mathcal{U}_{ad} \neq \emptyset$, namely, the problem (P) has admissible controls, then the existence of optimal controls for the problem (P) follows from a standard argument. Second, since the functional $\widetilde{J} : \mathcal{U}_{ad} \to R^+$, defined by $\widetilde{J}(u) = J(y(u), u)$, is strictly convex, the optimal control for the problem (P), if exists, is unique. Hence, it suffices to show the existence of admissible controls for the problem (P). We argue it as follows.

According to the definition of the finite codimension (see page 134 in [5]), there exists an element s_0 in the set S such that the space

$$\mathrm{span}\{S-s_0\} \triangleq \text{the closed subspace spanned by } \{s-s_0 :\ s \in S\}$$

is finite codimensional in $L^2(\Omega)$ and the set $\{S-s_0\}$ has a nonempty interior in this subspace. Hence, on one hand, there exist linearly independent vectors $\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_{m_0}$ in $L^2(\Omega)$ such that

$$\mathrm{span}\{S-s_0\} \oplus \mathrm{span}\{\tilde{z}_1, \cdots, \tilde{z}_{m_0}\} = L^2(\Omega), \tag{2.1}$$

while on the other hand, the space $L^2(\Omega)$ contains a closed ball $B(s^*-s_0, \varepsilon_0)$, centered at (s^*-s_0) with $s^* \in S$ and of radius $\varepsilon_0 > 0$, such that

$$B(s^*-s_0, \varepsilon_0) \cap \mathrm{span}\{S-s_0\} \subset S-s_0. \tag{2.2}$$

Now we write

$$\tilde{z}_i = \tilde{z}_{i1} + \tilde{z}_{i2},\ i = 1, 2, \cdots, m_0, \tag{2.3}$$

where $\tilde{z}_{i1} \in \mathrm{span}\{S-s_0\}$ and $\tilde{z}_{i2} \in (\mathrm{span}\{S-s_0\})^\perp$, $i = 1, 2, \cdots, m_0$. It follows from (2.1) and (2.3) that

$$\mathrm{span}\{S-s_0\} \oplus \mathrm{span}\{\tilde{z}_{12}, \cdots, \tilde{z}_{m_0,2}\} = L^2(\Omega)$$

and

$$(\mathrm{span}\{S-s_0\})^\perp = \mathrm{span}\{\tilde{z}_{12}, \cdots, \tilde{z}_{m_0,2}\}.$$

An application of Lemma 2.1 to the case, where $E = (\mathrm{span}\{S-s_0\})^\perp$, $z_0 = y_0$ and $z_1 = s^*$, gives the existence of such a control $u \in L^2(0, T; L^2(\Omega))$ that the corresponding solution $y(u)(\cdot)$ to the equation (1.1) holds the following properties:

$$\Pi_E(y(u)(T)) = \Pi_E(s^*) \text{ and } \|y(u)(T) - s^*\|_{L^2(\Omega)} \leq \frac{\varepsilon_0}{2}.$$

These imply that

$$\Pi_E(y(u)(T) - s_0) = \Pi_E(s^*-s_0) = 0 \text{ and } \|(y(u)(T)-s_0) - (s^*-s_0)\|_{L^2(\Omega)} \leq \frac{\varepsilon_0}{2}.$$

Therefore, we find that

$$y(u)(T)-s_0 \in \mathrm{span}\{S-s_0\} \text{ and } y(u)(T) - s_0 \in B(s^*-s_0, \varepsilon_0),$$

which, together with (2.2), yield that $y(u)(T) \in S$. This completes the proof. $\square$

Next, we state the Pontryagin's maximum principle for the problem (P) which is indeed a necessary and sufficient condition for the optimal control in this case, and will be frequently used in the rest of the paper. It can be proved by standard methods. For the sake of completeness, we will give its proof in Appendix of this paper.

Theorem 2.3. *Let $S \subset L^2(\Omega)$ be a closed and convex subset of finite codimension. Then u^* and $y(u^*)$ are the optimal control and the corresponding optimal state for the problem (P), respectively, if and only if there exist a function μ^* in $L^2(\Omega)$ and a function p^* in $H^1(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ such that the following properties hold:*

$$y(u^*)(T) \in S,\ \langle \mu^*, s - y(u^*)(T)\rangle \leq 0,\ \forall\ s \in S, \tag{2.4}$$

$$\begin{cases} \partial_t y(u^*) - \triangle y(u^*) = \chi_\omega u^* & in \quad \Omega \times (0, T), \\ y(u^*) = 0 & on \quad \partial\Omega \times (0, T), \\ y(u^*)(0) = y_0(x) & in \quad \Omega, \end{cases} \tag{2.5}$$

$$
\begin{cases} \partial_t p^* + \triangle p^* = y(u^*) - y_d & in \quad \Omega \times (0,T), \\ p^* = 0 & on \quad \partial\Omega \times (0,T), \\ p^*(T) = -\mu^* & in \quad \Omega \end{cases} \tag{2.6}
$$

and

$$
u^* = \chi_\omega p^* \ \ in\ \Omega \times (0,T). \tag{2.7}
$$

Remark 2.4. *When a constraint set S satisfies the conditions $(A1)$ and $(A2)$, it has nonempty interior in $H_1^{\perp}$, by Definition 1.5 of Chapter 4 in* [5]*, one can check that S is of finite codimension in $L^2(\Omega)$. Furthermore, it is a convex and closed subset. Hence, Theorem 2.2 and Theorem 2.3 hold for the optimal control problem (P) with such constraint sets that have the properties $(A1)$ and $(A2)$.*

Now, we are going to study the regularities of the multiplier $-\mu^*$, the adjoint state p^* and the optimal control u^* in Theorem 2.3, which are very important in the investigation of our error estimate.

Proposition 2.5. *Let S satisfy the properties $(A1)$ and $(A2)$. Then, it holds that $\mu^* \in H_0^1(\Omega)$, $p^* \in L^2(0,T; H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$ and $u^* \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\omega))$.*

Proof. Since the desired regularities for p^* and u^* are direct consequences of the $H_0^1(\Omega)-$regularity of μ^*, together with (2.6) and (2.7), respectively, it suffices to prove that $\mu^* \in H_0^1(\Omega)$.

For this purpose, we write

$$
\mu^* = \mu_1^* + \mu_2^*, \ \ \text{where } \mu_1^* \in H_1 \subset H_0^1(\Omega) \ \text{ and } \ \mu_2^* \in H_1^{\perp}. \tag{2.8}
$$

By $(A1)$ and (2.4), we get

$$
\langle \mu_2^*, s - y(u^*)(T) \rangle \leq 0, \quad \forall\, s \in S. \tag{2.9}
$$

Since $y(u^*)(T) \in S$, there are only two alternatives: either $y(u^*)(T) \in intS$ or $y(u^*)(T) \in \partial S$. In the first case, it follows at once from $(A2)$ and (2.9) that $\mu_2^* = 0$, which, together with (2.8), gives

$$
\mu^* = \mu_1^* \in H_0^1(\Omega). \tag{2.10}
$$

In the second case, we first assert that

$$
\mu_2^* \perp T_{y(u^*)(T)} \partial S \ \text{ and } \ F'(y(u^*)(T)) \perp T_{y(u^*)(T)} \partial S. \tag{2.11}
$$

Here $T_{y(u^*)(T)}\partial S$ denotes the tangent space of the manifold ∂S at the point $y(u^*)(T)$. The argument is as follows: Given a vector $v \in T_{y(u^*)(T)}\partial S$, we can find a C^1-para metrized curve on the manifold ∂S given by $\alpha(t) : [-1,1] \to \partial S$, such that

$$
\alpha(0) = y(u^*)(T) \ \text{ and } \ \alpha'(0) = v. \tag{2.12}
$$

On one hand, by (2.9) and (2.12), we get

$$
\langle \mu_2^*, \alpha(t) - \alpha(0) \rangle \leq 0, \ \ \forall\, t \in [-1,1],
$$

which implies that

$$
\langle \mu_2^*, t^{-1}(\alpha(t) - \alpha(0)) \rangle \leq 0, \ \ \forall\, t \in (0,1] \tag{2.13}
$$

and

$$
\langle \mu_2^*, t^{-1}(\alpha(t) - \alpha(0)) \rangle \geq 0, \ \ \forall\, t \in [-1,0). \tag{2.14}
$$

Passing to the limits for $t \to 0^+$ in (2.13) and for $t \to 0^-$ in (2.14), respectively, we derive that

$$
\langle \mu_2^*, \alpha'(0) \rangle = 0.
$$

This, together with (2.12), shows that $\langle \mu_2^*, v\rangle = 0$, for all v in the tangent space $T_{y(u^*)(T)}\partial S$. Hence, the first property in (2.11) holds. On the other hand, it follows from $(A2)$ that

$$F(\alpha(t)) \equiv 0, \ \ \forall\, t \in [-1, 1],$$

which leads to

$$\langle F'(\alpha(0)), \alpha'(0)\rangle = 0.$$

This, combined with (2.12), yields that $\langle F'(y(u^*)(T)), v\rangle = 0$ for all v in $T_{y(u^*)(T)}\partial S$, and the second property in (2.11) follows. Thus, we have proved the above-mentioned assertion. Next, we deduce from (2.11) and $(A2)$ that $\mu_2^* = kF'(y(u^*)(T))$, for some constant k, which, together with $(A2)$, gives $\mu_2^* \in H_0^1(\Omega)$. Then, we infer from (2.8) that $\mu^* \in H_0^1(\Omega)$ in the second case, and conclude that μ^* has the $H_0^1(\Omega)-$regularity. This completes the proof. □

3. Some notations, hypotheses and existing results on $(P_{h\tau})$

We begin with introducing some notations and certain existing results on finite element spaces, which will be used later. Associated with a positive parameter h, we consider a family $\mathcal{T}_h$ of triangulations in $\overline{\Omega}$. Let $\overline{\Omega}_h = \cup_{\mathcal{T}\in\mathcal{T}_h}\mathcal{T}$ be the polygonal approximation of $\overline{\Omega}$. Write Ω_h and $\partial\Omega_h$ for the interior and boundary of the set $\overline{\Omega}_h$, respectively. The vertices of $\mathcal{T}_h$, which are on the boundary $\partial\Omega_h$, belong to $\partial\Omega$. Corresponding to each element $\mathcal{T} \in \mathcal{T}_h$, we denote by $\rho(\mathcal{T})$ and $\sigma(\mathcal{T})$ the diameters of the set $\mathcal{T}$ and of the biggest ball included in $\mathcal{T}$, respectively. Let $h = \max_{\mathcal{T}\in\mathcal{T}_h}\rho(\mathcal{T})$. In the rest of this paper, the following hypotheses are effective:

(i) There exist two positive constants ρ and σ independent of h, such that

$$\frac{\rho(\mathcal{T})}{\sigma(\mathcal{T})} \le \sigma \ \ \textit{and} \ \ \frac{h}{\rho(\mathcal{T})} \le \rho, \ \ \textit{for each element } \mathcal{T} \in \mathcal{T}_h.$$

(ii) The subset ω is a polygon. Moreover, for any triangulation $\mathcal{T}_h$, there exists a subset $\tilde{\mathcal{T}}_h \subset \mathcal{T}_h$ such that $\omega = \cup_{\mathcal{T}\in\tilde{\mathcal{T}}_h}\mathcal{T}$.

We shall set up the discrete state space and control space to our problem in different manners. With regard to the state space, we define, corresponding to each triangulation $\mathcal{T}_h$, the following discrete space:

$$V_h = \{\varphi_h \in C(\overline{\Omega});\ \varphi_h|_{\mathcal{T}} \in P_1(\mathcal{T}), \text{ for every } \mathcal{T} \in \mathcal{T}_h, \text{ and } \varphi_h|_{\overline{\Omega}\setminus\Omega_h} = 0\},$$

where $P_1(\mathcal{T})$ is the space of all polynomials, defined on $\mathcal{T}$ and with the degree less than or equal to 1 on $\mathcal{T}$. It is obvious that $V_h \subset H_0^1(\Omega)$. Regarding the control space, we set

$$U_h = \{v \in L^2(\Omega) : v|_{\mathcal{T}} \text{ is a constant function for each } \mathcal{T} \in \mathcal{T}_h,\ v|_{\Omega\setminus\Omega_h} = 0\}.$$

Let Q_h be the L^2-projection from $L^2(\Omega)$ to V_h defined by

$$\langle Q_h\varphi, \varphi_h\rangle = \langle \varphi, \varphi_h\rangle,\ \forall\, \varphi \in L^2(\Omega),\ \varphi_h \in V_h. \tag{3.1}$$

The following well-known inequalities ([1] and [10]) will be frequently used in the rest of the paper:

$$\|\varphi_h\|_{H_0^1(\Omega)} \le Ch^{-1}\|\varphi_h\|_{L^2(\Omega)},\ \forall\, \varphi_h \in V_h, \tag{3.2}$$

$$\|Q_h\varphi\|_{L^2(\Omega)} \le \|\varphi\|_{L^2(\Omega)},\ \forall\, \varphi \in L^2(\Omega), \tag{3.3}$$

and

$$\begin{aligned}\|\varphi - Q_h\varphi\|_{L^2(\Omega)} + h\|\varphi - Q_h\varphi\|_{H_0^1(\Omega)} \\ \le \quad Ch^{m+1}\|\varphi\|_{H^{m+1}(\Omega)},\ \forall\, \varphi \in H^{m+1}(\Omega) \cap H_0^1(\Omega),\end{aligned} \tag{3.4}$$

for $m = 0, 1$. Here and throughout this section, C stands for several positive constants independent of h (and also τ), which may be different in different contexts.

