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A SINGULARLY PERTURBED CONVECTION–DIFFUSION PROBLEM WITH A MOVING INTERIOR LAYER

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Abstract. A singularly perturbed parabolic equation of convection-diffusion type with an interior layer in the initial condition is studied. The solution is decomposed into a discontinuous regular component, a continuous outflow boundary layer component and a discontinuous interior layer component. A priori parameter-explicit bounds are derived on the derivatives of these three components. Based on these bounds, a parameter-uniform Shishkin mesh is constructed for this problem. Numerical analysis is presented for the associated numerical method, which concludes by showing that the numerical method is a parameter-uniform numerical method. Numerical results are presented to illustrate the theoretical bounds on the error established in the paper.

Key Words. Singular perturbation, interior layer, Shishkin mesh.

1. Introduction

The solutions of singularly perturbed parabolic equations of convection-diffusion type typically contain boundary layers [1, 6, 16], which can appear along the boundary corresponding to the outflow boundary of the problem. Additional interior layers can form in the solution if either the coefficients, the inhomogenous term or the boundary/initial conditions are not sufficiently smooth [2]. In this paper, we examine a linear singularly perturbed parabolic problem with smooth data and an interior layer in the solution, which is created by artificially inserting a layer into the initial condition. This problem is motivated from studying a singularly perturbed parabolic problem of convection-diffusion type, with a singularity generated by a discontinuity between the boundary and initial conditions at the inflow corner. At some distance from this inflow corner, the solution is characterized by the presence of an interior layer moving in time along the characteristic of the reduced problem, which passes through the inflow corner. This paper formulates a related problem which captures this effect of an interior layer being transported in a convection-diffusion parabolic problem. Our interest is to design and analyse a parameter-uniform numerical method [6] for such a problem.

Parameter-uniform numerical methods for several classes of singularly perturbed parabolic problems of the form

$$-\varepsilon u_{xx} + au_x + bu + u_t = f, \quad (x,t) \in (0,1) \times (0,T],$$

with discontinuous coefficients (or discontinuous inhomogeneous term) have been constructed and analysed in [4, 15]. These methods are based on upwind discretizations combined with appropriate piecewise-uniform Shishkin meshes [6], which are aligned to the trajectory of the point where the data are discontinuous [15]. In

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[4, 15], it is assumed that the boundary/initial conditions are sufficiently smooth and compatible at the corners. The nature and width of any interior layers appearing in the solutions of these problems, is dictated mainly by the sign of the convective coefficient a either side of a discontinuity. In this paper, we examine a problem where the coefficient a is smooth and always positive, but the solution contains a strong interior layer, generated solely from the fact that the initial condition contains an internal layer.

Parameter-uniform numerical methods for singularly perturbed parabolic problems with a discontinuous initial condition have been examined by Shishkin et al. in a series of papers (see [7, 10, 11] and the references therein). Rather than using simple upwind finite difference operators on a piecewise-uniform mesh, Shishkin et al. use suitable fitted operator methods to capture the singularity in the neighbourhood of the discontinuity. In this paper, the initial solution is smooth, but contains an interior layer. We will see that it is not necessary to use a fitted operator method here, as a suitable Shishkin mesh combined with a standard upwind finite difference operator suffices to generate a parameter-uniform method.

This paper is structured as follows: In §2, we state the problem to be investigated. In §3, we employ a mapping [5] which is used to align the mesh to the location of the interior layer. The solution is decomposed into a sum of a regular component, a boundary layer component and an interior layer component. In §4, we examine the regular component and deduce parameter-explicit bounds on its derivatives. In §5 parameter-explicit bounds on the derivatives of the layer components are established, which are central to the design of a piecewise-uniform Shishkin mesh, given in §6. In §7, the associated numerical analysis is presented and in the final §8 some numerical results are given to illustrate the theoretical error bounds established in §7.

Notation. The space $\mathcal{C}^{0+\gamma}(D)$, where $D \subset \mathbf{R}^2$ is an open set, is the set of all functions that are Hölder continuous of degree γ with respect to the metric $\|\cdot\|$, where for all $\mathbf{u}=(u_1,u_2),\ \mathbf{v}=(v_1,v_2)\in\mathbf{R}^2$, $\|\mathbf{u}-\mathbf{v}\|^2=(u_1-v_1)^2+|u_2-v_2|$. For f to be in $\mathcal{C}^{0+\gamma}(D)$ the following semi-norm needs to be finite

$$\lceil f \rceil_{0+\gamma,D} = \sup_{\mathbf{u}\neq\mathbf{v},\ \mathbf{u},\mathbf{v}\in D} \frac{|f(\mathbf{u})-f(\mathbf{v})|}{\|\mathbf{u}-\mathbf{v}\|^{\gamma}}.$$

The space $\mathcal{C}^{n+\gamma}(D)$ is defined by

$$\mathcal{C}^{n+\gamma}(D)=\{z : \frac{\partial^{i+j}z}{\partial x^i \partial y^j}\in \mathcal{C}^{0+\gamma}(D),\ 0\le i+2j\le n\},$$

and $\|\cdot\|_{n+\gamma},\ \lceil\cdot\rceil_{n+\gamma}$ are the associated norms and semi-norms. Throughout the paper, c or C denotes a generic constant that is independent of the singular perturbation parameter ε and of all discretization parameters.

2. Continuous problem

In this paper, we examine the following singularly perturbed parabolic problem: Find $\hat{u}$ such that

$$\hat{\mathcal{L}}_\varepsilon \hat{u} := -\varepsilon \hat{u}_{ss} + \hat{a}(t)\hat{u}_s + \hat{u}_t = \hat{f}(s,t), \quad (s,t)\in Q := (0,1)\times(0,T], \tag{1a}$$

$$\hat{u}(s,0)=\phi(s;\varepsilon), 0\le s\le 1, \tag{1b}$$

$$\hat{u}(0,t)=\phi_L(t),\ \hat{u}(1,t)=\phi_R(t),\ 0<t\le T, \tag{1c}$$

$$\hat{a}(t)>\alpha>0, \tag{1d}$$

where the initial condition ϕ is smooth, but contains an interior layer in the vicinity of a point $s = d, 0 < d < 1$ and d is independent of ε. The function ϕ is defined as the solution of the singularly perturbed two point boundary value problem

$$-\varepsilon\phi'' + b(s)\phi = f_1(s) := q(s) + A\tanh(\frac{\sqrt{\alpha_2}(s-d)}{2\sqrt{\varepsilon}}), \tag{1e}$$

$$b(s) \geq \beta > 2\alpha,\ \alpha_2 \geq 2\alpha,\ \phi(0) = \phi_0(0),\ \phi(1) = \phi_0(1), \tag{1f}$$

where $\phi_0(s)$ is the reduced discontinuous initial condition defined by

$$b(s)\phi_0(s) := q(s) - A,\ s < d; \qquad b(s)\phi_0(s) := q(s) + A,\ s > d.$$

Our focus will be on the influence this interior layer has on the solution of the time dependent problem. Hence, the boundary values (1f) for the initial condition have been chosen to dampen boundary layers appearing in the initial condition. We also design q so that

$$\phi_0''(0) = \phi_0''(1) = 0,$$

which further reduces the amplitude of any boundary layers in the initial condition.

The zero level of compatibility required such that $\hat{u}$ is continuous on the boundary $\bar{Q} \setminus Q$ is specified by

$$\phi_L(0) = \phi(0),\ \phi_R(0) = \phi(1). \tag{1g}$$

We can increase the level of compatibility by assuming that

$$\big(-\varepsilon\frac{d^2\phi}{ds^2} + \hat{a}\frac{d\phi}{ds} + \frac{d\phi_L}{dt}\big)(0,0) = \hat{f}(0,0), \tag{1h}$$

and a similar condition is assumed to hold at the corner $(1, 0)$.

Differentiate (1a) with respect to time and let $\hat{y} = \hat{u}_t$ then

$$\begin{aligned} -\varepsilon\hat{y}_{ss} + \hat{a}\hat{y}_s + \hat{y}_t &= \hat{f}_t - \hat{a}_t\hat{u}_s,\ (s,t) \in Q, \\ \hat{y}(s,0) = \phi_1(s) &:= \hat{f}(s,0) + \varepsilon\phi''(s) - \hat{a}(0)\phi'(s), 0 \leq s \leq 1, \\ \hat{y}(0,t) = \phi_L'(t),\ \hat{y}(1,t) &= \phi_R'(t),\ 0 < t \leq T. \end{aligned}$$

Imposing the first level compatibility conditions on this problem leads to the second level compatibility conditions for problem (1). At the corner $(0, 0)$, we require that

$$-\varepsilon\frac{d^2\phi_1}{ds^2} + \hat{a}\frac{d\phi_1}{ds} + \frac{d^2\phi_L}{dt^2} = \frac{\partial\hat{f}}{\partial t} - \frac{\partial\hat{a}}{\partial t}\frac{d\phi}{ds}, \tag{1i}$$

and analogously at the corner $(1, 0)$. With all these compatibility conditions (1g), (1h),(1i), at both corners and if $\hat{a}, \hat{f} \in \mathcal{C}^{2+\gamma}(\bar{Q})$ then the solution of problem (1) $\hat{u} \in \mathcal{C}^{4+\gamma}(\bar{Q})$ [12, pg. 320].

The characteristic curve associated with the reduced differential equation (formally set $\varepsilon = 0$ in (1a)) can be described by the set of points

$$\Gamma^* := \{(d(t), t)|d'(t) = \hat{a}(t),\ 0 < d(0) = d < 1\}.$$

We also define the two subdomains of Q either side of Γ^* by

$$Q^- := \{(s,t)|s < d(t) < 1\} \quad \text{and} \quad Q^+ := \{(s,t)|s > d(t) > 0\}.$$

The solution of problem (1) will have an interior layer of width $\mathcal{O}(\sqrt{\varepsilon})$ (emanating from the initial condition) which travels along Γ^*. In general a boundary layer of width $\mathcal{O}(\varepsilon)$ will also appear in the vicinity of the edge $x = 1$. We restrict the size of the final time T so that the interior layer does not interact with this boundary layer. Since $\hat{a} > 0$, the function $d(t)$ is monotonically increasing. Thus, we limit the final time T such that $0 < c < 1 - d(T)$. We define the parameter

$$\delta := \frac{1 - d(T)}{1 - d} > 0, \tag{2}$$

which plays an important role throughout this paper.

In later sections, we construct a piecewise-uniform mesh, which is designed to be refined in the neighbourhood of the curve Γ^*. To analyse the parameter-uniform convergence of the resulting numerical approximations on such a mesh, it is more convenient to perform the analysis in a transformed domain where the location of the interior layer is fixed in time. As most of the paper deals with this transformed domain, we have adopted the notation $\hat{u}(s,t)$ for the solution in the original domain and we use the simpler notation of $u(x,t)$ for the solution in the transformed domain.

