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A HYPERBOLIC-ELLIPTIC MODEL OF TWO-PHASE FLOW IN POROUS MEDIA – EXISTENCE OF ENTROPY SOLUTIONS

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Dedicated to the memory of Magne S. Espedal

Abstract. We consider the flow of two-phases in a porous medium and propose a modified version of the fractional flow model for incompressible, two-phase flow based on a Helmholtz regularization of the Darcy phase velocities. We show the existence of global-in-time entropy solutions for this model with suitable assumptions on the boundary conditions. Numerical experiments demonstrating the approximation of the classical two-phase flow equations with the new model are presented.

Key Words. Porous media flow, conservation law, elliptic equation, weak solution, existence

1. The two Phase Flow Problem

Many geophysical and industrial processes like enhanced oil recovery and carbon dioxide sequestration involve the flow of two-phases, say oil and water, in a porous medium.

The variables of interest are the phase saturations s_w and s_o representing the saturation (volume fraction) of the water and oil phase respectively. We have the identity:

$$s_w + s_o \equiv 1. \tag{1.1}$$

Hence, we can describe the dynamics in terms of the saturation of either of the two-phases. We denote the water saturation as $s_w = s$ in the discussion below. Assuming a constant porosity ($\phi \equiv 1$), the two-phases are transported by [4]

$$(s_r)_t + \mathrm{div}_x(\mathbf{v}_r) = 0, \quad r \in \{w, o\}. \tag{1.2}$$

Here, the phase velocities are denoted by $\mathbf{v}_w$ and $\mathbf{v}_o$ respectively. In view of the identity (1.1), the two-phase velocities can be summed up to yield the *incompressibility* condition,

$$\mathrm{div}_x(\mathbf{v}) = 0, \quad \mathbf{v} = \mathbf{v}_w + \mathbf{v}_o. \tag{1.3}$$

The total velocity is denoted by $\mathbf{v}$.

The phase velocities in a homogeneous isotropic medium are described by the Darcy's law [4]:

$$\mathbf{v}_r = -\lambda_r \nabla_x p_r + \lambda_r \rho_r g \mathbf{k}, \quad r \in \{w, o\}. \tag{1.4}$$

Here, g is the constant acceleration due to gravity, $\mathbf{k}$ is the direction in which gravity acts and ρ_r is the (constant) density of the phase r. The quantity $\lambda_r = \lambda_r(s_r)$ is

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the phase mobility and p_r is the phase pressure. Assume that the *capillary pressure* i.e, $p_c = p_w - p_o$ is zero, we can sum (1.4) for both phases and obtain

$$\mathbf{v} = -\lambda_T(s)\nabla_x p + (\lambda_w\rho_w + \lambda_o\rho_o)g\mathbf{k}, \tag{1.5}$$

with $p = p_w = p_o$ being the pressure and $\lambda_T = \lambda_w + \lambda_o$ being the total mobility. Using (1.5), the gradient of pressure in (1.4) can be eliminated leading to

$$\mathbf{v}_w = \frac{\lambda_w(s)}{\lambda_T(s)}\mathbf{v} + \frac{\lambda_w(s)\lambda_o(s)}{\lambda_T(s)}(\rho_w - \rho_o)g\mathbf{k}.$$

Denoting the fractional flow function f as

$$f(s) = \frac{\lambda_w(s)}{\lambda_T(s)} = \frac{\lambda_w(s)}{\lambda_w(s) + \lambda_o(s)},$$

and the gravity function g as

$$g(s) = \frac{\lambda_w(s)\lambda_o(s)}{\lambda_T(s)}(\rho_w - \rho_o)g,$$

the saturation equation (1.2) for water can be written down as

$$s_t + \mathrm{div}_x(f(s)\mathbf{v} + g(s)\mathbf{k}) = 0. \tag{1.6}$$

Combining the saturation equation with the incompressibility condition (1.3) and the pressure equation, we obtain the evolution equations for two-phase flow in a porous medium:

$$\begin{aligned} s_t + \mathrm{div}_x(f(s)\mathbf{v} + g(s)\mathbf{k}) &= 0, \\ \mathrm{div}_x(\mathbf{v}) &= 0, \\ \mathbf{v} &= -\lambda_T(s)\nabla_x p + (\rho_w\lambda_w(s) + \rho_o\lambda_o(s))g\mathbf{k}. \end{aligned} \tag{1.7}$$

The above equations have to be augmented by suitable initial and boundary conditions.

The phase mobility $\lambda_w : [0,1] \mapsto \mathbb{R}$ is a monotone increasing function with $\lambda_w(0) = 0$ and the phase mobility $\lambda_o : [0,1] \mapsto \mathbb{R}$ is a monotone decreasing function with $\lambda_o(1) = 0$. Furthermore, the total mobility is strictly positive i.e, $\lambda_T \geq \lambda_* > 0$ for some λ_*.

The above equations are a hyperbolic-elliptic system as the saturation equation in (1.7) is a scalar hyperbolic conservation law in several space dimensions with a coefficient given by the velocity $\mathbf{v}$. The velocity can be obtained by solving an elliptic equation for the pressure p.

It is well known that solutions of hyperbolic conservation laws can develop discontinuities, even for smooth initial data, [8]. The presence of these discontinuities or shock waves implies that solutions of conservation laws are sought in a weak sense and are augmented with additional admissibility criteria or *entropy conditions* in order to ensure uniqueness.

As the two-phase flow equations involve a conservation law, we need to define a suitable concept of entropy solutions for these equations and show that these solutions are well-posed. The problem of proving well-posedness of global weak solutions of the two-phase flow equations (1.7) has remained open for many decades. The main challenge in showing existence is the fact that the velocity field $\mathbf{v}$ acts as a coefficient in the saturation equations. Although conservation laws with coefficients have been studied extensively in recent years, see [1, 11, 7, 2] and references therein, the state of the art results require that the coefficient is a function of bounded variation. Many attempts at showing that the velocity field $\mathbf{v}$ in (1.6) is sufficiently regular, for example is a BV function or has enough Sobolev regularity, have failed.

Partial results (with strong assumptions on the velocity field or on the solution) have been obtained in [14, 18] and references therein.

Another approach is to consider a modified version of the two-phase flow equations. Recalling that the two-phase flow equations (1.7) were derived under the assumption that the capillary pressure was zero. Adding small but non-zero capillary pressure leads to a viscous perturbation of the saturation equation, see [12]. The viscous problem has been shown to be well-posed in [12] (see also [5, 6, 16, 10] for mildly degenerate diffusion coefficients). However, the fact that the coefficient of viscosity can be very small leads to difficulties in numerical approximation of these equations as the viscous scales have to resolved. Furthermore, sharp saturation fronts might be artificially smeared due to the added viscosity.

Herein we consider a modified version of the two-phase flow equations and show that global weak solutions exist for this modified problem. We are motivated by the fact that the velocity field $\mathbf{v}$ in (1.6) needs to be regularized but the sharp fronts in saturation should not be diffused. Hence, we suggest the following modification of the Darcy's law (1.4):

$$\begin{aligned} \mathbf{v}_r &= -\Lambda_r \nabla_x p + \Lambda_r \rho_r g \mathbf{k}, \quad r \in \{w, o\}, \\ -\mu_r \Delta_x \Lambda_r + \Lambda_r &= \lambda_r(s). \end{aligned} \tag{1.8}$$

where $\mu_o, \mu_w \in (0,1)$ are small regularization parameters.

