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NUMERICAL COMPUTATION OF THE FIRST EIGENVALUE OF THE p-LAPLACE OPERATOR ON THE UNIT SPHERE

A. EL SOUFI, M. JAZAR, AND H. AZARI

Abstract. In this paper, we discuss a numerical approximation of the first eigenvalue of the p-Laplace operator on the sphere (S^n, g) of $\mathbb{R}^{n+1}$.

Key Words. First eigenvalue, p-Laplace operator, numerical approximation.

1. Introduction

The p-Laplace operator has been extensively studied in recent years, especially in the context of a bounded domain in $\mathbb{R}^n$ [12, 7, 6, 11, 5, 13, 2, 1]. Recently, there has been an increasing interest in the study of this operator - and in particular of its first eigenvalue - in the more general setting of Riemannian manifolds. The aim of this work is to provide numerical approximation of the first eigenvalue of the p-Laplace operator on the sphere (S^n, g) of $\mathbb{R}^{n+1}$, g being the standard Riemannian metric of the sphere, namely the first positive number λ^* such that the following problem admits a non trivial solution in $W^{1,p}(S^n)$

$$\Delta_p^g u = \lambda^* u|u|^{p-2} \text{ in } S^n, \tag{1.1}$$

where $p > 1$. It is well known that λ^* is the minimizer of the associated energy

$$\lambda^* := \min\{\int_{S^n} |\nabla f|^p, \ f \in W^{1,p}(S^n), \ \|f\|_{L^p} = 1, \ \int_{S^n} |f|^{p-2} f = 0\}. \tag{1.2}$$

That is, λ^* is the best constant such that the following Poincaré type inequality holds for any f such that $\int_{S^n} |f|^{p-2} f = 0$:

$$\int_{S^n} |\nabla f|^p \geq \lambda^* \int_{S^n} |f|^p.$$

By [10, Corollaire 3.1], we know that λ^* is also the first eigenvalue of the p-Laplace operator on a semi-sphere with Dirichlet boundary condition

$$\begin{cases} \Delta_p^g u = \lambda^* u|u|^{p-2} & \text{in } \ S_+^n, \\ u = 0 & \text{on } \ \partial S_+^n = S^{n-1}, \end{cases} \tag{1.3}$$

where S_+^n is the upper semi–sphere.

We know the following

(1) $\lambda^* \geq \left[\frac{n-1}{p-1}\right]^{p/2}$ for $p \geq 2$. [10, Theorem3.2]
(2) $\lambda^* = n$ in the case where $p = 2$.
(3) The first eigenfunction u of (1.3) can be chosen to be nonnegative.
(4) u is radial: $u = \varphi(\rho)$ where ρ is the geodesic distance from the north pole S_+^n.
(5) u is a non increasing function of $\rho \in [0, \pi/2]$, $\varphi(\pi/2) = 0$ and $\varphi'(0) = 0$.

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Of course, one can set the normalization $\varphi(0) = 1$.

From the expression of the spherical Laplacian in polar coordinates, the constant λ^* appears as the unique positive number such that the following problem admits a solution

$$
\begin{cases}
\varphi_* \in C^2(0, \pi/2) \\
[-\varphi_*']^{p-2}\left[(p-1)\varphi_*'' + (n-1)\dfrac{\cos\rho}{\sin\rho}\varphi_*'\right] = -\lambda^* \varphi_*^{p-1}, \qquad \rho \in (0, \pi/2) \\
\varphi_* \geq 0,\ \varphi_*(0) = 1,\ \varphi_*'(0) = 0,\ \varphi_*(\pi/2) = 0.
\end{cases}
\tag{1.4}
$$

Behavior of the eigenfunction near $\frac{\pi}{2}$ Let's look to the behavior of the solution of (1.4) near $\frac{\pi}{2}$. First, note that if $p < 2$ then $\varphi_*'(\pi/2) = 0$ implies that $\varphi_*''(\pi/2) = 0$ also. Now, if $p > 2$ then putting $t := \frac{\pi}{2} - \rho$ and writing $\varphi_*(\rho) = \varphi_*(\frac{\pi}{2} - t) = at^\alpha + O(t^\alpha)$, with $\alpha > 1$, we get

$$
(a\alpha t^{\alpha-1})^{p-2}\left[(p-1)a\alpha(\alpha-1)t^{\alpha-2} + (n-1)a\alpha t^\alpha\right] = -\lambda^*(\alpha t)^{p-1}.
$$

Then necessarily, one has $(\alpha-1)(p-2) + \alpha - 2 = p - 1$, i.e. $\alpha = \frac{2p-1}{p-1} > 2$. In both cases, $p > 2$ or $p < 2$, we have

$$
\varphi_*(\pi/2) = \varphi_*'(\pi/2) = \varphi_*''(\pi/2) = 0. \tag{1.5}
$$

2. Some monotony properties

By "first positive eigenvalue" problem it is classically meant: given a manifold $\mathcal{M}$, find a couple (λ, φ), λ the least positive possible such that the problem

$$
\begin{cases}
\Delta_p \varphi = \lambda \varphi |\varphi|^{p-2} & \text{in} \quad \mathcal{M}, \\
\varphi = 0 & \text{on} \quad \partial\mathcal{M}.
\end{cases}
\tag{2.1}
$$

Aiming to point out some monotony properties, we invert the order: given $\lambda > 0$, find a couple $(\mathcal{M}, \varphi)$ such that the associated problem admits a solution.