3. Mapping to fix the location of interior layer

Consider the map $X : (s,t) \to (x,t)$ given by

$$x(s,t) = \begin{cases} \dfrac{d}{d(t)} s, & s \le d(t), \\ 1 - \dfrac{1-d}{1-d(t)}(1-s), & s \ge d(t). \end{cases} \tag{3}$$

Note that $x = s$ at $t = 0$ and $x = d$ for all t such that $s = d(t)$. This maps

$$\bar{Q}^- \to \bar{\Omega}^- := [0,d] \times [0,T], \quad \bar{Q}^+ \to \bar{\Omega}^+ := [d,1] \times [0,T]. \tag{4}$$

Remark 3.1. *We employ the following notation in subsequent sections:*

$$u(x,t) := \hat{u}(X(s,t),t).$$

Noting that $d'(t) = \hat{a}(t), d(0) = d$, we have for $s < d(t)$ or $x < d$ that

$$\frac{\partial \hat{u}}{\partial t} = \frac{-a(t)}{d(t)} x \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}, \quad \frac{\partial \hat{u}}{\partial s} = \frac{d}{d(t)} \frac{\partial u}{\partial x}.$$

Using this map, the differential equation (1a) transforms into

$$\mathcal{L}_\varepsilon u := \big(-\varepsilon u_{xx} + \kappa(x,t) u_x\big) + g(x,t) u_t = g(x,t) f(x,t), \ x \ne d, \tag{5a}$$

$$u(x,0) = \phi(x;\varepsilon), \ 0 < x \le 1, u(0,t) = \phi_L(t), \ u(1,t) = \phi_R(t), \ 0 < t \le T, \tag{5b}$$

$$\kappa(x,t) := \begin{cases} \dfrac{d(t)a(t)}{d}\big(1 - \dfrac{x}{d}\big), & x < d, \\ \dfrac{(1-d(t))a(t)}{1-d}\big(1 - \dfrac{1-x}{1-d}\big), & x > d, \end{cases} \tag{5c}$$

$$g(x,t) := \begin{cases} \big(\dfrac{d(t)}{d}\big)^2, & x < d, \\ \big(\dfrac{1-d(t)}{1-d}\big)^2, & x > d. \end{cases} \tag{5d}$$

Note that for any t such $d(t) \ne d$, then $g(d^-,t) \ne g(d^+,t)$. Also for $x > d$,

$$g(x,t) \ge \delta^2 > 0, \qquad \alpha\delta \frac{x-d}{1-d} \le \kappa(x,t) \le \|a\| \frac{x-d}{1-d}.$$

In this transformed problem, the coefficient $\kappa(x,t)$ of the first derivative in space is positive, except along the internal line $x = d$ where it is zero. To have a well-posed problem for u, we seek a solution $u \in \mathcal{C}^1(\bar{\Omega})$ satisfying (5). As $\hat{u} \in \mathcal{C}^{4+\gamma}(\bar{Q})$ then $u \in \big(\mathcal{C}^{4+\gamma}(\bar{\Omega}^-) \cup \mathcal{C}^{4+\gamma}(\bar{\Omega}^+)\big) \cap \mathcal{C}^1(\bar{\Omega})$.

Note that, by using a comparison principle for the steady state differential operator $(-\varepsilon w'' + bw)(x)$, $x \in (0,1)$; $w(x)$, $x \in \{0,1\}$, we deduce that $\|\phi(x)\| \leq C$, where ϕ is defined in (1e). We also associate the following differential operator

$$\mathcal{L}'_\varepsilon \omega(x,t) := \begin{cases} \omega(x,t), & x = 0,1,\ t \geq 0, \\ -\varepsilon\omega_{xx}(x,0) + b(x)\omega(x,0), & x = (0,1), \\ -\varepsilon\omega_{xx} + \kappa(x,t)\omega_x + g(x,t)\omega_t, & x = (0,d)\cup(d,1),\ t > 0, \\ -[\omega_x], & x = d,\ t \geq 0, \end{cases}$$

with the problem (5) where $[z](d,t) := z(d^+,t) - z(d^-,t)$.

For the operator $\mathcal{L}'_\varepsilon$ a comparison principle holds.

Theorem 1. *Assume that a function* $\omega \in \mathcal{C}^0(\overline{\Omega^- \cup \Omega^+}) \cap \mathcal{C}^2(\Omega^- \cup \Omega^+)$ *satisfies* $\mathcal{L}'_\varepsilon \omega(x,t) \geq 0$, *for all* $(x,t) \in \overline{\Omega^- \cup \Omega^+}$ *then* $\omega(x,t) \geq 0$, *for all* $(x,t) \in \overline{\Omega^- \cup \Omega^+}$.

Proof. Consider the continuous function $\nu(x,t)$, defined by

$$\omega(x,t) = \begin{cases} e^{\frac{t}{2\varepsilon\delta^2}} e^{\frac{x-d}{2\varepsilon}} \nu(x,t), & (x,t) \in \bar{\Omega}^-, \\ e^{\frac{t}{2\varepsilon\delta^2}} e^{\frac{x-d}{4\varepsilon}} \nu(x,t), & (x,t) \in \bar{\Omega}^+. \end{cases}$$

Apply a standard proof by contradiction argument, where $x \neq d$ and d are treated seperately. □

From this, we deduce that the solution of problem (1) satisfies $\|\hat{u}\|_{\bar{Q}} \leq C$.

4. Discontinuous regular component

The initial function ϕ defined in (1e) is assumed to be sufficiently smooth and can be split into the sum of two discontinuous functions ϕ_v, ϕ_w, where $\phi = \phi_v + \phi_w$ and

$$-\varepsilon\phi_v'' + b\phi_v = b\phi_0, \quad s \neq d,$$
$$\phi_v(0) = \phi(0),\ b(d)\phi_v(d^\pm) = b(d)\phi_0(d^\pm) + \varepsilon\phi_0''(d^\pm),\ \phi_v(1) = \phi(1),$$
$$-\varepsilon\phi_w'' + b\phi_w = f_1(s) - b\phi_0, \quad s \neq d.$$

The discontinuous component ϕ_w satisfies a linear second order differential equation on the two subintervals $(0,d)$ and $(d,1)$. Hence, ϕ_w is well defined when we impose the additional four conditions:

$$[\phi_w](d) = -[\phi_v](d),\ [\phi_w'](d) = -[\phi_v'](d),\ \phi_w(0) = \phi_w(1) = 0.$$

We can further decompose the regular component ϕ_v such that

$$\phi_v(s) = \phi_0(s) + \varepsilon\phi_1(s) + \varepsilon^2\phi_2(s), \quad s \neq d,$$

where $b\phi_1 = \phi_0''$, $-\varepsilon\phi_2'' + b\phi_2 = \phi_1'', s \neq d, \phi_2(0) = \phi_2(1) = \phi_2(d) = 0$. For $s \neq d$, one can establish the following parameter-explicit bounds on the derivatives of the individual components $\phi_i, i = 0,1,2$

$$|\phi_0(s)|_k \leq C,\ |\phi_1(s)|_k \leq C,\ |\phi_2(s)|_k \leq C(1 + \varepsilon^{-k/2} e^{-\sqrt{\frac{2\alpha}{\varepsilon}}|s-d|}).$$

For the layer component ϕ_w, we see that on $(0,d)$

$$-\varepsilon\phi_w'' + b\phi_w = \frac{2Ae^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-s)}}{1 + e^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-s)}}, \quad |\phi_w(d^-)| \leq C.$$

Use the barrier function $Ce^{-\sqrt{\frac{2\alpha}{\varepsilon}}(d-x)}$ and a maximum principle to bound $\phi_w(x)$ on the interval $[0,d]$. Then following the arguments in [3], one can deduce that at all points $s \neq d$ and for all $k \leq 7$

$$|\phi_v(s)|_k \leq C(1+\varepsilon^{2-k/2}), \tag{6a}$$

$$|\phi_w(s)|_k \leq C\varepsilon^{-k/2}e^{-\sqrt{\frac{2\alpha}{\varepsilon}}|s-d|}. \tag{6b}$$

On the domain $\bar{Q}^-$ we will define the regular left component $\hat{v}^-(s,t)$ such that

$$\hat{\mathcal{L}}_\varepsilon \hat{v}^- = \hat{f},\ (s,t) \in Q^-, \hat{v}^-(0,t) = \phi_L(t),\ \hat{v}^-(s,0) = \phi_v(s) + \phi_c(s).$$

In the next theorem, we construct a function $r(t) = \hat{v}^-(d(t),t)$ satisfying second order compatibility at the corner $(d,0)$, so that there are no layers in $\hat{v}^-$ in the vicinity of $(d(t),t)$. Recall that ϕ, ϕ_L and $\hat{f}$ satisfy compatibility up to second order at the inflow corner $(0,0)$. The function $\phi_c(s)$ is a polynomial designed so that ϕ_v, ϕ_L and $\hat{f}$ satisfy compatibility up to second order and $\phi_c(s) \equiv 0, s \geq 0.5d$. From the bounds on ϕ_w in (6a) one can then deduce that

$$|\phi_c(s)|_k \leq Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}d},\ k \leq 7.$$

Theorem 2. *There exist functions $r_0(t), r_1(t), r_2(t)$ such that the solutions $\hat{v}^-$, $\hat{v}^+$ of the problems*

$$\hat{\mathcal{L}}_\varepsilon \hat{v}^- = \hat{f},\ (s,t) \in Q^-,\ \hat{v}^-(0,t) = \phi_L(t),\ \hat{v}^-(s,0) = \phi_v(s) + \phi_c(s),$$
$$\hat{\mathcal{L}}_\varepsilon \hat{v}^+ = \hat{f},\ (s,t) \in Q^+,\ \hat{v}^+(s,0) = \phi_v(s),$$
$$\hat{v}^-(d(t),t) = r_0(t),\ \hat{v}^+(d(t),t) = r_1(t),\ \hat{v}^+(1,t) = r_2(t),$$

satisfy $\hat{v}^- \in \mathcal{C}^{4+\gamma}(\bar{Q}^-)$, $\hat{v}^+ \in \mathcal{C}^{4+\gamma}(\bar{Q}^+)$ and the bounds

$$\|\hat{v}^-\|_{Q^-} \leq C,\ \|\hat{v}^+\|_{Q^+} \leq C, \tag{7a}$$

$$\left\|\frac{\partial^{j+m}\hat{v}^-}{\partial s^j \partial t^m}\right\|_{Q^-} \leq C(1+\varepsilon^{2-(j+m)}), \quad 1 \leq j+2m \leq 4, \tag{7b}$$

$$\left\|\frac{\partial^{j+m}\hat{v}^+}{\partial s^j \partial t^m}\right\|_{Q^+} \leq C(1+\varepsilon^{2-(j+m)}), \quad 1 \leq j+2m \leq 4. \tag{7c}$$

