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A FRONT-FIXING FINITE ELEMENT METHOD FOR THE VALUATION OF AMERICAN PUT OPTIONS ON ZERO-COUPON BONDS

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Abstract. A front-fixing finite element method is developed for the valuation of American put options on zero-coupon bonds under a class of one-factor models of short interest rates. Numerical results are presented to examine our method and to compare it with the usual finite element method. A conjecture concerning the behavior of the early exercise boundary near the option expiration date is proposed according to the numerical results.

Key Words. American put option, zero-coupon bond, free boundary problem, front-fixing method, finite element method

1. Introduction

Consider a class of one-factor models of the short interest rate process:

$$r(t) = \zeta(X(t)), \quad dX(t) = (\phi(t) - \psi(t)X(t))dt + \sigma(t)dW(t), \tag{1.1}$$

where $\zeta(x)$ is an invertible function on $(-\infty, +\infty)$, $\phi(t)$, $\psi(t)$ and $\sigma(t)$ are some known functions of t, and $W(t)$ is a standard Brownian motion under the risk-neutral measure. For $\zeta(x) = x$ and $\zeta(x) = e^x$, we have the popular Hull-White model ([7]) and Black and Karasinski model ([4]), respectively.

Let $x = \eta(r)$ be the inverse function of $r = \zeta(x)$. Assume that $\zeta(x)$ is twice continuously differentiable. By using Ito's formula, we can obtain the stochastic differential equation (SDE) for the interest rate process $r(t)$:

$$dr(t) = a(r(t), t)dt + b(r(t), t)dW(t), \tag{1.2}$$

where

$$a(r,t) = \zeta'(\eta(r))(\phi(t) - \psi(t)\eta(r)) + \frac{1}{2}\sigma(t)^2\zeta''(\eta(r)), \quad b(r,t) = \sigma(t)\zeta'(\eta(r)).$$

Then we have the following fundamental partial differential equation (PDE) for the rational price $V(r,t)$ of an interest rate derivative at time t ([3][14]):

$$V_t + \frac{1}{2}b(r,t)^2V_{rr} + a(r,t)V_r - rV = 0. \tag{1.3}$$

Since $\zeta(x)$ is invertible, we can rewrite the above PDE into the PDE for $\widetilde{V}(x,t) = V(\zeta(x), t)$:

$$\widetilde{V}_t + \frac{1}{2}\sigma(t)^2\widetilde{V}_{xx} + (\phi(t) - \psi(t)x)\widetilde{V}_x - \zeta(x)\widetilde{V} = 0.$$

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The assumption that $\zeta(x)$ is invertible is necessary to derive the equivalent SDE (1.2) for $r(t)$ and the PDE (1.3) . For example, when $\zeta(x) = x^2$, we have the well-known quadratic model. Since $\zeta(x) = x^2$ is not invertible on $(-\infty, +\infty)$, we do not have an SDE for the interest rate process $r(t) = X(t)^2$ and can not express the interest rate derivative price as a function of interest rate r. It should be pointed out that $\zeta(x)$ can be chosen to be any bounded invertible function from $(-\infty, +\infty)$ to $(0, 1)$, e.g.,

$$\zeta(x) = \frac{e^x}{1 + e^x}. \tag{1.4}$$

For such a choice of $\zeta(x)$, the interest rates will not take unrealistic values more than 1. We are referred to [8] and [9] for other possible choices of $\zeta(x)$ and the calibration of one-factor models.

Now let us consider an American put option on a T^*-maturity zero-coupon bond. The option expiration date is T $(< T^*)$ and its exercise price is K. Since the option can be exercised at any time up to its expiration date, there is a critical interest rate $r^*(t)$ which is referred to as the early exercise interest rate. Denote the option price by $p(r, t)$. Let $x^*(t) = \eta(r^*(t))$ and $\widetilde{p}(x, t) = p(\zeta(x), t)$. According to the above argument, we can show that $\widetilde{p}(x, t)$ and $x^*(t)$ solve the following free boundary problem:

$$(1.5)\, \widetilde{p}_t + \frac{1}{2}\sigma(t)^2 \widetilde{p}_{xx} + (\phi(t) - \psi(t)x)\widetilde{p}_x - \zeta(x)\widetilde{p} = 0,\ -\infty < x < x^*(t),\ 0 \le t \le T,$$
$$(1.6)\, \widetilde{p}(x^*(t), t) = g(x^*(t), t), \quad 0 \le t \le T,$$
$$(1.7)\, \widetilde{p}_x(x^*(t), t) = g_x(x^*(t), t), \quad 0 \le t \le T,$$
$$(1.8)\, \widetilde{p}(x, T) = g(x, T), \quad -\infty < x < \infty,$$

where $g(x, t) = \max(K - \widetilde{P}(x, t; T^*), 0)$ is the payoff of the put option and $\widetilde{P}(x, t; T^*)$ is the bond price when $r = \zeta(x)$ at time t.

Front-fixing/front-tracking methods have been applied for numerical valuation of American options. Their favorable feature is that the early exercise boundaries and option prices can be computed simultaneously and with higher accuracy. We are referred to [6, 11, 12, 13, 15, 16] for recent work in this aspect for American stock options. For the usual front-fixing method, the Landau transformation $y = (x + L)/(x^*(t) + L)$ will be employed after restricting the problem on the bounded domain $(-L, x^*(t))$ for a sufficiently large positive number L. Here we shall use the linear transformation $y = x + L - x^*(t)$ while the problem is truncated on the variable domain $(x^*(t) - L, x^*(t))$. The transformation will not affect the coefficient of the leading term in the partial differential equation (1.5). This approach is first proposed for American options on stocks in [2], and our numerical results show that it produces much more accurate approximations of early exercise boundaries and option prices. In this paper we shall consider such a front-fixing finite element method (FFEM) for the free boundary problem (1.5)–(1.8).

The outline of the paper is as follows. In §2 we develop a FFEM for the free boundary problem (1.5)–(1.8) and establish its stability with an appropriate assumption. In §3 we give details for the implementation of our method and show how to compute bond prices and their derivatives when analytic formulas are not available. In §4 numerical results are presented to examine our method and to compare it with the usual finite element method in [17]. In particular, we shall analyze the behavior of early exercise interest rates near the option expiration dates numerically. We conclude the paper with remarks in the last section, §5.

2. A front-fixing finite element method

In this section we consider a front-fixing finite element method for the free boundary value problem (1.5)–(1.8) and analyze its stability with an appropriate assumption about the approximate free boundaries near the option expiration date.