For our purpose, we limit ourselves to geodesic balls, i.e., $\mathcal{M} = B_g(N, \rho)$, where N is the north pole on the unit sphere and $\rho \in (0, \pi)$. The problem can then be formulated as follows: given $\lambda > 0$, find $(\rho_\lambda, \varphi_\lambda)$ so that φ_λ is the unique solution, up to the multiplication by a constant, of the problem (2.1) on $B_g(N, \rho_\lambda)$. This gives directly the following

Proposition 2.1. *For all $\lambda > 0$ there exists a unique $\rho_\lambda \in (0, \pi)$ such that the problem (2.1) admits a unique solution φ_λ on $B_g(N, \rho_\lambda)$ satisfying $\varphi_\lambda(N) = 1$. Moreover, the mapping $\lambda \longmapsto \rho_\lambda$ is continuous decreasing and* $\lim_{\lambda\to 0} \rho_\lambda = \pi$ *and* $\lim_{\lambda\to\infty} \rho_\lambda = 0$.

3. Approximate problem

Fix $\lambda > 0$, $\rho_\lambda \in (0, \pi)$ and φ_λ solution of the following

$$
\begin{cases}
\varphi_\lambda \in C^2(0, \rho_\lambda), \\
[-\varphi_\lambda']^{p-2}\left[(p-1)\varphi_\lambda'' + (n-1)\dfrac{\cos\rho}{\sin\rho}\varphi_\lambda'\right] = -\lambda \varphi_\lambda^{p-1}, \qquad \rho \in (0, \rho_\lambda), \\
\varphi_\lambda \geq 0, \quad \varphi_\lambda(0) = 1, \quad \varphi_\lambda'(0) = 0, \quad \varphi_\lambda(\rho_\lambda) = 0.
\end{cases}
\tag{3.1}
$$

In order to study Problem (3.1) we transform it into an initial condition problem. Since we have a problem at zero, using development into fractional Taylor series, one find that $\varphi(\rho) = 1 - a\rho^{2+\alpha} + O(\rho^{2+\alpha})$, for ρ near zero, where

$$
\alpha := \frac{2-p}{p-1} \qquad \text{and} \qquad a := \frac{p-1}{p}[\lambda/n]^{\frac{1}{p-1}}.
$$

Therefore, instead of considering the problem (3.1), one consider, for $\varepsilon > 0$, the approximated problem (3.2)

$$\begin{cases} \varphi_{\lambda,\varepsilon} \in C^2(\varepsilon, \rho_{\lambda,\varepsilon}) \\ [-\varphi'_{\lambda,\varepsilon}]^{p-2}\left[(p-1)\varphi''_{\lambda,\varepsilon} + (n-1)\dfrac{\cos\rho}{\sin\rho}\varphi'_{\lambda,\varepsilon}\right] = -\lambda\varphi^{p-1}_{\lambda,\varepsilon}, \qquad \rho \in (\varepsilon, \rho_{\lambda,\varepsilon}) \\ \varphi_{\lambda,\varepsilon}(\varepsilon) = 1 - a\varepsilon^{2+\alpha}, \qquad \varphi'_{\lambda,\varepsilon}(\varepsilon) = -a(2+\alpha)\varepsilon^{1+\alpha}, \end{cases} \tag{3.2}$$

where $\rho_{\lambda,\varepsilon} := \sup\{0 < r < \pi \text{ s.t. } \varphi_{\lambda,\varepsilon} > 0 \quad \text{on} \quad [\varepsilon, r]\}$.

We have the following

Proposition 3.1. *For all $\varepsilon < \rho < \min(\frac{\pi}{2}, \rho_{\lambda,\varepsilon})$, $\varphi'_{\lambda,\varepsilon}(\rho)$ is negative.*

Proof. Assume that there exists $\varepsilon < \rho_0 < \min(\frac{\pi}{2}, \rho_{\lambda,\varepsilon})$ such that $\varphi'_{\lambda,\varepsilon}(\rho_0) = 0$. If $p > 2$ then $\varphi_{\lambda,\varepsilon}(\rho_0) = 0$ and hence $\rho_0 = \rho_{\lambda,\varepsilon}$. Now, if $p < 2$ then writing $\rho = \rho_0 - t$, $t > 0$ we have, by using the Taylor series of $\varphi_{\lambda,\varepsilon}$ near $t = 0$, $\varphi_{\lambda,\varepsilon}(\rho) = \varphi_{\lambda,\varepsilon}(\rho_0) + at^\alpha$, we get

$$(a\alpha t^{\alpha-1})^{p-2}\left[(p-1)a\alpha(\alpha-1)t^{\alpha-2} + (n-1)(-a\alpha)\cot\rho_0 t^{\alpha-1}\right] = -\lambda(\varphi_{\lambda,\varepsilon}(\rho_0))^{p-1}.$$