Proof. The main argument is similar to the proofs given in [13, Theorem 3.3] and [15, Theorem 3.3]. Using smooth extensions of the data, the problem $\hat{\mathcal{L}}_\varepsilon \hat{v}^- = \hat{f}, (s,t) \in Q^-$ can be extended to the domain $(0,2) \times (0,T]$ so that

$$\hat{\mathcal{L}}^*_\varepsilon \hat{v}^* = \hat{f}^*,\ \hat{v}^*(0,t) = \phi_L(t),\ \hat{v}^*(s,0) = (\phi^*_v + \phi^*_c)(s),\ \hat{v}^*(2,t) = r^*(t),$$

where ϕ^*_v, ϕ^*_c are extensions of ϕ_v, ϕ_c to the interval $[0,2]$ so that the bounds (6a) are applicable for all $s \in [0,1]$. Then boundary data $r^*(t)$ can be specified so that $\hat{v}^* \in \mathcal{C}^{4+\gamma}([0,2] \times [0,T])$. Using the expansion $\hat{v}^* = \hat{v}^*_0 + \varepsilon\hat{v}^*_1 + \varepsilon^2\hat{v}^*_2$, where $\hat{v}^*_0$ is the solution of the extended reduced problem

$$\hat{\mathcal{L}}^*_0 \hat{v}^*_0 := (\hat{a}^*\partial_s + \partial_t)v^*_0 = \hat{f}^*, \hat{v}^*_0(0,t) = \hat{v}^*(0,t), \hat{v}^*_0(s,0) = \hat{v}^*(s,0),$$

and $\hat{\mathcal{L}}^*_0 \hat{v}^*_1 = (\hat{v}^*_0)''$ then the bounds in (7a)-(7b) hold. We now choose

$$r_0(t) := \hat{v}^*(d(t),t).$$

In the case of $\hat{v}^+$, we again extend the problem to the domain $(-2,2) \times (0,T]$ so that

$$\hat{\mathcal{L}}^{**}_\varepsilon \hat{v}^{**} = \hat{f}^{**},\ \hat{v}^{**}(s,0) = \phi^{**}_v(s).$$

Boundary data can be chosen on the boundaries $s = -2, s = 2$ so that the bounds (6a) apply to the extension $\hat{v}^{**}$. Then we choose

$$r_1(t) := \hat{v}^{**}(d(t), t), \qquad r_2(t) := \hat{v}^{**}(1, t).$$

The fact that the initial condition ϕ depends on ε introduces additional issues into the proof of this result. These technical issues are addressed in the appendix. Note that for $\hat{v}^-$ we use a transformation in space of the form $[0, d] \to [0, 1]$ (and for $\hat{v}^+$, $[d, 1] \to [0, 1]$) before using the arguments from the appendix. $\square$

5. Layer Components

The numerical method presented below in §6 is specified in the transformed domains $\Omega^- \cup \Omega^+$. In the subsequent numerical analysis section, it is more convenient to have bounds on the derivatives of the layer components in the transformed variables. Note that $u, v^+ \in \mathcal{C}^{4+\gamma}(\bar{\Omega}^+)$ and so

$$\mathcal{L}_\varepsilon(u - v^+) = 0, \quad \text{and} \quad u - v^+ \in \mathcal{C}^{4+\gamma}(\bar{\Omega}^+).$$

Hence, $(u - v^+)$ and $(\phi - \phi_v)$ satisfy second order compatibility conditions at the corner $(1, 0)$. We further decompose the function $u - v^+$ into two subcomponents $u - v^+ = w + z$, where

$$\mathcal{L}_\varepsilon w = \mathcal{L}_\varepsilon z = 0, \quad \text{and} \quad w(d, t) = z(1, t) = 0.$$

The boundary layer function w is the solution of

(8a) $$\mathcal{L}_\varepsilon w = 0, \quad (x, t) \in \Omega^+; \qquad w \equiv 0, \quad (x, t) \in \bar{\Omega}^-;$$

(8b) $$w(d, t) = 0, \; w(x, 0) = \phi_T(x) e^{-\frac{\alpha\delta}{\varepsilon}(1-x)}, \quad w(1, t) = u(1, t) - v^+(1, t),$$

where $\phi_T(x)$ is a polynomial function such that w satisfies the second order compatibility conditions at the corners $(d, 0), (1, 0)$. At the corner $(1, 0)$, we specify the values of $\phi_T^{(k)}(1), 0 \le k \le 4$ so that $\phi_T(1) = 0$,

$$-\varepsilon\phi_w''(1) + a(0)\phi_w'(1) = -\varepsilon(\phi_T(x)e^{-\frac{\alpha\delta}{\varepsilon}(1-x)})''(1) + a(0)(\phi_T(x)e^{-\frac{\alpha\delta}{\varepsilon}(1-x)})'(1),$$

and, in addition, second order compatibility is also satisfied. For example, by setting $\phi_T^{(k)}(d) = 0, 0 \le k \le 2$, $\phi_T'(1) = \phi_w'(1)$ and $\phi_T''(1) = (\phi_w'' - \frac{2\alpha\delta}{\varepsilon}\phi_w')(1)$ we ensure that the first order compatibility condition is satisfied at both corners. Using the bounds on the derivatives of ϕ_w at $s = 1$, we can deduce that $|\phi_T^{(k)}(1)| \le Ce^{-\sqrt{\frac{3\alpha}{2\varepsilon}}(1-d)}, 0 \le k \le 5$. Hence, we can have that

$$|\phi_T(x)|_k \le Ce^{-\sqrt{\frac{3\alpha}{2\varepsilon}}(1-d)}, \quad 0 \le k \le 5.$$

Theorem 3. *The solution of (8) $w \in \mathcal{C}^0(\bar{\Omega}) \cap \mathcal{C}^{4+\gamma}(\bar{\Omega}^+)$ and*

$$\begin{aligned}
|w(x, t)| &\le Ce^{-\frac{\alpha\delta}{\varepsilon}\int_{s=x}^{1}\frac{s-d}{1-d}\,ds}e^{\frac{\alpha t}{(1-d)\delta}}, \; (x, t) \in \Omega^+, \\
\left|\frac{\partial^j w}{\partial x^j}(x, t)\right| &\le C\varepsilon^{-j}e^{-\frac{\alpha\delta(1-x)}{2\varepsilon}}, \; 1 \le j \le 4, \; (x, t) \in \Omega^+, \\
\left|\frac{\partial^m w}{\partial t^m}(x, t)\right| &\le C\varepsilon^{1-m}, \; (x, t) \in \Omega^+, \quad m = 1, 2.
\end{aligned}$$

Proof. Consider the barrier function

$$\Phi_1(x, t) := Ce^{-\frac{\alpha\delta}{\varepsilon}\int_{s=x}^{1}\frac{s-d}{1-d}\,ds}e^{\frac{\alpha t}{(1-d)\delta}}.$$

Note first that for $x \ge d$,

$$|w(x, 0)| \le Ce^{-\frac{\alpha\delta}{\varepsilon}(1-x)} \le Ce^{-\frac{\alpha\delta}{\varepsilon}\int_{s=x}^{1}\frac{s-d}{1-d}\,ds},$$

and for $(x,t) \in \Omega^+$,

$$\mathcal{L}_\varepsilon \Phi_1 \geq \alpha\delta\big(\kappa\frac{x-d}{\varepsilon} - \alpha\delta\frac{(x-d)^2}{(1-d)\varepsilon}\big)\frac{\Phi_1}{1-d} \geq 0.$$

This yields the first bound on $|w(x,t)|$. Introduce the stretched variables $\zeta = (1-x)/\varepsilon$, $\tau = t/\varepsilon$. Then $\tilde{w}(\zeta,t)$ satisfies

$$-\tilde{w}_{\zeta\zeta} - \tilde{\kappa}\tilde{w}_\zeta + \tilde{g}\tilde{w}_\tau = 0,\ (\zeta,\tau) \in (0,\frac{1-d}{\varepsilon}) \times (0,\frac{T}{\varepsilon}),$$
$$\tilde{w}(\zeta,0) = \tilde{\phi}_T(\zeta)e^{-\alpha\delta\zeta}, \quad \tilde{w}(\frac{1-d}{\varepsilon},\tau) = 0,\ \tilde{w}(0,\tau) = u(1,\tau) - v^+(1,\tau).$$

We employ the a priori interior estimates in [8, pg. 123] or [12, pg. 352], where we also split the region between $\zeta \ \geq 1$ and $\zeta \leq 1$ (see also the argument in [13, Theorem 4]). Transforming back to the original variables and using

$$|w(x,t)| \leq Ce^{-\frac{\alpha\delta(1-x)}{2\varepsilon}}, \quad |\frac{\partial^i \tilde{w}(\zeta,0)}{\partial\zeta^i}| \leq Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}(1-d)}e^{-\alpha\delta\zeta}, \quad 0 \leq i \leq 5,$$

we deduce that

$$\Big|\frac{\partial^{j+m}w}{\partial x^j \partial t^m}(x,t)\Big| \leq C\varepsilon^{-j}\varepsilon^{-m}e^{-\frac{\alpha\delta(1-x)}{2\varepsilon}}, \quad 1 \leq j+2m \leq 4,\ (x,t) \in \Omega^+. \tag{9}$$

To obtain the final bound on the time derivatives, we follow the argument in [14, Lemma 3.9]. Note that

$$w(x,t) = w(x,0) + (w(1,t) - w(1,0))\phi(x,t) + \varepsilon R(x,t),$$

where $-\varepsilon\phi_{xx} + \kappa(1,t)\phi_x = 0,\ \phi(d,t) = 0, \phi(1,t) = 1$. As in [14], one can deduce that $\varepsilon\|R_t\| \leq C, \varepsilon^2\|R_{tt}\| \leq C$. □

The discontinuous multi-valued interior layer function z is defined as the solution of

$$\mathcal{L}_\varepsilon z = 0, \quad (x,t) \in \Omega^- \cup \Omega^+, \tag{10a}$$
$$[z](d,t) = -[v](d,t),\ [z_x](d,t) = -[v_x + w_x](d,t),\ t \geq 0, \tag{10b}$$
$$z(0,t) = z(1,t) = 0, \quad z(x,0) = \phi_w(x) - \phi_c(x) - w(x,0),\ x \neq d. \tag{10c}$$

Theorem 4. *The solution of (10) $z \in \mathcal{C}^{4+\gamma}(\bar{\Omega}^-) \cup \mathcal{C}^{4+\gamma}(\bar{\Omega}^+)$ and we have*

$$|z(x,t)| \ \leq \ Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-x)}, (x,t) \in \Omega^-, \tag{11a}$$
$$|z(x,t)| \ \leq \ Ce^{-e^{-\frac{\|a\|}{\delta(1-d)}T}\sqrt{\frac{\alpha}{\varepsilon}}(x-d)}, \quad (x,t) \in \Omega^+. \tag{11b}$$