Let L be a positive number large enough such that

$$g(x,t)=0, \quad \widetilde{p}(x,t)\le \epsilon, \quad \forall\, x\le x^*(t)-L,$$

where ϵ is a given error tolerance. Then we can truncate the free boundary problem (1.5)–(1.8) into the following problem:

$$\begin{aligned}
&\widetilde{p}_t+\frac{1}{2}\sigma(t)^2\widetilde{p}_{xx}+(\phi(t)-\psi(t)x)\widetilde{p}_x-\zeta(x)\widetilde{p}=0, \quad x^*(t)-L<x<x^*(t), \quad 0\le t\le T,\\
&\widetilde{p}(x^*(t)-L,t)=g(x^*(t)-L,t), \quad 0\le t\le T,\\
&\widetilde{p}(x^*(t),t)=g(x^*(t),t), \quad 0\le t\le T,\\
&\widetilde{p}_x(x^*(t),t)=g_x(x^*(t),t), \quad 0\le t\le T,\\
&\widetilde{p}(x,T)=g(x,T), \quad -\infty<x<\infty.
\end{aligned}$$

Consider the variable transforms:

$$\begin{aligned}
\tau=T-t, \quad y=x-x^*(T-\tau)+L,\\
u(x,\tau)=e^{-\beta\tau}\widetilde{p}(x,T-\tau), \quad \varphi(\tau)=x^*(T-\tau).
\end{aligned}$$

where β is a positive constant. Then we have the following nonlinear problem for φ and u:

$$\begin{aligned}
&(2.1) \quad u_\tau-\gamma(\tau)u_{yy}+c(y,\tau,\varphi,\varphi')u_y+d(y,\tau;\varphi)u=0, \quad 0\le\tau\le T,\ 0<y<L,\\
&(2.2) \quad u(0,\tau)=f(0,\tau;\varphi), \quad 0\le\tau\le T,\\
&(2.3) \quad u(L,\tau)=f(L,\tau;\varphi), \quad 0\le\tau\le T,\\
&(2.4) \quad u_y(L,\tau)=f_y(L,\tau;\varphi), \quad 0\le\tau\le T,\\
&(2.5) \quad u(y,0)=u_0(y), \quad 0\le y\le L,
\end{aligned}$$

where

$$\begin{aligned}
&\gamma(\tau)=\frac{1}{2}\sigma(T-\tau)^2,\\
&c(y,\tau;\varphi,\varphi')=\psi(T-\tau)(y+\varphi(\tau)-L)-\phi(T-\tau)-\varphi'(\tau),\\
&d(y,\tau;\varphi)=\zeta(y+\varphi(\tau)-L)+\beta,\\
&f(y,\tau;\varphi)=e^{-\beta\tau}g(y+\varphi(\tau)-L,T-\tau),\\
&v_0(y)=g(y+\varphi(0)-L,T).
\end{aligned}$$

Notice that we have two boundary conditions at $y=L$. The Neumann boundary condition will be integrated into the variational problem and the Dirichlet boundary condition will be used as a nonlinear equation for $\varphi(\tau)$. Define the bilinear form $\mathcal{A}$ as follows:

$$\mathcal{A}(v,w;\tau,\varphi,\varphi')=\gamma(\tau)(v_y,w_y)+(c(y,\tau;\varphi,\varphi')v_y,w)+(d(y,\tau;\varphi)v,w),$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$, the space of square integrable functions on $\Omega=(0,L)$. Let $H^1_E(\Omega)$ be the closure of $\left\{v\in C^\infty\left(\overline{\Omega}\right): v(0)=0\right\}$ in the usual Sobolev space $H^1(\Omega)$, and let $H^{-1}(\Omega)$ be the dual space of $H^1_E(\Omega)$ (see [1][10]). The variational form for problem (2.1)–(2.5) is: Find $u\in L^2\left(0,T;H^1_E(\Omega)\right)$

and $\varphi \in C\left([0,T]\right) \cap C^1((0,T])$ such that $u_\tau \in L^2(0,T;H^{-1}(\Omega))$, $u(0)=u_0$, and

$$(2.6)\quad (u_\tau, w) + \mathcal{A}(u,w;\tau,\varphi,\varphi') = G(\tau;\varphi)w(L), \quad \forall\, w \in H^1_E(\Omega), \quad \text{a.e. } 0 \le \tau \le T,$$

$$(2.7)\quad u(L,\tau) = f(L,\tau;\varphi), \quad 0 \le \tau \le T,$$

where $G(\tau;\varphi) = -\gamma(\tau) f_y(L,\tau;\varphi)$.

Let $\Delta_\tau : 0 = \tau_0 < \tau_1 < \cdots < \tau_M = T$ and $\Delta_y : 0 = y_0 < y_1 < \cdots < y_N = L$ be partitions of $[0,T]$ and $[0,L]$, respectively, where M and N are positive integers. Let V_h be the piecewise linear element subspace of $H^1_E(\Omega)$ with respect to partition Δ_y where $h = \max_{1\le j\le N}(y_j - y_{j-1})$. Denote the natural basis functions of V_h by $\omega_1, \omega_2, \ldots, \omega_N$, i.e., $\omega_j \in V_h$ such that $\omega_j(y_i) = \delta_{ij}$ for $j = 1,2,\ldots,N$ and $i = 0,1,\ldots,N$, where δ_{ij} is the Kronecker delta.

Recall that $\varphi(0)$ is the solution of the following equation for x:

$$\widetilde{P}(x,T;T^*) = K.$$

Let

$$u_h^0 = \sum_{j=1}^{N} u_0(y_j)\omega_j(y), \quad \varphi_0 = \varphi(0).$$

The finite element approximation to the variational problem (2.6)–(2.7) is: For $m = 1,2,\cdots,N$, find $u_h^m \in V_h$ and $\varphi_m > 0$ such that

$$(2.8)\qquad (\delta_\tau u_h^m, w) + \mathcal{A}_m\left(u_h^{m-\frac{1}{2}}, w\right) = G_m w(L), \quad \forall w \in V_h$$

$$(2.9)\qquad u_h^m(L) = f(L,\tau_m;\varphi_m).$$

where

$$\mathcal{A}_m(u,w) = \mathcal{A}\left(u,w;\tau_{m-\frac{1}{2}},\varphi_m,\delta_\tau\varphi_m\right), \quad G_m = G\left(\tau_{m-\frac{1}{2}};\varphi_{m-\frac{1}{2}}\right),$$

$$u_h^{m-\frac{1}{2}} = \frac{u_h^m + u_h^{m-1}}{2}, \quad \tau_{m-\frac{1}{2}} = \frac{\tau_m + \tau_{m-1}}{2}, \quad \varphi_{m-\frac{1}{2}} = \frac{\varphi_m + \varphi_{m-1}}{2}.$$

$$\delta_\tau u_h^m = \frac{u_h^m - u_h^{m-1}}{k_m}, \quad \delta_\tau\varphi_m = \frac{\varphi_m - \varphi_{m-1}}{k_m}, \qquad k_m = \tau_m - \tau_{m-1}.$$

Here we only considered the the Crank-Nicolson scheme in time.