Define a L^2-projection operator $\tilde{\Pi}_h$ from $L^2(\Omega)$ to U_h by

$$\langle \tilde{\Pi}_h v, v_h\rangle = \langle v, v_h\rangle,\ \forall\, v \in L^2(\Omega),\ v_h \in U_h. \tag{3.5}$$

Clearly, it follows that

$$\tilde{\Pi}_h v|_{\mathcal{T}} = \frac{1}{|\mathcal{T}|}\int_{\mathcal{T}} v\, dx,\ \forall\, v \in L^2(\Omega),\ \forall\, \mathcal{T} \in \mathcal{T}_h \tag{3.6}$$

and

$$\|\tilde{\Pi}_h v\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)},\ \forall\, v \in L^2(\Omega). \tag{3.7}$$

Moreover, we have (see page 164 in [3]),

$$\|\tilde{\Pi}_h v - v\|_{L^2(\mathcal{T})} \le C\rho(\mathcal{T})\|v\|_{H^1(\mathcal{T})} \le Ch\|v\|_{H^1(\mathcal{T})},\ \forall\, v \in H^1(\mathcal{T}),\ \forall\, \mathcal{T} \in \mathcal{T}_h. \tag{3.8}$$

Next, we turn to the time discretization. We divide the time interval $(0, T)$ into N equally-spaced subintervals by the nodal points:

$$0 = t_0 < t_1 < \cdots < t_N = T.$$

Here $t_i = i\tau$ with $i = 0, 1, \cdots, N$, and $\tau = \frac{T}{N}$. For a sequence of functions $\{Z^i\}_{i=0}^N$ given in the space $L^2(\Omega)$, we denote by $\partial_\tau Z^i$ the difference quotient $\frac{Z^i - Z^{i-1}}{\tau}$, where $i = 1, 2, \cdots, N$.

Now, we consider the semi-discrete equation

$$\left\{\begin{array}{ll} \langle \partial_t z_h, \varphi_h\rangle + \langle \nabla z_h, \nabla\varphi_h\rangle = \langle v, \varphi_h\rangle, & \forall\, \varphi_h \in V_h,\ t \in (0, T), \\ z_h(0) = z_{0h} & \text{in } \Omega \end{array}\right. \tag{3.9}$$

and the fully discrete equation

$$\left\{\begin{array}{ll} \langle \partial_\tau Z_h^i, \varphi_h\rangle + \langle \nabla Z_h^i, \nabla\varphi_h\rangle = \langle U^i, \varphi_h\rangle, & \forall\, \varphi_h \in V_h,\ 1 \le i \le N, \\ Z_h^0 = z_{0h} & \text{in } \Omega, \end{array}\right. \tag{3.10}$$

respectively. The following results are quoted from [12] and will be used later.

Lemma 3.1. *Let $z_{0h} \in V_h$ and $v \in L^2(Q)$. Then the equation (3.9) has a unique solution z_h in the space $H^1(0, T; V_h)$ with the following estimate:*

$$\|z_h\|^2_{C([0,T];H_0^1(\Omega))} + \|\partial_t z_h\|^2_{L^2(Q)} \le C(\|z_{0h}\|^2_{H_0^1(\Omega)} + \|v\|^2_{L^2(Q)}).$$

Lemma 3.2. *Let $z_{0h} \in V_h$ and $(U^1, U^2, \cdots, U^N) \in (L^2(\Omega))^N$. Then the equation (3.10) has a unique solution $Z_{h\tau} = (Z_h^1, Z_h^2, \cdots, Z_h^N) \in (V_h)^N$. Moreover, the following estimate holds:*

$$\max_{1\le i\le N}\|Z_h^i\|^2_{H_0^1(\Omega)} + \tau\sum_{i=1}^N \|\partial_\tau Z_h^i\|^2_{L^2(\Omega)} \le C\Big(\|z_{0h}\|^2_{H_0^1(\Omega)} + \tau\sum_{i=1}^N \|U^i\|^2_{L^2(\Omega)}\Big).$$

Lemma 3.3. *Let $z_0 \in H_0^1(\Omega)$ and $v \in L^2(Q)$. Write z and $z_h \in H^1(0, T; V_h)$ for the solutions to the equation*

$$\left\{\begin{array}{ll} \langle \partial_t z(t), \varphi\rangle + \langle \nabla z(t), \nabla\varphi\rangle = \langle v, \varphi\rangle, & \forall\, \varphi \in H_0^1(\Omega),\ t \in (0, T), \\ z(0) = z_0(x) & \textit{in } \Omega \end{array}\right.$$

and the equation

$$\begin{cases} \langle \partial_t z_h(t), \varphi_h \rangle + \langle \nabla z_h(t), \nabla \varphi_h \rangle = \langle v, \varphi_h \rangle, & \forall\ \varphi_h \in V_h,\ t \in (0,T), \\ z_h(0) = Q_h z_0(x) & \text{in } \Omega, \end{cases}$$

respectively. Then it holds that

$$\begin{aligned} & \|z - z_h\|_{L^2(Q)} + h(\|z - z_h\|_{C([0,T];L^2(\Omega))} + \|z - z_h\|_{L^2(0,T;H_0^1(\Omega))}) \\ \leq \quad & Ch^2(\|z_0\|_{H_0^1(\Omega)} + \|v\|_{L^2(Q)}). \end{aligned}$$

Lemma 3.4. *Let $v \in H^1(0,T;L^2(\Omega)), (U^1, U^2, \cdots, U^N) \in (L^2(\Omega))^N$ and $z_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. Write $z_h \in H^1(0,T;V_h)$ and $Z_{h\tau} = (Z_h^1, Z_h^2, \cdots, Z_h^N) \in (V_h)^N$ for the solutions to equations:*

$$\begin{cases} \langle \partial_t z_h(t), \varphi_h \rangle + \langle \nabla z_h(t), \nabla \varphi_h \rangle = \langle \chi_\omega v, \varphi_h \rangle, & \forall\ \varphi_h \in V_h,\ t \in (0,T), \\ z_h(0) = Q_h z_0(x) & \text{in } \Omega \end{cases}$$

and

$$\begin{cases} \langle \partial_\tau Z_h^i, \varphi_h \rangle + \langle \nabla Z_h^i, \nabla \varphi_h \rangle = \langle \chi_\omega U^i, \varphi_h \rangle, & \forall\ \varphi_h \in V_h,\ 1 \leq i \leq N, \\ Z_h^0 = Q_h z_0 & \text{in } \Omega, \end{cases}$$

respectively. Then it holds that

$$\begin{aligned} & \max_{1 \leq i \leq N} \|z_h(t_i) - Z_h^i\|_{L^2(\Omega)}^2 + \tau \sum_{i=1}^N \|z_h(t_i) - Z_h^i\|_{H_0^1(\Omega)}^2 \\ \leq \quad & C\Big(\tau \sum_{i=1}^N \Big\| \frac{1}{\tau} \int_{t_{i-1}}^{t_i} v\,dt - U^i \Big\|_{L^2(\Omega)}^2 + \tau^2 (\|v\|_{H^1(0,T;L^2(\Omega))}^2 + \|z_0\|_{H^2(\Omega)}^2)\Big) \end{aligned}$$

and

$$\max_{t \in [0,T]} \|\partial_t z_h\|_{L^2(\Omega)}^2 + \int_0^T \|\partial_t z_h\|_{H_0^1(\Omega)}^2\,dt \leq C(\|v\|_{H^1(0,T;L^2(\Omega))}^2 + \|z_0\|_{H^2(\Omega)}^2).$$

4. Approximating scheme for the problem (P)

We start with building a discrete problem according to the problem (P). Write $d_S(\cdot)$ for the distance function from $\cdot$ to S in $L^2(\Omega)$. We define a penalty functional $J_{h\tau}$ from $(V_h)^N \times (U_h)^N$ to R^+ by setting

$$\begin{aligned} \text{(4.1)} \qquad & J_{h\tau}(Y_{h\tau}, U_{h\tau}) \\ = \quad & \frac{[d_S(Y_h^N) + h + \tau^{\frac{1}{2}}]^2}{2(h + \tau^{\frac{1}{2}})} + \frac{\tau}{2} \sum_{i=1}^N (\|Y_h^i - y_d(t_i)\|_{L^2(\Omega)}^2 + \|U_h^i\|_{L^2(\Omega)}^2), \end{aligned}$$

where $Y_{h\tau}=(Y_h^1, Y_h^2, \cdots, Y_h^N) \in (V_h)^N$, $U_{h\tau}=(U_h^1, U_h^2, \cdots, U_h^N) \in (U_h)^N$. The first right hand term in (4.1) is responsible for the elimination of the end-point state constraint, while the combination $2(h + \tau^{\frac{1}{2}})$ of the mesh size and the time step plays the role of the penalized parameter. Consider the following discrete problem:

$$(P_{h\tau}) \qquad \text{Min} J_{h\tau}(Y_{h\tau}, U_{h\tau}),$$

over all such pairs $(Y_{h\tau}, U_{h\tau}) \in (V_h)^N \times (U_h)^N$ that

$$\text{(4.2)} \quad \begin{cases} \langle \partial_\tau Y_h^i, \varphi_h \rangle + \langle \nabla Y_h^i, \nabla \varphi_h \rangle = \langle \chi_\omega U_h^i, \varphi_h \rangle, & \forall\ \varphi_h \in V_h,\ i = 1, 2, \cdots, N, \\ Y_h^0 = Q_h y_0(x) & \text{in } \Omega. \end{cases}$$

When $(Y_{h\tau}^*, U_{h\tau}^*)$ solves the problem $(P_{h\tau})$, it will be called an optimal pair, while $U_{h\tau}^*$ and $Y_{h\tau}^*$ are called an optimal control and an optimal state, respectively.

With regard to the problem $(P_{h\tau})$, the existence and uniqueness of the optimal control and the Pontryagin maximum principle are given in order.

Lemma 4.1. *The problem $(P_{h\tau})$ has a unique optimal control.*
Proof. Let

$$d^* = \inf J_{h\tau}(Y_{h\tau}, U_{h\tau}),$$

where the infimum is taken over all pairs $(Y_{h\tau}, U_{h\tau})$, with $Y_{h\tau} = (Y_h^1, Y_h^2, \cdots, Y_h^N) \in (V_h)^N$ and $U_{h\tau} = (U_h^1, U_h^2, \cdots, U_h^N) \in (U_h)^N$, satisfying the equation (4.2). It is obvious that $d^* \geq 0$. Hence, there exists a sequence $\{(Y_{h\tau,m}, U_{h\tau,m})\}_{m=1}^\infty$, with $Y_{h\tau,m} = (Y_{h,m}^1, Y_{h,m}^2, \cdots, Y_{h,m}^N)$ and $U_{h\tau,m} = (U_{h,m}^1, U_{h,m}^2, \cdots, U_{h,m}^N)$, such that

$$(4.3) \qquad d^* \leq \frac{[d_S(Y_{h,m}^N) + h + \tau^{\frac{1}{2}}]^2}{2(h + \tau^{\frac{1}{2}})} + \frac{\tau}{2}\sum_{i=1}^N (\|Y_{h,m}^i - y_d(t_i)\|_{L^2(\Omega)}^2 + \|U_{h,m}^i\|_{L^2(\Omega)}^2) \leq d^* + \frac{1}{m}$$

and

$$(4.4) \quad \begin{cases} \langle \partial_\tau Y_{h,m}^i, \varphi_h \rangle + \langle \nabla Y_{h,m}^i, \nabla \varphi_h \rangle = \langle \chi_\omega U_{h,m}^i, \varphi_h \rangle, & \forall\, \varphi_h \in V_h,\ 1 \leq i \leq N, \\ Y_{h,m}^0 = Q_h y_0(x) & \text{in } \Omega. \end{cases}$$

According to Lemma 3.2, and by (4.4), (3.4) and (4.3), we have the estimate:

$$(4.5) \qquad \max_{1\leq i\leq N} \|Y_{h,m}^i\|_{H_0^1(\Omega)}^2 \leq C\Big(\|y_0\|_{H_0^1(\Omega)}^2 + \tau\sum_{i=1}^N \|U_{h,m}^i\|_{L^2(\Omega)}^2\Big) \leq C.$$

Here, C stands for two different positive constants independent of m. Then by (4.3) and (4.5), we can take a subsequence of $\{m\}_{m=1}^\infty$, still denoted in the same way, such that when $m \to \infty$,

$$U_{h,m}^i \to U_h^{*i} \text{ weakly in } L^2(\Omega),\ Y_{h,m}^i \to Y_h^{*i} \text{ weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega),$$

where $i = 1, 2, \cdots, N$. Furthermore, one can easily check that for all $i = 1, 2, \cdots, N$, $U_h^{*i} \in U_h$ and $Y_h^{*i} \in V_h$. Therefore, by passing to the limit for $m \to \infty$ in (4.3) and (4.4), respectively, we derive that

$$(4.6) \qquad \frac{[d_S(Y_h^{*N}) + h + \tau^{\frac{1}{2}}]^2}{2(h + \tau^{\frac{1}{2}})} + \frac{\tau}{2}\sum_{i=1}^N (\|Y_h^{*i} - y_d(t_i)\|_{L^2(\Omega)}^2 + \|U_h^{*i}\|_{L^2(\Omega)}^2) \leq d^*$$

and

$$(4.7) \quad \begin{cases} \langle \partial_\tau Y_h^{*i}, \varphi_h \rangle + \langle \nabla Y_h^{*i}, \nabla \varphi_h \rangle = \langle \chi_\omega U_h^{*i}, \varphi_h \rangle, & \forall\, \varphi_h \in V_h,\ 1 \leq i \leq N, \\ Y_h^{*0} = Q_h y_0(x) & \text{in } \Omega. \end{cases}$$

Now, we write $Y_{h\tau}^* = (Y_h^{*1}, \cdots, Y_h^{*N})$ and $U_{h\tau}^* = (U_h^{*1}, \cdots, U_h^{*N})$. Then, according to Lemma 3.2, it follows at once from (4.6) and (4.7) that $(Y_{h\tau}^*, U_{h\tau}^*)$ is an optimal pair to the problem $(P_{h\tau})$.