If $d(1-\frac{d}{d(T)}) < x \leq d$ and $0 \leq t \leq T$, then

$$\Big|\frac{\partial^j z}{\partial x^j}(x,t)\Big| \ \leq \ C\varepsilon^{-j/2}e^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-x)},\ 1 \leq j \leq 4, \tag{11c}$$
$$\Big|\frac{\partial^m z}{\partial t^m}(x,t)\Big| \ \leq \ C, \quad m = 1,2. \tag{11d}$$

If $d \leq x < d + (1-d(T))$ and $0 \leq t \leq T$, then

$$\Big|\frac{\partial^j z}{\partial x^j}(x,t)\Big| \ \leq \ C\varepsilon^{-j/2}e^{-\frac{1-d(t)}{1-d}\sqrt{\frac{\alpha}{\varepsilon}}(x-d)},\ 1 \leq j \leq 4, \tag{11e}$$
$$\Big|\frac{\partial^m z}{\partial t^m}(x,t)\Big| \ \leq \ C, \quad m = 1,2. \tag{11f}$$

Proof. Recall that $z = u - (v + w)$ and so $z \in \mathcal{C}^{4+\gamma}(\bar{\Omega}^-) \cup \mathcal{C}^{4+\gamma}(\bar{\Omega}^+)$. Since u, v, w are bounded, we have that $\|z\| \le C$. Note that $|z(x,0)| \le Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-x)}, 0 < x < d$ and

$$\mathcal{L}_\varepsilon\big(e^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-x)\frac{d(t)}{d}}e^{\alpha t}\big) = 0.$$

Thus we have established

$$|z(x,t)| \le Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-x)\frac{d(t)}{d}}e^{\alpha t} \le Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-x)}, (x,t) \in \Omega^-. \tag{12}$$

To the right of the interface $x = d$ we introduce the barrier function,

$$\Phi_2(x,t) := Ce^{-e^{-C^*t}\sqrt{\frac{\alpha}{\varepsilon}}(x-d)}e^{\frac{C^*t}{\delta}}, \quad (x,t) \in \Omega^+.$$

Note first that $|z(x,0)| \le \Phi_2(x,0), d < x < 1$ and for $\delta C^*(1-d) \ge \|a\|$,

$$\begin{aligned}
\mathcal{L}_\varepsilon\Phi_2 &\ge \big(\delta C^* - \alpha e^{-2C^*t} + \sqrt{\frac{\alpha}{\varepsilon}}\frac{(1-d(t))}{(1-d)^2}((1-d(t))C^* - \|a\|)(x-d)e^{-C^*t}\big)\Phi_2 \\
&\ge \big(\delta C^* - \alpha e^{-2C^*t} + \sqrt{\frac{\alpha}{\varepsilon}}\frac{(1-d(t))}{(1-d)^2}(\delta(1-d)C^* - \|a\|)(x-d)e^{-C^*t}\big)\Phi_2 \\
&\ge 0.
\end{aligned}$$

Hence, $|z(x,t)| \le \Phi_2(x,t),\ (x,t) \in \Omega^+$.

We now again use the interior estimates from [8, pg. 123] or [12, pg. 352]. Let us first determine bounds on the time derivatives of the solution u of (5) along the line $x = d$. We introduce the time dependent stretched variable

$$\zeta := \sqrt{\alpha}\frac{(d-x)d(t)}{d\sqrt{\varepsilon}} = \sqrt{\alpha}\frac{s-d(t)}{\sqrt{\varepsilon}}. \tag{13}$$

Note that with $\tilde{u}(\zeta,t) := \hat{u}(s,t)$ then

$$\begin{aligned}
&\frac{\partial\hat{u}}{\partial s} = \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}\frac{\partial\tilde{u}}{\partial\zeta} \quad \text{and} \quad \frac{\partial\hat{u}}{\partial t} = -\sqrt{\alpha}\frac{a(t)}{\sqrt{\varepsilon}}\frac{\partial\tilde{u}}{\partial\zeta} + \frac{\partial\tilde{u}}{\partial t}, \\
&\frac{\partial u}{\partial x} = -\sqrt{\alpha}\frac{d(t)}{d\sqrt{\varepsilon}}\frac{\partial\tilde{u}}{\partial\zeta} \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{a(t)}{d(t)}\zeta\frac{\partial\tilde{u}}{\partial\zeta} + \frac{\partial\tilde{u}}{\partial t}.
\end{aligned} \tag{14}$$

Hence the function $\tilde{u}(\zeta,t)$ satisfies the Poisson equation

$$-\alpha\tilde{u}_{\zeta\zeta} + \tilde{u}_t = \tilde{f}.$$

On the rectangle $R := (-\sqrt{\alpha}, \sqrt{\alpha}) \times (0,T]$ use Ladyzhenskaya estimates [12, pg. 352]. Hence

$$|\tilde{u}(\zeta,t)|_{4+\gamma} \le C, \quad (\zeta,t) \in R.$$

Transforming back to the variables (x,t) we have

$$\left|\frac{\partial^m u}{\partial t^m}(d,t)\right| = \left|\frac{\partial^m\tilde{u}}{\partial t^m}(d,t)\right| \le C, \quad m \le 2+\gamma. \tag{15}$$

On the left domain Ω^-, we use the same time dependent stretched variable (13). The function $\tilde{z}(\zeta,t) := z(x,t)$ satisfies the heat equation $-\alpha\tilde{z}_{\zeta\zeta} + \tilde{z}_t = 0$. Consider the rectangular region $(\zeta,t) \in S := (0,K) \times (0,T]$, where $K \le \sqrt{\alpha}\frac{d}{\sqrt{\varepsilon}}$. Recall that from (12) we have that

$$|\tilde{z}(\zeta,t)| \le Ce^{-\zeta}, \quad (\zeta,t) \in S.$$

Using (7b), (15) and

$$\frac{\partial^m z}{\partial t^m}(d,t) = \frac{\partial^m\tilde{z}}{\partial t^m}(0,t), \quad \frac{\partial^m\hat{v}^-}{\partial t^m}(d(t),t) = \frac{\partial^m\tilde{v}^-}{\partial t^m}(0,t), \quad m \le 2+\gamma,$$

we have

$$\left|\frac{\partial^m \tilde{z}}{\partial t^m}(0,t)\right| \leq C, \quad 0 \leq t \leq T, \quad m \leq 2+\gamma. \tag{16}$$

Note further that

$$|\frac{\partial^j \tilde{z}}{\partial \zeta^j}(\zeta,0)| \quad \leq \quad Ce^{-\zeta} + Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}d}, \quad j \leq 4+\gamma, \tag{17}$$

where the second term is due to the presence of the function ϕ_c.

If $0 \leq \zeta \leq 1$, from (16) and (17), [12, (10.5)] yields

$$\left|\frac{\partial^{j+m} \tilde{z}}{\partial \zeta^j \partial t^m}(\zeta,t)\right| \leq Ce^{-\zeta} + C \leq Ce^{-\zeta}, \quad 0 \leq \zeta \leq 1, \quad 0 \leq j+2m \leq 4,$$

since $\zeta \leq 1$. If $1 < \zeta \leq K$, use [12, (10.5)] again but now

$$\left|\frac{\partial^{j+m} \tilde{z}}{\partial \zeta^j \partial t^m}(\zeta,t)\right| \leq Ce^{-\zeta}, \quad 1 < \zeta \leq K,\ t \geq 0, \quad 0 \leq j+2m \leq 4.$$

Hence

$$\left|\frac{\partial^{j+m} \tilde{z}}{\partial \zeta^j \partial t^m}(\zeta,t)\right| \leq Ce^{-\zeta}, \quad 0 \leq \zeta \leq K,\ t \geq 0, \quad 0 \leq j+2m \leq 4.$$

Thus, in particular, the space derivatives satisfy

$$\left|\frac{\partial^j z}{\partial x^j}(x,t)\right| \leq C\varepsilon^{-j/2} e^{-\sqrt{\frac{\alpha}{\varepsilon}}(d-x)}, \quad d(1-\frac{d}{d(T)}) < x \leq d,\ t \geq 0, \quad 0 \leq j \leq 4.$$

For the time derivatives, noting (14) for the first order time derivative and

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 \tilde{z}}{\partial \zeta^2}\Big(\frac{\partial \zeta}{\partial t}\Big)^2 + 2\frac{\partial^2 \tilde{z}}{\partial \zeta \partial t}\frac{\partial \zeta}{\partial t} + \frac{\partial \tilde{z}}{\partial \zeta}\frac{\partial^2 \zeta}{\partial t^2} + \frac{\partial^2 \tilde{z}}{\partial t^2},$$

for the second order time derivative, we deduce that

$$\left|\frac{\partial z}{\partial t}(x,t)\right| \leq C(\zeta+1)e^{-\zeta} \leq C, \ \left|\frac{\partial^2 z}{\partial t^2}(x,t)\right| \leq C(\zeta^2+\zeta+1)e^{-\zeta} \leq C,$$

for $d(1-\frac{d}{d(T)}) < x \leq d$ and $0 \leq t \leq T$. An analogous argument applies on the region $(d, d+(1-d(T))) \times (0,T] \subset \Omega^+$. $\square$

6. Numerical method

Let N and M be two positive integers. To approximate the solution of problem (5) we use a uniform mesh in time $\{t_j = j\Delta t, \mid \Delta t = T/M\}$ and a piecewise uniform mesh of Shishkin type in space $\{x_i\}_{i=1}^N$ (described below) in the transformed variables (x,t). The grid is given by

$$\overline{\Omega}^{N,M} = \{t_j\}_{j=0}^M \times \{x_i\}_{i=0}^N, \quad \overline{\Gamma}^{N,M} = \overline{\Omega}^{N,M} \cap (\overline{\Omega} \setminus \Omega), \quad \Omega^{N,M} = \overline{\Omega}^{N,M} \setminus \overline{\Gamma}^{N,M}.$$

The local spatial mesh sizes are denoted by $h_i = x_i - x_{i-1}$, $1 \leq i \leq N$.