Concerning the stability of the finite element approximations, we need to assume that the system (2.8)–(2.9) has a unique solution and that

$$(2.10)\qquad |\delta_\tau\varphi_m| \le C\tau_m^{\nu-1}$$

where $0 < \nu < 1$ and C is a generic positive constant. The above inequality has been verified by numerical tests, but has not been mathematically proven. Substituting w by $u_h^{m-\frac{1}{2}}$ in (2.8), we get

$$(2.11)\qquad \left(\delta_\tau u_h^m, u_h^{m-\frac{1}{2}}\right) + \mathcal{A}_m\left(u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}}\right) = G_m u_h^{m-\frac{1}{2}}(L).$$

Let

$$\gamma_m = \gamma\left(T - \tau_{m-\frac{1}{2}}\right), \quad \psi_m = \psi\left(T - \tau_{m-\frac{1}{2}}\right),$$

$$c_m(y) = c\left(y,\tau_{m-\frac{1}{2}};\varphi_m,\delta_\tau\varphi_m\right), \quad d_m(y) = d\left(y,\tau_{m-\frac{1}{2}};\varphi_m\right).$$

Then we have

$$\mathcal{A}_m\left(u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}}\right) = \gamma_m \left\|u_{hy}^{m-\frac{1}{2}}\right\|^2 + \left(c_m(y)u_{hy}^{m-\frac{1}{2}} + d_m(y)u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}}\right)$$
$$= \gamma_m \left\|u_{hy}^{m-\frac{1}{2}}\right\|^2 + \frac{1}{2}c_m(L)\left|u_h^{m-\frac{1}{2}}(L)\right|^2 - \frac{\psi_m}{2}\left\|u_h^{m-\frac{1}{2}}\right\|^2 + \left(d_m(y)u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}}\right)$$
$$\geq \gamma_m \left\|u_{hy}^{m-\frac{1}{2}}\right\|^2 + \frac{1}{2}c_m(L)\left|u_h^{m-\frac{1}{2}}(L)\right|^2 + \left(\beta + \min_{0\leq y\leq L}\zeta\left(y+\varphi_{m-\frac{1}{2}}-L\right) - \frac{\psi_m}{2}\right)\left\|u_h^{m-\frac{1}{2}}\right\|^2.$$

Thus, when β is sufficiently large, we can get

$$\mathcal{A}_m\left(u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}}\right) \geq \overline{\gamma}\left\|u_{hy}^{m-\frac{1}{2}}\right\|^2 + \frac{1}{2}c_m(L)\left|u_h^{m-\frac{1}{2}}(L)\right|^2.$$

where $\overline{\gamma} = \min_{0\leq\tau\leq T}\gamma(\tau)$. Hence it follows from (2.11) that

$$\|u_h^m\|^2 - \left\|u_h^{m-1}\right\|^2 + 2\gamma_m k_m\left\|u_{hy}^{m-\frac{1}{2}}\right\|^2 + k_m c_m(L)\left|u_h^{m-\frac{1}{2}}(L)\right|^2 \leq 2k_m G_m u_h^{m-\frac{1}{2}}(L).$$

The assumption (2.10) implies that the sequence $\{\varphi_m\}$ is bounded. Thus, the sequences $\{G_m\}$ and $\{u_h^m(L)\} = \{f(L,\tau_m;\varphi_m)\}$ are also bounded. Therefore, we have

$$\|u_h^m\|^2 - \left\|u_h^{m-1}\right\|^2 + 2\overline{\gamma}k_m\left\|u_{hy}^{m-\frac{1}{2}}\right\|^2 \leq Ck_m\left(1+\tau_m^{\nu-1}\right).$$

By summation, we obtain

$$\max_{m=1}^{M}\|u_h^m\|^2 + 2\overline{\gamma}k_m\sum_{m=1}^{M}\left\|u_{hy}^{m-\frac{1}{2}}\right\|^2 \leq \left\|u_h^0\right\|^2 + C\sum_{m=1}^{M}k_m\left(1+\tau_m^{\nu-1}\right).$$

which leads to the following stability estimate

$$\max_{m=1}^{M}\|u_h^m\| + 2\overline{\gamma}k_m\sum_{m=1}^{M}\left\|u_{hy}^{m-\frac{1}{2}}\right\|^2 \leq \left\|u_h^0\right\|^2 + C.$$

3. Implementation

In this section we discuss the implementation of our FFEM in detail. Since there are no known analytic formulas for the model other than the Hull-White model (including the Vasicek model), we first consider how to compute zero-coupon bond prices and their derivatives with respect to r, for example, for the Black-Karasinski model and the model (1.4). Then we consider how to solve the nonlinear system (2.8)–(2.9) and give an algorithm to implement our method.

As discussed in §1, we can see that $\widetilde{P}(x,t;T^*)$ is the solution of the final value problem:

$$\widetilde{P}_t + \frac{1}{2}\sigma(t)^2\widetilde{P}_{xx} + (\phi(t)-\psi(t)x)\widetilde{P}_x - \zeta(x)\widetilde{P} = 0, \quad -\infty < x < \infty,\ 0 \leq t < T^*,$$
$$\widetilde{P}(x,T^*;T^*) = 1, \quad -\infty < x < \infty.$$

In order to solve this problem on a bounded domain, we consider the transforms:

$$\tau = T^* - t, \quad y = \zeta_0(x) \equiv \frac{e^x}{1+e^x}, \quad v(y,\tau) = \widetilde{P}(\zeta_0^{-1}(y), T^*-\tau; T^*).$$

Then we have the following initial value problem for $v(y,\tau)$ on the interval $\Omega = (0,1)$:

$$v_\tau - c_0(y,\tau)v_{yy} + c_1(y,\tau)v_y + c_2(y,\tau)v = 0, \quad 0 < y < 1,\ 0 < \tau \leq T^*, \tag{3.1}$$
$$v(y,T^*) = 1, \quad 0 < y < 1, \tag{3.2}$$

where

$$c_0(y,\tau) = \frac{1}{2}\sigma(T^* - \tau)^2 y^2 (1-y)^2,$$
$$c_1(y,\tau) = y(1-y)\left(\psi(T^*-\tau)\zeta_0^{-1}(y) - \phi(T^*-\tau) - \frac{1}{2}\sigma(T^*-\tau)^2(1-2y)\right),$$
$$c_2(y) = \zeta\left(\zeta_0^{-1}(y)\right).$$

Notice that $y = r$ for the model (1.4). The new function $v(y,\tau)$ is simply the bond price at time $T^* - \tau$ for this model.