Next, we shall prove the uniqueness of the optimal control to the problem $(P_{h\tau})$. For this purpose, we define a functional $\tilde{J}_{h\tau} : (U_h)^N \to R^+$ by setting $\tilde{J}_{h\tau}(U_{h\tau}) = J_{h\tau}(Y_{h\tau}, U_{h\tau})$, where $Y_{h\tau} \in (V_h)^N$ is the unique solution of (4.2) corresponding to $U_{h\tau}$. One can easily check that the functional $\tilde{J}_{h\tau}$ is strictly convex. Therefore, the above-mentioned uniqueness follows immediately. This completes the proof. □

Theorem 4.2. *Let $(Y_{h\tau}^*, U_{h\tau}^*) \in (V_h)^N \times (U_h)^N$ be the optimal pair for the problem $(P_{h\tau})$, where $Y_{h\tau}^* = (Y_h^{*1}, Y_h^{*2}, \cdots, Y_h^{*N})$ and $U_{h\tau}^* = (U_h^{*1}, U_h^{*2}, \cdots, U_h^{*N})$. Then* *there exist a positive constant* $\lambda_{h\tau}$, *functions* $p^*_{h\tau} = (p_h^{*0}, p_h^{*1}, \cdots, p_h^{*N-1}) \in (V_h)^N$ *and* $a_{h\tau} \in L^2(\Omega)$ *satisfying:*

$$\text{(4.8)} \quad \begin{cases} \langle \partial_\tau Y_h^{*i}, \varphi_h \rangle + \langle \nabla Y_h^{*i}, \nabla \varphi_h \rangle = \langle \chi_\omega U_h^{*i}, \varphi_h \rangle, & \forall\ \varphi_h \in V_h,\ 1 \le i \le N, \\ Y_h^{*0} = Q_h y_0(x) & in\ \Omega, \end{cases}$$

$$\text{(4.9)} \begin{cases} \langle \partial_\tau p_h^{*i}, \varphi_h \rangle - \langle \nabla p_h^{*i-1}, \nabla \varphi_h \rangle = \langle Y_h^{*i} - y_d(t_i), \varphi_h \rangle, & \forall\ \varphi_h \in V_h, 1 \le i \le N, \\ p_h^{*N} = -\mu^*_{h\tau} \triangleq -\dfrac{Q_h a_{h\tau}}{\lambda_{h\tau}} & in\ \Omega, \end{cases}$$

$$\text{(4.10)} \qquad U_h^{*i} = \tilde{\Pi}_h \chi_\omega p_h^{*i-1},\ 1 \le i \le N,$$

$$\text{(4.11)} \qquad a_{h\tau} \in \partial d_S(Y_h^{*N}),$$

$$\text{(4.12)} \qquad \|a_{h\tau}\|_{L^2(\Omega)} = \begin{cases} 1 & if\ Y_h^{*N} \notin S, \\ 0 & if\ Y_h^{*N} \in S, \end{cases}$$

and

$$\text{(4.13)} \qquad \lambda_{h\tau} = \frac{h + \tau^{\frac{1}{2}}}{h + \tau^{\frac{1}{2}} + d_S(Y_h^{*N})}.$$

Proof. Corresponding to each $v_{h\tau} = (v_h^1, v_h^2, \cdots, v_h^N) \in (U_h)^N$ and $\lambda > 0$, we let $Y_{h\tau,\lambda} \equiv (Y^1_{h,\lambda}, Y^2_{h,\lambda}, \cdots, Y^N_{h,\lambda})$ be the solution to the following equation:

$$\text{(4.14)} \quad \begin{cases} \langle \partial_\tau Y^i_{h,\lambda}, \varphi_h \rangle + \langle \nabla Y^i_{h,\lambda}, \nabla \varphi_h \rangle = \langle \chi_\omega (U_h^{*i} + \lambda v_h^i), \varphi_h \rangle, & \forall\ \varphi_h \in V_h, \\ & 1 \le i \le N, \\ Y^0_{h,\lambda} = Q_h y_0(x) & \text{in } \Omega. \end{cases}$$

Then, we write

$$\text{(4.15)} \qquad z_h^i = \frac{Y^i_{h,\lambda} - Y_h^{*i}}{\lambda},\ 0 \le i \le N.$$

Noticing that the optimal pair $(Y^*_{h\tau}, U^*_{h\tau})$ solves the equation (4.8), we infer from (4.14) that

$$\text{(4.16)} \quad \begin{cases} \langle \partial_\tau z_h^i, \varphi_h \rangle + \langle \nabla z_h^i, \nabla \varphi_h \rangle = \langle \chi_\omega v_h^i, \varphi_h \rangle, & \forall\ \varphi_h \in V_h,\ 1 \le i \le N, \\ z_h^0 = 0 & \text{in } \Omega. \end{cases}$$

Let $U_{h\tau,\lambda} = (U_h^{*1} + \lambda v_h^1, U_h^{*2} + \lambda v_h^2, \cdots, U_h^{*N} + \lambda v_h^N)$. Since the pair $(Y^*_{h\tau}, U^*_{h\tau})$ is optimal to the problem $(P_{h\tau})$, we find that

$$\frac{J_{h\tau}(Y_{h\tau,\lambda}, U_{h\tau,\lambda}) - J_{h\tau}(Y^*_{h\tau}, U^*_{h\tau})}{\lambda} \ge 0.$$

By (4.1) and (4.15), we can pass to the limit for $\lambda \to 0^+$ in the above inequality to get

$$\text{(4.17)} \quad \frac{d_S(Y_h^{*N}) + h + \tau^{\frac{1}{2}}}{h + \tau^{\frac{1}{2}}} \langle a_{h\tau}, z_h^N \rangle + \tau \sum_{i=1}^{N} [\langle Y_h^{*i} - y_d(t_i), z_h^i \rangle + \langle U_h^{*i}, v_h^i \rangle] \ge 0,$$

where

$$\text{(4.18)} \qquad a_{h\tau} \in \partial d_S(Y_h^{*N}) \text{ and } \|a_{h\tau}\|_{L^2(\Omega)} = \begin{cases} 1 & \text{if } Y_h^{*N} \notin S, \\ 0 & \text{if } Y_h^{*N} \in S. \end{cases}$$

Let $\lambda_{h\tau}$ be the number given by (4.13). Then it follows from (4.17) that

$$\text{(4.19)} \qquad \langle a_{h\tau}, z_h^N \rangle + \lambda_{h\tau} \cdot \tau \sum_{i=1}^{N} [\langle Y_h^{*i} - y_d(t_i), z_h^i \rangle + \langle U_h^{*i}, v_h^i \rangle] \ge 0.$$

Write $(p_h^0, \cdots, p_h^{N-1}) \in (V_h)^N$ for the solution to the following equation:

$$
(4.20) \quad \begin{cases} \langle \partial_\tau p_h^i, \varphi_h \rangle - \langle \nabla p_h^{i-1}, \nabla \varphi_h \rangle = \langle -\lambda_{h\tau}(Y_h^{*i} - y_d(t_i)), \varphi_h \rangle, & \forall\ \varphi_h \in V_h, \\ & 1 \le i \le N, \\ p_h^N = Q_h a_{h\tau} & \text{in } \Omega. \end{cases}
$$

By taking $\varphi_h = p_h^{i-1}$ in (4.16), we obtain that

$$
\langle z_h^i, p_h^{i-1} \rangle - \langle z_h^{i-1}, p_h^{i-1} \rangle + \tau \langle \nabla z_h^i, \nabla p_h^{i-1} \rangle = \tau \langle \chi_\omega v_h^i, p_h^{i-1} \rangle,\ 1 \le i \le N.
$$

Summing the above equalities over $i = 1, 2, \cdots, N$, after some calculations, we conclude that

$$
(4.21) \quad \langle z_h^N, p_h^{N-1} \rangle - \tau \sum_{i=1}^{N-1} \langle z_h^i, \partial_\tau p_h^i \rangle + \tau \sum_{i=1}^{N} \langle \nabla z_h^i, \nabla p_h^{i-1} \rangle = \tau \sum_{i=1}^{N} \langle \chi_\omega v_h^i, p_h^{i-1} \rangle.
$$

Taking $\varphi_h = z_h^i$ in (4.20) and then summing them over $i = 1, 2, \cdots, N$, after some calculations, we derive that

$$
\tau \sum_{i=1}^{N} \langle \partial_\tau p_h^i, z_h^i \rangle - \tau \sum_{i=1}^{N} \langle \nabla p_h^{i-1}, \nabla z_h^i \rangle = -\tau \sum_{i=1}^{N} \lambda_{h\tau} \langle Y_h^{*i} - y_d(t_i), z_h^i \rangle,
$$

which, together with (4.21) and the second equation in (4.20), gives

$$
\langle z_h^N, Q_h a_{h\tau} \rangle = \tau \sum_{i=1}^{N} \langle \chi_\omega v_h^i, p_h^{i-1} \rangle - \tau \sum_{i=1}^{N} \lambda_{h\tau} \langle Y_h^{*i} - y_d(t_i), z_h^i \rangle.
$$

This, combined with (3.1), leads to the equality:

$$
(4.22) \qquad \langle z_h^N, a_{h\tau} \rangle = \tau \sum_{i=1}^{N} \langle \chi_\omega p_h^{i-1}, v_h^i \rangle - \tau \sum_{i=1}^{N} \lambda_{h\tau} \langle Y_h^{*i} - y_d(t_i), z_h^i \rangle.
$$

Now, we derive from (4.19) and (4.22) that

$$
(4.23) \qquad \sum_{i=1}^{N} \langle \chi_\omega p_h^{i-1} + \lambda_{h\tau} U_h^{*i}, v_h^i \rangle = 0,\ \forall\ v_{h\tau} = (v_h^1, v_h^2, \cdots, v_h^N) \in (U_h)^N.
$$

Next, we let p_h^{*i} denote $-p_h^i / \lambda_{h\tau}$, for $i = 0, 1, 2, \cdots, N$. Then, the equation (4.9) follows at once from (4.20). Moreover, by (4.23), we deduce that

$$
(4.24) \qquad \sum_{i=1}^{N} \langle \chi_\omega p_h^{*i-1} - U_h^{*i}, v_h^i \rangle = 0,\ \forall\ v_{h\tau} = (v_h^1, v_h^2, \cdots, v_h^N) \in (U_h)^N.
$$

The remainder is to prove (4.10). For this purpose, we arbitrarily fix an element $\mathcal{T}$ in $\mathcal{T}_h$. By taking $v_{h\tau} = (0, \cdots, 0, v_h^i, 0, \cdots, 0)$ with $v_h^i = \chi_{\mathcal{T}}, i = 1, 2, \cdots, N$, in (4.24), we get

$$
0 = \int_{\mathcal{T}} (\chi_\omega p_h^{*i-1} - U_h^{*i})\, dx = \int_{\mathcal{T}} \chi_\omega p_h^{*i-1}\, dx - |\mathcal{T}| \cdot U_h^{*i}|_{\mathcal{T}}.
$$

Hence, it holds that

$$
U_h^{*i}|_{\mathcal{T}} = \frac{1}{|\mathcal{T}|} \int_{\mathcal{T}} \chi_\omega p_h^{*i-1}\, dx,
$$

which gives (4.10). This completes the proof. $\square$

5. An error estimate between the solutions of (P) and $(P_{h\tau})$

In this section, we shall give an error estimate between the optimal controls to problems (P) and $(P_{h\tau})$. In what follows, C stands for several positive constants independent of h and τ, which may be different in different contexts.

Lemma 5.1. *Let $(Y^*_{h\tau}, U^*_{h\tau})$ be the optimal pair for the problem $(P_{h\tau})$, where $Y^*_{h\tau} = (Y_h^{*1}, \cdots, Y_h^{*N})$ and $U^*_{h\tau} = (U_h^{*1}, \cdots, U_h^{*N})$. Then, the following estimates hold for sufficiently small h and τ:*

$$d_S(Y_h^{*N}) \le C(h + \tau^{\frac{1}{2}})^{\frac{1}{2}}, \tag{5.1}$$

$$\tau \sum_{i=1}^{N} (\|Y_h^{*i} - y_d(t_i)\|^2_{L^2(\Omega)} + \|U_h^{*i}\|^2_{L^2(\Omega)}) \le C \tag{5.2}$$

and

$$\max_{1 \le i \le N} \|Y_h^{*i}\|^2_{H^1_0(\Omega)} + \tau \sum_{i=1}^{N} \|\partial_\tau Y_h^{*i}\|^2_{L^2(\Omega)} \le C, \tag{5.3}$$

here and throughout proof of this lemma, C denotes several positive constants dependent on y_0, u^* and y_d.

Proof. Write $u_h^{*i} = \tilde{\Pi}_h(\frac{1}{\tau}\int_{t_{i-1}}^{t_i} u^*\, dt) \in U_h$, $i = 1, 2, \cdots, N$, where $\tilde{\Pi}_h$ is the operator defined by (3.5). For each $i = 1, 2, \cdots, N$, we let $y_h^{*i} \in V_h$ denote the solution of the discrete equation:

$$\begin{cases} \langle \partial_\tau y_h^{*i}, \varphi_h \rangle + \langle \nabla y_h^{*i}, \nabla \varphi_h \rangle = \langle \chi_\omega u_h^{*i}, \varphi_h \rangle, & \forall\, \varphi_h \in V_h, \\ y_h^{*0} = Q_h y_0(x) & \text{in } \Omega. \end{cases} \tag{5.4}$$

Then, by the optimality of the pair $(Y^*_{h\tau}, U^*_{h\tau})$ for the problem $(P_{h\tau})$, we find that

$$\begin{aligned} & J_{h\tau}(Y^*_{h\tau}, U^*_{h\tau}) \\ = \;& \frac{1}{2(h+\tau^{\frac{1}{2}})}[d_S(Y_h^{*N}) + h + \tau^{\frac{1}{2}}]^2 + \frac{\tau}{2}\sum_{i=1}^{N}(\|Y_h^{*i} - y_d(t_i)\|^2_{L^2(\Omega)} + \|U_h^{*i}\|^2_{L^2(\Omega)}) \\ \le \;& \frac{1}{2(h+\tau^{\frac{1}{2}})}[d_S(y_h^{*N}) + h + \tau^{\frac{1}{2}}]^2 + \frac{\tau}{2}\sum_{i=1}^{N}(\|y_h^{*i} - y_d(t_i)\|^2_{L^2(\Omega)} + \|u_h^{*i}\|^2_{L^2(\Omega)}). \end{aligned} \tag{5.5}$$