To describe the numerical method we use the following notation for the finite difference approximations of the derivatives

$$D_t^- \Upsilon(x_i,t_j) := \frac{\Upsilon(x_i,t_j) - \Upsilon(x_i,t_{j-1})}{\Delta t}, \ D_x^- \Upsilon(x_i,t_j) := \frac{\Upsilon(x_i,t_j) - \Upsilon(x_{i-1},t_j)}{h_i},$$

$$D_x^+ \Upsilon(x_i,t_j) := \frac{\Upsilon(x_{i+1},t_j) - \Upsilon(x_i,t_j)}{h_{i+1}},$$

$$\delta_x^2 \Upsilon(x_i,t_j) := \frac{2}{h_i + h_{i+1}}(D_x^+ \Upsilon(x_i,t_j) - D_x^- \Upsilon(x_i,t_j)).$$

Discretize problem (5) using an Euler method to approximate the time variable and an upwind finite difference operator to approximate in space. The finite difference equation associated with each grid point is given by

(18a) $-\varepsilon\delta_x^2 U + \kappa(x_i,t_j)D_x^- U + g(x_i,t_j)D_t^- U = g(x_i,t_j)f(x_i,t_j),\ x_i \neq d,\ t_j > 0,$
(18b) $D_x^- U(d,t_j) = D_x^+ U(d,t_j), t_j > 0,$
(18c) $-\varepsilon\delta_x^2 U(x_i,0) + b(x_i)U(x_i,0) = f_1(x_i),\ 0 < x_i < 1,$
(18d) $U(0,t_j) = \phi_L(t_j), U(1,t_j) = \phi_R(t_j),\ t_j \geq 0.$

The space domain is discretized using a piecewise uniform mesh which splits the space domain $[0,1]$ into four subintervals

$$[0,d-\tau_1] \cup [d-\tau_1, d+\tau_2] \cup [d+\tau_2, 1-\sigma] \cup [1-\sigma, 1], \tag{18e}$$

where

$$\tau_1 = \min\{\frac{d^2}{d(T)}, \sqrt{\frac{2\varepsilon}{\alpha}}\ln N\}, \quad \tau_2 = \min\{1-d(T), e^{C^*T}\sqrt{\frac{2\varepsilon}{\alpha}}\ln N\}, \tag{18f}$$

$$\sigma = \min\{\frac{1-(d+\tau_2)}{2}, \frac{4\varepsilon}{\alpha\delta}\ln N\}, \quad C^* = \frac{\|a\|}{\delta(1-d)}. \tag{18g}$$

The grid points are uniformly distributed within each subinterval such that

$$x_0 = 0,\ x_{N/4} = d-\tau_1,\ x_{N/2} = d,\ x_{5N/8} = d+\tau_2,\ x_{7N/8} = 1-\sigma,\ x_N = 1.$$

In the next section, the error analysis will concentrate on the case when

$$\tau_1 = \sqrt{\frac{2\varepsilon}{\alpha}}\ln N, \quad \tau_2 = e^{C^*T}\sqrt{\frac{2\varepsilon}{\alpha}}\ln N, \quad \sigma = \frac{4\varepsilon}{\alpha\delta}\ln N. \tag{19}$$

The other possibilities for the mesh parameters will be dealt with using a classical argument.

7. Numerical Analysis

Define the following finite difference operator $L^{N,M}$ by

$$L^{N,M}\Upsilon(x_i,t_j) = \begin{cases} \Upsilon(x_i,t_j), & x_i = 0,1,\ t_j \geq 0, \\ -\varepsilon\delta_x^2\Upsilon(x_i,0) + b(x_i)\Upsilon(x_i,0), & x_i \in (0,1), \\ \big(-\varepsilon\delta_x^2\Upsilon + \kappa D_x^-\Upsilon + gD_t^-\Upsilon\big)(x_i,t_j), & x_i \in (0,d)\cup(d,1),\ t_j > 0, \\ (D_x^- - D_x^+)\Upsilon(x_i,t_j), & x_i = d,\ t_j > 0. \end{cases}$$

Theorem 5. *Let Υ be any mesh function defined on $\overline{\Omega}^{N,M}$. If $L^{N,M}\Upsilon(x_i,t_j) \geq 0,\ \forall (x_i,t_j) \in \overline{\Omega}^{N,M}$ then $\Upsilon(x_i,t_j) \geq 0,\ \forall (x_i,t_j) \in \overline{\Omega}^{N,M}$.*

Proof. It is a standard proof by contradiction argument. First establish that $\Upsilon(x_i,0) \geq 0$ and then apply the standard argument for any $t_j > 0$. The cases of $x_i = d$ and $x_i \neq d$ are treated separately. □

Noting that $\|U(x_i,0)\| \leq C$, we can conclude from the discrete maximum principle that $\|U\| \leq C$. Let $U = V+W+Z$, where V, W, Z are the discrete counterparts to the continuous components v, w, z. The discrete regular component V is multi-valued and defined separately on Ω^- and Ω^+ by

$$\begin{aligned}
\big(-\varepsilon\delta_x^2+\kappa D_x^-+gD_t^-\big)V^- &= f, \quad (x_i,t_j)\in\Omega^-,\\
-\varepsilon\delta_x^2V^-(x_i,0)+b(x_i)V^-(x_i,0) &= q(x_i)-A,\ x_i\le d,\\
V^-(0,t_j)=\phi_L(t_j),\quad V^-(d,t_j) &= v^-(d^-,t_j),\ t_j\ge 0,\\
\big(-\varepsilon\delta_x^2+\kappa D_x^-+gD_t^-\big)V^+ &= f, \quad (x_i,t_j)\in\Omega^+,\\
-\varepsilon\delta_x^2V^+(x_i,0)+b(x_i)V^+(x_i,0) &= q(x_i)+A,\ x_i\ge d,\\
V^+(d,t_j)=v^+(d^+,t_j),\quad V^+(1,t_j) &= v^+(1,t_j),\ t_j\ge 0.
\end{aligned}$$

Note that

$$|V^-(x_i,t_j)|\le C,\quad |V^+(x_i,t_j)|\le C.$$

The discrete boundary layer function W is defined by

$$W\equiv 0,\ (x_i,t_j)\in\bar\Omega^-\cap\bar\Omega^{N,M}, \tag{20a}$$

$$\big(-\varepsilon\delta_x^2+\kappa D_x^-+gD_t^-\big)W=0,\ (x_i,t_j)\in\Omega^+\cap\Omega^{N,M}, \tag{20b}$$

$$W(d,t_j)=0,\ W(x_i,0)=0,\ W(1,t_j)=w(1,t_j). \tag{20c}$$

Observe that $|w(x_i,0)|\le Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}(1-d)}\le Ce^{-\sqrt{\frac{\alpha}{\varepsilon}}\tau_1}\le CN^{-1}$. The discrete interior layer function Z is also multi-valued and defined by:

$$\big(-\varepsilon\delta_x^2+\kappa D_x^-+gD_t^-\big)Z^- = 0,\ x_i<d, t_j>0,\ Z^-(0,t_j)=0, \tag{21a}$$

$$\big(-\varepsilon\delta_x^2+\kappa D_x^-+gD_t^-\big)Z^+ = 0,\ x_i>d, t_j>0,\ Z^+(1,t_j)=0, \tag{21b}$$

$$-\varepsilon\delta_x^2Z^-(x_i,0)+b(x_i)Z^-(x_i,0) = A(\tanh(\frac{\sqrt{\alpha_2}(x_i-d)}{2\sqrt{\varepsilon}})+1),\ x_i<d, \tag{21c}$$

$$-\varepsilon\delta_x^2Z^+(x_i,0)+b(x_i)Z^+(x_i,0) = A(\tanh(\frac{\sqrt{\alpha_2}(x_i-d)}{2\sqrt{\varepsilon}})-1),\ x_i>d, \tag{21d}$$

and for all $t_j\ge 0$

$$(Z^++V^+)(d,t_j) = (Z^-+V^-)(d,t_j), \tag{21e}$$

$$(D_x^+Z^+-D_x^-Z^-)(d,t_j) = (D_x^-V^--D_x^+(V^++W))(d,t_j). \tag{21f}$$

Theorem 6. *For sufficiently large M, the solution of (20) satisfies the bound*

$$|W(x_i,t_j)|\le C\frac{\Pi_{k=N/2}^{i}\big(1+\frac{\alpha\delta(x_k-d)h_k}{2\varepsilon(1-d)}\big)}{\Pi_{k=N/2}^{N}\big(1+\frac{\alpha\delta(x_k-d)h_k}{2\varepsilon(1-d)}\big)},\quad N/2\le i\le N.$$

Proof. Consider the following discrete barrier function

$$\Phi_3(x_i,t_j):=C(1-\frac{C_1T}{\delta^2M})^{-j}\frac{\Pi_{k=N/2}^{i}\big(1+\frac{C_1(x_k-d)h_k}{2\varepsilon}\big)}{\Pi_{k=N/2}^{N}\big(1+\frac{C_1(x_k-d)h_k}{2\varepsilon}\big)},\quad C_1:=\frac{\alpha\delta}{1-d},$$

where M is sufficiently large so that

$$0<c<1-\frac{C_1T}{\delta^2M}. \tag{22}$$

Note first that

$$\Phi_3(x_i,0)\ge 0,\quad \Phi_3(d,t_j)\ge 0,\quad \Phi_3(1,t_j)\ge C(1-\frac{C_1T}{\delta^2M})^{-M}\ge C>0.$$

We also observe that

$$2\varepsilon D_x^+\Phi_3(x_i,t_j) = C_1(x_{i+1}-d)\Phi_3(x_i,t_j), \quad D_t^-\Phi_3(x_i,t_j) = \frac{C_1}{\delta^2}\Phi_3(x_i,t_j),$$
$$2\varepsilon\big(1+\frac{C_1(x_i-d)h_i}{2\varepsilon}\big)D_x^-\Phi_3(x_i,t_j) = C_1(x_i-d)\Phi_3(x_i,t_j),$$
$$-\varepsilon\delta_x^2\Phi_3(x_i,t_j) = -\frac{2}{h_i+h_{i+1}}\big(\frac{C_1h_{i+1}}{2}+\frac{C_1^2(x_i-d)^2h_i}{4\varepsilon+2C_1(x_i-d)h_i}\big)\Phi_3(x_i,t_j).$$

From this we deduce that

$$-\varepsilon\delta_x^2\Phi_3 + C_1(x_i-d)D_x^-\Phi_3 + \delta^2 D_t^-\Phi_3 \geq 0.$$

Hence,

$$\big(-\varepsilon\delta_x^2+\kappa D_x^-+gD_t^-\big)(\Phi_3\pm W) \geq \big(\kappa - C_1(x_i-d)\big)D_x^-\Phi_3 + (g-\delta^2)D_t^-\Phi_3 \geq 0.$$

Using a discrete comparison principle over the region $\overline{\Omega}^{N,M}\cap\bar{\Omega}^+$ will complete the proof. $\square$

We remark that

$$\frac{(x_i-d)}{2(1-d)} \geq \frac{1}{4}, \quad \text{if} \quad x_i \geq 1-\sigma,$$

and so, from the bound in Theorem 6, we have that

$$|W(1-\sigma,t_j)| \leq C\big(1+\frac{2\alpha\delta\sigma}{N\varepsilon}\big)^{-N/8} \leq CN^{-1}.$$