The system (1.8) amounts to a Helmholtz regularization of the velocity field, or more precisely of the phase mobilities via $(1-\mu_r\Delta_x)^{-1}[\lambda_r(s)]$. Observe that this kind of regularization is quite different from a viscous regularization which makes the saturation smooth by dissipating energy at small scales. With the Helmholtz regularization, the saturation equation will still possess shock wave (discontinuous) solutions, while it is the velocity field that becomes more regular. Consequently, the new "Helmholtz regularized" two-phase flow model is expected to correctly predict the underlying flow phenomena. A chief feature of this new model will be that one can prove rigorously that there exists global-in-time solutions; this is still an open problem for (1.7).

1.0.1. Motivation for (1.8)**.** In [17] Neumann derived Darcy's law for single phase flow in porous media by an averaging the potential flow (Navier-Stokes) equations

$$\Delta \mathbf{v} = \nabla_x p,$$

where p denotes pressure. We now briefly recap Neumann's derivation of Darcy's law. We assume that the flow takes place in a system of small channels (pores) in the rock. The continuity equation

$$\mathrm{div}_{\mathbf{x}}(\mathbf{v}) = 0,$$

means that the pressure solves the Laplace equation

$$\Delta p = 0, \text{ for } \mathbf{x} \in \text{the porous space}.$$

The boundary of the porous space consists of the outer boundary, on which we can impose boundary conditions, and the pore walls, on which it is natural to impose a no flow condition. Let Ω denote the domain enclosed by the outer boundary, and Ω_ϕ the porous space.

Let N be some averaging kernel, and let χ_ϕ denote the characteristic function of the pore space, and set

$$\phi = \int_\Omega N(\mathbf{x}-\mathbf{y})\chi_\phi(\mathbf{y})\,d\mathbf{y}.$$

For simplicity, we assume that the porous medium is homogeneous. Hence, the porosity ϕ is independent of $\mathbf{x}$. Next, define the averages

$$\langle f\rangle = \int_\Omega N(\mathbf{x}-\mathbf{y})\chi_\phi(\mathbf{y})f(\mathbf{y})\,d\mathbf{y}, \text{ and } \langle f\rangle^* = \frac{\langle f\rangle}{\phi}.$$

One key point in the derivation of Darcy's law is the relation

$$\langle \nabla_x f\rangle(\mathbf{x}) = \nabla\langle f\rangle(\mathbf{x}) + \int_{\partial\Omega_\phi} N(\mathbf{x}-\mathbf{y})f(\mathbf{y})\mathbf{n}(\mathbf{y})\,dS(\mathbf{y}).$$

Since the pressure p solves Laplace's equation in Ω_ϕ, we have the solution formula

$$p(\mathbf{x}) = \int_{\partial\Omega\cap\Omega_\phi} p_b(\mathbf{y})\nabla_x H(\mathbf{x}-\mathbf{v})\cdot\mathbf{n}(\mathbf{y})\,dS(\mathbf{y}),$$

where H is the Green's function on $\Omega_\phi\cap\Omega$ with boundary conditions as indicated above. Now, p, and thus also $\mathbf{v}$, is linearly dependent on the imposed boundary condition p_b. Neumann then uses a scaling argument to show that

$$\mathbf{v} = A\langle\mathbf{v}\rangle^*,$$

for some symmetric matrix A which is *independent of the boundary conditions* Φ_b. Thus

$$\langle\Delta\mathbf{v}\rangle^* = \left\langle\Delta\left(A\langle\mathbf{v}\rangle^*\right)\right\rangle^* \approx \langle\Delta A\rangle^*\langle\mathbf{v}\rangle^*.$$

From this we get

$$\langle\Delta A\rangle^*\langle\mathbf{v}\rangle^* = \langle\nabla_x p\rangle^* = \nabla_x\langle p\rangle^* + \int_{\partial\Omega_\phi} N(\mathbf{x}-\mathbf{y})p(\mathbf{y})\mathbf{n}(\mathbf{y})\,dS(\mathbf{y}).$$

It turns out that $\langle\Delta A\rangle^*$ is invertible, and we call its inverse, K, *the permeability.* Furthermore, in many cases, the integral along the boundary of the porous media vanishes, due to the many twists and turns of the pores. Then we are left with Darcy's law

$$\langle\mathbf{v}\rangle = K\nabla_x\langle p\rangle.$$

The matrix K is called the rock permeability.

For two-phase flow, it is not possible to derive Darcy's law at this level of rigor. However, one can motivate it by the following considerations. Assume now that we have two-phases; oil and water. These are chemically inert, and do not dissolve in one another, but we assume that on the scale of the pores, oil and water are mixed well enough to define saturation. On a very small scale, much smaller that the width of the pore walls, one phase will act similarly to the way the rock acts on the fluid in the single phase case. See Figure 1 for an illustration of the three scales. Now we have two velocities $\mathbf{v}^{\mathrm{o}}$ and $\mathbf{v}^{\mathrm{w}}$, each of which satisfies the Navier-Stokes

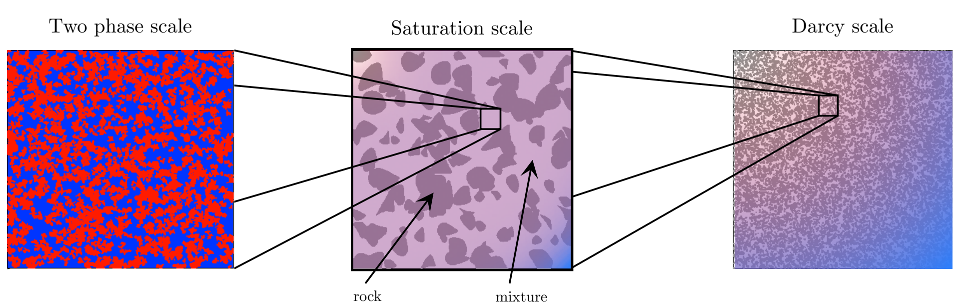


FIGURE 1. The three scales, in the middle scale one can define saturation.

equation. For a given phase i the other phase will act similarly to the rock in one phase flow. Under this assumption, Darcy's law for each phase reads

$$\left\langle \mathbf{v}^i \right\rangle_N = K_i \nabla_x \left\langle p_i \right\rangle_N , \quad i = \mathrm{w, o},$$

where we have chosen to indicate the dependence on the averaging kernel N. Next we assume that the two fluids are uniformly mixed in each direction, so that $K_i = \lambda_i I$. Thus on the saturation scale we have two velocities and two pressures in each point of the pore space. The continuity equation for each phase $\mathrm{div}_{\mathbf{x}} \left(\mathbf{v}^i\right) = 0$ now reads

$$\mathrm{div}_{\mathbf{x}} \left(\lambda\right)_i \nabla_x \left\langle p_i \right\rangle_N = 0, \quad \mathbf{x} \in \Omega_\phi,$$

with boundary conditions

$$\begin{cases} \lambda_i \nabla_x \left\langle p_i \right\rangle_N = 0, & \mathbf{x} \in \partial\Omega_\phi, \\ \lambda_i \nabla_x \left\langle p_i \right\rangle_N = f_i, & \mathbf{x} \in \partial\Omega \cap \Omega_\phi. \end{cases}$$