Since it follows from (3.7) that

$$\tau \sum_{i=1}^{N} \|u_h^{*i}\|^2_{L^2(\Omega)} \le \tau \sum_{i=1}^{N} \left\| \frac{1}{\tau}\int_{t_{i-1}}^{t_i} u^*\, dt \right\|^2_{L^2(\Omega)} \le \int_0^T \|u^*\|^2_{L^2(\Omega)}\, dt, \tag{5.6}$$

we can apply Lemma 3.2 and use (5.4) and (3.4) to get the estimate:

$$\begin{aligned} & \max_{1 \le i \le N} \|y_h^{*i}\|^2_{H^1_0(\Omega)} + \tau \sum_{i=1}^{N} \|\partial_\tau y_h^{*i}\|^2_{L^2(\Omega)} \\ \le \;& C\Big(\|y_0\|^2_{H^1_0(\Omega)} + \tau \sum_{i=1}^{N} \|u_h^{*i}\|^2_{L^2(\Omega)}\Big) \le C. \end{aligned} \tag{5.7}$$

Now, we write $y_h(u^*)$ for the solution to the following semi-discrete equation:

$$\begin{cases} \langle \partial_t y_h(u^*), \varphi_h \rangle + \langle \nabla y_h(u^*), \nabla \varphi_h \rangle = \langle \chi_\omega u^*, \varphi_h \rangle, & \forall\, \varphi_h \in V_h, \\ y_h(u^*)(0) = Q_h y_0(x) & \text{in } \Omega. \end{cases} \tag{5.8}$$

According to Lemma 3.4 and Proposition 2.5, it follows from (5.4) and (5.8) that

$$\begin{aligned}
&\max_{1\le i\le N}\|y_h(u^*)(t_i)-y_h^{*i}\|^2_{L^2(\Omega)}+\tau\sum_{i=1}^N\|y_h(u^*)(t_i)-y_h^{*i}\|^2_{H_0^1(\Omega)}\\
\le\ & C\Big(\tau\sum_{i=1}^N\Big\|\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt-u_h^{*i}\Big\|^2_{L^2(\Omega)}+\tau^2(\|u^*\|^2_{H^1(0,T;L^2(\Omega))}+\|y_0\|^2_{H^2(\Omega)})\Big)\\
\le\ & C\Big(\tau\sum_{i=1}^N\Big\|\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt-\tilde{\Pi}_h\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big\|^2_{L^2(\Omega)}+\tau^2\Big).
\end{aligned}$$

Since $u^*\in L^2(0,T;H^1(\omega))$ (by Proposition 2.5), we deduce from the latter estimate, (2.7), the assumption (ii), (3.6) and (3.8) that

$$\begin{aligned}
&\max_{1\le i\le N}\|y_h(u^*)(t_i)-y_h^{*i}\|^2_{L^2(\Omega)}+\tau\sum_{i=1}^N\|y_h(u^*)(t_i)-y_h^{*i}\|^2_{H_0^1(\Omega)}\\
\le\ & C\Big(\tau\sum_{i=1}^N\Big\|\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt-\tilde{\Pi}_h\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big\|^2_{L^2(\omega)}+\tau^2\Big)\\
\le\ & C\Big(\tau\sum_{i=1}^N h^2\Big\|\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big\|^2_{H^1(\omega)}+\tau^2\Big)\\
\le\ & C(\tau^2+h^2).
\end{aligned}\tag{5.9}$$

On the other hand, an application of Lemma 3.3 to the equations (2.5) and (5.8) yields

$$\|y(u^*)-y_h(u^*)\|_{C([0,T];L^2(\Omega))}\le Ch(\|y_0\|_{H_0^1(\Omega)}+\|u^*\|_{L^2(Q)})\le Ch.\tag{5.10}$$

Finally, putting the estimates (5.5), (5.6), (5.7), (5.9) and (5.10) together, we conclude that

$$\begin{aligned}
&J_{h\tau}(Y^*_{h\tau},U^*_{h\tau})\\
=\ & \frac{1}{2(h+\tau^{\frac12})}[d_S(Y_h^{*N})+h+\tau^{\frac12}]^2+\frac{\tau}{2}\sum_{i=1}^N\Big(\|Y_h^{*i}-y_d(t_i)\|^2_{L^2(\Omega)}+\|U_h^{*i}\|^2_{L^2(\Omega)}\Big)\\
\le\ & \frac{1}{2(h+\tau^{\frac12})}[\|y_h^{*N}-y(u^*)(T)\|_{L^2(\Omega)}+h+\tau^{\frac12}]^2+C\\
\le\ & \frac{1}{2(h+\tau^{\frac12})}[\|y_h^{*N}-y_h(u^*)(T)\|_{L^2(\Omega)}\\
&\qquad +\|y_h(u^*)(T)-y(u^*)(T)\|_{L^2(\Omega)}+h+\tau^{\frac12}]^2+C\\
\le\ & C(h+\tau^{\frac12})+C\\
\le\ & C.
\end{aligned}$$

The desired estimates (5.1) and (5.2) follow at once from the latter inequality, while the estimate (5.2), together with (4.8), Lemma 3.2 and (3.4), gives

$$\max_{1\le i\le N}\|Y_h^{*i}\|^2_{H_0^1(\Omega)}+\tau\sum_{i=1}^N\|\partial_\tau Y_h^{*i}\|^2_{L^2(\Omega)}\le C\Big(\|y_0\|^2_{H_0^1(\Omega)}+\tau\sum_{i=1}^N\|U_h^{*i}\|^2_{L^2(\Omega)}\Big)\le C.$$

Thus, we complete the proof of the lemma. $\square$

Lemma 5.2. *Suppose that $(Y^*_{h\tau}, U^*_{h\tau})$ is the optimal pair for the problem $(P_{h\tau})$, where $Y^*_{h\tau} = (Y_h^{*1}, Y_h^{*2}, \cdots, Y_h^{*N}) \in (V_h)^N$ and $U^*_{h\tau} = (U_h^{*1}, U_h^{*2}, \cdots, U_h^{*N}) \in (U_h)^N$. Then, it holds that*

$$\tau \sum_{i=1}^{N} \|Y_h^{*i} - Y_h^{*i-1}\|^2_{H_0^1(\Omega)} \leq C[\tau + \tau^2 h^{-2}(h + \tau^{\frac{1}{2}})^{-1}],$$

here and throughout proof of this lemma, C denotes several positive constants dependent on y_0, u^* and y_d.

Proof. Let $Z_h^{*i} = Y_h^{*i} - Y_h^{*i-1}$, where $i = 1, 2, \cdots, N$. Then, by subtracting two consecutive equations in (4.8), we deduce that

$$\langle \partial_\tau Z_h^{*i}, \varphi_h \rangle + \langle \nabla Z_h^{*i}, \nabla \varphi_h \rangle = \langle \chi_\omega (U_h^{*i} - U_h^{*i-1}), \varphi_h \rangle, \ \forall \ \varphi_h \in V_h, \ 2 \leq i \leq N.$$

Taking $\varphi_h = \tau Z_h^{*i}$ in the above equality, after some simple calculations, we get

$$\|Z_h^{*i}\|^2_{L^2(\Omega)} - \|Z_h^{*i-1}\|^2_{L^2(\Omega)} + \tau \|\nabla Z_h^{*i}\|^2_{L^2(\Omega)} \leq C\tau \|U_h^{*i} - U_h^{*i-1}\|^2_{L^2(\Omega)}, \ 2 \leq i \leq N,$$

Summing the above inequalities over $i = 2, \cdots, N$, we obtain the estimate:

$$\|Z_h^{*N}\|^2_{L^2(\Omega)} - \|Z_h^{*1}\|^2_{L^2(\Omega)} + \tau \sum_{i=2}^{N} \|\nabla Z_h^{*i}\|^2_{L^2(\Omega)} \leq C\tau \sum_{i=2}^{N} \|U_h^{*i} - U_h^{*i-1}\|^2_{L^2(\Omega)}.$$

This implies that

$$\begin{aligned} \tag{5.11} & \tau \sum_{i=1}^{N} \|\nabla Z_h^{*i}\|^2_{L^2(\Omega)} \\ \leq \ & \tau \|\nabla Z_h^{*1}\|^2_{L^2(\Omega)} + \|Z_h^{*1}\|^2_{L^2(\Omega)} + C\tau \sum_{i=2}^{N} \|U_h^{*i} - U_h^{*i-1}\|^2_{L^2(\Omega)}. \end{aligned}$$

Now, we shall estimate the right hand terms in (5.11) by the following two steps.

Step 1. To prove the estimate:

$$\tag{5.12} \tau \sum_{i=2}^{N} \|U_h^{*i} - U_h^{*i-1}\|^2_{L^2(\Omega)} \leq C\tau^2 (h + \tau^{\frac{1}{2}})^{-1} h^{-2}.$$

For this purpose, we first infer from Lemma 3.2 and (4.9) that

$$\begin{aligned} \tag{5.13} & \max_{1 \leq i \leq N} \|p_h^{*i-1}\|^2_{H_0^1(\Omega)} + \tau \sum_{i=1}^{N} \|\partial_\tau p_h^{*i}\|^2_{L^2(\Omega)} \\ \leq \ & C\left(\lambda_{h\tau}^{-2} \|Q_h a_{h\tau}\|^2_{H_0^1(\Omega)} + \tau \sum_{i=1}^{N} \|Y_h^{*i} - y_d(t_i)\|^2_{L^2(\Omega)} \right). \end{aligned}$$

Here, we recall that $\lambda_{h\tau}$ is the positive constant given by Theorem 4.2. Because of (4.13) and (5.1), we find that

$$\tag{5.14} \lambda_{h\tau}^{-1} = \frac{h + \tau^{\frac{1}{2}} + d_S(Y_h^{*N})}{h + \tau^{\frac{1}{2}}} \leq C(h + \tau^{\frac{1}{2}})^{-\frac{1}{2}}.$$

Hence, putting the estimates (5.13), (3.2), (5.2), (3.3) and (4.12) together gives

$$
\begin{aligned}
(5.15)\qquad & \max_{1\le i\le N}\|p_h^{*i-1}\|^2_{H_0^1(\Omega)}+\tau\sum_{i=1}^{N}\|\partial_\tau p_h^{*i}\|^2_{L^2(\Omega)}\\
\le\ & C[(h+\tau^{\frac12})^{-1}h^{-2}\|Q_h a_{h\tau}\|^2_{L^2(\Omega)}+1]\\
\le\ & C[(h+\tau^{\frac12})^{-1}h^{-2}\|a_{h\tau}\|^2_{L^2(\Omega)}+1]\\
\le\ & C(h+\tau^{\frac12})^{-1}h^{-2},
\end{aligned}
$$

which, together with (4.10) and (3.7), yields that

$$
\begin{aligned}
& \tau\sum_{i=2}^{N}\|U_h^{*i}-U_h^{*i-1}\|^2_{L^2(\Omega)}=\tau\sum_{i=2}^{N}\|\tilde\Pi_h\chi_\omega p_h^{*i-1}-\tilde\Pi_h\chi_\omega p_h^{*i-2}\|^2_{L^2(\Omega)}\\
\le\ & \tau\sum_{i=2}^{N}\|p_h^{*i-1}-p_h^{*i-2}\|^2_{L^2(\Omega)}=\tau\sum_{i=1}^{N-1}\|p_h^{*i}-p_h^{*i-1}\|^2_{L^2(\Omega)}\\
=\ & \tau^3\sum_{i=1}^{N-1}\|\partial_\tau p_h^{*i}\|^2_{L^2(\Omega)}\\
\le\ & C\tau^2(h+\tau^{\frac12})^{-1}h^{-2}.
\end{aligned}
$$

Step 2. To show estimates:

$$
(5.16)\qquad \tau\|\nabla Z_h^{*1}\|^2_{L^2(\Omega)}\le C\tau \quad \textit{and} \quad \|Z_h^{*1}\|^2_{L^2(\Omega)}\le C\tau.
$$

To this end, we observe first that the second estimate in (5.16) is a direct consequence of the estimate (5.3). In order to prove the first estimate in (5.16), we write $y_h(\cdot)$ and $(\tilde Y_h^{*1},\tilde Y_h^{*2},\cdots,\tilde Y_h^{*N})$ for the solutions of the equation

$$
\begin{cases}
\langle\partial_t y_h(t),\varphi_h\rangle+\langle\nabla y_h(t),\nabla\varphi_h\rangle=\langle\chi_\omega U_h^{*1},\varphi_h\rangle, & \forall\ \varphi_h\in V_h,\ t\in(0,T),\\
y_h(0)=Q_h y_0(x) & \text{in } \Omega
\end{cases}
$$

and the equation

$$
\begin{cases}
\langle\partial_\tau \tilde Y_h^{*i},\varphi_h\rangle+\langle\nabla \tilde Y_h^{*i},\nabla\varphi_h\rangle=\langle\chi_\omega U_h^{*1},\varphi_h\rangle, & \forall\ \varphi_h\in V_h,\ i=1,2,\cdots,N\\
\tilde Y_h^{*0}=Q_h y_0 & \text{in } \Omega,
\end{cases}
$$

respectively. It is clear that $\tilde Y_h^{*1}=Y_h^{*1}$, hence, an application of Lemma 3.4 gives

$$
\|y_h(\tau)-Y_h^{*1}\|^2_{L^2(\Omega)}+\tau\|y_h(\tau)-Y_h^{*1}\|^2_{H_0^1(\Omega)}\le C\tau^2(\|U_h^{*1}\|^2_{L^2(\Omega)}+\|y_0\|^2_{H^2(\Omega)})
$$

and

$$
\max_{t\in[0,\tau]}\|\partial_t y_h\|^2_{L^2(\Omega)}+\int_0^\tau\|\partial_t y_h\|^2_{H_0^1(\Omega)}\,dt\le C(\|U_h^{*1}\|^2_{L^2(\Omega)}+\|y_0\|^2_{H^2(\Omega)}).
$$

These two inequalities, together with (5.2), yield that

$$\begin{aligned}
&\tau\|\nabla Z_h^{*1}\|^2_{L^2(\Omega)} = \tau\|\nabla Y_h^{*1} - \nabla Y_h^{*0}\|^2_{L^2(\Omega)}\\
\leq\ & C\tau\|\nabla Y_h^{*1} - \nabla y_h(\tau)\|^2_{L^2(\Omega)} + C\tau\|\nabla y_h(\tau) - \nabla Q_h y_0\|^2_{L^2(\Omega)}\\
\leq\ & C\tau(\tau\|U_h^{*1}\|^2_{L^2(\Omega)} + \tau) + C\tau\Big\|\nabla\int_0^\tau (y_h)_t\,dt\Big\|^2_{L^2(\Omega)}\\
\leq\ & C\tau + C\tau^2\int_0^\tau \|\nabla(y_h)_t\|^2_{L^2(\Omega)}\,dt\\
\leq\ & C\tau + C\tau(\tau\|U_h^{*1}\|^2_{L^2(\Omega)} + \tau)\\
\leq\ & C\tau.
\end{aligned}$$

Thus we have proved the first estimate in (5.16) and reached the aim of Step 2.