In passing we note that $|W| \leq C(x_i-d),\ x_i \geq d$, and so $|D_x^+W(d,t_j)| \leq C$. Since $|U|, |W|$ and $|V|$ are all bounded we have that

$$\|Z^-(d,t_j)\|,\ \|Z^+(d,t_j)\| \leq C.$$

Theorem 7. *For sufficiently large $M \geq \mathcal{O}(\ln(N))$, the solution of (21) satisfies the bounds*

$$(a) \qquad |Z(x_i,t_j)| \leq C\frac{\Pi_{k=1}^{i}\big(1+\frac{\sqrt{\alpha}h_k}{\sqrt{2\varepsilon}}\big)}{\Pi_{k=1}^{N/2}\big(1+\frac{\sqrt{\alpha}h_k}{\sqrt{2\varepsilon}}\big)}, \quad x_i \leq d,$$

$$(b) \qquad |Z(x_i,t_j)| \leq C\Pi_{n=N/2}^{i}\big(1+\frac{e^{-C^*T}\sqrt{\alpha}h_n}{\sqrt{2\varepsilon}}\big)^{-1} + CN^{-1}\ln N, \quad x_i \geq d.$$

Proof. (a) For $x_i \leq d$, consider the following barrier function

$$\Phi_4(x_i,t_j) := C\big(1-\frac{\alpha T}{\delta^2 M}\big)^{-j}\frac{\Pi_{k=1}^{i}\big(1+\frac{\sqrt{\alpha}h_k}{\sqrt{2\varepsilon}}\big)}{\Pi_{k=1}^{N/2}\big(1+\frac{\sqrt{\alpha}h_k}{\sqrt{2\varepsilon}}\big)},$$

where M is sufficiently large so that

$$0 < c < 1-\frac{\alpha T}{\delta^2 M}. \tag{23}$$

Note that

$$\sqrt{2\varepsilon}D_x^+\Phi_4(x_i,t_j) = \sqrt{\alpha}\Phi_4(x_i,t_j), \quad D_t^-\Phi_4(x_i,t_j) = \frac{\alpha}{\delta^2}\Phi_4(x_i,t_j),$$
$$\sqrt{2\varepsilon}\big(1+\frac{\sqrt{\alpha}h_i}{\sqrt{2\varepsilon}}\big)D_x^-\Phi_4(x_i,t_j) = \sqrt{\alpha}\Phi_4(x_i,t_j),$$
$$-\varepsilon\delta_x^2\Phi_4(x_i,t_j) = -\frac{\alpha h_i}{h_i+h_{i+1}}\big(1+\frac{\sqrt{\alpha}h_i}{\sqrt{2\varepsilon}}\big)^{-1}\Phi_4(x_i,t_j) \geq -\alpha\Phi_4(x_i,t_j),$$

and so, it follows that,

$$(-\varepsilon\delta_x^2 + \kappa D_x^- + gD_t^-)\Phi_4(x_i,t_j) \geq 0.$$

Note also that $\Phi_4(0,t_j) \geq 0, \Phi_4(d,t_j) \geq C > 0$ and

$$-\varepsilon\delta_x^2\Phi_4(x_i,0) + b\Phi_4(x_i,0) \geq (b-\alpha)\Phi_4(x_i,0).$$

Also, we have

$$\Phi_4(x_i,0) = C\frac{\Pi_{k=1}^i\big(1+\frac{\sqrt{\alpha}h_k}{\sqrt{2\varepsilon}}\big)}{\Pi_{k=1}^{N/2}\big(1+\frac{\sqrt{\alpha}h_k}{\sqrt{2\varepsilon}}\big)} \geq Ce^{\sqrt{\alpha}(x_i-d)/\sqrt{2\varepsilon}} \geq Ce^{\sqrt{\alpha_2}(x_i-d)/\sqrt{\varepsilon}}.$$

Thus,

$$-\varepsilon\delta_x^2\Phi_4(x_i,0) + b\Phi_4(x_i,0) \geq |-\varepsilon\delta_x^2 Z(x_i,0) + bZ(x_i,0)|.$$

Finish using a discrete comparison principle.

(b) For $x_i \geq d$, consider the following barrier function

$$B(x_i,t_j) := C\Phi_5(x_i)\Psi_1(t_j) + C(N^{-1}\ln N)t_j,$$

where

$$\Phi_5(x_i) := \Pi_{n=N/2}^i(1+\frac{\sqrt{\rho}h_n}{\sqrt{2\varepsilon}})^{-1}, \quad \Psi_1(t_j) := (1-\frac{\theta T\ln N}{\delta^2 M})^{-j},$$

and $M(N)$ is chosen sufficiently large so that

$$0 < c < (1-\frac{\theta T\ln N}{\delta^2 M}). \tag{24}$$

The parameters ρ,θ are specified below. Note first that

$$B(d,t_j) \geq C\Psi_1(t_j) \geq C(1-\frac{\theta T\ln N}{\delta^2 M})^{-j} \geq C > 0, \quad B(1,t_j) \geq 0.$$

In addition we have that

$$\sqrt{2\varepsilon}D_x^-\Phi_5(x_i) = -\sqrt{\rho}\Phi_5(x_i),\ \Phi_5(d)=1, \quad -\varepsilon\delta_x^2\Phi_5(x_i) \geq -\rho\Phi_5(x_i),$$

and

$$D_t^-\Psi_1(t_j) = \frac{\theta\ln N}{\delta^2}\Psi_1(t_j), \quad \Psi_1(0)=1.$$

Initially at $t_j=0$, $\Phi_5(x_i) \geq Z^+(x_i,0)$, when $\rho \leq \alpha$. For $d < x_i \leq d+\tau_2$,

$$\begin{aligned}\big(-\varepsilon\delta_x^2 + \kappa D_x^- + gD_t^-)\Phi_5(x_i)\Psi_1(t_j) &\geq \big(-\rho - \frac{\|a\|\sqrt{\rho}(x_i-d)}{(1-d)\sqrt{2\varepsilon}} + \theta\ln N\big)\Phi_5(x_i)\Psi_1(t_j)\\ &\geq \big(-\rho - \frac{\|a\|\sqrt{\rho}e^{C^*T}\ln N}{(1-d)\sqrt{\alpha}} + \theta\ln N\big)\Phi_5(x_i)\Psi_1(t_j).\end{aligned}$$

By defining, $\sqrt{\rho} := e^{-C^*T}\sqrt{\alpha} \leq \sqrt{\alpha},\ \theta := 1+\|a\|(1-d)^{-1}$, we have that for N sufficiently large (independently of ε)

$$\big(-\varepsilon\delta_x^2 + \kappa D_x^- + gD_t^-)\Phi_5(x_i)\Psi_1(t_j) \geq \Phi_5(x_i)\Psi_1(t_j), \quad d < x_i \leq x_i+\tau_2.$$

If $d+\tau_2 < x_i \leq 1-\sigma$, then using the inequality $nt \leq (1+t)^n, t\geq 0$,

$$\begin{aligned}\frac{(x_i-d)}{\sqrt{2\varepsilon}}\Phi_5(x_i) &= \frac{\tau_2}{\sqrt{2\varepsilon}}\Phi_5(x_i) + \Phi_5(d+\tau_2)\frac{(x_i-\tau_2-d)}{\sqrt{2\varepsilon}}(1+\frac{\sqrt{\rho}H_\star}{\sqrt{2\varepsilon}})^{-(i-5N/8)}\\ &\leq CN^{-1}\ln N,\end{aligned}$$

where $NH_\star := 8(1-\sigma-(d+\tau_2))$. If $1-\sigma < x_i < 1$, then

$$\frac{(x_i-d)}{\sqrt{2\varepsilon}}\Phi_5(x_i) \le \frac{1-\sigma-d}{\sqrt{2\varepsilon}}\Phi_5(1-\sigma) + \frac{x_i-(1-\sigma)}{\sqrt{2\varepsilon}}\Phi_5(x_i) \le CN^{-1}.$$

Then for sufficiently large N and $d+\tau_2 < x_i < 1$,

$$\begin{aligned}\big(-\varepsilon\delta_x^2+\kappa D_x^- + gD_t^-\big)\Phi_5(x_i)\Psi_1(t_j) &\ge \Big(-\rho - \frac{\|a\|\sqrt{\rho}(x_i-d)}{(1-d)\sqrt{2\varepsilon}} + \theta\ln N\Big)\Phi_5(x_i)\Psi_1(t_j)\\ &\ge \big(-\rho+\theta\ln N\big)\Phi_5(x_i)\Psi_1(t_j) - CN^{-1}\ln N.\end{aligned}$$

Then $C\Phi_5(x_i)\Psi_1(t_j) + C(N^{-1}\ln N)t_j$ is a suitable barrier function for Z. □

Note by assuming (24) then the other constraints (22), (23) are automatically satisfied. Constraint (24) requires M sufficiently large so that

$$M > (\ln N)\frac{(\|a\|+1-d)T}{(1-d)\delta^2}, \tag{25}$$

which will be more restrictive as T increases and δ decreases.

Theorem 8. *Assume (19) applies. For M sufficiently large so that (25) is satisfied,*

$$\begin{aligned}&(a)\quad \|V-v\| \le CN^{-1}+CM^{-1},\\ &(b)\quad \|W-w\| \le C(N^{-1}\ln N + M^{-1})\ln N,\\ &(c)\quad \|U-u\| \le C(N^{-1}\ln N + M^{-1})\ln N.\end{aligned}$$

Proof. (a) For the regular component, we derive the necessary bound on Ω^-, Ω^+ separately, noting that

$$|L^{N,M}\big(V^\pm - v^\pm\big)(x_i,0)| \le CN^{-1}.$$

For all other time levels,

$$\begin{aligned}\|L^{N,M}(V^- - v^-)\|_{\Omega^-} &\le C(\varepsilon N^{-1}\|v_{xxx}^-\| + N^{-1}\|v_{xx}^-\| + M^{-1}\|v_{tt}^-\|)\\ &\le C(N^{-1}+M^{-1}).\end{aligned}$$

Use the barrier function $C(N^{-1}+M^{-1})(1+t_j)$ to bound $\|V^\pm - v^\pm\|$.

(b) Firstly, for $d \le x_i \le 1-\sigma$, use the exponential character of this component and the bound in Theorem 6 to get that

$$|W(x_i,t_j)| \le |W(1-\sigma,t_j)| \le CN^{-1}.$$

Hence, $|(W-w)(x_i,t_j)| \le |W(x_i,t_j)| + |w(x_i,t_j)| \le CN^{-1},\ x_i \le 1-\sigma$. Secondly, for $1-\sigma < x_i < 1$ and $t_j > 0$, we have

$$|L^{N,M}(W-w)(x_i,t_j)| \le C(N^{-1}\ln N + M^{-1})\varepsilon^{-1}.$$

Initially, we have $W(x_i,0)=0$ and $|w(x_i,0)| \le Ce^{-\frac{\alpha\delta}{2\varepsilon}(1-x)} \le CN^{-1}$. Use a discrete maximum principle on $[1-\sigma,1]$ with the barrier function

$$(N^{-1}\ln N + M^{-1})\ln N(x_i-(1-\sigma))\sigma^{-1}$$

to prove the result.