Thus, if $\varphi_{\lambda,\varepsilon}(\rho_0) \neq 0$ we have $(\alpha-1)(p-2) + \alpha - 2 = 0$, i.e. $\alpha = \frac{p}{p-1} > 1$ and $(a\alpha)^{p-1} = -\lambda(\varphi_{\lambda,\varepsilon}(\rho_0))^{p-1}$. In both cases $\varphi_{\lambda,\varepsilon}(\rho_0) = 0$. Contradiction. ☐

Proposition 3.2. *Let $0 < \lambda_1, \lambda_2 < \Lambda$ and $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$, ε_0 small. Then for all $\varepsilon_2 \leq \rho_0 < \rho_1 < \min\{\rho_{\lambda_1,\varepsilon_1}, \rho_{\lambda_2,\varepsilon_2}\}$ there exists $C = C(n, p, \Lambda, \varepsilon_0, \rho_1) > 0$ such that*

$$\|\varphi_{\lambda_1,\varepsilon_1} - \varphi_{\lambda_2,\varepsilon_2}\|_{C^1([\rho_0,\rho_1])} \leq C\max\{\left|\lambda_1^{1+\alpha} - \lambda_2^{1+\alpha}\right|, \left|\varepsilon_1^{1+\alpha} - \varepsilon_2^{1+\alpha}\right|\}. \tag{3.3}$$

Proof. Write the ODE 3.2 as $\Phi' = F(\Phi, \lambda, \rho)$, where $\Phi := (\varphi, \varphi')$ and $F(x, y, \lambda, \rho) := \left(y, \frac{1}{p-1}\left[-(n-1)y\cot\rho - \lambda x^{p-1}(-y)^{2-p}\right]\right)$. Denote by $\varphi_1 := \varphi_{\lambda_1,\varepsilon_1}$ and $\varphi_2 := \varphi_{\lambda_2,\varepsilon_2}$. Since $\varphi_1, \varphi_2 \in C^\infty(\varepsilon_2, \rho_m)$ where $\rho_m := \min\{\rho_{\lambda_1,\varepsilon_1}, \rho_{\lambda_2,\varepsilon_2}\}$, let $L := \max\{\|F_x\|_{L^\infty(A,B)}, \|F_y\|_{L^\infty(A,B)}\}$, where A and B such that $\varphi([\rho_0, \rho_1]) \subset [A, B]$ and $\varphi'([\rho_0, \rho_1]) \subset [A, B]$. Then by the general theory of the ODE one has,

$$\|\Phi_1 - \Phi_2\|_{L^\infty(\varepsilon_2, b)} \leq e^{Lb}\|\Phi_1(\varepsilon_2) - \Phi_2(\varepsilon_2)\|.$$

Remains to estimate $\|\Phi_1(\varepsilon_2) - \Phi_2(\varepsilon_2)\|$. We have

$$|\varphi_1(\varepsilon_2) - \varphi_1(\varepsilon_1)| \leq |\varepsilon_2 - \varepsilon_1|\,\|\varphi'_1\|_{L^\infty(\varepsilon_1\varepsilon_2)} \leq C|\varepsilon_2 - \varepsilon_1|,$$

as $|\varphi'|$ is increasing and $\varphi'_1(\varepsilon_1) = a(2+\alpha)\varepsilon_1^{\alpha+1}$. Now, setting $a(\lambda) := (\lambda/n)^{\frac{1}{p-1}}(p-1)/p$, one has

$$\begin{aligned} |\varphi_1(\varepsilon_2) - \varphi_2(\varepsilon_2)| &\leq |\varphi_1(\varepsilon_1) - \varphi_2(\varepsilon_2)| + C\,|\varepsilon_1 - \varepsilon_2| \\ &\leq |a(\lambda_1)\varepsilon_1^{2+\alpha} - a(\lambda_2)\varepsilon_2^{2+\alpha}| + C|\varepsilon_1 - \varepsilon_2| \\ &\leq C\max\left\{|\varepsilon_1 - \varepsilon_2|, \left|\lambda_1^{1+\alpha} - \lambda_2^{1+\alpha}\right|\right\}. \end{aligned}$$

Let's show that

$$\|\varphi''_1\|_{L^\infty(\varepsilon_1,\varepsilon_2)} \leq C\varepsilon_1^\alpha. \tag{3.4}$$

Using (3.5) and the fact that $\varphi'' < 0$ near zero, we have

$$\begin{aligned} |\varphi''_1| &\leq C\cot\varepsilon_1|\varphi'(\varepsilon_2)| + C(1 - \varphi_1(\varepsilon_1))^{p-1}\max(\varepsilon_1^\alpha, \varepsilon_2\alpha) \\ &\leq C\cot\varepsilon_1|\varphi'_1(\varepsilon_1)| + C\cot\varepsilon_1|\varphi'_1(\varepsilon_2) - \varphi'_1(\varepsilon_1)| + C\varepsilon_1\max(\varepsilon_1^\alpha, \varepsilon_2\alpha) \\ &\leq C\varepsilon_1^\alpha + C\cot\varepsilon_1|\varepsilon_2 - \varepsilon_1|\|\varphi''_1\|_{L^\infty(\varepsilon_1,\varepsilon_2)} + C\varepsilon_1\max(\varepsilon_1^\alpha, \varepsilon_2^\alpha). \end{aligned}$$