The phase velocities $\mathbf{v}^i$ are linearly dependent on f_i, so again scale arguments lead to the relation

$$\left\langle \mathbf{v}^i \right\rangle_N = A_i \left\langle \left\langle \mathbf{v}^i \right\rangle_N \right\rangle_M^* ,$$

where M is a different averaging kernel from N. We now assume that the matrices A_o and A_w are equal. This amounts to saying that the rock does not discriminate between water and oil. We assume that A is invertible, and can write

$$\begin{aligned} \left\langle \left\langle \mathbf{v}^i \right\rangle_N \right\rangle_M^* &= \left\langle A^{-1} \lambda_i \nabla_x \left\langle p_i \right\rangle_N \right\rangle_M^* \\ &= \left\langle A^{-1} \lambda \right\rangle_M^* \left\langle \nabla_x \left\langle p_i \right\rangle_N \right\rangle_M^* \ \text{(yet another assumption)} \\ &= \left\langle A^{-1} \lambda \right\rangle_M^* \nabla_x \left\langle \left\langle p_i \right\rangle_N \right\rangle_M^* + \left\langle A^{-1} \lambda \right\rangle_M^* \underbrace{\int_{\partial\Omega_\phi} \left\langle p_i \right\rangle_N (\mathbf{y}) M(\mathbf{x}-\mathbf{y}) \mathbf{n}(\mathbf{y})\, d\mathbf{y}}_{=0 \text{ by twists and turns}} . \end{aligned}$$

Now assume that the saturation, and thus λ_i, changes rapidly compared with the size of the support of M. We are interested in $\tilde{p}_i = \left\langle \left\langle p \right\rangle_N \right\rangle_M$, the continuity equation for $\left\langle \mathbf{v}^i \right\rangle_N$ implies that

$$\mathrm{div}_{\mathbf{x}} \left(\left\langle A^{-1} \lambda \right\rangle_M^* \nabla_x \tilde{p}_i \right) = 0.$$

For simplicity, we assume that $A^{-1} = I$. We choose M to be the Helmholtz kernel $\left(1 - \mu\Delta_x\right)^{-1}$, with μ being a small parameter, and finally we end up with the usual equations describing two-phase flow, but where the equation for the velocity (1.8) is more regular than is commonly assumed.

1.0.2. Scope of the current paper. In this paper, we consider the two-phase flow equations (1.7) but the modified version of the Darcy's law (1.8). The notion of weak solutions for this modified problem is defined and weak solutions are shown to exist. Our existence proof proceeds in two steps. First, we define vanishing viscosity approximations of the modified two-phase flow equations and derive a priori bounds on the approximation solutions. In particular, compactness results for the pressure and velocity fields are obtained. The second step is to obtain compactness for the approximate saturations. Here, we employ a suitably adapted form of the kinetic formulation for conservation laws (see [20] for a related approach) and derive compactness for the saturation equation establishing the existence of global weak solutions for the modified two-phase model.

In addition to the heuristic motivation for the modified Darcy's law (1.8), we provide numerical evidence for the robustness of this approximation. In particular, the numerical experiments show that the solutions obtained with the modified problem are *close* to those of (1.7). Furthermore, these modified solutions converge to the solution of the classical two-phase flow equations as the regularization parameter μ_r in (1.8) vanishes.

The rest of this paper is organized as follows: in section 2, we state the modified two-phase flow model and define weak solutions. A priori estimates on the approximate solutions are obtained in section 3 and compactness for the saturation in terms of the kinetic formulation is obtained in section 4. Numerical experiments comparing the classical and modified form of the two-phase flow equations are presented in section 5.

2. Statement of problem

The modified model for two-phase flows in a porous medium leads to the following elliptic-hyperbolic system

$$
\begin{cases}
\partial_t s + \mathrm{div}_{\mathbf{x}}\,(f(s)\mathbf{v} + g(s)\mathbf{k}(\mathbf{x})) = 0, & t > 0,\, \mathbf{x} \in \Omega,\\
\mathrm{div}_{\mathbf{x}}\,(\mathbf{v}) = 0, & t > 0,\, \mathbf{x} \in \Omega,\\
\mathbf{v} = -\Lambda_T \nabla_{\mathbf{x}} p + (\rho_w \Lambda_w + \rho_o \Lambda_o)\mathbf{k}, & t > 0,\, \mathbf{x} \in \Omega,\\
-\mu_w \Delta_{\mathbf{x}} \Lambda_w + \Lambda_w = \lambda_w(s), & t > 0,\, \mathbf{x} \in \Omega,\\
-\mu_o \Delta_{\mathbf{x}} \Lambda_o + \Lambda_o = \lambda_o(s), & t > 0,\, \mathbf{x} \in \Omega,\\
\Lambda_T = \Lambda_w + \Lambda_o, & t > 0,\, \mathbf{x} \in \Omega,\\
\big(f(s)\mathbf{v} + g(s)\mathbf{k}\big) \cdot \nu = h(t, \mathbf{x}), & t > 0,\, \mathbf{x} \in \partial\Omega,\\
\partial_\nu p(t, \mathbf{x}) = \pi(t, \mathbf{x}),\ \ \Lambda_w(t, \mathbf{x}) = \Lambda_o(t, \mathbf{x}) = \frac{\lambda_*}{2}, & t > 0,\, \mathbf{x} \in \partial\Omega,\\
\int_\Omega p(t, \mathbf{x}) d\mathbf{x} = 0, & t > 0,\\
s(0, \mathbf{x}) = s_0(\mathbf{x}), & \mathbf{x} \in \Omega,
\end{cases}
\tag{2.1}
$$

where

(H.1) Ω is an open connected subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary and ν is the unit outer normal;

(H.2) f and g are smooth functions, $\mathbf{k} : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth vector field, μ_w and μ_o are positive constants, and $h, \pi : (0, \infty) \times \partial\Omega \to \mathbb{R}$ are smooth bounded maps such that

$$
\begin{aligned}
f(0) = g(0) = g(1) &= 0,\\
h(t, \mathbf{x}) &\leq 0,\\
h(t, \mathbf{x}) + \lambda_* f(1) \left(\pi(t, \mathbf{x}) - \frac{\rho_w + \rho_o}{2} \mathbf{k} \cdot \nu \right) &\geq 0,\\
-\lambda_* f'(\xi) \left(\pi(t, \mathbf{x}) - \frac{\rho_w + \rho_o}{2} \mathbf{k} \cdot \nu \right) + g'(\xi)\mathbf{k}(\mathbf{x}) \cdot \nu(\mathbf{x}) &\leq 0,
\end{aligned}
$$

for every $t \geq 0$, $\mathbf{x} \in \partial\Omega$, $\xi \in \mathbb{R}$;

(H.3) λ_w and λ_o are smooth and non-negative, $\lambda_* > 0$, and $\lambda_w(\cdot) + \lambda_o(\cdot) \geq \lambda_*$;

(H.4) the initial datum satisfies the condition: $0 \leq s_0 \leq 1$.

If $\mu_w = \mu_o = 0$ we recover the classical two-phase problem (1.7).