Since $c_0(0,t) = c_0(1,t) = 0$, we essentially do not require any boundary conditions at $y = 0$ and $y = 1$. For the Black-Karasinski model, we have $c_2(y) = y/(1-y)$ which is singular at $y = 1$. However, we can show that $\widetilde{P}(x,t;T^*) \to 0$ and $\widetilde{P}_x(x,t;T^*) \to 0$ as $x \to \infty$ for this model. Thus $V = \{w \in H^1(\Omega) : \ w(1) = 0\}$ is the natural choice of the space for the weak form of the corresponding problem. Since $c_2(y) = y$ for the model (1.4) is bounded, we simply set $V = H^1(\Omega)$. Define the bilinear form $\mathcal{B}$ by

$$\mathcal{B}(\tau; v, w) = (c_0(y,\tau)v_y, w_y) + (c_4(y,\tau)v_y, w) + (c_2(y)v, w),$$

where

$$c_4(y,\tau) = y(1-y)\left(\psi(T^*-\tau)\zeta_0^{-1}(y) - \phi(T^*-\tau) + \frac{1}{2}\sigma(T^*-\tau)^2(1-2y)\right).$$

The variational problem for the initial value problem (3.1)–(3.2) is: Find $v \in L^2(0,T^*;V)$ such that $v_\tau \in L^2(0,T^*;V')$, $v(0) \equiv 1$, and

$$(v_\tau, w) + \mathcal{B}(\tau, v, w) = 0, \quad \forall w \in V, \quad \text{a.e. }, \ 0 < \tau \le T^* \tag{3.3}$$

Furthermore, we can form the following variational problem for $q(y,\tau) = \widetilde{P}_x(\zeta_0^{-1}(y), T^*-\tau; T^*)$: Find Find $q \in L^2(0,T^*;V)$ such that $q_\tau \in L^2(0,T^*;V')$, $q(0) = 0$, and

$$(q_\tau, w) + \mathcal{B}(\tau; q, w) = F(\tau; w), \quad \forall w \in V, \quad \text{a.e. }, \ 0 < \tau \le T^* \tag{3.4}$$

where

$$F(\tau; w) = -\left(\zeta'\left(\zeta_0^{-1}(y)\right)v, w\right) - \psi(T^*-\tau)(y(1-y)v_y, w).$$

The above two variational problems can be solved by the same finite element method simultaneously, e.g., using a piecewise linear element space.

Now let us consider how to solve the nonlinear system (2.8)–(2.9). Write

$$v_h^m(y) = \sum_{j=1}^{N} v_j^m \omega_j(y)$$

We can rewrite (2.8)–(2.9) into the following matrix form:

$$\left(A + \frac{1}{2}k_m B_m\right)V^m = \left(A - \frac{1}{2}k_m B_m\right)V^{m-1} + k_m F^m, \tag{3.5}$$

$$v_N^m = f(L, \tau_m; \varphi_m) \tag{3.6}$$

where

$$A = (\omega_j, \omega_i)_{N\times N}, \quad B_m = \left(\mathcal{A}_m\left(\omega_j, \omega_i\right)\right)_{N\times N},$$
$$V^m = \left(v_1^m, \ldots, v_N^m\right), \quad F^m = \left(0, \ldots, G_m\right).$$

For saving computational time, we divide B_m into

$$B_m = B_m^{(1)} + \varphi_m B_m^{(2)},$$

where $B_m^{(1)}$ and $B_m^{(2)}$ are independent of φ_m. Notice that (3.5) defines an implicit vector function V^m of φ_m. We can consider (3.6) as a nonlinear equation of φ_m which can be rewritten as:

$$H(\varphi_m) = e^{\beta\tau_m} v_N^m - K + P(\varphi_m, T - \tau_m; T^*) = 0,$$

where we have used the fact that $P(\varphi_m, T - \tau_m; T^*) \leq K$. In order to solve this equation by Newton's method, we need $\dot{v}_N^m = \frac{\partial v_N^m}{\partial \varphi_m}$. Differentiating (3.5) with respect to φ_m, we get the linear system for the derivative $\dot{V}^m = \frac{\partial V^m}{\partial \varphi_m}$:

$$\left(A + \frac{1}{2}k_m B_m\right)\dot{V}^m = k_m \frac{\partial F^m}{\partial \varphi_m} - \frac{1}{2}k_m B_m^{(2)} V^{m-\frac{1}{2}}. \tag{3.7}$$

Systems (3.5) and (3.7) have the same tridiagonal coefficient matrix and thus can be solved simultaneously by using the Thomas algorithm.

To sum up, for a given tolerance ϵ, our front-fixing finite element method is implemented as follows:

For $m = 1, 2, \ldots, M$, *do*

1. *Compute* $B_m^{(1)}$ *and* $B_m^{(2)}$.
2. *Let* $\varphi_m^{(0)} = \varphi_{m-1}$. *For* $j = 1, 2, \ldots$
   - 2.1. *Solve (3.3) and (3.4) by the same finite element method.*
   - 2.2. *Build the systems (3.5) and (3.7).*
   - 2.3. *Solve the systems (3.5) and (3.7) by using the Thomas algorithm.*
   - 2.4. *Compute* $\varphi_m^{(j)}$ *by*

$$\varphi_m^{(j)} = \varphi_m^{(j-1)} - \frac{H\left(\varphi_m^{(j-1)}\right)}{H'\left(\varphi_m^{(j-1)}\right)}.$$

   - 2.5. *If*

$$\left|\varphi_m^{(j-1)} - \varphi_m^{(j)}\right| < \epsilon,$$

   then let $\varphi_m = \varphi_m^{(j)}$ *and terminate the* j*-loop. Otherwise, go to 2.1.*

4. *Solve system (3.5) for a better approximation of* V^m.