Therefore, taking $\varepsilon_2 - \varepsilon_1 \le \frac{1}{2C}\varepsilon_1$ we get

$$\|\varphi_1''\|_{L^\infty(\varepsilon_1,\varepsilon_2)} \le C\varepsilon_1^\alpha + C\varepsilon_1^{1+\alpha}.$$

Using this, we have

$$|\varphi_1'(\varepsilon_2) - \varphi_1'(\varepsilon_1)| \le C|\varepsilon_2 - \varepsilon_1|\varepsilon_1^\alpha \le C|\varepsilon_2^{1+\alpha} - \varepsilon_1^{1+\alpha}|.$$

Then

$$\begin{aligned} |\varphi_1'(\varepsilon_2) - \varphi_2'(\varepsilon_2)| &\le |\varphi_1'(\varepsilon_1) - \varphi_2'(\varepsilon_2)| + C\,|\varepsilon_2^{1+\alpha} - \varepsilon_1^{1+\alpha}| \\ &\le (2+\alpha)|a(\lambda_1)\varepsilon_1^{1+\alpha} - a(\lambda_2)\varepsilon_2^{1+\alpha}| + C|\varepsilon_2^{1+\alpha} - \varepsilon_1^{1+\alpha}| \\ &\le C\max\left\{|\varepsilon_1^{1+\alpha} - \varepsilon_2^{1+\alpha}|, \left|\lambda_1^{1+\alpha} - \lambda_2^{1+\alpha}\right|\right\}. \end{aligned}$$

□

Lemma 3.3. *There exists $\varepsilon_0 > 0$ small such that for $\varepsilon \le \varepsilon_0$ the function $\lambda \mapsto A(\lambda,\varepsilon) := \varphi''_{\lambda,\varepsilon}(\varepsilon)$ is decreasing, and $\varphi''_{\lambda,\varepsilon}(\varepsilon) \le 0$.*

Proof. We have

$$(p-1)\varphi''_{\lambda,\varepsilon}(\varepsilon) = a(2+\alpha)\varepsilon^\alpha\left[(n-1)\varepsilon\cot\varepsilon - n(1-a\varepsilon^{2+\alpha})^{p-1}\right]. \tag{3.5}$$

A direct calculation gives

$$\begin{aligned} (p-1)\frac{\partial}{\partial a}A(\lambda,\varepsilon)(\varepsilon) &= (2+\alpha)\varepsilon^\alpha\big[(n-1)\varepsilon\cot\varepsilon - n(1-a\varepsilon^{2+\alpha})^{p-1} \\ &\quad + n(p-1)\varepsilon^{2+\alpha}a(1-a\varepsilon^{2+\alpha})^{p-2}\big] \\ &= (2+\alpha)\varepsilon^\alpha\left[(n-1)\varepsilon\cot\varepsilon - n(1-a\varepsilon^{2+\alpha})^{p-2}(1-pa\varepsilon^{2+\alpha})\right], \end{aligned}$$

which is negative for small ε. □

Proposition 3.4. *Let for $\varepsilon > 0$ and $\lambda > 0$, $\varphi_{\lambda,\varepsilon}$ the solution of (3.2). Assume that $\rho_{\lambda,\varepsilon} < \pi$, then $\varphi_{\lambda,\varepsilon}(\rho_{\lambda,\varepsilon}) = 0$. Moreover, we have*

1. *The function $\lambda \longmapsto \rho_{\lambda,\varepsilon}$ is continuous and decreasing.*
2. *There exists a unique $\lambda_\varepsilon > 0$ such that $\rho_{\lambda_\varepsilon,\varepsilon} = \frac{\pi}{2}$.*
3. *The function $\varepsilon \longmapsto \lambda_\varepsilon$ is increasing and for all $\rho \in [\varepsilon, \rho_{\lambda_\varepsilon,\varepsilon}]$ the function $\varepsilon \longmapsto \varphi_{\lambda_\varepsilon,\varepsilon}(\rho)$ is increasing and*

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon = \lambda^*,$$

and

$$\lim_{\varepsilon\to 0}\varphi_{\lambda_\varepsilon,\varepsilon} = \varphi^*.$$

Moreover, there exists a constant $C > 0$ such that

$$|\lambda_\varepsilon - \lambda^*| \le C\varepsilon^{1+\alpha} \tag{3.6}$$

and

$$\|\varphi_* - \varphi_{\lambda_\varepsilon,\varepsilon}\|_{C^1([\varepsilon,\rho_{\lambda_\varepsilon,\varepsilon}])} \le C\varepsilon^{1+\alpha}. \tag{3.7}$$

Proof. **1.** For the first point we will prove that the function $\lambda \longmapsto \rho_{\lambda,\varepsilon}$ is decreasing. For this we observe that $\lambda \mapsto 1 - a\varepsilon^{2+\alpha}$ and $\lambda \mapsto -a(2+\alpha)\varepsilon^{1+\alpha}$ are decreasing.