The term $g(s)\mathbf{k}$ takes in account the gravitational effects.

Regarding the assumption (**H.2**) we remind that in the physical model (1.7), we have that

$$\xi \in (0,1) \Rightarrow f'(\xi) > 0, \qquad f'(0) = f'(1) = 0, \qquad \frac{g'}{f'} \in L^\infty(0,1), \qquad \mathbf{k}\,\text{constant},$$

hence the conditions in (**H.2**) are satisfied if, for example, π is big compared to h and $\mathbf{k}\cdot\nu$ on $(0,\infty)\times\partial\Omega$.

Definition 2.1. *Let s, Λ_w, Λ_o, $p : (0,\infty)\times\Omega \to \mathbb{R}$ and $\mathbf{v} : (0,\infty)\times\Omega \to \mathbb{R}^N$ be functions. We say that $(s, \Lambda_w, \Lambda_o, p, \mathbf{v})$ is an entropy solution of* (2.1) *if*

i) $s \in L^\infty((0,\infty)\times\Omega)$, Λ_w, $\Lambda_o \in L^\infty((0,\infty)\times\Omega) \cap L^\infty(0,\infty; W^{2,r}(\Omega))$, $1 \le r < \infty$, $p \in L^\infty(0,\infty; W^{3,2}(\Omega))$, $\mathbf{v} \in L^\infty(0,\infty; W^{2,2}(\Omega))$;

ii) Λ_w, Λ_o, p, $\mathbf{v}$ *are distributional solutions of the corresponding equations in* (2.1) *and satisfy the corresponding initial and boundary conditions in the sense of traces;*

iii) for every test function $\varphi \in C^\infty([0,\infty)\times\mathbb{R}^N)$ *with compact support*

$$\int_0^\infty \int_\Omega \big(s\partial_t\varphi + \big(f(s)\mathbf{v} + g(s)\mathbf{k}(\mathbf{x})\big)\nabla_{\mathbf{x}}\varphi\big)dtd\mathbf{x} - \int_0^\infty \int_{\partial\Omega} h\varphi dtd\sigma + \int_\Omega s_0(\mathbf{x})\varphi(0,\mathbf{x})d\mathbf{x} = 0;$$

iv) for every test function $\varphi \in C^\infty([0,\infty)\times\Omega)$ *with compact support and any* C^2 *convex entropy* η

$$\int_0^\infty \int_\Omega \big(\eta(s)\partial_t\varphi + \big(\mathcal{F}(s)\mathbf{v} + \mathcal{G}(s)\mathbf{k}(\mathbf{x})\big)\nabla_{\mathbf{x}}\varphi\big)dtd\mathbf{x} + \int_\Omega \eta(s_0(\mathbf{x}))\varphi(0,\mathbf{x})d\mathbf{x} \ge 0,$$

where $\mathcal{F}$ *and* $\mathcal{G}$ *are the corresponding entropy fluxes defined as follows*

$$\mathcal{F}(s) = \int^s \eta'(\xi)f'(\xi)d\xi, \qquad \mathcal{G}(s) = \int^s \eta'(\xi)g'(\xi)d\xi, \qquad s\in\mathbb{R}. \tag{2.2}$$

Let us point out the fact that the test functions considered in *iii)* are not supposed to have support contained in Ω.

Our main result is the following theorem.

Theorem 2.1. *Assume (**H.1**), (**H.2**), (**H.3**), and (**H.4**). The initial boundary value problem* (2.1) *has a solution in the sense of Definition 2.1.*

We use the following approximation of (2.1)

$$\begin{cases}
\partial_t s_\varepsilon + \mathrm{div}_{\mathbf{x}}\,(f(s_\varepsilon)\mathbf{v}_\varepsilon + g(s_\varepsilon)\mathbf{k}) = \varepsilon\Delta_{\mathbf{x}} s_\varepsilon, & t>0,\ \mathbf{x}\in\Omega,\\
\mathrm{div}_{\mathbf{x}}\,(\mathbf{v}_\varepsilon) = 0, & t>0,\ \mathbf{x}\in\Omega,\\
\mathbf{v}_\varepsilon = -\Lambda_{T,\varepsilon}\nabla_{\mathbf{x}} p_\varepsilon + (\rho_w\Lambda_{w,\varepsilon} + \rho_o\Lambda_{o,\varepsilon})\mathbf{k}, & t>0,\ \mathbf{x}\in\Omega,\\
-\mu_w\Delta_{\mathbf{x}}\Lambda_{w,\varepsilon} + \Lambda_{w,\varepsilon} = \lambda_w(s_\varepsilon), & t>0,\ \mathbf{x}\in\Omega,\\
-\mu_o\Delta_{\mathbf{x}}\Lambda_{o,\varepsilon} + \Lambda_{o,\varepsilon} = \lambda_o(s_\varepsilon), & t>0,\ \mathbf{x}\in\Omega,\\
\Lambda_{T,\varepsilon} = \Lambda_{w,\varepsilon} + \Lambda_{o,\varepsilon}, & t>0,\ \mathbf{x}\in\Omega,\\
\big(f(s_\varepsilon)\mathbf{v}_\varepsilon + g(s_\varepsilon)\mathbf{k}\big)\cdot\nu - \varepsilon\partial_\nu s_\varepsilon = h(t,\mathbf{x}), & t>0,\ \mathbf{x}\in\partial\Omega,\\
\partial_\nu p_\varepsilon(t,\mathbf{x}) = \pi(t,\mathbf{x}),\ \ \Lambda_{w,\varepsilon}(t,\mathbf{x}) = \Lambda_{o,\varepsilon}(t,\mathbf{x}) = \frac{\lambda_*}{2}, & t>0,\ \mathbf{x}\in\partial\Omega,\\
\int_\Omega p_\varepsilon(t,\mathbf{x})d\mathbf{x} = 0, & t>0,\\
s_\varepsilon(0,\mathbf{x}) = s_{0,\varepsilon}(\mathbf{x}), & \mathbf{x}\in\Omega,
\end{cases} \tag{2.3}$$

where ε is a positive parameter and $s_{0,\varepsilon}$ is a smooth approximation of s_0.

The existence of a smooth solution $(s_\varepsilon, \mathbf{v}_\varepsilon, \Lambda_{w,\varepsilon}, \Lambda_{o,\varepsilon}, p_\varepsilon)$ for the approximate problem (2.3) can be proved using the same argument of [12, 18].

3. A Priori Estimates and Basic Compactness

This section is devoted to some a priori estimates uniform with respect to ε on the solution $(s_\varepsilon, \mathbf{v}_\varepsilon, \Lambda_{w,\varepsilon}, \Lambda_{o,\varepsilon}, p_\varepsilon)$ of (2.3).