End do

We conclude this section by some remarks on the partitions Δ_τ and Δ_y. As in [2], we shall use variable step sizes in time by setting

$$\tau_m = \frac{Tm^2}{M^2}, \quad m = 0, 1, \ldots, M.$$

In this way, we have relatively more steps near the option expiration date in order to capture the singularity of the early exercise interest rates. For example, we have more than 3% of steps in the interval $[0, 0.001]$ for one-year options. We shall use uniform partitions in spatial variable y such that the mesh size h in y is almost the same as $\min_{1\leq m\leq M}\sqrt{k_m} = \sqrt{T}/M$. We just simply set $\beta = 0$ since our tests show that the value of β does not affect the accuracy of computations.

4. Numerical Results

In this section, we shall examine our front-fixing finite element method (FFEM) numerically and compare it with the usual finite element method (FEM) in [17]. Our programs were written in C++ and run on a computer with an Intel Core 2 CPUs of 3.0 GHz. Here we only consider the Vasicek model and the Hull-White model for simplicity. For the Vasicek model, the values of the parameters are given in Table 1 in which $\theta = \phi/\psi$ is the long-term expected interest rate. For the Hull-White model, we assume that the initial term structure is determined by the two-factor CIR model as in [7, 18]:

$$r(t) = x_1(t) + x_2(t),$$
$$dx_i = \kappa_i(\theta_i - x_i)dt + \sigma_i\sqrt{x_i}dW_i(t), \quad i = 1, 2.$$

where $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motion under the risk-neutral probability. The parameters are specified in Table 2. We are referred to [7, 17] for how to calibrate the Hull-White model.

Table 1. Parameters for the Vasicek model

Case	σ	ψ	θ	$r(0)$
VAS1	0.06	0.40	0.08	0.08
VAS2	0.10	0.30	0.10	0.10

Table 2. Parameters for the TCIR model

Case	σ_1	κ_1	θ_1	$x_1(0)$	σ_2	κ_2	θ_2	$x_2(0)$
TCIR1	0.05	0.2	0.06	0.06	0.10	0.4	0.04	0.04
TCIR2	0.20	0.3	0.06	0.06	0.30	0.4	0.04	0.04

In the following, one-year American put options written on 5-year and 30-year bonds will be considered. The option exercise prices are chosen to be the same as the current forward bond prices. First, we check the dependency of truncation errors on L. In Table 3 and Table 4, we display the maximum of the maximum absolute errors (MMAE) for today's option prices and early exercise interest rates (EEIR) for the 4 put options. The maximum absolute errors are computed between the approximate values for $L = 2.0$ and the L as in the tables when the error tolerance for Newton's Method is set to be $1.0e - 10$. The maximum numbers (I) of iterations for Newton's method is also given in these tables. The results suggest that $L = 1.0$ is sufficiently large enough for the eight options. It also shows that Newton's method attains the desired accuracy within at most 7 iterations.

Table 3. Dependency of MAEs on L: the Vasicek model

M	100			1000			10000		
L	Price	EEIR	I	Price	EEIR	I	Price	EEIR	I
0.8	$3.3e-10$	$1.8e-13$	5	$4.7e-10$	$3.9e-11$	6	$4.6e-09$	$1.5e-09$	7
1.0	$1.0e-12$	$1.9e-13$	5	$4.9e-11$	$4.8e-11$	6	$4.6e-09$	$3.0e-09$	7
1.2	$4.8e-13$	$1.8e-13$	5	$9.1e-11$	$4.1e-11$	6	$5.2e-09$	$1.7e-09$	7
1.4	$7.0e-13$	$1.7e-13$	5	$8.8e-11$	$3.6e-11$	6	$4.5e-09$	$2.6e-09$	7
1.6	$1.2e-12$	$1.7e-13$	5	$4.2e-11$	$3.4e-11$	6	$6.4e-09$	$2.2e-09$	7

Now we investigate the convergence of our method with $L = 1.0$ and $\epsilon = 1.0e-8$. We display the the L^2-norm and H^1-norm of the error $u_h^M - u_{h/2}^M$ in Figure 1

TABLE 4. Dependency of MAEs on L: the Hull-White model

M	100			1000			10000		
L	Price	EEIR	I	Price	EEIR	I	Price	EEIR	I
0.8	$2.0e-12$	$3.8e-14$	5	$2.1e-11$	$9.8e-12$	5	$1.0e-09$	$1.5e-09$	6
1.0	$1.1e-12$	$4.9e-14$	5	$2.9e-11$	$7.3e-12$	5	$3.8e-10$	$1.1e-09$	6
1.2	$8.7e-13$	$5.5e-14$	5	$2.5e-11$	$1.1e-11$	5	$2.9e-10$	$1.1e-09$	6
1.4	$1.1e-12$	$3.9e-14$	5	$2.2e-11$	$6.4e-12$	5	$1.0e-09$	$1.3e-09$	6
1.6	$7.6e-13$	$3.6e-14$	5	$5.4e-11$	$7.4e-12$	5	$2.3e-10$	$1.1e-09$	6

and Figure 3. We also display the the L^2-norm and maximum norm of the error $\varphi_M - \varphi_{2M}$ in Figure 2 and Figure 4, where φ_M is the piecewise linear interpolation of φ^m $(m = 0, 1, \ldots, M)$. We can observe that the Crank-Nicolson scheme converges linearly and quadratically in the L^2-norm and H^1-norm as expected. The rates of convergence for the early exercise interest rates are more than 1 and 0.5 in the L^2-norm and maximum norm, respectively.