(c) From Theorem 4 and Theorem 7, we have that

$$|z(x_i,t_j)| \le CN^{-1}, \quad |Z(x_i,t_j)| \le CN^{-1}, \quad \text{for} \quad x_i \le d-\tau_1.$$

From Theorem 4, $|z(x_i,t_j)| \le CN^{-1}$, for $x_i \ge d+\tau_2$ and from Theorem 7 we deduce that $|Z(x_i,t_j)| \le CN^{-1}\ln N$, for $x_i \ge d+\tau_2$. Then

$$|(U-u)(x_i,t_j)| \le C(N^{-1}\ln N + M^{-1})\ln N, \quad \text{for } x_i \in\ [0,d-\tau_1]\cup\ [d+\tau_2,1].$$

We now examine the truncation error $L^{N,M}(U-u)$ in the layer region $(d-\tau_1, d+\tau_2)\times[0,T]$. Initially, we have

$$|L^{N,M}(U-u)(x_i,0)| \le C\varepsilon h_i \|u_{xxx}(x,0)\| \le CN^{-1}\ln N.$$

Standard discrete comparison principle yields

$$|U(x_i,0)-u(x_i,0)| \le CN^{-1}\ln N, \quad x_i \in (d-\tau_1, d+\tau_2).$$

Note that

$$\begin{aligned} |u_x(d^+,t_j) - D_x^+ u(d,t_j)| &\le |v_x(d^+,t_j) - D_x^+ v(d,t_j)| \\ &+ |w_x(d^+,t_j) - D_x^+ w(d,t_j)| + |z_x(d^+,t_j) - D_x^+ z(d,t_j)| \\ &\le CN^{-1}\tau_2 + CN^{-1}\tau_2 + C\frac{N^{-1}\tau_2}{\varepsilon}, \end{aligned}$$

with a similar bound to the left of d. Hence,

$$\begin{aligned} |(D_x^+ - D_x^-)(U-u)(d,t_j)| &= |[u_x] - (D_x^+ - D_x^-)u| \\ &\le C\frac{N^{-1}(\tau_1+\tau_2)}{\varepsilon} \le C\frac{N^{-1}\ln N}{\sqrt{\varepsilon}}. \end{aligned}$$

In the fine mesh region around the points $x=d$ the truncation error will be of the form

$$\varepsilon N^{-2}\tau^2|u_{xxxx}| + \kappa(x_i,t_j)N^{-1}\tau|u_{xx}| + M^{-1}|u_{tt}|, \quad \tau := \max\{\tau_1,\tau_2\}.$$

From (9), $\|w_t\|_{[d-\tau_1,d+\tau_2]} \le C,\ \|w_{tt}\|_{[d-\tau_1,d+\tau_2]} \le C$. Here we also use the bounds in Theorem 4 and the fact that (19) allows the use of the derivative estimates (11c) and (11e) on the component z in the region $(d-\tau_1, d+\tau_2)$. So the truncation error in the fine mesh (for $x_i \ne d$) is bounded by

$$CN^{-2}\tau^2\varepsilon^{-1} + CN^{-1}\tau^2\varepsilon^{-1} + CM^{-1} \le CN^{-1}(\ln N)^2 + M^{-1}.$$

To complete the proof use the barrier function

$$\Phi_6(x_i,t_j) = C(N^{-1}\ln N + M^{-1})\ln N(t_j+1) + CN^{-1}(\ln N)^2\Phi^*(x_i,t_j),$$

where Φ^* is a piecewise linear function defined by

$$\Phi^*(d-\tau_1,t_j) = 0 = \Phi^*(d+\tau_2,t_j), \qquad \Phi^*(d,t_j) = 1.$$

Note that

$$-(D_x^+ - D_x^-)\Phi^*(d,t_j) = \frac{\tau_2+\tau_1}{\tau_2\tau_1} = \frac{\sqrt{\alpha}(1-e^{-C^*T})}{\sqrt{2\varepsilon}\ln N}.$$

□

Remark 7.1. *When $\sqrt{\varepsilon}\ln N \ge C.$, we can establish that*

$$\|U-u\| \le C(N^{-1}+M^{-1})(\ln N)^5. \tag{26}$$

Applying the argument given in the appendix to the original formulation of the problem (1), one can deduce that for $i=2,3$,

$$\Big\|\frac{\partial^i u}{\partial x^i}\Big\|_{\Omega^-\cup\Omega^+} \le C\varepsilon^{-\frac{(i+1)}{2}}, \quad \Big\|\frac{\partial^2 u}{\partial t^2}\Big\|_{\Omega^-\cup\Omega^+} \le C\varepsilon^{-5/2}.$$

Combining these bounds with a standard truncation error argument yields the required bound (26). Note we do not employ a decomposition of the solution u in this case where $\sqrt{\varepsilon}\ln N \ge C$.

The nodal error estimate established in Theorem 8 (and in (26)) is easily extended to a global error estimate using simple linear interpolation. Define

$$\bar{U}(x,t) := \sum_{i=0,j=1}^{N,M} U(x_i,t_j)\varphi_i(x)\psi_j(t),$$

where $\varphi_i(x)$ is the standard hat function centered at $x = x_i$ and $\psi_j(t) = M(t - t_{j-1}), t \in [t_{j-1}, t_j)$.

Theorem 9. *For M sufficiently large so that (25) is satisfied,*

$$\|\bar{U} - \hat{u}\|_{\bar{Q}} \le C(N^{-1} + M^{-1})(\ln N)^5,$$

where U is the discrete solution of (18) and $\hat{u}$ is the solution of the continuous problem (1).

Proof. Combine the arguments in [6, Theorem 3.12] with the interpolation bounds in [17, Lemma 4.1] and the bounds on the derivatives of the components v, w, z established in §4,5. Note that from [17, Lemma 4.1], we only require the first time derivative of u to be uniformly bounded. In the interior layer we use [17, Lemma 4.1] and outside of the layer that the continuous singular component z satisfies $|z(x,t)| \le CN^{-1}$. □

8. Numerical experiments

Consider the following test problem

$$\begin{aligned} &-\varepsilon\hat{u}_{ss} + \hat{u}_s + \hat{u}_t = 4s(1-s), \quad (s,t) \in (0,1) \times (0,0.5], \\ &\hat{u}(s,0) = \phi(s;\varepsilon), 0 < s \le 1, \quad \hat{u}(0,t) = 0,\ \hat{u}(1,t) = 2,\ 0 < t \le 1, \end{aligned} \tag{27a}$$

where the initial condition ϕ is the solution of the following problem

$$-\varepsilon\phi'' + \phi = 1 + \tanh(\frac{s - 0.25}{2\sqrt{\varepsilon}}), \quad \phi(0) = 0, \phi(1) = 2. \tag{27b}$$

For this problem, the parameters in the numerical method (18) are taken to be $\alpha = 1$ and $\delta = 1/3$.

As the exact solution is unknown, we estimate the errors using a variant of the double mesh principle, where the finest mesh contains the grid points of the original mesh and its midpoints. Denoting by U and Υ the solutions associated with the coarser mesh $\bar{\Omega}^{N,M}$ and the twice finer mesh respectively, we compute the maximum two–mesh differences $d_\varepsilon^{N,M}$ and the uniform differences $d^{N,M}$ from

$$d_\varepsilon^{N,M} := \max_{0\le j\le M} \max_{0\le i\le N} |(U - \Upsilon)(x_i,t_j)|, \quad d^{N,M} := \max_{S_\varepsilon} d_\varepsilon^{N,M},$$

where $S_\varepsilon = \{2^0, 2^{-1}, \ldots, 2^{-30}\}$. From these values we calculate the corresponding computed orders of convergence $q_\varepsilon^{N,M}$ and the computed orders of uniform convergence $q^{N,M}$ using

$$q_\varepsilon^{N,M} := \log_2\left(d_\varepsilon^{N,M}/d_\varepsilon^{2N,2M}\right), \quad q^{N,M} := \log_2\left(d^{N,M}/d^{2N,2M}\right).$$

The uniform differences and the computed orders of uniform convergence for test problem (27) are given in Table 1. We observe uniform convergence of the finite difference approximations, which is in agreement with the theory. In Figure 1 we display the solution and the corresponding two mesh differences for $\varepsilon = 10^{-8}$ with $N = 64$, and $M = 64$, observing clearly the interior layer and that the largest differences appear in the interior layer region. Note that we compute the solution in the transformed domain, but we display the computed solution and differences in the original domain.

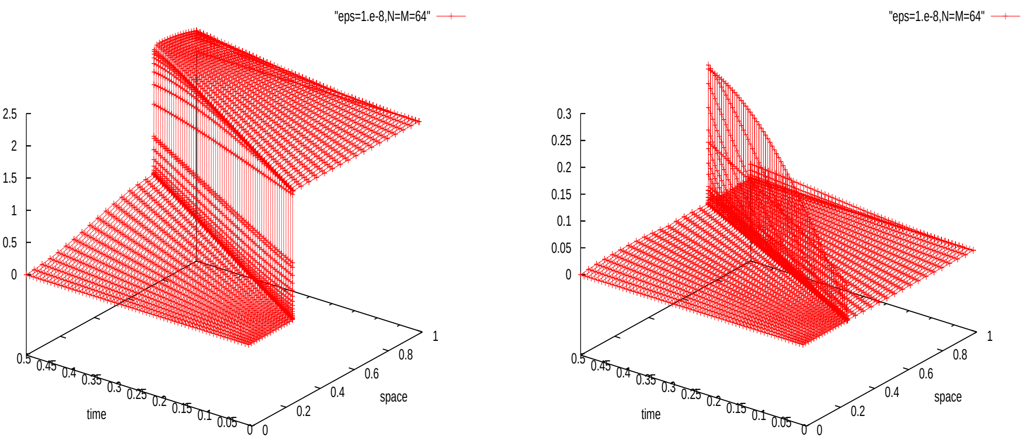

FIGURE 1. Test problem (27): Solution U and two–mesh differences $|(U-\Upsilon)(x_i,t_j)|$ for $\varepsilon = 10^{-8}$ and $N = M = 64$. The final time is $T = 0.5$.