Lemma 3.5. *Consider two solutions φ_1 and φ_2 corresponding respectively to $\lambda_1 < \lambda_2$. Then $\varphi_1(\rho) > \varphi_2(\rho)$ for all $\rho \in [\varepsilon, \rho_m]$, where $\rho_m := \min(\rho_{\lambda_1,\varepsilon}, \rho_{\lambda_2,\varepsilon})$.*

Proof. Since $\varphi_1(\varepsilon) > \varphi_2(\varepsilon)$, $\varphi_1'(\varepsilon) > \varphi_2'(\varepsilon)$ and, by Lemma 3.3, $\varphi_1''(\varepsilon) > \varphi_2''(\varepsilon)$, let $\varphi := \varphi_1 - \varphi_2$ and

$$\rho_0 := \sup\{\rho \in (\varepsilon, \rho_m),\ \varphi''(\rho) > 0\},$$
$$\rho_1 := \sup\{\rho \in (\varepsilon, \rho_m),\ \varphi'(\rho) > 0\},$$

and

$$\rho_2 := \sup\{\rho \in (\varepsilon, \rho_m),\ \varphi(\rho) > 0\},$$

then, obviously, $\rho_0 < \rho_1 < \rho_2$. In order to prove that $\varphi_1(\rho) > \varphi_2(\rho)$ for all $\rho \in [\varepsilon, \rho_m]$ it is sufficient to show that $\rho_m < \rho_2$. Assume that $\rho_2 \leq \rho_m$. To get a contradiction, evaluate the expression of the second derivative

$$(p-1)\varphi''(\rho) = (n-1)(-\varphi'(\rho))\cot\rho - \lambda(-\varphi')^{2-p}(\rho)\varphi^{p-1}(\rho) \tag{3.8}$$

at ρ_1. Using $\varphi_1'(\rho_1) = \varphi_2'(\rho_1)$ and $-\lambda_1\varphi_1^{p-1}(\rho_1) > -\lambda_2\varphi_2^{p-1}(\rho_1)$, we get $\varphi_1''(\rho_1) > \varphi_2''(\rho_1)$ which is a contradiction. Henceforth, $\rho_m < \rho_2$. □

2. We will show that $\lim_{\lambda\to 0}\rho_{\lambda,\varepsilon} > \frac{\pi}{2}$ and $\lim_{\lambda\to\lambda_0}\rho_{\lambda,\varepsilon} < \frac{\pi}{2}$ where $\lambda_0 := n(\frac{p}{p-1})^{p-1}\varepsilon^{-p}$.
For this write the differential equation as

$$(p-1)\psi'' = -(n-1)\cot\rho\psi' + \lambda(1-\psi)^{p-1}(\psi')^{2-p}, \tag{3.9}$$

where $\psi := 1 - \varphi$. This implies directly the following differential inequality

$$(p-1)\psi'' \leq -(n-1)\cot\rho\psi' + \lambda(\psi')^{2-p},$$

which, setting $z := \psi'$ can be writing as

$$(p-1)z' \leq -(n-1)\cot\rho z + \lambda z^{2-p},$$

with $z(\varepsilon) = C\lambda^{\frac{1}{p-1}}$, $C > 0$ is a constant. By Proposition 3.1 $z > 0$ for all $\rho \in (\varepsilon, \rho_0)$, with $\rho_0 := \min(\pi/2, \rho_{\lambda,\varepsilon})$, thus $(p-1)z' \leq \lambda z^{2-p}$ which implies that $z^{p-1}(\rho) \leq \lambda[\rho - \varepsilon + C^{p-1}] \leq C'^{p-1}\lambda$ for $\rho \in (\varepsilon, \rho_0)$. Therefore, $\psi'(\rho) \leq C'\lambda^{\frac{1}{p-1}}$ for all $\rho \in (\varepsilon, \rho_0)$ and hence, for $\rho \in (\varepsilon, \rho_0)$,

$$1 - \varphi(\rho) = \psi(\rho) \leq \lambda^{\frac{1}{p-1}}[C + C'\rho],$$

or

$$\varphi(\rho) \geq 1 - C''\lambda^{\frac{1}{p-1}} \qquad \forall\rho \in (\varepsilon, \rho_0). \tag{3.10}$$

This obviously implies that $\lim_{\lambda\to 0}\rho_{\lambda,\varepsilon} > \frac{\pi}{2}$.
Let's show the second limit: $\lim_{\lambda\to\lambda_0}\rho_{\lambda,\varepsilon} < \frac{\pi}{2}$. Equation (3.9) implies

$$\begin{aligned}(p-1)\psi'' &\geq -(n-1)\cot\rho\psi' \\ (p-1)z' &\geq -(n-1)\cot\rho z \geq -(n-1)\cot\varepsilon z \\ z' &\geq -\gamma z,\end{aligned}$$

where $\gamma := \frac{n-1}{p-1}\cot\varepsilon$. Integrating this differential inequality gives

$$-\varphi'(\rho) = \psi'(\rho) \geq \psi'(\varepsilon)e^{-\gamma(\rho-\varepsilon)}$$

then

$$\varphi'(\rho) \leq -\varphi'(\varepsilon)e^{-\gamma(\rho-\varepsilon)},$$

which gives

$$\varphi(\rho) \leq \varphi(\varepsilon) + \frac{\varphi'(\varepsilon)}{\gamma}\left[e^{-\gamma(\rho-\varepsilon)} - 1\right].$$

This implies that φ vanishes before $\rho = \varepsilon - \gamma\ln\left[1 - \gamma\frac{\varphi(\varepsilon)}{\varphi'(\varepsilon)}\right]$ which is obviously less than $\frac{\pi}{2}$ as λ is near λ_0 (since in that case $\varphi(\varepsilon)$ is close to 0).