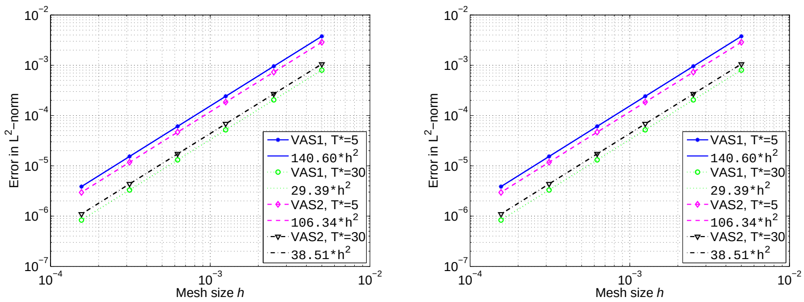

FIGURE 1. Convergence of option prices: the Vasicek model

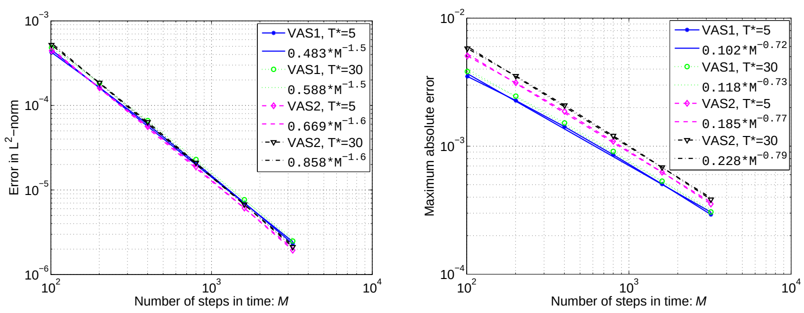

FIGURE 2. Convergence of early exercise interest rates: the Vasicek model

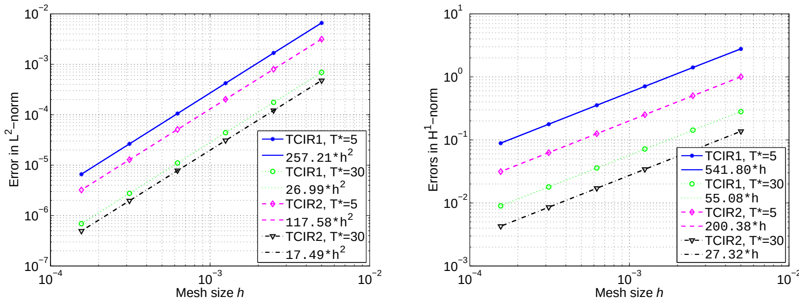

FIGURE 3. Convergence of option prices: the Hull-White model

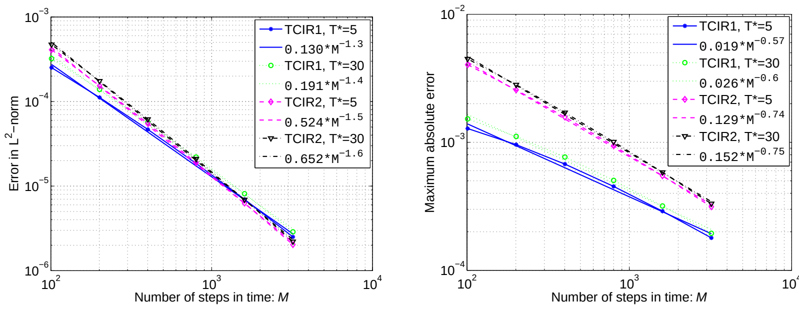

FIGURE 4. Convergence of early exercise interest rates: the Hull-White model

Next, we verify the assumption (2.10) numerically. To this end, we use a very small step size in time by taking $M = 20000$. In Figures 5– 6, we depict the graphs of φ_M, φ'_M over the interval $[0, 0.001]$ for which there are about 630 time steps (only 20 time steps for the uniform partition). The exponential functions in the figures are obtained by fitting the data to the model $a + b * \tau^c$. These figures demonstrate that the assumption (2.10) holds. Especially, we can observe that the sums of the exponents for the fitted exponential functions for φ_M and φ'_M are very close to 1, which also supports the assumption. Furthermore, we can propose a conjecture about the behavior of φ near $\tau = 0$: There are some constants $c_1 > 0$, $c_2 \geq 0$, $\nu > 0$, and $\mu \geq 0$ such that

$$\varphi(\tau) \sim \varphi(0) + \tau^\nu (c_1 - c_2 \log(\tau))^\mu, \quad \tau \to 0^+. \tag{4.1}$$

It seems that we do not have $\nu = \mu = \frac{1}{2}$ as for American options on stocks (see [5]).

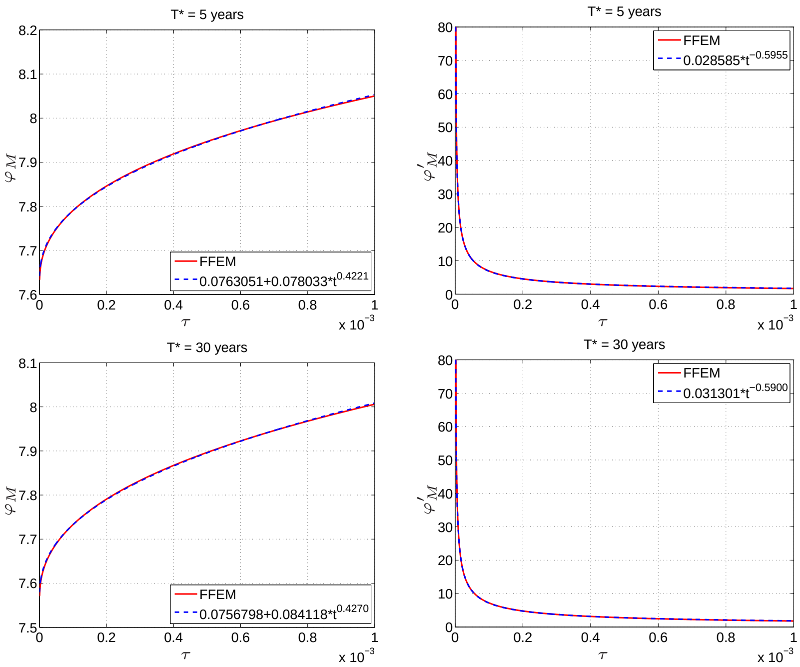

FIGURE 5. Early exercise interest rates near the option expiration date: the Vasicek model & Case VAS1