Lemma 3.1 (L^∞ estimate on $\{s_\varepsilon\}_{\varepsilon>0}$). *We have that*

$$0 \le s_\varepsilon(t, \mathbf{x}) \le 1 \qquad \text{for each } t > 0,\ \mathbf{x} \in \Omega.$$

Proof. The maps with constant values 0 and 1 are a sub and a super solution for the first equation in (2.3). Indeed both 0 and 1 solve the equation

$$\partial_t s + \operatorname{div}_{\mathbf{x}} \left(f(s)\mathbf{v}_\varepsilon + g(s)\mathbf{k}\right) = \varepsilon \Delta_{\mathbf{x}} s,$$

and on the boundary $(0, \infty) \times \partial\Omega$, thanks to (**H.2**), we have

$$\begin{aligned}
\Big[\varepsilon\partial_\nu s - \big(f(s)\mathbf{v}_\varepsilon + g(s)\mathbf{k}\big)\cdot\nu + h\Big]\Big|_{s=0} =& h \le 0,\\
\Big[\varepsilon\partial_\nu s - \big(f(s)\mathbf{v}_\varepsilon + g(s)\mathbf{k}\big)\cdot\nu + h\Big]\Big|_{s=1} =& -f(1)\mathbf{v}_\varepsilon\cdot\nu + h\\
=& \lambda_* f(1)\partial_\nu p_\varepsilon - \frac{\lambda_*}{2} f(1)(\rho_w + \rho_o)\mathbf{k}\cdot\nu + h\\
=& \lambda_* f(1)\left(\pi - \frac{\rho_w + \rho_o}{2}\mathbf{k}\cdot\nu\right) + h \ge 0.
\end{aligned}$$

The claim follows from (**H.4**) and the comparison principle for parabolic equations. □

Lemma 3.2 (L^∞ estimate on $\{\Lambda_{w,\varepsilon}\}_{\varepsilon>0}$, $\{\Lambda_{o,\varepsilon}\}_{\varepsilon>0}$, $\{\Lambda_{T,\varepsilon}\}_{\varepsilon>0}$). *We have that*

$$\begin{aligned}
&0 < \frac{\lambda_*}{2} \le \Lambda_{w,\varepsilon}(t, \mathbf{x}) \le \|\lambda_w\|_{L^\infty(0,1)},\\
&0 < \frac{\lambda_*}{2} \le \Lambda_{o,\varepsilon}(t, \mathbf{x}) \le \|\lambda_o\|_{L^\infty(0,1)},\\
&0 < \lambda_* \le \Lambda_{T,\varepsilon}(t, \mathbf{x}) \le \|\lambda_w\|_{L^\infty(0,1)} + \|\lambda_o\|_{L^\infty(0,1)},
\end{aligned} \tag{3.1}$$

for each $t > 0$, $\mathbf{x} \in \Omega$.

Proof. The maps with constant values $\frac{\lambda_*}{2}$ and $\|\lambda_w\|_{L^\infty(0,1)}$ are respectively sub and super solutions for the fourth equation in (2.3): the first inequality is consequence of the monotonicity of the elliptic operator $-\mu_w\Delta + 1$. The same argument works also for the second inequality. The last estimate follows from the other two and the definition of $\Lambda_{T,\varepsilon}$. □

Lemma 3.3 (Sobolev estimate on $\{\Lambda_{w,\varepsilon}\}_{\varepsilon>0}$, $\{\Lambda_{o,\varepsilon}\}_{\varepsilon>0}$, $\{\Lambda_{T,\varepsilon}\}_{\varepsilon>0}$). *Let $1 \le r < \infty$ be fixed. The following inequalities hold*

$$\begin{aligned}
&\|\Lambda_{w,\varepsilon}\|_{L^\infty(0,\infty;W^{2,r}(\Omega))} \le C_r \frac{\|\lambda_w\|_{L^\infty(0,1)}}{\mu_w},\\
&\|\Lambda_{o,\varepsilon}\|_{L^\infty(0,\infty;W^{2,r}(\Omega))} \le C_r \frac{\|\lambda_o\|_{L^\infty(0,1)}}{\mu_o},\\
&\|\Lambda_{T,\varepsilon}\|_{L^\infty(0,\infty;W^{2,r}(\Omega))} \le C_r \frac{\|\lambda_w\|_{L^\infty(0,1)}}{\mu_w} + \frac{\|\lambda_o\|_{L^\infty(0,1)}}{\mu_o},
\end{aligned} \tag{3.2}$$

where C_r is a positive constant dependent on r but not on μ_w, μ_o, and ε.

Proof. From (2.3), we know that

$$-\Delta\Lambda_{w,\varepsilon} = \frac{\lambda_w(s_\varepsilon) - \Lambda_{w,\varepsilon}}{\mu_w}, \qquad -\Delta\Lambda_{o,\varepsilon} = \frac{\lambda_o(s_\varepsilon) - \Lambda_{o,\varepsilon}}{\mu_o}.$$

Thanks to [3, Theorem 8.2], Lemmas 3.1, 3.2, and (**H.3**)

$$\begin{aligned}
\|\Lambda_{w,\varepsilon}(t,\cdot)\|_{W^{2,r}(\Omega)} \leq & c_1 \left\| \frac{\lambda_w(s_\varepsilon(t,\cdot)) - \Lambda_{w,\varepsilon}(t,\cdot)}{\mu_w} \right\|_{L^\infty(\Omega)} \\
\leq & c_1 \frac{\|\lambda_w(s_\varepsilon)\|_{L^\infty((0,\infty)\times\Omega)} + \|\Lambda_{w,\varepsilon}\|_{L^\infty((0,\infty)\times\Omega)}}{\mu} \leq 2c_1 \frac{\|\lambda_w\|_{L^\infty(0,1)}}{\mu_w}, \\
\|\Lambda_{o,\varepsilon}(t,\cdot)\|_{W^{2,r}(\Omega)} \leq & c_2 \left\| \frac{\lambda_o(s_\varepsilon(t,\cdot)) - \Lambda_{o,\varepsilon}(t,\cdot)}{\mu_o} \right\|_{L^\infty(\Omega)} \\
\leq & c_2 \frac{\|\lambda_o(s_\varepsilon)\|_{L^\infty((0,\infty)\times\Omega)} + \|\Lambda_{o,\varepsilon}\|_{L^\infty((0,\infty)\times\Omega)}}{\mu_o} \leq 2c \frac{\|\lambda_o\|_{L^\infty(0,1)}}{\mu_o},
\end{aligned}$$

where $c_!$ and c_2 are positive constants depending only on r. That proves the first two inequalities. The last one follows from the definition of $\Lambda_{T,\varepsilon}$. □

Lemma 3.4 (Sobolev estimate on $\{p_\varepsilon\}_{\varepsilon>0}$). *There exists $\rho > 2$ independent on μ and ε such that*

$$\|p_\varepsilon\|_{L^\infty(0,\infty;W^{1,\rho}(\Omega))} \leq \kappa, \tag{3.3}$$

$$\|p_\varepsilon\|_{L^\infty(0,\infty;W^{3,r}(\Omega))} \leq \frac{K_r}{\min\{\mu_w,\, \mu_o\}}, \qquad 1 \leq r < \rho \tag{3.4}$$

for some positive constants K_r and κ independent on μ_w, μ_o, and ε.