TABLE 1. The maximum two–mesh differences $d_\varepsilon^{N,M}$, the uniform differences $d^{N,M}$, the computed orders of convergence $q_\varepsilon^{N,M}$ and the computed orders of uniform convergence $q^{N,M}$ for test problem (27)

	N=64 M=64	N=128 M=128	N=256 M=256	N=512 M=512	N=1024 M=1024	N=2048 M=2048	N=4096 M=4096	N=8192 M=8192
$\varepsilon = 2^{0}$	0.111E-02 0.950	0.576E-03 0.976	0.293E-03 0.974	0.149E-03 0.886	0.807E-04 0.931	0.423E-04 0.960	0.218E-04 0.977	0.111E-04
$\varepsilon = 2^{-2}$	0.288E-02 0.969	0.147E-02 0.985	0.745E-03 0.992	0.374E-03 0.996	0.188E-03 0.998	0.939E-04 0.999	0.470E-04 1.000	0.235E-04
$\varepsilon = 2^{-4}$	0.565E-02 0.995	0.284E-02 0.998	0.142E-02 0.999	0.710E-03 1.000	0.355E-03 1.000	0.178E-03 1.000	0.889E-04 1.000	0.444E-04
$\varepsilon = 2^{-6}$	0.163E-01 0.881	0.885E-02 0.936	0.462E-02 0.966	0.237E-02 0.983	0.120E-02 0.991	0.603E-03 0.996	0.302E-03 0.998	0.151E-03
$\varepsilon = 2^{-8}$	0.301E-01 0.614	0.197E-01 0.603	0.130E-01 0.776	0.758E-02 0.865	0.416E-02 0.918	0.220E-02 0.957	0.113E-02 0.978	0.576E-03
$\varepsilon = 2^{-10}$	0.412E-01 0.797	0.237E-01 0.572	0.159E-01 0.665	0.100E-01 0.738	0.602E-02 0.791	0.348E-02 0.829	0.196E-02 0.859	0.108E-02
$\varepsilon = 2^{-12}$	0.946E-01 1.202	0.411E-01 1.113	0.190E-01 0.948	0.985E-02 0.721	0.597E-02 0.779	0.348E-02 0.826	0.197E-02 0.857	0.109E-02
$\varepsilon = 2^{-14}$	0.202E+00 1.259	0.846E-01 1.181	0.373E-01 1.027	0.183E-01 0.985	0.924E-02 0.986	0.467E-02 0.992	0.235E-02 0.996	0.118E-02
$\varepsilon = 2^{-16}$	0.259E+00 0.848	0.144E+00 1.025	0.707E-01 0.962	0.363E-01 1.012	0.180E-01 0.984	0.911E-02 0.986	0.460E-02 0.992	0.231E-02
	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
$\varepsilon = 2^{-30}$	0.259E+00 0.848	0.144E+00 1.025	0.707E-01 0.933	0.370E-01 0.866	0.203E-01 0.851	0.113E-01 0.861	0.620E-02 0.876	0.338E-02
$d^{N,M}$ $q^{N,M}_{uni}$	0.259E+00 0.848	0.144E+00 1.025	0.707E-01 0.934	0.370E-01 0.866	0.203E-01 0.851	0.113E-01 0.862	0.620E-02 0.876	0.338E-02

In Table 2 we examine if the restriction (25) is needed in practice. In this table, we display the numerical results for problem (27) associated with different values of the final time T and $\varepsilon = 10^{-8}$. Here, M^* denotes the minimum value of M such that (25) holds. From these results we conclude that this theoretical constraint appears to be required in practice.

More extensive numerical experiments for other test problems are given in [9].

TABLE 2. Test problem (27): Maximum two–mesh differences and computed orders of convergence for $\varepsilon = 10^{-8}$ and different values of T. M^* denotes the minimum value of M satisfying (25)

$T = 0.5$	N,M=64 ($M^* = 44$)	N,M=128 ($M^* = 51$)	N,M=256 ($M^* = 59$)	N,M=512 ($M^* = 66$)	N,M=1024 ($M^* = 73$)	N,M=2048 ($M^* = 81$)
$d_\varepsilon^{N,M}$	0.259E+00	0.144E+00	0.707E-01	0.370E-01	0.203E-01	0.113E-01
$q_\varepsilon^{N,M}$	0.848	1.025	0.933	0.866	0.851	
$T = 0.53$	N,M=64 ($M^* = 60$)	N,M=128 ($M^* = 70$)	N,M=256 ($M^* = 80$)	N,M=512 ($M^* = 90$)	N,M=1024 ($M^* = 100$)	N,M=2048 ($M^* = 110$)
$d_\varepsilon^{N,M}$	0.325E+00	0.224E+00	0.118E+00	0.599E-01	0.321E-01	0.177E-01
$q_\varepsilon^{N,M}$	0.534	0.927	0.976	0.901	0.861	
$T = 0.6$	N,M=64 ($M^* = 146$)	N,M=128 ($M^* = 170$)	N,M=256 ($M^* = 195$)	N,M=512 ($M^* = 219$)	N,M=1024 ($M^* = 243$)	N,M=2048 ($M^* = 267$)
$d_\varepsilon^{N,M}$	0.370E+00	0.342E+00	0.321E+00	0.282E+00	0.188E+00	0.990E-01
$q_\varepsilon^{N,M}$	0.114	0.090	0.190	0.582	0.927	
$T = 0.7$	N,M=64 ($M^* = 1529$)	N,M=128 ($M^* = 1784$)	N,M=256 ($M^* = 2038$)	N,M=512 ($M^* = 2293$)	N,M=1024 ($M^* = 2548$)	N,M=2048 ($M^* = 2803$)
$d_\varepsilon^{N,M}$	0.379E+00	0.350E+00	0.329E+00	0.291E+00	0.187E+00	0.925E-01
$q_\varepsilon^{N,M}$	0.118	0.089	0.178	0.639	1.011	

Appendix. A priori bounds on the solution of the continuous problem

In this appendix, we restate the standard arguments (see for example [11]) to derive a priori parameter explicit bounds on the solution of a singularly perturbed parabolic problem of convection-diffusion type. However, in this appendix, we allow the initial condition to depend on the singular perturbation parameter and we track the influence of this parameter dependence in the derivation of the bounds on the derivatives of the solution. In addition, we also examine the effect such an initial condition has on the bounds of the regular component in a standard Shishkin decomposition of the solution into regular and layer components.

In this appendix, let L_ε denote the differential operator defined by

$$L_\varepsilon \omega(x,t) := -\varepsilon \omega_{xx} + a(x,t)\omega_x + b(x,t)\omega + \omega_t,$$

where $a(x,t) \geq \alpha > 0$, $b \geq 0$. We also define

$$L_0 \omega(x,t) := a(x,t)\omega_x + b(x,t)\omega + \omega_t.$$

Consider the following parabolic problem

$$L_\varepsilon u = f,\ (x,t) \in (0,1) \times (0,T] =: \Omega, \tag{28a}$$

$$u(0,t) = \phi_L(t), u(1,t) = \phi_R(t), \tag{28b}$$

$$u(x,0) = \phi(x;\varepsilon), \quad |\phi|_i \leq C(1 + \varepsilon^{k-\frac{i}{2}}), i \leq n+3. \tag{28c}$$

Note that the initial condition for the above problem is not as in the main part of the paper, but it does correspond to the initial condition used in the definition of the regular component v^- defined in Theorem 2 after transforming in space from $[0,d]$ to $[0,1]$. Throughout this appendix we assume sufficient compatibility and regularity so that the solution of (28) $u \in \mathcal{C}^{n+\gamma}(\bar{\Omega})$. We can write

$$z = u - \phi(x) - (1-x)(\phi_L(t) - \phi_L(0)) - x(\phi_R(t) - \phi_R(0)).$$

Then, z has zero initial/boundary conditions and

$$L_\varepsilon z = F := f - L_\varepsilon(\phi(x) + (1-x)(\phi_L(t) - \phi_L(0)) + x(\phi_R(t) - \phi_R(0))).$$

Note that

$$\|F\|_i \leq C(1+\varepsilon^{k-\frac{(i+1)}{2}}), \quad i \leq n+1.$$

From the maximum principle, we have that

$$\|z\| \leq C(1+\varepsilon^{k-\frac{1}{2}}).$$

Introduce the stretched variables $\varsigma := x/\varepsilon$, $\tau := t/\varepsilon$, $\tilde{g}(\varsigma,\tau) := g(x,t)$. Then

$$\tilde{z}_{\varsigma\varsigma} + \tilde{a}\tilde{z}_\varsigma - \varepsilon\tilde{b}\tilde{z} - \tilde{d}\tilde{z} = \varepsilon\tilde{F}.$$

Using [8, pg. 65] or [12, pg. 320], we have

$$\|\tilde{z}\|_{1+\gamma} \leq C\|\tilde{z}\| + C\varepsilon\|\tilde{F}\|_{0+\gamma}, \quad \|\tilde{z}\|_{n+\gamma} \leq C\|\tilde{z}\| + C\varepsilon\|\tilde{F}\|_{n-2+\gamma}, \quad n = i+2j \geq 2.$$

Observe that $\|\tilde{z}\|_n \leq \|\tilde{z}\|_{n+\gamma},\ n = i+2j$, and

$$\|\tilde{F}\|_n = \sum_{i=0}^{n} |\tilde{F}|_i = \sum_{i=0}^{n} \varepsilon^i |F|_i.$$

Note that $\|\tilde{F}\|_n \leq C$, for $k \geq 0.5$. In the unstretched variables, for $k \geq 0.5$,

$$\|z\|_1 \leq C\varepsilon^{-1}, \quad \|z\|_n \leq C\varepsilon^{-n} + C\varepsilon^{-(n-1)}\|\tilde{F}\|_{n-2+\gamma} \leq C\varepsilon^{-n}.$$

It easily follows that, for $k \geq 0.5$, $\|u\|_n \leq C\varepsilon^{-n}$. Let the regular component v of the solution to problem (28) be further decomposed by $v := v_0 + \varepsilon v_1 + \varepsilon^2 v_2$. Here v_0 is the solution

$$L_0 v_0 = f, \quad v_0(0,t) = \phi_L(t), \quad v_0(x,0) = \phi(x;\varepsilon).$$

From the explicit closed form solution representation given in [1, pg.396]

$$|v_0|_m \leq C(1+\varepsilon^{k-\frac{m}{2}}), \quad m = i+j \leq n+3.$$

The second term v_1 in the expansion of v is defined by

$$L_0 v_1 = (v_0)_{xx}, \quad v_1(x,0) = 0,\ v_1(0,t) = 0.$$

Then using [1, pg.396] again $|v_1|_m \leq C(1+\varepsilon^{k-1-\frac{m}{2}}),\ m = i+j \leq n+1$. Finally, define v_2 as the solution of

$$L_\varepsilon v_2 = (v_1)_{xx}, \quad v_2(x,0) = v_2(0,t) = v_2(1,t) = 0.$$

Note that $|(v_1)_{xx}|_m \leq C(1+\varepsilon^{k-2-\frac{m}{2}}),\ m = i+j \leq n-1$. Hence, using the arguments for the parabolic problem given above,

$$\|v_2\| \leq C\varepsilon^{k-2}, \quad |v_2|_1 \leq C\varepsilon^{k-3},$$
$$|v_2|_m \leq C\varepsilon^{k-2}\varepsilon^{-m} + C\varepsilon^{k-\frac{3+m}{2}}, \quad m = i+2j \leq n.$$

To conclude, we have established the following bounds

$$|v|_m \leq C(1+\varepsilon^{k-\frac{(m-1)}{2}} + \varepsilon^{k-m}) \leq C(1+\varepsilon^{k-m}), \quad m = i+2j \leq n.$$

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