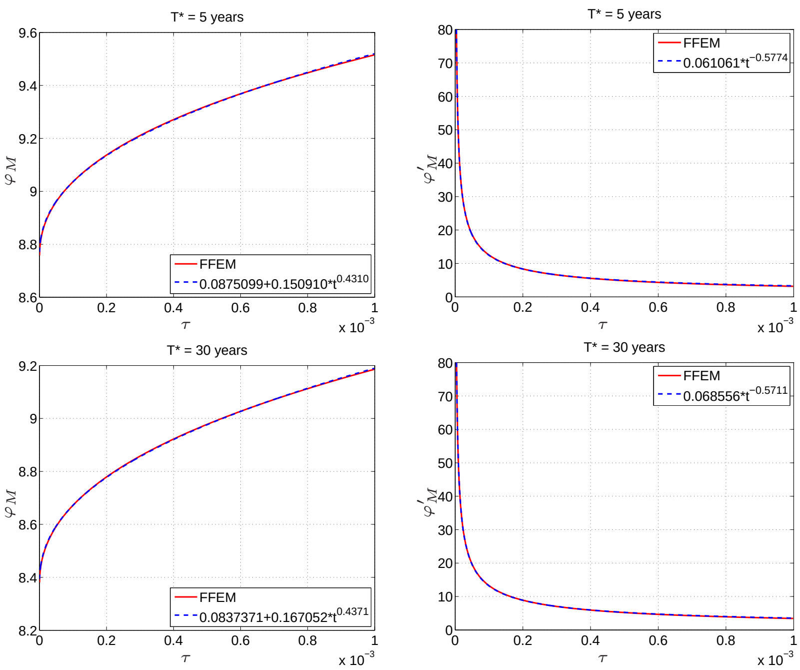

FIGURE 6. Early exercise interest rates near the option expiration date: the Vasicek model & Case VAS2

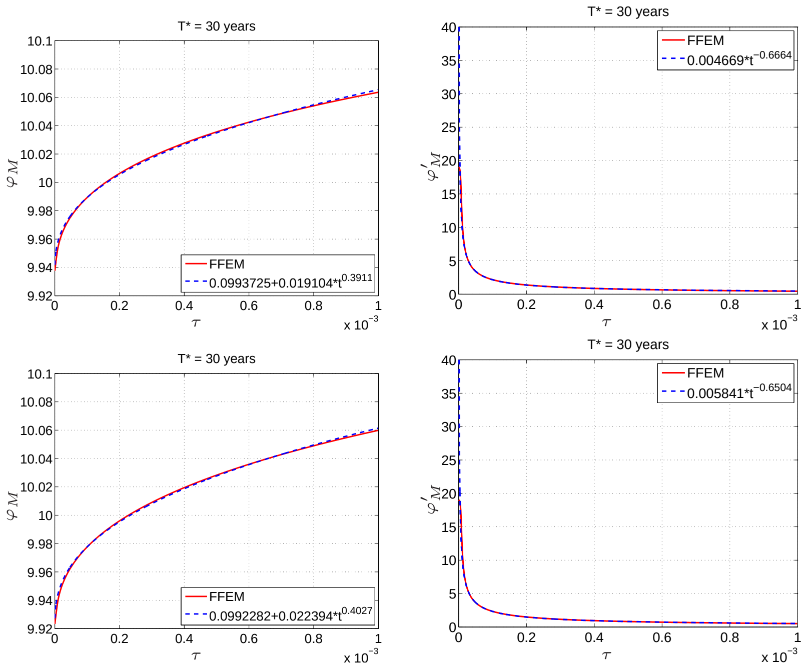

FIGURE 7. Early exercise interest rates near the option expiration date: the Hull-White model & Case TCIR1

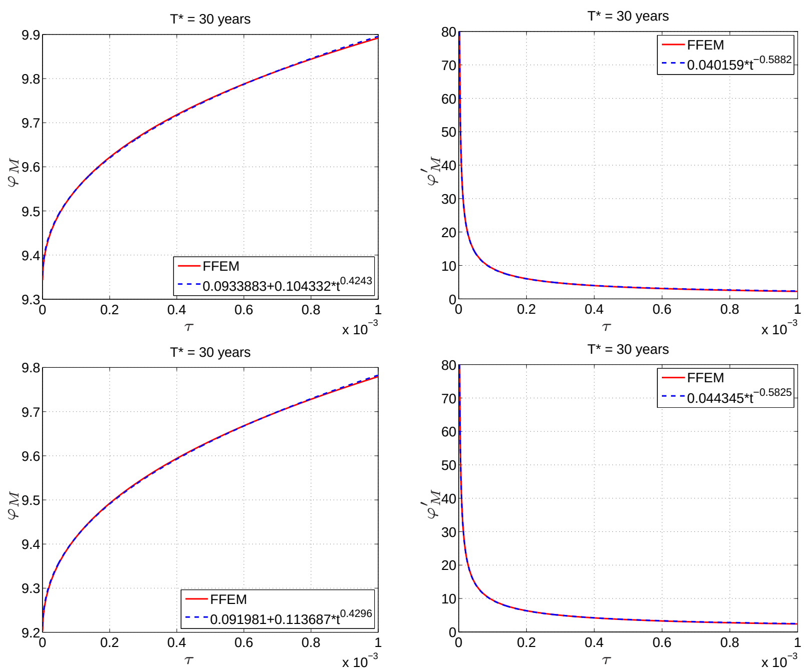

FIGURE 8. Early exercise interest rates near the option expiration date: the Hull-White model & Case TCIR2

Finally, we want to compare our front-fixing finite element method (FFEM) with the finite element method (FEM) in [17]. We consider one-year American put options written on bonds with expiration dates 5 years, 10 years, 15 years, and 20 years. The option exercise prices are given as the percentage of the current forward bond price: 87%, 88%, 89%, 90%, and 91%. We display the maximum absolute errors for 20 option prices, hedge ratios ($\partial p/\partial P$), and early exercise interest rates in Table 5 – Table 10. In these tables, the first and second rows are the maximum absolute errors when the "exact values" are the approximate values computed by FEM and FFEM with 25600 time steps, respectively, for each given number M of time steps. As expected, FFEM provides more accurate approximations, especially for early exercise interest rates.