3. The monotony of $\varepsilon \longmapsto \lambda_\varepsilon$ is a direct consequence of the following lemma

Lemma 3.6. *Consider $0 < \varepsilon_1 < \varepsilon_2$ small and denote by $\varphi_1 := \varphi_{\lambda_{\varepsilon_1},\varepsilon_1}$ and $\varphi_2 := \varphi_{\lambda_{\varepsilon_2},\varepsilon_2}$ the corresponding solutions respectively. Then $\lambda_1 := \lambda_{\varepsilon_1} < \lambda_2 := \lambda_{\varepsilon_1}$ and $\varphi_1 \leq \varphi_2$ on $[\varepsilon_2, \frac{\pi}{2})$.*

Proof. In order to find a contradiction let us assume that $\lambda_1 \geq \lambda_2$. Since $\varphi_1''(\varepsilon_1) < 0$ and $\varepsilon_2 - \varepsilon_1$ is small, one have by direct computation, $\varphi_1(\varepsilon_2) < \varphi_2(\varepsilon_2)$ and $\varphi_1'(\varepsilon_2) < \varphi_2'(\varepsilon_2)$. Now, since $\lambda_1 \geq \lambda_2$ then, by Lemma 3.3, $A(\lambda_1, \varepsilon_2) \leq A(\lambda_2, \varepsilon_2)$. Using the continuity of the mappings $\varepsilon \mapsto \varphi_1''(\varepsilon)$ and $\varepsilon \mapsto A(\lambda_2, \varepsilon)$ at ε_1 one has $\varphi_1''(\varepsilon_2) \leq \varphi_2''(\varepsilon_2)$. Using a similar argument of that of the first point one gets that $\varphi_1 < \varphi_2$ on $[\varepsilon_2, \rho_m]$. This is a contradiction since here $\rho_m = \rho_{\lambda_1,\varepsilon_1} = \rho_{\lambda_2,\varepsilon_2} = \frac{\pi}{2}$ and by definition $\varphi_1(\rho_{\lambda_1,\varepsilon_1}) = 0 = \varphi_2(\rho_{\lambda_2,\varepsilon_2})$. This proves the lemma and the monotony. □

This monotony implies that the limit $\lambda_0 := \lim_{\varepsilon \to 0} \lambda_\varepsilon$ exists and for all $\rho \in (0, \pi/2)$, $\varphi_{\lambda_\varepsilon,\varepsilon}(\rho)$ converges to some function $\varphi_0(\rho)$ and the same is true for the first and second derivatives. Obviously this function φ_0 is solution of (3.1) on the interval $(0, \pi/2)$. Moreover, $\varphi_0(\pi/2) = 0$ by construction, $\varphi_0(0) = \lim_{\varepsilon \to 0} \varphi_{\lambda_\varepsilon,\varepsilon}(\varepsilon) = 1 - a\varepsilon^{2+\alpha} \to 1$ and by the same way $\varphi_0'(0) = 0$. In other words $\lambda^* = \lambda_0$ and $\varphi^* = \varphi_0$. Now, for $\varepsilon > 0$ small enough, one have, by construction,

$$\| (\varphi_*(\varepsilon), \varphi_*'(\varepsilon)) - \left(\varphi_{\lambda_\varepsilon,\varepsilon}(\varepsilon), \varphi_{\lambda_\varepsilon,\varepsilon}'(\varepsilon)\right) \| \leq C(\lambda)\varepsilon^{1+\alpha}.$$

To terminate notice that, using the continuous dependence of the solution in terms of the initial data, one have, for all $\varepsilon < r < \pi/2$

$$\|\varphi_* - \varphi_{\lambda_\varepsilon,\varepsilon}\|_{C^1([\varepsilon,\rho_{\lambda_\varepsilon,\varepsilon}])} \leq C(\lambda)\varepsilon^{1+\alpha} \leq C\varepsilon^{1+\alpha}.$$

Writing (3.8) in the form

$$\varphi'' = F(\rho, \varphi, \varphi'),$$

where $F(\rho, x, y) := \frac{1}{p-1}\left[-(n-1)\cot\rho y - \lambda x^{p-1}(-y)^{2-p}\right]$, one can get the estimate (3.6) by using the previous inequality (3.7) and calculating the difference at a specific value of ρ, $\pi/4$ for example. □

4. Numerical procedures

For each value of possible λ, solve the ode with the following two conditions:
φ is positive on $[0, \pi/2)$,
$\varphi(\pi/2) = 0$.

Proposed algorithm:

(1) Fix $\varepsilon > 0$.
(2) Take λ equal to an initial starting value.
(3) Solve the ODE $(P_{\lambda_\varepsilon})$ and determine $\rho_{\lambda,\varepsilon}$.
(4) Test $(\rho_{\lambda,\varepsilon} = \pi/2)$
(5) If test false then
  - if $\rho_{\lambda,\varepsilon} > \phi/2$ then increase λ.
  - elsewhere then decrease λ.

By "decrease" one means decrease half of the last step.