Finally, by (5.11), (5.12) and (5.16), we obtain that

$$\tau\sum_{i=1}^N \|\nabla Z_h^{*i}\|^2_{L^2(\Omega)} \leq C[\tau + \tau^2 h^{-2}(h + \tau^{\frac{1}{2}})^{-1}].$$

This completes the proof of this lemma. □

Now, we turn to the main result of this paper.

Theorem 5.3. *Suppose that u^* and $U_{h\tau}^*$ are the optimal controls for the problems (P) and $(P_{h\tau})$, respectively, where $U_{h\tau}^* = (U_h^{*1}, U_h^{*2}, \cdots, U_h^{*N}) \in (U_h)^N$. Then the following error estimate holds:*

$$\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|u^* - U_h^{*i}\|^2_{L^2(\Omega)}\,dt \leq C(h + \tau^{\frac{1}{2}})^{\frac{1}{2}} + C\tau h^{-1}(h + \tau^{\frac{1}{2}})^{-\frac{1}{2}}.$$

Furthermore, it holds that

$$\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|u^* - U_h^{*i}\|^2_{L^2(\Omega)}\,dt \leq Ch^{\frac{1}{2}}, \quad \text{whenever } \tau \approx O(h^2),$$

here and through proof of this theorem, C denotes several positive constants dependent on y_0, u^*, y_d and μ^*.

Proof. We shall write $Y_{h\tau}^* = (Y_h^{*1}, Y_h^{*2}, \cdots, Y_h^{*N})$ for the optimal state to the problem $(P_{h\tau})$. Namely, $(Y_{h\tau}^*, U_{h\tau}^*)$ is the optimal pair to the problem $(P_{h\tau})$. Note that the second estimate in the theorem is obviously a consequence of the first one. Hence, it suffices to show the first estimate. We shall carry out the proof by several stages as follows:

Stage 1. To prove the estimate:

$$\max_{1\leq i\leq N}\|p_h^{*i-1}\|^2_{L^2(\Omega)} + \tau\sum_{i=1}^N \|\nabla p_h^{*i-1}\|^2_{L^2(\Omega)} \leq C(h + \tau^{\frac{1}{2}})^{-1}. \tag{5.17}$$

Here, we recall that $p_h^{*i} \in V_h,\ i = 0, 1, \cdots, N-1$, are given by (4.9).

For this purpose, we take $\varphi_h = p_h^{*i-1}$ in the first equation of (4.9). After some calculations, we find that

$$\|p_h^{*i-1}\|^2_{L^2(\Omega)} - \|p_h^{*i}\|^2_{L^2(\Omega)} + \tau\|\nabla p_h^{*i-1}\|^2_{L^2(\Omega)} \leq C\tau\|Y_h^{*i} - y_d(t_i)\|^2_{L^2(\Omega)},\ 1 \leq i \leq N.$$

Summing up the above inequalities over $i = k, \cdots, N$, with $1 \leq k \leq N$, we get

$$\|p_h^{*k-1}\|_{L^2(\Omega)}^2 + \tau \sum_{i=k}^{N} \|\nabla p_h^{*i-1}\|_{L^2(\Omega)}^2 \leq \|p_h^{*N}\|_{L^2(\Omega)}^2 + C\tau \sum_{i=1}^{N} \|Y_h^{*i} - y_d(t_i)\|_{L^2(\Omega)}^2.$$

This, together with (5.2), the second equation in (4.9), (3.3), (4.12) and (5.14), yields that

$$\begin{aligned} & \|p_h^{*k-1}\|_{L^2(\Omega)}^2 + \tau \sum_{i=k}^{N} \|\nabla p_h^{*i-1}\|_{L^2(\Omega)}^2 \leq C + \lambda_{h\tau}^{-2} \|Q_h a_{h\tau}\|_{L^2(\Omega)}^2 \\ \leq \quad & C + \lambda_{h\tau}^{-2} \|a_{h\tau}\|_{L^2(\Omega)}^2 \leq C + C(h + \tau^{\frac{1}{2}})^{-1} \\ \leq \quad & C(h + \tau^{\frac{1}{2}})^{-1}, \qquad\qquad \forall\, 1 \leq k \leq N, \end{aligned}$$

which leads to the estimate (5.17).

Stage 2. To show the equality:

$$\begin{aligned} & \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|u^* - U_h^{*i}\|_{L^2(\Omega)}^2 \, dt \\ = \quad & \tau \sum_{i=1}^{N} \Big\langle U_h^{*i} - \chi_\omega p_h^{*i-1}, \tilde{\Pi}_h \Big(\frac{1}{\tau} \int_{t_{i-1}}^{t_i} u^* \, dt \Big) - \frac{1}{\tau} \int_{t_{i-1}}^{t_i} u^* \, dt \Big\rangle \\ & + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle \chi_\omega p^*, u^* \rangle \, dt + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle \chi_\omega p_h^{*i-1}, U_h^{*i} \rangle \, dt \\ & - \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle \chi_\omega p^*, U_h^{*i} \rangle \, dt - \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle \chi_\omega p_h^{*i-1}, u^* \rangle \, dt \\ \triangleq \quad & \sum_{i=1}^{5} J_i. \end{aligned} \tag{5.18}$$

Here, we recall that p^* is given by (2.6) and $\tilde{\Pi}_h$ is the operator defined by (3.5).

To this end, we first derive from (2.7) and (4.10) that

$$\begin{aligned} & \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|u^* - U_h^{*i}\|_{L^2(\Omega)}^2 \, dt \\ = \quad & \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle u^*, u^* - U_h^{*i} \rangle \, dt - \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle U_h^{*i}, u^* - U_h^{*i} \rangle \, dt \\ = \quad & \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle \chi_\omega p^*, u^* - U_h^{*i} \rangle \, dt - \tau \sum_{i=1}^{N} \Big\langle U_h^{*i}, \frac{1}{\tau} \int_{t_{i-1}}^{t_i} u^* \, dt - U_h^{*i} \Big\rangle \\ & + \tau \sum_{i=1}^{N} \Big\langle U_h^{*i} - \chi_\omega p_h^{*i-1}, \tilde{\Pi}_h \Big(\frac{1}{\tau} \int_{t_{i-1}}^{t_i} u^* \, dt \Big) - U_h^{*i} \Big\rangle. \end{aligned} \tag{5.19}$$

Since it clearly holds that

$$\begin{aligned}
& \tau\sum_{i=1}^{N}\Big\langle U_h^{*i}-\chi_\omega p_h^{*i-1},\tilde{\Pi}_h\Big(\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big)-U_h^{*i}\Big\rangle \\
= & \tau\sum_{i=1}^{N}\Big\langle U_h^{*i}-\chi_\omega p_h^{*i-1},\tilde{\Pi}_h\Big(\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big)-\frac{1}{\tau}\int_{t_i-1}^{t_i}u^*\,dt\Big\rangle \\
& +\tau\sum_{i=1}^{N}\Big\langle U_h^{*i}-\chi_\omega p_h^{*i-1},\ \frac{1}{\tau}\int_{t_i-1}^{t_i}u^*\,dt-U_h^{*i}\Big\rangle,
\end{aligned}$$

we infer from from (5.19) that

$$\begin{aligned}
& \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\|u^*-U_h^{*i}\|^2_{L^2(\Omega)}\,dt \\
= & \tau\sum_{i=1}^{N}\Big\langle U_h^{*i}-\chi_\omega p_h^{*i-1},\tilde{\Pi}_h\Big(\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big)-\frac{1}{\tau}\int_{t_i-1}^{t_i}u^*\,dt\Big\rangle \\
& +\sum_{i=1}^{N}\Big\{\int_{t_i-1}^{t_i}\Big\langle\chi_\omega(p^*-p_h^{*i-1}),u^*-U_h^{*i}\Big\rangle\,dt \\
& +\int_{t_i-1}^{t_i}\Big\langle\chi_\omega p_h^{*i-1},\ u^*-\frac{1}{\tau}\int_{t_i-1}^{t_i}u^*\,dt\Big\rangle\,dt\Big\}.
\end{aligned}$$

Notice that the last term on the above equality is identically zero because $\chi_\omega p_h^{*i-1}$ is independent of the t variable. Thus, the equality (5.18) follows at once.

Stage 3. To estimate the right hand sum $\sum_{i=1}^{5}J_i$ in (5.18).
This will be done by several steps as follows:
Step 3.1. To prove the estimate:

$$J_1\le Ch(h+\tau^{\frac{1}{2}})^{-\frac{1}{2}}. \tag{5.20}$$

Indeed, it follows from (2.7), the assumption (ii), (3.8), (5.2) and (5.17) that

$$\begin{aligned}
J_1 \equiv\ & \tau\sum_{i=1}^{N}\Big\langle U_h^{*i}-\chi_\omega p_h^{*i-1},\tilde{\Pi}_h\Big(\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big)-\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big\rangle \\
\le\ & C\tau h\sum_{i=1}^{N}\|U_h^{*i}-\chi_\omega p_h^{*i-1}\|_{L^2(\Omega)}\Big\|\frac{1}{\tau}\int_{t_{i-1}}^{t_i}u^*\,dt\Big\|_{H^1(\omega)} \\
\le\ & Ch\sum_{i=1}^{N}\tau^{\frac{1}{2}}\|U_h^{*i}-\chi_\omega p_h^{*i-1}\|_{L^2(\Omega)}\Big(\int_{t_{i-1}}^{t_i}\|u^*\|^2_{H^1(\omega)}\,dt\Big)^{\frac{1}{2}} \\
\le\ & Ch\Big(\sum_{i=1}^{N}\tau\|U_h^{*i}-\chi_\omega p_h^{*i-1}\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\
\le\ & Ch(h+\tau^{\frac{1}{2}})^{-\frac{1}{2}}.
\end{aligned}$$

Step 3.2. Studies on the sum $\sum_{i=2}^{5}J_i$.

About the term J_2, we have

$$(5.21)\quad \begin{aligned} J_2 &\equiv \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\langle \chi_\omega p^*, u^*\rangle\,dt = \int_0^T \langle p^*, \partial_t y(u^*) - \triangle y(u^*)\rangle\,dt \\ &= \langle y(u^*)(T), p^*(T)\rangle - \langle y_0, p^*(0)\rangle \\ &\quad -\int_0^T \langle \partial_t p^*, y(u^*)\rangle\,dt - \int_0^T \langle \triangle p^*, y(u^*)\rangle\,dt \\ &= -\langle y(u^*)(T), \mu^*\rangle - \langle y_0, p^*(0)\rangle - \int_0^T \langle y(u^*), y(u^*) - y_d\rangle\,dt. \end{aligned}$$

Concerning the term J_3, we infer from (4.8) and (4.9) that

$$(5.22)\quad \begin{aligned} J_3 &\equiv \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\langle \chi_\omega p_h^{*i-1}, U_h^{*i}\rangle\,dt = \tau\sum_{i=1}^{N}\langle p_h^{*i-1}, \chi_\omega U_h^{*i}\rangle \\ &= \tau\left[\sum_{i=1}^{N}\langle \partial_\tau Y_h^{*i}, p_h^{*i-1}\rangle + \sum_{i=1}^{N}\langle \nabla Y_h^{*i}, \nabla p_h^{*i-1}\rangle\right] \\ &= -\langle Q_h y_0, p_h^{*0}\rangle + \langle Y_h^{*N}, p_h^{*N}\rangle \\ &\quad -\tau\sum_{i=1}^{N}\langle Y_h^{*i}, \partial_\tau p_h^{*i}\rangle + \tau\sum_{i=1}^{N}\langle \nabla Y_h^{*i}, \nabla p_h^{*i-1}\rangle \\ &= -\Big\langle Y_h^{*N}, \frac{Q_h a_{h\tau}}{\lambda_{h\tau}}\Big\rangle - \langle Q_h y_0, p_h^{*0}\rangle - \tau\sum_{i=1}^{N}\langle Y_h^{*i}, Y_h^{*i} - y_d(t_i)\rangle. \end{aligned}$$

As regards the term J_4, we shall prove the estimate:

$$(5.23)\quad \begin{aligned} J_4 &\le Ch + \langle Y_h^{*N}, \mu^*\rangle + \langle Q_h y_0, p^*(0)\rangle + \int_0^T \langle \overline{Y}^*_{h\tau}, y(u^*) - y_d\rangle\,dt \\ &\quad + \int_0^T \Big\|\nabla(\overline{Y}^*_{h\tau} - \hat{\overline{Y}}^*_{h\tau})\Big\|_{L^2(\Omega)} \|\nabla \tilde{p}_h^*\|_{L^2(\Omega)}\,dt. \end{aligned}$$

Here, the function $\overline{Y}^*_{h\tau} \in H^1(0,T;V_h)$ is given by $\overline{Y}^*_{h\tau}(t) = Y_h^{*i-1} + \frac{t-t_{i-1}}{\tau}(Y_h^{*i} - Y_h^{*i-1})$ when $t \in (t_{i-1}, t_i]$, $i = 1, 2, \cdots, N$, $\hat{\overline{Y}}^*_{h\tau}$ is the step function taking value Y_h^{*i} over the interval $(t_{i-1}, t_i]$ for $i = 1, 2, \cdots, N$, and $\tilde{p}_h^* \in H^1(0,T;V_h)$ is the solution to the equation:

$$(5.24)\quad \begin{cases} \langle \partial_t \tilde{p}_h^*, \varphi_h\rangle - \langle \nabla \tilde{p}_h^*, \nabla\varphi_h\rangle = \langle y(u^*) - y_d, \varphi_h\rangle, & \forall\ \varphi_h \in V_h, \\ \tilde{p}_h^*(T) = Q_h p^*(T) & \text{in } \Omega. \end{cases}$$