Table 5. MAEs for put prices: The Vasicek model

Case	VAS1		VAS2	
M	FEM	FFEM	FEM	FFEM
100	$1.27e-2$	$8.63e-3$	$7.19e-3$	$1.81e-3$
	$1.27e-2$	$8.63e-3$	$7.20e-3$	$1.81e-3$
200	$3.42e-3$	$2.12e-3$	$1.49e-3$	$4.25e-4$
	$3.42e-3$	$2.12e-3$	$1.49e-3$	$4.25e-4$
400	$8.05e-4$	$5.25e-4$	$4.05e-4$	$1.01e-4$
	$8.05e-4$	$5.25e-4$	$4.05e-4$	$1.01e-4$
800	$2.19e-4$	$1.30e-4$	$1.06e-4$	$2.40e-5$
	$2.20e-4$	$1.30e-4$	$1.07e-4$	$2.43e-5$
1600	$5.38e-5$	$3.21e-5$	$2.59e-5$	$5.57e-6$
	$5.40e-5$	$3.22e-5$	$2.62e-5$	$5.89e-6$

Table 6. MAEs for hedge ratios: The Vasicek model

Case	VAS1		VAS2	
M	FEM	FFEM	FEM	FFEM
100	$1.39e-3$	$7.34e-4$	$7.17e-4$	$4.69e-5$
	$1.39e-3$	$7.34e-4$	$7.17e-4$	$4.69e-5$
200	$3.04e-4$	$1.74e-4$	$1.79e-4$	$9.85e-6$
	$3.04e-4$	$1.74e-4$	$1.79e-4$	$9.86e-6$
400	$8.48e-5$	$4.38e-5$	$2.81e-5$	$2.27e-6$
	$8.48e-5$	$4.38e-5$	$2.81e-5$	$2.28e-6$
800	$2.28e-5$	$1.11e-5$	$5.70e-6$	$5.06e-7$
	$2.28e-5$	$1.11e-5$	$5.69e-6$	$5.14e-7$
1600	$5.77e-6$	$2.73e-6$	$1.11e-6$	$1.17e-7$
	$5.78e-6$	$2.74e-6$	$1.10e-6$	$1.23e-7$

Table 7. MAEs for early exercise interest rates: The Vasicek model

Case	VAS1		VAS2	
M	FEM	FFEM	FEM	FFEM
100	$5.04e-3$	$6.33e-5$	$4.61e-3$	$3.80e-5$
	$5.03e-3$	$4.96e-5$	$4.61e-3$	$2.24e-5$
200	$2.58e-3$	$2.92e-5$	$2.27e-3$	$2.17e-5$
	$2.58e-3$	$1.29e-5$	$2.27e-3$	$5.93e-6$
400	$1.25e-3$	$2.07e-5$	$1.25e-3$	$1.73e-5$
	$1.24e-3$	$3.31e-6$	$1.26e-3$	$1.54e-6$
800	$5.47e-4$	$1.90e-5$	$6.25e-4$	$1.72e-5$
	$5.36e-4$	$8.45e-7$	$6.17e-4$	$3.95e-7$
1600	$3.12e-4$	$1.86e-5$	$3.12e-4$	$1.74e-5$
	$3.14e-4$	$2.14e-7$	$3.04e-4$	$1.01e-7$

Table 8. MAEs for put prices: The Hull-White model

Case	TCIR1		TCIR2	
M	FEM	FFEM	FEM	FFEM
100	$1.01e-1$	$8.01e-2$	$9.38e-3$	$5.09e-3$
	$1.01e-1$	$8.01e-2$	$9.38e-3$	$5.09e-3$
200	$2.05e-2$	$1.93e-2$	$2.23e-3$	$1.25e-3$
	$2.05e-2$	$1.93e-2$	$2.23e-3$	$1.25e-3$
400	$5.72e-3$	$4.29e-3$	$5.90e-4$	$3.06e-4$
	$5.73e-3$	$4.29e-3$	$5.90e-4$	$3.07e-4$
800	$1.74e-3$	$1.05e-3$	$1.55e-4$	$7.55e-5$
	$1.74e-3$	$1.06e-3$	$1.55e-4$	$7.57e-5$
1600	$4.26e-4$	$2.62e-4$	$3.96e-5$	$1.85e-5$
	$4.26e-4$	$2.62e-4$	$3.98e-5$	$1.87e-5$

Table 9. MAEs for hedge ratios: The Hull-white model

Case	TCIR1		TCIR2	
M	FEM	FFEM	FEM	FFEM
100	$5.04e-2$	$7.87e-2$	$1.22e-3$	$1.72e-4$
	$5.04e-2$	$7.87e-2$	$1.22e-3$	$1.72e-4$
200	$2.14e-2$	$2.61e-2$	$1.74e-4$	$3.35e-5$
	$2.14e-2$	$2.61e-2$	$1.74e-4$	$3.35e-5$
400	$1.91e-3$	$2.43e-3$	$4.63e-5$	$8.86e-6$
	$1.91e-3$	$2.43e-3$	$4.63e-5$	$8.86e-6$
800	$5.82e-4$	$7.48e-4$	$1.05e-5$	$2.01e-6$
	$5.83e-4$	$7.47e-4$	$1.05e-5$	$2.01e-6$
1600	$1.58e-4$	$1.82e-4$	$2.66e-6$	$5.01e-7$
	$1.58e-4$	$1.81e-4$	$2.66e-6$	$5.01e-7$

5. Conclusions

In this paper we have applied the front-fixing finite element method proposed in [2] to American put option problems on zero-coupon bonds. Our numerical

TABLE 10. MAEs for early exercise interest rates: The Hull-White model

Case	TCIR1		TCIR2	
M	FEM	FFEM	FEM	FFEM
100	$6.17e-3$	$2.97e-5$	$5.16e-3$	$5.49e-5$
	$6.17e-3$	$2.50e-5$	$5.16e-3$	$3.74e-5$
200	$2.77e-3$	$1.46e-5$	$2.38e-3$	$2.76e-5$
	$2.76e-3$	$6.19e-6$	$2.40e-3$	$9.76e-6$
400	$1.17e-3$	$1.75e-5$	$1.17e-3$	$2.05e-5$
	$1.17e-3$	$1.57e-6$	$1.18e-3$	$2.52e-6$
800	$7.03e-4$	$1.82e-5$	$5.86e-4$	$1.91e-5$
	$7.03e-4$	$4.00e-7$	$5.98e-4$	$6.44e-7$
1600	$3.13e-4$	$1.84e-5$	$3.12e-4$	$1.95e-5$
	$3.28e-4$	$1.01e-7$	$3.01e-4$	$1.63e-7$

results show that the method with the Crank-Nicolson scheme converges linearly and quadratically in L^2 and H^1 norms. Together with variable step sizes in time, it produces very accurate approximations of early exercise interest rates, which enables us to propose a conjecture (4.1) about the asymptotic expansion of early exercise interest rates near option expiration dates for the Hull-white model. We shall try to establish this conjecture. Now we are developing C++ programs for the calibration of model (1.1) and the valuation of the bond prices in the general case. Empirical testing of model (1.1) will also be considered in our near future work.

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