Another, more general, trick is to treat the constant λ as another dependent variable, adding the ODE [3, 4]

$$\lambda' = 0. \tag{4.1}$$

The problem (3.2) is again in standard form. Now, any of our standard methods can be used to solve the BVP (3.2), (4.1) because its solution, which are eigenvalue–eigenfunction pairs, are isolated.

This approach can be extremely convenient in the common case where only the first one or two eigenvalues and eigenfunctions are desired.

Finite difference methods can also be applied directly to solve our problems. After discretization the problem, it can be solved using standard linear algebra routines. We can use the QR algorithm, note that this gives us all of eigenvalues and eigenvectors.

There are other well–studied methods for calculating a set of eigenvalue approximations. The Rayleigh–Ritz method and collocation method, utilizes a variational characterization of the eigenvalues and, after selection of an approximation space for the eigenfunctions (e.g., piecewise polynomials) involves solving a resulting matrix eigenvalue problem just as above.

We solving BVPs for ODEs in MATLAB with `bvp4c` [9], `bvp4c` implements a collocation method for the solution of BVPs of the form

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}, \mathbf{p}), \qquad a \le x \le b,$$

subject to general nonlinear, two–point boundary conditions

$$\mathbf{g}(\mathbf{y}(a), \mathbf{y}(b), \mathbf{p}) = \mathbf{0}.$$

Here $\mathbf{p}$ is a vector of unknown parameters. For simplicity it is suppressed in the expressions that follow. The approximate solution $\mathbf{y}_h(x)$ is a continuous function that is a cubic polynomial on each subinterval $[x_n, x_{n+1}]$ of a mesh $a = x_0 < x_1 < \cdots < x_N = b$. It satisfies the boundary condition

$$\mathbf{g}(\mathbf{y}_h(a), \mathbf{y}_h(b)) = \mathbf{0}$$

and it satisfies the differential equations (collocates) at both ends and the midpoint of each subinterval

$$\mathbf{y}_h'(x_n) = \mathbf{f}(x_n, \mathbf{y}_h(x_n)),$$

$$\mathbf{y}_h'((x_n + x_{n+1})/2) = \mathbf{f}((x_n + x_{n+1})/2, \mathbf{y}_h((x_n + x_{n+1})/2)),$$

$$\mathbf{y}_h'(x_{n+1}) = \mathbf{f}(x_{n+1}, \mathbf{y}_h(x_{n+1})).$$

These conditions result in a system of nonlinear algebraic equations for the coefficients defining $\mathbf{y}(x)$. In contrast to shooting, the solution $\mathbf{y}(x)$ is approximated over the whole interval $[a, b]$ and the boundary conditions are taken into account at all times. The nonlinear algebraic equations are solved iteratively by linearization, so this approach relies upon the linear equation solvers of MATLAB rather than its IVP codes. The basic method of `bvp4c`, which we call Simpson's method, is well–known and is found in a number of codes. It can be shown [8] that with modest assumptions, $\mathbf{y}_h(x)$ is a fourth order approximation to an isolated solution $\mathbf{y}(x)$, i.e., $\|\mathbf{y}(x) - \mathbf{y}_h(x)\| \le Ch^4$. Here h is the maximum of the step sizes $h_n = x_{n+1} - x_n$ and C is a constant. Because it is not true of some popular collocation methods, we stress the important fact that this bound hold for all x in $[a, b]$.

5. Numerical results

In this section we will present some numerical examples by using the numerical procedures discussed in the previous sections.

Example 5.1. *This example illustrates the solution of a boundary value problem involving an unknown parameter. It also shows how to evaluate the solution anywhere in the interval of integration. The task is to compute the eigenvalue of the problem (3.2) with* $n = 3, p = 2$,

$$\frac{d^2}{dt^2}\varphi(\rho) + 2\,\frac{\cos(\rho)\,\frac{d}{dt}\varphi(\rho)}{\sin(\rho)} + \lambda\,\varphi(\rho) = 0$$

on $[\varepsilon, \pi/2]$, *where an initial guess of* $\lambda_0 = 2.5$ *and* $\varepsilon = 0.001$. *The computed value for the unknown parameter is* $\lambda = 3.000000213508$. *The Numerical solution of* $\varphi(\rho)$ *is shown in Figure 1.*

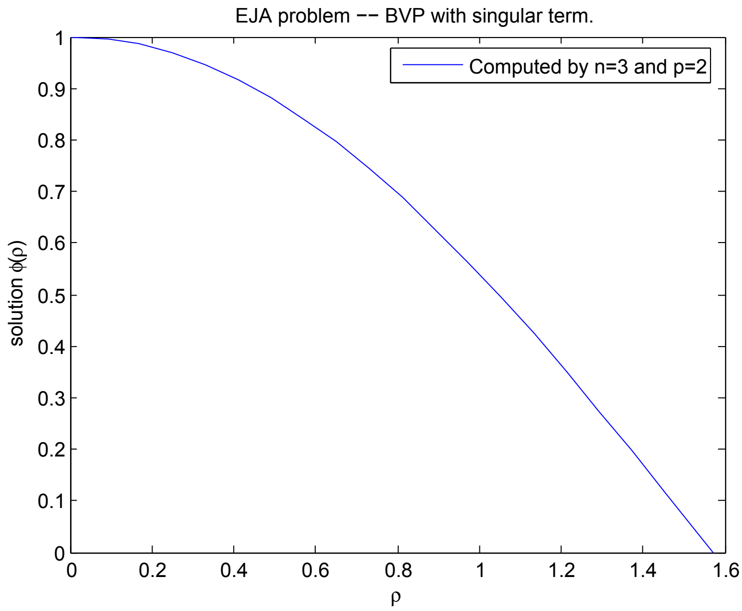

FIGURE 1. Numerical solution obtained with $n = 3, p = 2$

Example 5.2. *The boundary value problem (3.2) involving an unknown parameter and a singular point with* $n = 3, p = 1.5$,