For this purpose, we let $\overline{U}^*_{h\tau}$ be the step function that takes value U_h^{*i} on the interval $(t_{i-1}, t_i]$ for $i = 1, 2, \cdots, N$. Clearly, $\overline{Y}^*_{h\tau}, \hat{\overline{Y}}^*_{h\tau}$ and $\overline{U}^*_{h\tau}$ satisfy the following equation:

$$(5.25)\quad \begin{cases} \langle \partial_t \overline{Y}^*_{h\tau}, \varphi_h\rangle + \langle \nabla \hat{\overline{Y}}^*_{h\tau}, \nabla\varphi_h\rangle = \langle \chi_\omega \overline{U}^*_{h\tau}, \varphi_h\rangle, & \forall\ \varphi_h \in V_h, \\ & \text{a.e. } t \in (0,T), \\ \overline{Y}^*_{h\tau}(0) = Q_h y_0(x) & \text{in } \Omega. \end{cases}$$

According to Proposition 2.5 and Lemma 3.3, we derive from (2.6) and (5.24) that

$$\begin{aligned}(5.26)\qquad &\|\tilde{p}_h^* - p^*\|_{L^2(Q)} + h(\|\tilde{p}_h^* - p^*\|_{C([0,T];L^2(\Omega))} + \|\tilde{p}_h^* - p^*\|_{L^2(0,T;H_0^1(\Omega))})\\ \leq\ & Ch^2(\|p^*(T)\|_{H_0^1(\Omega)} + \|y(u^*) - y_d\|_{L^2(Q)})\\ \leq\ & Ch^2.\end{aligned}$$

Notice that

$$\begin{aligned}(5.27)\qquad J_4 \ &\equiv\ -\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\langle\chi_\omega p^*, U_h^{*i}\rangle\,dt\\ &=\ -\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\langle p^* - \tilde{p}_h^*, \chi_\omega U_h^{*i}\rangle\,dt - \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\langle\tilde{p}_h^*, \chi_\omega U_h^{*i}\rangle\,dt\\ &\triangleq\ J_{41} + J_{42}.\end{aligned}$$

In what follows, we shall estimate the terms J_{41} and J_{42} separately. First, it follows at once from (5.2) and (5.26) that

$$\begin{aligned}(5.28)\qquad J_{41}\ &\leq\ \sum_{i=1}^{N}\|U_h^{*i}\|_{L^2(\Omega)}\tau^{\frac{1}{2}}\Big(\int_{t_{i-1}}^{t_i}\|p^* - \tilde{p}_h^*\|^2_{L^2(\Omega)}\,dt\Big)^{\frac{1}{2}}\\ &\leq\ \Big(\sum_{i=1}^{N}\tau\|U_h^{*i}\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}\Big(\int_0^T\|p^* - \tilde{p}_h^*\|^2_{L^2(\Omega)}\,dt\Big)^{\frac{1}{2}}\\ &\leq\ Ch^2.\end{aligned}$$

Then, we turn to study the term J_{42}. By (5.25) and (5.24), we find that

$$\begin{aligned}J_{42}\ &\equiv\ -\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\langle\tilde{p}_h^*, \chi_\omega U_h^{*i}\rangle\,dt = -\int_0^T\langle\tilde{p}_h^*, \chi_\omega\overline{U}^*_{h\tau}\rangle\,dt\\ &=\ -\int_0^T\langle\partial_t\overline{Y}^*_{h\tau}, \tilde{p}_h^*\rangle\,dt - \int_0^T\langle\nabla\hat{\overline{Y}}^*_{h\tau}(t), \nabla\tilde{p}_h^*\rangle\,dt\\ &=\ -\int_0^T\langle\partial_t\overline{Y}^*_{h\tau}, \tilde{p}_h^*\rangle\,dt - \int_0^T\langle\nabla\overline{Y}^*_{h\tau}, \nabla\tilde{p}_h^*\rangle\,dt + \int_0^T\langle\nabla(\overline{Y}^*_{h\tau} - \hat{\overline{Y}}^*_{h\tau}), \nabla\tilde{p}_h^*\rangle\,dt\\ &=\ -\langle\overline{Y}^*_{h\tau}(T), \tilde{p}_h^*(T)\rangle + \langle\overline{Y}^*_{h\tau}(0), \tilde{p}_h^*(0)\rangle + \int_0^T\langle\overline{Y}^*_{h\tau}, \partial_t\tilde{p}_h^*\rangle\,dt\\ &\quad -\int_0^T\langle\nabla\overline{Y}^*_{h\tau}, \nabla\tilde{p}_h^*\rangle\,dt + \int_0^T\langle\nabla(\overline{Y}^*_{h\tau} - \hat{\overline{Y}}^*_{h\tau}), \nabla\tilde{p}_h^*\rangle\,dt\\ &=\ -\langle Y_h^{*N}, Q_h p^*(T)\rangle + \langle Q_h y_0, \tilde{p}_h^*(0)\rangle + \int_0^T\langle\overline{Y}^*_{h\tau}, y(u^*) - y_d\rangle\,dt\\ &\quad +\int_0^T\langle\nabla(\overline{Y}^*_{h\tau} - \hat{\overline{Y}}^*_{h\tau}), \nabla\tilde{p}_h^*\rangle\,dt.\end{aligned}$$

This, combined with (3.1), (2.6), (3.3) and (5.26), gives

$$\begin{aligned}J_{42}\ &=\ -\langle Y_h^{*N}, p^*(T)\rangle + \langle Q_h y_0, \tilde{p}_h^*(0) - p^*(0)\rangle + \langle Q_h y_0, p^*(0)\rangle\\ &\quad +\int_0^T\langle\overline{Y}^*_{h\tau}, y(u^*) - y_d\rangle\,dt + \int_0^T\langle\nabla(\overline{Y}^*_{h\tau} - \hat{\overline{Y}}^*_{h\tau}), \nabla\tilde{p}_h^*\rangle\,dt\\ &\leq\ Ch + \langle Y_h^{*N}, \mu^*\rangle + \langle Q_h y_0, p^*(0)\rangle\\ &\quad +\int_0^T\langle\overline{Y}^*_{h\tau}, y(u^*) - y_d\rangle\,dt + \int_0^T\Big\|\nabla(\overline{Y}^*_{h\tau} - \hat{\overline{Y}}^*_{h\tau})\Big\|_{L^2(\Omega)}\|\nabla\tilde{p}_h^*\|_{L^2(\Omega)}\,dt.\end{aligned}$$

Thus, the above estimate, together with (5.27) and (5.28), leads to the estimate (5.23).

With regard to the term J_5, we first observe that

$$J_5 \equiv -\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\langle \chi_\omega p_h^{*i-1}, u^*\rangle\, dt = -\sum_{i=1}^{N}\Big\langle p_h^{*i-1}, \chi_\omega \int_{t_{i-1}}^{t_i} u^*\, dt\Big\rangle. \tag{5.29}$$

Then, by integrating the first equation of (5.8) from t_{i-1} to t_i, we obtain that

$$\begin{aligned} &\langle y_h(u^*)(t_i) - y_h(u^*)(t_{i-1}), \varphi_h\rangle + \Big\langle \nabla \int_{t_{i-1}}^{t_i} y_h(u^*)\, dt, \nabla\varphi_h\Big\rangle \\ = \;& \Big\langle \chi_\omega \int_{t_{i-1}}^{t_i} u^*\, dt, \varphi_h\Big\rangle, \;\forall\; \varphi_h \in V_h, \end{aligned}$$

which gives

$$\begin{aligned} &\langle y_h(u^*)(t_i) - y_h(u^*)(t_{i-1}), p_h^{*i-1}\rangle + \Big\langle \nabla \int_{t_{i-1}}^{t_i} y_h(u^*)\, dt, \nabla p_h^{*i-1}\Big\rangle \\ = \;& \Big\langle \chi_\omega \int_{t_{i-1}}^{t_i} u^*\, dt, p_h^{*i-1}\Big\rangle. \end{aligned} \tag{5.30}$$

Now, by (5.29), (5.30) and (5.8), we get

$$\begin{aligned} J_5 \;=\;& -\sum_{i=1}^{N}\langle y_h(u^*)(t_i) - y_h(u^*)(t_{i-1}), p_h^{*i-1}\rangle - \sum_{i=1}^{N}\Big\langle \nabla \int_{t_{i-1}}^{t_i} y_h(u^*)\, dt, \nabla p_h^{*i-1}\Big\rangle \\ =\;& \langle Q_h y_0, p_h^{*0}\rangle - \langle y_h(u^*)(T), p_h^{*N}\rangle + \tau\sum_{i=1}^{N}\langle y_h(u^*)(t_i), \partial_\tau p_h^{*i}\rangle \\ & -\sum_{i=1}^{N}\Big\langle \nabla \int_{t_{i-1}}^{t_i} y_h(u^*)\, dt, \nabla p_h^{*i-1}\Big\rangle. \end{aligned}$$

This, together with (4.9), yields that

$$\begin{aligned} J_5 \;=\;& \langle y_h(u^*)(T), \lambda_{h\tau}^{-1} Q_h a_{h\tau}\rangle + \tau\sum_{i=1}^{N}\langle \nabla y_h(u^*)(t_i), \nabla p_h^{*i-1}\rangle \\ & +\langle Q_h y_0, p_h^{*0}\rangle + \tau\sum_{i=1}^{N}\langle y_h(u^*)(t_i), Y_h^{*i} - y_d(t_i)\rangle \\ & -\sum_{i=1}^{N}\Big\langle \nabla \int_{t_{i-1}}^{t_i} y_h(u^*)\, dt, \nabla p_h^{*i-1}\Big\rangle \\ =\;& \langle y_h(u^*)(T), \lambda_{h\tau}^{-1} Q_h a_{h\tau}\rangle + \langle Q_h y_0, p_h^{*0}\rangle \\ & +\tau\sum_{i=1}^{N}\langle y_h(u^*)(t_i) - y(u^*)(t_i), Y_h^{*i} - y_d(t_i)\rangle \\ & +\tau\sum_{i=1}^{N}\langle y(u^*)(t_i), Y_h^{*i} - y_d(t_i)\rangle \\ & +\tau\sum_{i=1}^{N}\Big\langle \nabla p_h^{*i-1}, \nabla\Big(y_h(u^*)(t_i) - \frac{1}{\tau}\int_{t_{i-1}}^{t_i} y_h(u^*)\, dt\Big)\Big\rangle. \end{aligned} \tag{5.31}$$

Putting (5.21), (5.22), (5.23) and (5.31) together, and by (3.1), we conclude that

$$
\begin{aligned}
(5.32)\qquad & J_2+J_3+J_4+J_5\\
\le\ & Ch+\langle\mu^*,Y_h^{*N}-y(u^*)(T)\rangle+\langle p^*(0),Q_hy_0-y_0\rangle\\
&+\lambda_{h\tau}^{-1}\langle a_{h\tau},y_h(u^*)(T)-Y_h^{*N}\rangle\\
&-\int_0^T\langle y(u^*),y(u^*)-y_d\rangle\,dt-\tau\sum_{i=1}^N\langle Y_h^{*i},Y_h^{*i}-y_d(t_i)\rangle\\
&+\int_0^T\langle \overline{Y}^*_{h\tau},y(u^*)-y_d\rangle\,dt+\int_0^T\left\|\nabla(\overline{Y}^*_{h\tau}-\hat{\overline{Y}}^*_{h\tau})\right\|_{L^2(\Omega)}\|\nabla\tilde{p}_h^*\|_{L^2(\Omega)}\,dt\\
&+\tau\sum_{i=1}^N\langle y_h(u^*)(t_i)-y(u^*)(t_i),Y_h^{*i}-y_d(t_i)\rangle\\
&+\tau\sum_{i=1}^N\langle y(u^*)(t_i),Y_h^{*i}-y_d(t_i)\rangle\\
&+\tau\sum_{i=1}^N\Big\langle\nabla p_h^{*i-1},\nabla\Big(y_h(u^*)(t_i)-\frac{1}{\tau}\int_{t_{i-1}}^{t_i}y_h(u^*)\,dt\Big)\Big\rangle.
\end{aligned}
$$

Step 3.3. Further studies on the right hand terms in (5.32).