Proof. Since ρ_ε satisfies the equation (see (2.3))

$$\begin{aligned}
-\mathrm{div}_{\mathbf{x}} (\Lambda_{T,\varepsilon} \nabla_{\mathbf{x}} p_\varepsilon) &= -\,\mathrm{div}_{\mathbf{x}} ((\rho_w \Lambda_{w,\varepsilon} + \rho_o \Lambda_{o,\varepsilon})\mathbf{k}) \\
&= -\,(\rho_w \Lambda_{w,\varepsilon} + \rho_o \Lambda_{o,\varepsilon}) \mathrm{div}_{\mathbf{x}} (\mathbf{k}) - (\rho_w \nabla_{\mathbf{x}} \Lambda_{w,\varepsilon} + \rho_o \nabla_{\mathbf{x}} \Lambda_{o,\varepsilon}) \cdot \mathbf{k},
\end{aligned} \tag{3.5}$$

the $(\mu_w,\, \mu_o,\, \varepsilon)$-independent estimate in Lemma 3.2, (**H.2**), and [15, Theorem 1] give (3.3).

Since (3.5) gives

$$\Delta p_\varepsilon = \frac{1}{\Lambda_{T,\varepsilon}} \Big((\rho_w \Lambda_{w,\varepsilon} + \rho_o \Lambda_{o,\varepsilon}) \mathrm{div}_{\mathbf{x}} (\mathbf{k}) + (\rho_w \nabla_{\mathbf{x}} \Lambda_{w,\varepsilon} + \rho_o \nabla_{\mathbf{x}} \Lambda_{o,\varepsilon}) \cdot \mathbf{k} - \nabla_{\mathbf{x}} \Lambda_{T,\varepsilon} \cdot \nabla p_\varepsilon \Big),$$

and, from (3.2) and (3.3), $\nabla_{\mathbf{x}} \Lambda_{T,\varepsilon} \cdot \nabla p_\varepsilon$ is uniformly bounded in $L^\infty(0,\infty; L^r(\Omega))$, $1 \leq r < \rho$, from [3, Theorem 8.2], (3.1), and (3.2) we have

$$\|p_\varepsilon\|_{L^\infty(0,\infty;W^{2,r}(\Omega))} \leq \frac{K_r}{\min\{\mu_w,\, \mu_o\}}, \qquad 1 \leq r < \rho. \tag{3.6}$$

Differentiating (3.5) with respect to x_i, $i \in \{1, \ldots, N\}$, we get

$$\begin{aligned}
\Delta \partial_{x_i} p_\varepsilon = & \frac{1}{\Lambda_{T,\varepsilon}} \Big((\rho_w \partial_{x_i} \Lambda_{w,\varepsilon} + \rho_o \partial_{x_i} \Lambda_{o,\varepsilon}) \mathrm{div}_{\mathbf{x}} (\mathbf{k}) + (\rho_w \Lambda_{w,\varepsilon} + \rho_o \Lambda_{o,\varepsilon}) \mathrm{div}_{\mathbf{x}} (\partial_{x_i} \mathbf{k}) \\
& + (\rho_w \nabla_{\mathbf{x}} \partial_{x_i} \Lambda_{w,\varepsilon} + \rho_o \nabla_{\mathbf{x}} \partial_{x_i} \Lambda_{o,\varepsilon}) \cdot \mathbf{k} + (\rho_w \nabla_{\mathbf{x}} \Lambda_{w,\varepsilon} + \rho_o \nabla_{\mathbf{x}} \Lambda_{o,\varepsilon}) \cdot \partial_{x_i} \mathbf{k} \\
& - \partial_{x_i} \Lambda_\varepsilon \Delta p_\varepsilon - \nabla \partial_{x_i} \Lambda_\varepsilon \cdot \nabla p_\varepsilon - \nabla \Lambda_\varepsilon \cdot \nabla \partial_{x_i} p_\varepsilon \Big).
\end{aligned}$$

Since, from (3.2) and (3.6), the right hand side is uniformly bounded in $L^\infty(0,\infty; L^r(\Omega))$, $1 \leq r < \rho$, from [3, Theorem 8.2], (3.1), and (3.2) we have (3.4). □

Lemma 3.5 (Sobolev estimate on $\{\mathbf{v}_\varepsilon\}_{\varepsilon>0}$). *The following inequality holds*

$$\|\mathbf{v}_\varepsilon\|_{L^\infty(0,\infty;L^\rho(\Omega))} \le \Gamma_\rho, \qquad \|\mathbf{v}_\varepsilon\|_{L^\infty(0,\infty;W^{2,r}(\Omega))} \le \frac{\Gamma_r}{\min\{\mu_w,\,\mu_o\}}, \qquad 1 \le r < \rho,$$

where ρ is the one introduced in Lemma 3.4 and Γ_r is a positive constant dependent on r but not on μ_w, μ_o, and ε.

Proof. Since

$$\mathbf{v}_\varepsilon = -\Lambda_{T,\varepsilon}\nabla_{\mathbf{x}} p_\varepsilon + (\rho_w\Lambda_{w,\varepsilon} + \rho_o\Lambda_{o,\varepsilon})\mathbf{k},$$

and, for every $i,\, j \in \{1, ..., N\}$,

$$\begin{aligned}\partial^2_{x_ix_j}\mathbf{v}_\varepsilon =& -\partial^2_{x_ix_j}\Lambda_{T,\varepsilon}\nabla p_\varepsilon - \partial_{x_i}\Lambda_{T,\varepsilon}\nabla\partial_{x_j}p_\varepsilon - \partial_{x_j}\Lambda_{T,\varepsilon}\nabla\partial_{x_i}p_\varepsilon - \Lambda_{T,\varepsilon}\nabla\partial^2_{x_ix_j}p_\varepsilon\\ &+ \partial^2_{x_ix_j}(\rho_w\Lambda_{w,\varepsilon} + \rho_o\Lambda_{o,\varepsilon})\mathbf{k} + \partial_{x_i}(\rho_w\Lambda_{w,\varepsilon} + \rho_o\Lambda_{o,\varepsilon})\partial_{x_j}\mathbf{k}\\ &+ \partial_{x_j}(\rho_w\Lambda_{w,\varepsilon} + \rho_o\Lambda_{o,\varepsilon})\partial_{x_i}\mathbf{k} + (\rho_w\Lambda_{w,\varepsilon} + \rho_o\Lambda_{o,\varepsilon})\partial^2_{x_ix_j}\mathbf{k},\end{aligned}$$

the claim follows directly from Lemmas 3.3 and 3.4. ☐

Lemma 3.6 (Entropy Dissipation). *Let $\varepsilon > 0$ and $\eta \in C^2(\mathbb{R})$. The following inequality holds*

$$\begin{aligned}\int_\Omega \eta(s_\varepsilon(t,\mathbf{x}))d\mathbf{x} + &\varepsilon\int_0^t\int_\Omega \eta''(s_\varepsilon(\tau,\mathbf{x}))|\nabla_{\mathbf{x}}s_\varepsilon(\tau,\mathbf{x})|^2 d\tau d\mathbf{x}\\ &\le \int_\Omega \eta(s_{0,\varepsilon})d\mathbf{x} + \|\eta'\|_{L^\infty(0,1)}\|h\|_{L^1((0,t)\times\partial\Omega)} + K(\eta)t,\end{aligned} \tag{3.7}$$

for each $t \ge 0$, where

$$\begin{aligned}K(\eta) =& \Big(\|g\|_{L^\infty(0,1)}\|\eta'\|_{L^\infty(0,1)} + \|g'\eta'\|_{L^1(0,1)}\Big)\|\mathbf{k}\|_{W^{1,\infty}(\Omega)}(|\Omega| + |\partial\Omega|)\\ &+ \Big(\|f\|_{L^\infty(0,1)}\|\eta'\|_{L^\infty(0,1)} + \|f'\eta'\|_{L^1(0,1)}\Big)\times\\ &\times\lambda_*\left(\|\pi\|_{L^\infty((0,\infty)\times\partial\Omega)} + \frac{\rho_w+\rho_o}{2}\|h\|_{L^\infty(\Omega)}\right)|\partial\Omega|.\end{aligned}$$