$$\left(-\frac{d}{dt}\varphi(\rho)\right)^{-0.5}\left(0.5\,\frac{d^2}{dt^2}\varphi(\rho) + 2\,\frac{\cos(\rho)\,\frac{d}{dt}\varphi(\rho)}{\sin(\rho)}\right) + \lambda\,(\varphi(\rho))^{0.5} = 0$$

on $[\varepsilon, \pi/2]$ *where an initial guess of* $\lambda_0 = 2$ *and* $\varepsilon = 0.001$. *The computed value for the unknown parameter is* $\lambda = 2.401797995328$. *Figure 2 show the numerical solution of* $\varphi(\rho)$.

Example 5.3. *This example introduces BVP (3.2) with* $n = 3, p = 3$. *The differential equation is*

$$-\left(\frac{d}{dt}\varphi(\rho)\right)\left(2\,\frac{d^2}{dt^2}\varphi(\rho) + 2\,\frac{\cos(\rho)\,\frac{d}{dt}\varphi(\rho)}{\sin(\rho)}\right) + \lambda\,(\varphi(\rho))^2 = 0$$

on $[\varepsilon, \pi/2]$ *where an initial guess of* $\lambda_0 = 3$ *and* $\varepsilon = 0.001$. *The computed value for the unknown parameter is* $\lambda = 3.708216401664$. *We plot* $\varphi(\rho)$ *from 0 to* $\pi/2$ *(Figure 3).*

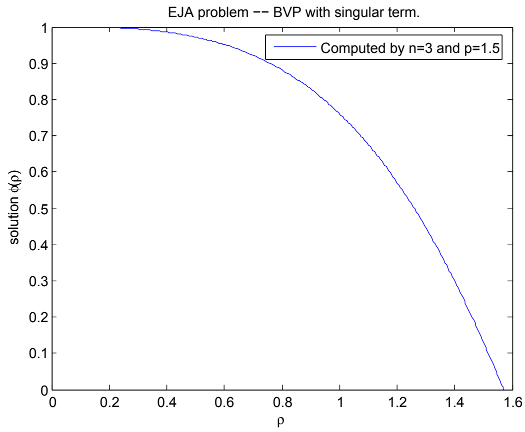

FIGURE 2. Numerical solution obtained with $n = 3, p = 1.5$

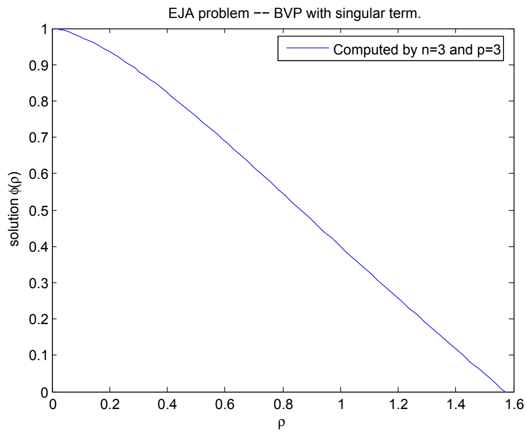

FIGURE 3. Numerical solution obtained with $n = 3, p = 3$

The numerical solution of $\lambda(n, p)$ is shown in Figure 4.

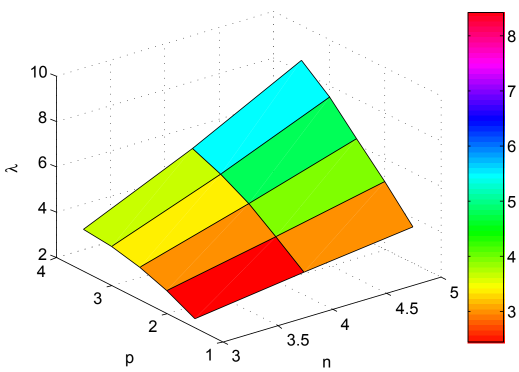

FIGURE 4. Problem with behavior of $\lambda(n, p)$

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Facult des Sciences et Techniques, University Franois Rabelais, 37200 TOURS, France
E-mail: `mailto:ahmad.elsoufi@univ-tours.frahmad.elsoufi@univ-tours.fr`

LaMA-Liban, Azm Research Center, EDST, Lebanese University, Tripoli, Lebanon
E-mail: `mailto:mjazar@ul.edu.lbmjazar@ul.edu.lb`

Department of Mathematical Science , Shahid Beheshti University, Tehran, Iran
E-mail: `mailto:h_azari@sbu.ac.irh_azari@sbu.ac.ir`