On one hand, because S is a closed subset in $L^2(\Omega)$, we infer from (5.1) that there exists an element $s_{h\tau}$ in S, such that

$$\|Y_h^{*N}-s_{h\tau}\|_{L^2(\Omega)}=d_S(Y_h^{*N})\le C(h+\tau^{\frac12})^{\frac12}.$$

This, together with (2.4), yields that

$$(5.33)\quad \langle\mu^*,Y_h^{*N}-y(u^*)(T)\rangle=\langle\mu^*,Y_h^{*N}-s_{h\tau}\rangle+\langle\mu^*,s_{h\tau}-y(u^*)(T)\rangle\le C(h+\tau^{\frac12})^{\frac12}.$$

Moreover, it follows from (3.4) that

$$(5.34)\qquad \langle p^*(0),Q_hy_0-y_0\rangle\le\|p^*(0)\|_{L^2(\Omega)}\|Q_hy_0-y_0\|_{L^2(\Omega)}\le Ch^2.$$

On the other hand, because of (4.11), we find that

$$\langle a_{h\tau},s-Y_h^{*N}\rangle\le 0,\ \forall\ s\in S.$$

This, together with (5.10), (4.12) and (5.14), gives

$$
\begin{aligned}
(5.35)\qquad & \lambda_{h\tau}^{-1}\langle a_{h\tau},y_h(u^*)(T)-Y_h^{*N}\rangle\\
=\ & \lambda_{h\tau}^{-1}\langle a_{h\tau},y_h(u^*)(T)-y(u^*)(T)\rangle+\lambda_{h\tau}^{-1}\langle a_{h\tau},y(u^*)(T)-Y_h^{*N}\rangle\\
\le\ & \lambda_{h\tau}^{-1}\langle a_{h\tau},y_h(u^*)(T)-y(u^*)(T)\rangle\\
\le\ & Ch(h+\tau^{\frac12})^{-\frac12}.
\end{aligned}
$$

Furthermore, by (5.10) and (5.2), we obtain that

$$
\begin{aligned}
&\tau\sum_{i=1}^N\langle y_h(u^*)(t_i)-y(u^*)(t_i),Y_h^{*i}-y_d(t_i)\rangle\\
\le\ & \Big(\sum_{i=1}^N\tau\|y_h(u^*)(t_i)-y(u^*)(t_i)\|^2_{L^2(\Omega)}\Big)^{\frac12}\Big(\sum_{i=1}^N\tau\|Y_h^{*i}-y_d(t_i)\|^2_{L^2(\Omega)}\Big)^{\frac12}\\
\le\ & Ch.
\end{aligned}
$$

Now, it follows from (5.32)-(5.35) and the above inequality that

$$
\begin{aligned}
(5.36)\qquad & J_2+J_3+J_4+J_5\\
\le\ & C(h+\tau^{\frac12})^{\frac12}-\int_0^T\langle y(u^*),y(u^*)-y_d\rangle\,dt-\tau\sum_{i=1}^N\langle Y_h^{*i},Y_h^{*i}-y_d(t_i)\rangle\\
&+\int_0^T\langle\overline{Y}_{h\tau}^{*},y(u^*)-y_d\rangle\,dt+\int_0^T\Big\|\nabla(\overline{Y}_{h\tau}^{*}-\hat{\overline{Y}}_{h\tau}^{*})\Big\|_{L^2(\Omega)}\|\nabla\tilde{p}_h^*\|_{L^2(\Omega)}\,dt\\
&+\tau\sum_{i=1}^N\langle y(u^*)(t_i),Y_h^{*i}-y_d(t_i)\rangle\\
&+\tau\sum_{i=1}^N\Big\langle\nabla p_h^{*i-1},\nabla\Big(y_h(u^*)(t_i)-\frac{1}{\tau}\int_{t_{i-1}}^{t_i}y_h(u^*)\,dt\Big)\Big\rangle.
\end{aligned}
$$

Step 3.4. Estimates on the right hand terms in (5.36).

According to Lemma 5.2, by (5.26) and after some simple calculations, we obtain that

$$
\begin{aligned}
(5.37)\qquad & \int_0^T\Big\|\nabla(\overline{Y}_{h\tau}^{*}-\hat{\overline{Y}}_{h\tau}^{*})\Big\|_{L^2(\Omega)}\|\nabla\tilde{p}_h^*\|_{L^2(\Omega)}\,dt\\
\le\ & \sum_{i=1}^N\int_{t_{i-1}}^{t_i}\|\nabla Y_h^{*i}-\nabla Y_h^{*i-1}\|_{L^2(\Omega)}\|\nabla\tilde{p}_h^*\|_{L^2(\Omega)}\,dt\\
\le\ & \Big(\sum_{i=1}^N\tau\|Y_h^{*i}-Y_h^{*i-1}\|^2_{H_0^1(\Omega)}\Big)^{\frac12}\Big(\int_0^T\|\nabla\tilde{p}_h^*\|^2_{L^2(\Omega)}\,dt\Big)^{\frac12}\\
\le\ & C[\tau^{\frac12}+\tau h^{-1}(h+\tau^{\frac12})^{-\frac12}].
\end{aligned}
$$

On the other hand, by (5.8), and applying Lemma 3.4, we find that

$$
\int_0^T\|\partial_t y_h(u^*)\|^2_{H_0^1(\Omega)}\,dt\le C(\|u^*\|^2_{H^1(0,T;L^2(\Omega))}+\|y_0\|^2_{H^2(\Omega)})\le C,
$$

which, combined with (5.17), yields that

$$
\begin{aligned}
&\tau\sum_{i=1}^N\Big\langle\nabla p_h^{*i-1},\nabla\Big(y_h(u^*)(t_i)-\frac{1}{\tau}\int_{t_{i-1}}^{t_i}y_h(u^*)\,dt\Big)\Big\rangle\\
\le\ & \sum_{i=1}^N\tau^{\frac32}\Big(\int_{t_{i-1}}^{t_i}\|\partial_t y_h(u^*)\|^2_{H_0^1(\Omega)}\,dt\Big)^{\frac12}\|\nabla p_h^{*i-1}\|_{L^2(\Omega)}\\
\le\ & \tau\Big(\sum_{i=1}^N\tau\|\nabla p_h^{*i-1}\|^2_{L^2(\Omega)}\Big)^{\frac12}\Big(\int_0^T\|\partial_t y_h(u^*)\|^2_{H_0^1(\Omega)}\,dt\Big)^{\frac12}\\
\le\ & C\tau(h+\tau^{\frac12})^{-\frac12}.
\end{aligned}
$$

Hence, it follows from (5.36), (5.37) and the aforementioned inequality that

$$
\begin{aligned}
(5.38)\qquad & J_2+J_3+J_4+J_5 \\
\le\;& C(h+\tau^{\frac{1}{2}})^{\frac{1}{2}}+C\tau h^{-1}(h+\tau^{\frac{1}{2}})^{-\frac{1}{2}}-\int_0^T\langle y(u^*),y(u^*)-y_d\rangle\,dt \\
& -\tau\sum_{i=1}^N\langle Y_h^{*i},Y_h^{*i}-y_d(t_i)\rangle+\int_0^T\langle \overline{Y}^*_{h\tau},y(u^*)-y_d\rangle\,dt \\
& +\tau\sum_{i=1}^N\langle y(u^*)(t_i),Y_h^{*i}-y_d(t_i)\rangle.
\end{aligned}
$$

Step 3.5. Estimates on the right hand terms in (5.38).

We first observe that

$$
\begin{aligned}
& -\int_0^T\langle y(u^*),y(u^*)-y_d\rangle\,dt \\
=\;& -\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle y(u^*)-Y_h^{*i},y(u^*)-Y_h^{*i}\rangle\,dt-\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle y(u^*)-Y_h^{*i},Y_h^{*i}-y_d\rangle\,dt \\
& -\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle Y_h^{*i},y(u^*)-y_d\rangle\,dt \\
\le\;& -2\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle Y_h^{*i},y(u^*)\rangle\,dt+\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle y(u^*),y_d\rangle\,dt+\tau\sum_{i=1}^N\|Y_h^{*i}\|^2_{L^2(\Omega)},
\end{aligned}
$$

which gives

$$
\begin{aligned}
(5.39)\qquad & -\int_0^T\langle y(u^*),y(u^*)-y_d\rangle\,dt-\tau\sum_{i=1}^N\langle Y_h^{*i},Y_h^{*i}-y_d(t_i)\rangle \\
\le\;& -2\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle Y_h^{*i},y(u^*)\rangle\,dt+\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle y(u^*),y_d\rangle\,dt+\tau\sum_{i=1}^N\langle Y_h^{*i},y_d(t_i)\rangle.
\end{aligned}
$$

Then, we rewrite the sum of the last two terms in (5.38) as:

$$
\begin{aligned}
& \int_0^T\langle \overline{Y}^*_{h\tau},y(u^*)-y_d\rangle\,dt+\tau\sum_{i=1}^N\langle y(u^*)(t_i),Y_h^{*i}-y_d(t_i)\rangle \\
=\;& \sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle \overline{Y}^*_{h\tau}-Y_h^{*i},y(u^*)\rangle\,dt+\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle Y_h^{*i},y(u^*)\rangle\,dt \\
& -\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle \overline{Y}^*_{h\tau},y_d-y_d(t_i)\rangle\,dt-\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle \overline{Y}^*_{h\tau},y_d(t_i)\rangle\,dt \\
& +\sum_{i=1}^N\int_{t_{i-1}}^{t_i}\langle y(u^*)(t_i),Y_h^{*i}-y_d(t_i)\rangle\,dt.
\end{aligned}
$$

This, along with (5.39), implies that

$$
\begin{aligned}
(5.40)\quad & -\int_0^T \langle y(u^*), y(u^*) - y_d\rangle\,dt - \tau\sum_{i=1}^N \langle Y_h^{*i}, Y_h^{*i} - y_d(t_i)\rangle \\
& + \int_0^T \langle \overline{Y}^*_{h\tau}, y(u^*) - y_d\rangle\,dt + \tau\sum_{i=1}^N \langle y(u^*)(t_i), Y_h^{*i} - y_d(t_i)\rangle \\
\le\ & \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \overline{Y}^*_{h\tau} - Y_h^{*i}, y(u^*)\rangle\,dt + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle y(u^*)(t_i) - y(u^*), Y_h^{*i}\rangle\,dt \\
& + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle y(u^*) - y(u^*)(t_i), y_d\rangle\,dt + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle y(u^*)(t_i), y_d - y_d(t_i)\rangle\,dt \\
& + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle Y_h^{*i} - \overline{Y}^*_{h\tau}, y_d(t_i)\rangle\,dt - \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \overline{Y}^*_{h\tau}, y_d - y_d(t_i)\rangle\,dt \\
\triangleq\ & \sum_{i=1}^6 Q_i.
\end{aligned}
$$

Next, we shall estimate terms $Q_i, 1 \le i \le 6$ in (5.40).

With regard to the term Q_1, we infer from (5.3) that

$$
\begin{aligned}
(5.41)\quad Q_1 \ \equiv\ & \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle \overline{Y}^*_{h\tau} - Y_h^{*i}, y(u^*)\rangle\,dt \\
=\ & \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (t - t_i)\langle \partial_\tau Y_h^{*i}, y(u^*)\rangle\,dt \le C\sum_{i=1}^N \tau^2 \|\partial_\tau Y_h^{*i}\|_{L^2(\Omega)} \\
\le\ & C\tau\left(\sum_{i=1}^N \tau\|\partial_\tau Y_h^{*i}\|^2_{L^2(\Omega)}\right)^{\frac12} \\
\le\ & C\tau.
\end{aligned}
$$

Concerning the term Q_2, we derive from (5.3) that

$$
\begin{aligned}
(5.42)\quad Q_2 \ \equiv\ & \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle y(u^*)(t_i) - y(u^*), Y_h^{*i}\rangle\,dt \\
\le\ & \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \tau^{\frac12}\left(\int_{t_{i-1}}^{t_i} \|\partial_t y(u^*)\|^2_{L^2(\Omega)}\,dt\right)^{\frac12} \|Y_h^{*i}\|_{L^2(\Omega)}\,dt \\
\le\ & \tau\left(\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\partial_t y(u^*)\|^2_{L^2(\Omega)}\,dt\right)^{\frac12} \Big(\sum_{i=1}^N \tau\|Y_h^{*i}\|^2_{L^2(\Omega)}\Big)^{\frac12} \\
\le\ & C\tau.
\end{aligned}
$$

About terms Q_i *with* $i = 3, \cdots, 6$, we can utilize the similar methods to get

$$(5.43)\qquad Q_3 + Q_4 + Q_5 + Q_6 \le C\tau.$$

Now, by utilizing (5.38) and (5.40)-(5.43), we find that

$$(5.44)\qquad J_2 + J_3 + J_4 + J_5 \le C(h + \tau^{\frac12})^{\frac12} + C\tau h^{-1}(h + \tau^{\frac12})^{-\frac12}.$$

Finally, putting (5.18), (5.20) and (5.44) together, we complete the proof of the theorem.

□

Remark 5.4. *Although the control, state and adjoint state have the same regularity as those in* [12]*, estimates about the multiplier* $-\mu^*_{h\tau}$ *corresponding to* $(P_{h\tau})$ *are weaker than those in* [12]*. For example, by the proof of (5.13)-(5.15), we know*

$$\|\mu^*_{h\tau}\|_{H^1_0(\Omega)} = \|p_h^{*N}\|_{H^1_0(\Omega)} \le C\lambda_{h\tau}^{-1}\|Q_h a_{h\tau}\|_{H^1_0(\Omega)} \le C(h+\tau^{\frac{1}{2}})^{-\frac{1}{2}}h^{-1}.$$

However, in [12]*,* $\|\mu^*_{h\tau}\|_{H^1_0(\Omega)}$ *is bounded by a constant independent of* h *and* τ*. Weak estimates about* $-\mu^*_{h\tau}$ *are due to the construction of penalty functional in* $(P_{h\tau})$*, and lead to weak error estimate about optimal controls between* (P) *and* $(P_{h\tau})$*.*

Appendix

Proof of Theorem 2.3. The proof of the "if" part is standard. We aim to show the "only if" part. For this purpose, we shall first build an approximation problem (P_ε), with $\varepsilon > 0$, to the problem (P). Write $Y = L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and recall that $d_S(\cdot)$ denotes the distance function from $\cdot$ to S in $L^2(\Omega)$. Let J_ε be the penalty functional from $Y \times L^2(0,T;L^2(\Omega))$ to R^+, defined by

$$J_\varepsilon(y,u) = [d_S(y(T))+\varepsilon]^2/2\varepsilon + \frac{1}{2}\int_0^T\int_\Omega[(y-y_d)^2+u^2]\,dx\,dt. \tag{1}$$

Consider the following optimal control problem (P_ε):

(P_ε) Min$J_\varepsilon(y,u)$, over all such pairs $(y,u)\in Y\times L^2(0,T;L^2(\Omega))$ that (1.1) holds.

When an pair $(y_\varepsilon,u_\varepsilon)$ solves the problem (P_ε), it will be called an optimal pair, while u_ε and y_ε are called an optimal control and an optimal state, respectively. Now, we shall carry out the proof with several stages.