In particular, if $\eta(s) = \frac{s^2}{2}$, we have

$$\|s_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)} + 2\varepsilon\int_0^t \|\nabla_{\mathbf{x}}s_\varepsilon(\tau,\cdot)\|^2_{L^2(\Omega)}d\tau \le \|s_{0,\varepsilon}\|^2_{L^2(\Omega)} + 2\|h\|_{L^1((0,t)\times\partial\Omega)} + \widetilde{K}t, \tag{3.8}$$

for each $t \ge 0$, where

$$\begin{aligned}\widetilde{K} =2\Big(&\|g\|_{W^{1,1}(0,1)}\|\mathbf{k}\|_{W^{1,\infty}(\Omega)}(|\Omega| + |\partial\Omega|)\\ &+ \|f\|_{W^{1,1}(0,1)}\lambda_*\left(\|\pi\|_{L^\infty((0,\infty)\times\partial\Omega)} + \frac{\rho_w+\rho_o}{2}\|h\|_{L^\infty(\Omega)}\right)|\partial\Omega|\Big).\end{aligned}$$

Proof. In light of (2.3)

$$\begin{aligned}&\partial_t\eta(s_\varepsilon) + \mathrm{div}_{\mathbf{x}}(\mathcal{F}(s_\varepsilon)\mathbf{v}_\varepsilon + \mathcal{G}(s_\varepsilon)\mathbf{k})\\ &\qquad + \big(g(s_\varepsilon)\eta'(s_\varepsilon) - \mathcal{G}(s_\varepsilon)\big)\mathrm{div}_{\mathbf{x}}(\mathbf{k}) = \varepsilon\Delta_{\mathbf{x}}\eta(s_\varepsilon) - \varepsilon\eta''(s_\varepsilon)|\nabla_{\mathbf{x}}s_\varepsilon|^2,\end{aligned}$$

where $\mathcal{F}$ and $\mathcal{G}$ are defined in (2.2). Integrating on Ω, since the boundary conditions in (2.3) say,

$$\mathbf{v}_\varepsilon\cdot\nu = -\lambda_*\pi + \lambda_*\frac{\rho_w+\rho_o}{2}\mathbf{k}\cdot\nu, \qquad \text{on } \partial\Omega$$

using Lemmas 3.1, 3.5,

$$\frac{d}{dt}\int_\Omega \eta(s_\varepsilon)d\mathbf{x} = -\varepsilon\int_\Omega \eta''(s_\varepsilon)|\nabla_{\mathbf{x}}s_\varepsilon|^2 d\mathbf{x} - \int_\Omega \big(g(s_\varepsilon)\eta'(s_\varepsilon) - \mathcal{G}(s_\varepsilon)\big)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,d\mathbf{x}$$

$$
\begin{aligned}
&+\int_{\partial\Omega}\big(\mathcal{F}(s_\varepsilon)\mathbf{v}_\varepsilon+\mathcal{G}(s_\varepsilon)\mathbf{k}\big)\cdot\nu d\sigma-\varepsilon\int_{\partial\Omega}\partial_\nu\eta(s_\varepsilon)d\sigma\\
=&-\varepsilon\int_\Omega\eta''(s_\varepsilon)|\nabla_{\mathbf{x}}s_\varepsilon|^2d\mathbf{x}-\int_\Omega\big(g(s_\varepsilon)\eta'(s_\varepsilon)-\mathcal{G}(s_\varepsilon)\big)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,d\mathbf{x}\\
&+\int_{\partial\Omega}\big(\mathcal{F}(s_\varepsilon)-f(s_\varepsilon)\eta'(s_\varepsilon)\big)\mathbf{v}_\varepsilon\cdot\nu d\sigma+\int_{\partial\Omega}\big(\mathcal{G}(s_\varepsilon)-g(s_\varepsilon)\eta'(s_\varepsilon)\big)\mathbf{k}\cdot\nu d\sigma\\
&+\int_{\partial\Omega}h\eta'(s_\varepsilon)d\sigma\\
\leq&-\varepsilon\int_\Omega\eta''(s_\varepsilon)|\nabla_{\mathbf{x}}s_\varepsilon|^2d\mathbf{x}\\
&+\big(\|g\|_{L^\infty(0,1)}\|\eta'\|_{L^\infty(0,1)}+\|\mathcal{G}\|_{L^\infty(0,1)}\big)\|\mathbf{k}\|_{W^{1,\infty}(\Omega)}(|\Omega|+|\partial\Omega|)\\
&+\big(\|f\|_{L^\infty(0,1)}\|\eta'\|_{L^\infty(0,1)}+\|\mathcal{F}\|_{L^\infty(0,1)}\big)\times\\
&\quad\times\lambda_*\left(\|\pi\|_{L^\infty((0,\infty)\times\partial\Omega)}+\lambda_*\frac{\rho_w+\rho_o}{2}\|\mathbf{k}\|_{L^\infty(\Omega)}\right)|\partial\Omega|\\
&+\|\eta'\|_{L^\infty(0,1)}\|h(t,\cdot)\|_{L^1(\partial\Omega)}.
\end{aligned}
$$

An integration with respect to t gives (3.7).

Finally, by choosing $\eta(s)=\frac{s^2}{2}$, (3.7) gives (3.8). □

As a consequence of Lemmas 3.3, 3.4, 3.5 we have the following compactness result.

Lemma 3.7. *Let ρ be the exponent introduced in Lemma 3.4. There exist a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}\subset(0,\infty)$, $\varepsilon_k\to0$, and some functions*

$$
\begin{aligned}
&s\in L^\infty((0,\infty)\times\Omega),\\
\Lambda_w,\,\Lambda_o\in{}&L^\infty((0,\infty)\times\Omega)\cap L^\infty(0,\infty;W^{2,r}(\Omega)),\quad 1\leq r<\infty,\\
&p\in L^\infty(0,\infty;W^{1,\rho}(\Omega)\cap W^{3,r}(\Omega)),\quad 1\leq r<\rho,\\
&\mathbf{v}\in L^\infty(0,\infty;L^\rho(\Omega)\cap W^{2,r}(\Omega)),\quad 1\leq r<\rho,
\end{aligned}
$$

such that

$$
\begin{aligned}
s_{\varepsilon_k}&\rightharpoonup s\quad\text{weakly in } L^r((0,\infty)\times\Omega),\ 1\leq r<\infty,\\
\Lambda_{w,\varepsilon_k}&\rightharpoonup\Lambda_w\quad\text{weakly in } L^r((0,\infty)\times\Omega)\cap L^{r'}(0,\infty;W^{2,r}(\Omega)),\ 1\leq r',\ r<\infty,\\
\Lambda_{o,\varepsilon_k}&\rightharpoonup\Lambda_o\quad\text{weakly in } L^r((0,\infty)\times\Omega)\cap L^{r'}(0,\infty;W^{2,r}(\Omega)),\ 1\leq r',\ r<\infty,\\
p_{\varepsilon_k}&\rightharpoonup p\quad\text{weakly in } L^{r'}(0,\infty;W^{1,\rho}(\Omega)\cap W^{3,r}(\Omega)),\ 1\leq r',\ r<\rho,\\
\mathbf{v}_{\varepsilon_k}&\rightharpoonup\mathbf{v}\quad\text{weakly in } L^{r'}(0,\infty;L^\rho(\Omega)\cap W^{2,r}(\Omega)),\ 1\leq r',\ r<\rho.
\end{aligned}
$$

4. Kinetic Formulation

Let us pass to the kinetic formulation of the first equation in (2.3).