Stage 1. The existence of optimal pairs $(y_\varepsilon,u_\varepsilon)$ *to the problem* (P_ε)*.*

Let $d^* = \inf J_\varepsilon(y,u)$, where the infimum is taken over all pairs $(y,u)\in Y\times L^2(0,T;L^2(\Omega))$ satisfying the equation (1.1). It is obvious that $d^*\ge 0$. Hence, there exists a sequence $\{(y_m,u_m)\}_{m=1}^\infty$ in $Y\times L^2(0,T;L^2(\Omega))$, such that

$$[d_S(y_m(T))+\varepsilon]^2/2\varepsilon + \frac{1}{2}\int_0^T\int_\Omega[(y_m-y_d)^2+u_m^2]\,dx\,dt \le d^*+\frac{1}{m} \tag{2}$$

and

$$\begin{cases} \partial_t y_m - \triangle y_m = \chi_\omega u_m & \text{in} \quad \Omega\times(0,T),\\ y_m = 0 & \text{on} \quad \partial\Omega\times(0,T),\\ y_m(0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{3}$$

By (2), we see that the sequence $\{u_m\}_{m=1}^\infty$ is bounded in $L^2(0,T;L^2(\Omega))$. Then, we can use the equation (3) to get the following estimate:

$$\|y_m\|_{L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))} + \|y'_m\|_{L^2(0,T;L^2(\Omega))} + \|u_m\|_{L^2(0,T;L^2(\Omega))} \le C.$$

Here, C stands for a positive constant independent of m. Thus, we can take a subsequence from $\{m\}_{m=1}^{\infty}$, still denoted in the same way, such that when $m \to \infty$,

$$\begin{aligned} y_m &\to \tilde{y} \quad \text{weakly in} \quad L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega)),\\ y_m &\to \tilde{y} \quad \text{strongly in} \quad C([0,T];L^2(\Omega)) \end{aligned}$$

and

$$u_m \to \tilde{u} \ \text{ weakly in } \ L^2(0,T;L^2(\Omega)).$$

Therefore, by passing to the limit for $m \to \infty$ in (2) and (3), respectively, we derive that

$$[d_S(\tilde{y}(T))+\varepsilon]^2/2\varepsilon + \frac{1}{2}\int_0^T\int_\Omega [(\tilde{y}-y_d)^2+\tilde{u}^2]\,dx\,dt \le d^* \tag{4}$$

and

$$\begin{cases} \partial_t\tilde{y}-\triangle\tilde{y}=\chi_\omega\tilde{u} & \text{in} \quad \Omega\times(0,T),\\ \tilde{y}=0 & \text{on} \quad \partial\Omega\times(0,T),\\ \tilde{y}(0)=y_0 & \text{in} \quad \Omega. \end{cases} \tag{5}$$

It follows at once from (4) and (5) that $(\tilde{y},\tilde{u})$ is an optimal pair to the problem (P_ε).

Stage 2. The convergence of the problem (P_ε). More precisely, there exists a subsequence of the family $\{\varepsilon\}_{\varepsilon>0}$, still denoted in the same way, such that when $\varepsilon \to 0^+$,

$$J_\varepsilon(y_\varepsilon,u_\varepsilon) \to J(y(u^*),u^*),$$

$$y_\varepsilon \to y(u^*) \ \textit{weakly in} \ L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$$

and

$$u_\varepsilon \to u^* \ \textit{weakly in} \ L^2(0,T;L^2(\Omega)).$$

Since the pair $(y_\varepsilon,u_\varepsilon)$ is optimal for the problem (P_ε), we infer from (1) that

$$J_\varepsilon(y_\varepsilon,u_\varepsilon) \le J_\varepsilon(y(u^*),u^*) = \frac{\varepsilon}{2}+J(y(u^*),u^*),$$

which gives

$$\overline{\lim}_{\varepsilon\to 0} J_\varepsilon(y_\varepsilon,u_\varepsilon) \le J(y(u^*),u^*). \tag{6}$$

It also yields that

$$\int_0^T\int_\Omega u_\varepsilon^2\,dx\,dt \le C \quad \text{and} \quad d_S(y_\varepsilon(T)) \le C\varepsilon^{\frac{1}{2}}. \tag{7}$$

Here, C denotes a positive constant independent of ε. By the first estimate in (7) and (1.1), we can utilize the same arguments as those in Stage 1 to find a subsequence of the family $\{\varepsilon\}_{\varepsilon>0}$, still denoted in the same way, such that when $\varepsilon \to 0$,

$$\begin{aligned} y_\varepsilon &\to \overline{y} \quad \text{weakly in} \quad L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega)),\\ y_\varepsilon &\to \overline{y} \quad \text{strongly in} \quad C([0,T];L^2(\Omega)),\\ u_\varepsilon &\to \overline{u} \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). \end{aligned} \tag{8}$$

Furthermore, one can easily check that $(\overline{y},\overline{u})$ solves the equation:

$$\begin{cases} \partial_t\overline{y}-\triangle\overline{y}=\chi_\omega\overline{u} & \text{in} \quad \Omega\times(0,T),\\ \overline{y}=0 & \text{on} \quad \partial\Omega\times(0,T),\\ \overline{y}(0)=y_0 & \text{in} \quad \Omega. \end{cases} \tag{9}$$

It follows from (1), (8) and (9) that

$$\underline{\lim}_{\varepsilon\to 0} J_\varepsilon(y_\varepsilon,u_\varepsilon) \geq \underline{\lim}_{\varepsilon\to 0}\frac{1}{2}\int_0^T\int_\Omega [(y_\varepsilon - y_d)^2 + u_\varepsilon^2]\,dx\,dt \geq J(\overline{y},\overline{u}). \tag{10}$$

Note that the second estimate in (7) and (8) yield that $\overline{y}(T) \in S$. Hence, $\overline{u}$ is admissible for the problem (P). Therefore, from the optimality of the pair $(y(u^*),u^*)$ to the problem (P), it follows that $J(\overline{y},\overline{u}) \geq J(y(u^*),u^*)$, which, together with (10) and (6) gives

$$\lim_{\varepsilon\to 0} J_\varepsilon(y_\varepsilon,u_\varepsilon) = J(\overline{y},\overline{u}) = J(y(u^*),u^*).$$

Hence, $(\overline{y},\overline{u})$ is an optimal pair for the problem (P). However, according to Theorem 2.2, the problem (P) has a unique optimal control. Thus, we must have $(\overline{y},\overline{u}) = (y(u^*),u^*)$. This, together with (8), gives the desired convergence of $\{(y_\varepsilon,u_\varepsilon)\}$.

Stage 3. Necessary conditions for an optimal pair $(y_\varepsilon,u_\varepsilon)$. Namely, there exist a positive constant λ_ε, functions $a_\varepsilon \in L^2(\Omega)$ and $p_\varepsilon \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;(H_0^1(\Omega))^*)$ satisfying:

$$\begin{cases} \partial_t y_\varepsilon - \triangle y_\varepsilon = \chi_\omega u_\varepsilon & \text{in} \quad \Omega\times(0,T), \\ y_\varepsilon = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ y_\varepsilon(0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{11}$$

$$\begin{cases} \partial_t p_\varepsilon + \triangle p_\varepsilon = -\lambda_\varepsilon(y_\varepsilon - y_d) & \text{in} \quad \Omega\times(0,T), \\ p_\varepsilon = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ p_\varepsilon(T) = a_\varepsilon & \text{in} \quad \Omega, \end{cases} \tag{12}$$

$$\chi_\omega p_\varepsilon = -\lambda_\varepsilon u_\varepsilon, \tag{13}$$

$$a_\varepsilon \in \partial d_S(y_\varepsilon(T)), \tag{14}$$

$$\|a_\varepsilon\|_{L^2(\Omega)} = \begin{cases} 1 & \text{if } y_\varepsilon(T) \notin S, \\ 0 & \text{if } y_\varepsilon(T) \in S, \end{cases} \tag{15}$$

and

$$\lambda_\varepsilon = \frac{\varepsilon}{\varepsilon + d_S(y_\varepsilon(T))}. \tag{16}$$

Corresponding to each $v \in L^2(0,T;L^2(\Omega))$ and $\lambda > 0$, we let $y_{\lambda,v}$ be the solution to the following equation:

$$\begin{cases} \partial_t y_{\lambda,v} - \triangle y_{\lambda,v} = \chi_\omega(u_\varepsilon + \lambda v) & \text{in} \quad \Omega\times(0,T), \\ y_{\lambda,v} = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ y_{\lambda,v}(0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{17}$$

Then, we write

$$z = \frac{y_{\lambda,v} - y_\varepsilon}{\lambda}. \tag{18}$$

Since $(y_\varepsilon,u_\varepsilon)$ solves the equation (11), we infer from (17) that

$$\begin{cases} \partial_t z - \triangle z = \chi_\omega v & \text{in} \quad \Omega\times(0,T), \\ z = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ z(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{19}$$

Because the pair $(y_\varepsilon, u_\varepsilon)$ is optimal to the problem (P_ε), we find that

$$\frac{J_\varepsilon(y_{\lambda,v}, u_\varepsilon + \lambda v) - J_\varepsilon(y_\varepsilon, u_\varepsilon)}{\lambda} \geq 0.$$

By (1) and (18), we can pass to the limit for $\lambda \to 0^+$ in the above inequality to get

$$\frac{d_S(y_\varepsilon(T)) + \varepsilon}{\varepsilon}\langle a_\varepsilon, z(T)\rangle + \int_0^T \int_\Omega (y_\varepsilon - y_d) \cdot z \, dx \, dt + \int_0^T \int_\Omega u_\varepsilon \cdot v \, dx \, dt \geq 0, \tag{20}$$

where a_ε satisfies (14) and (15). Let λ_ε be the number given by (16). Then it follows from (20) that for each $v \in L^2(0,T;L^2(\Omega))$,

$$\langle a_\varepsilon, z(T)\rangle + \lambda_\varepsilon \int_0^T \int_\Omega (y_\varepsilon - y_d) \cdot z \, dx \, dt + \lambda_\varepsilon \int_0^T \int_\Omega u_\varepsilon \cdot v \, dx \, dt \geq 0. \tag{21}$$

Write p_ε for the solution given by (12). Then, multiplying both sides of (19) by p_ε and integrating it over $\Omega \times (0,T)$, we obtain the identity:

$$\begin{aligned} &\langle z(T), p_\varepsilon(T)\rangle - \int_0^T \int_\Omega (p_\varepsilon' + \triangle p_\varepsilon) z \, dx \, dt \\ = \;& \int_0^T \int_\Omega \chi_\omega v \cdot p_\varepsilon \, dx \, dt, \qquad \forall\, v \in L^2(0,T;L^2(\Omega)). \end{aligned}$$

This, combined with (12) and (21), leads to the inequality:

$$\int_0^T \int_\Omega (\chi_\omega p_\varepsilon + \lambda_\varepsilon u_\varepsilon) \cdot v \, dx \, dt \geq 0, \;\; \forall\, v \in L^2(0,T;L^2(\Omega)),$$

which gives (13).

Stage 4. Passing to the limit for $\varepsilon \to 0$ in (12)-(14).

We first observe that the property (14) is equivalent to the following:

$$\langle a_\varepsilon, y_\varepsilon(T) - s\rangle \geq 0, \;\; \forall\, s \in S. \tag{22}$$

By (15) and (16), we get

$$1 \leq \|a_\varepsilon\|^2_{L^2(\Omega)} + \lambda_\varepsilon^2 \leq 2. \tag{23}$$

This, combined with (12) and the convergence results established in Stage 2, yields the estimate:

$$\begin{aligned} &\|p_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} + \|p_\varepsilon'\|_{L^2(0,T;(H_0^1(\Omega))^*)} \\ \leq \;& C(\lambda_\varepsilon\|y_\varepsilon - y_d\|_{L^2(0,T;L^2(\Omega))} + \|a_\varepsilon\|_{L^2(\Omega)}) \leq C. \end{aligned}$$

Here, C stands for two different positive constants independent of ε. Hence, it follows from (23) and the above estimate that there exists a subsequence of $\{\varepsilon\}_{\varepsilon>0}$, still denoted in the same way, such that when $\varepsilon \to 0$,

$$a_\varepsilon \to a_0 \;\text{ weakly in }\; L^2(\Omega), \;\lambda_\varepsilon \to \lambda_0 \tag{24}$$

and

$$p_\varepsilon \to p_0 \;\text{ weakly in }\; L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;(H_0^1(\Omega))^*). \tag{25}$$

Then, by (24), (25) and the convergence results obtained in Stage 2, we can pass to the limit for $\varepsilon \to 0$ in (12), (13) and (22) to get

$$\begin{cases} \partial_t p_0 + \triangle p_0 = -\lambda_0[y(u^*) - y_d] & \text{in} \quad \Omega \times (0,T), \\ p_0 = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ p_0(T) = a_0 & \text{in} \quad \Omega, \end{cases} \tag{26}$$

$$\chi_\omega p_0 = -\lambda_0 u^*, \tag{27}$$

$$\langle a_0, y(u^*)(T) - s\rangle \geq 0, \quad \forall\, s \in S. \tag{28}$$

Stage 5. Non-triviality of λ_0.

To prove that $\lambda_0 \neq 0$, we may assume that $\lambda_0 = 0$ and arrive at a contradiction. On one hand, we infer from (27), (26) and the uniqueness continuation for the heat equation ([9]) that $p_0 = 0$ a.e. in $\Omega \times (0, T)$. Hence,

$$a_0 = p_0(T) = 0. \tag{29}$$

On the other hand, it follows from (23) and (24) that there exists a positive constant ε_0, such that when $\varepsilon \leq \varepsilon_0$,

$$\|a_\varepsilon\|_{L^2(\Omega)} \geq \frac{1}{2}. \tag{30}$$

Moreover, by (22), (23) and the convergence results obtained in Stage 2, we deduce that

$$\langle a_\varepsilon, y(u^*)(T) - s\rangle \geq \langle a_\varepsilon, y(u^*)(T) - y_\varepsilon(T)\rangle \to 0 \text{ when } \varepsilon \to 0. \tag{31}$$

Since S is a set of finite codimension in $L^2(\Omega)$, it follows from Proposition 3.4 of Chapter 4 in [5] that so does the set $y(u^*)(T) - S \triangleq \{y(u^*)(T) - s : \ s \in S\}$. Therefore, according to Lemma 3.6 of Chapter 4 in [5], we infer from (31), (30) and (24) that $a_0 \neq 0$. It contradicts to (29) and the non-triviality of λ_0 is proved.

Stage 6. End of the proof. Because of (16), (24), we necessarily have $\lambda_0 > 0$. Let $p^* = -p_0/\lambda_0$ and $\mu^* = a_0/\lambda_0$. Then (2.4), (2.6) and (2.7) follow directly from (28), (26) and (27). This completes the proof.

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School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China.
E-mail: wanggs62@yeah.net and ljwang.math@whu.edu.cn