From (2.3), we know

$$
\partial_t s_\varepsilon+f'(s_\varepsilon)\nabla_{\mathbf{x}}s_\varepsilon\cdot\mathbf{v}_\varepsilon+g'(s_\varepsilon)\nabla_{\mathbf{x}}s_\varepsilon\cdot\mathbf{k}+g(s_\varepsilon)\mathrm{div}_{\mathbf{x}}(\mathbf{k})=\varepsilon\Delta_{\mathbf{x}}s_\varepsilon.\tag{4.1}
$$

Let η be an entropy. From (4.1) we get

$$
\begin{aligned}
&\partial_t\eta(s_\varepsilon)+\nabla_{\mathbf{x}}\mathcal{F}(s_\varepsilon)\cdot\mathbf{v}_\varepsilon+\nabla_{\mathbf{x}}\mathcal{G}(s_\varepsilon)\cdot\mathbf{k}\\
&\qquad+g(s_\varepsilon)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\eta'(s_\varepsilon)-\varepsilon\Delta_{\mathbf{x}}\eta(s_\varepsilon)=-\varepsilon\eta''(s_\varepsilon)|\nabla_{\mathbf{x}}s_\varepsilon|^2,
\end{aligned}\tag{4.2}
$$

where the entropy fluxes $\mathcal{F}$ and $\mathcal{G}$ are defined in (2.2)

Lemma 4.1. *The following identity holds in the sense of distributions*

$$\begin{aligned}\partial_t \chi_\varepsilon + \mathrm{div}_{\mathbf{x}}\left(f'(\xi)\chi_\varepsilon \mathbf{v}_\varepsilon\right) + \mathrm{div}_{\mathbf{x}}\left(g'(\xi)\chi_\varepsilon \mathbf{k}\right)\\ + \partial_\xi\big(g(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\chi_\varepsilon\big) - 2g'(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\chi_\varepsilon - \varepsilon\Delta_{\mathbf{x}}\chi_\varepsilon = \partial_\xi m_\varepsilon,\end{aligned}\tag{4.3}$$

where

$$\chi_\varepsilon(t,\mathbf{x},\xi) = \chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}, \quad m_\varepsilon = \varepsilon\delta_{\{\xi=s_\varepsilon(t,\mathbf{x})\}}|\nabla_{\mathbf{x}} s_\varepsilon|^2, \quad t>0, \mathbf{x}\in\Omega, \xi\in\mathbb{R},\tag{4.4}$$

and $\chi_{\{\xi<s\}}$ and $\delta_{\{\xi=s\}}$ are the characteristic function and the Dirac delta associated to the sets $\{\xi<s\}$ and $\{\xi=s\}$, respectively.

Proof. Let us consider the entropy

$$\eta(s) = (s-\xi)_+, \qquad s,\,\xi\in\mathbb{R}.$$

Since

$$\begin{aligned}\eta'(s) = \chi_{\{\xi<s\}}, \qquad \eta''(s) = \delta_{\{\xi=s\}},\\ \mathcal{F}(s) = \chi_{\{\xi<s\}}(f(s)-f(\xi)), \qquad \mathcal{G}(s) = \chi_{\{\xi<s\}}(g(s)-g(\xi)),\end{aligned}$$

(4.2) becomes

$$\begin{aligned}\partial_t(s_\varepsilon-\xi)_+ + \nabla_{\mathbf{x}}\big(\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}(f(s_\varepsilon)-f(\xi))\big)\cdot\mathbf{v}_\varepsilon\\ + \nabla_{\mathbf{x}}\big(\chi_{\{\xi<s_\varepsilon(t,x)\}}(g(s_\varepsilon)-g(\xi))\big)\cdot\mathbf{k}\\ + g(s_\varepsilon)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}} - \varepsilon\Delta_{\mathbf{x}}(s_\varepsilon-\xi)_+ = -m_\varepsilon.\end{aligned}\tag{4.5}$$

Since

$$\begin{aligned}&\partial_\xi\partial_t(s_\varepsilon-\xi)_+ = -\partial_t\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}} = -\partial_t\chi_\varepsilon,\\ &\partial_\xi\Big(\nabla_{\mathbf{x}}\big(\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}(f(s_\varepsilon)-f(\xi))\big)\cdot\mathbf{v}_\varepsilon\Big)\\ &\quad= \Big(\nabla_{\mathbf{x}}\partial_\xi\big(\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}(f(s_\varepsilon)-f(\xi))\big)\Big)\cdot\mathbf{v}_\varepsilon\\ &\quad= -\underbrace{\Big(\nabla_{\mathbf{x}}\big(\delta_{\{\xi=s_\varepsilon(t,\mathbf{x})\}}(f(s_\varepsilon)-f(\xi))\big)\Big)\cdot\mathbf{v}_\varepsilon}_{=0} - \nabla_{\mathbf{x}}\big(\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}f'(\xi)\big)\cdot\mathbf{v}_\varepsilon\\ &\quad= -f'(\xi)\nabla_{\mathbf{x}}\chi_\varepsilon\cdot\mathbf{v}_\varepsilon = -\mathrm{div}_{\mathbf{x}}\left(f'(\xi)\chi_\varepsilon\mathbf{v}_\varepsilon\right),\\ &\partial_\xi\Big(\nabla_{\mathbf{x}}\big(\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}(g(s_\varepsilon)-g(\xi))\big)\cdot\mathbf{k}\Big)\\ &\quad= \Big(\nabla_{\mathbf{x}}\partial_\xi\big(\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}(g(s_\varepsilon)-g(\xi))\big)\Big)\cdot\mathbf{k}\\ &\quad= -\underbrace{\Big(\nabla_{\mathbf{x}}\big(\delta_{\{\xi=s_\varepsilon(t,\mathbf{x})\}}(g(s_\varepsilon)-g(\xi))\big)\Big)\cdot\mathbf{k}}_{=0} - \nabla_{\mathbf{x}}\big(\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}g'(\xi)\big)\cdot\mathbf{k}\\ &\quad= -g'(\xi)\nabla_{\mathbf{x}}\chi_\varepsilon\cdot\mathbf{k},\\ &\partial_\xi\Big(g(s_\varepsilon)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}}\Big) = -g(s_\varepsilon)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\delta_{\{\xi=s_\varepsilon(t,\mathbf{x})\}} = -g(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\partial_\xi\chi_\varepsilon,\\ &\partial_\xi\varepsilon\Delta_{\mathbf{x}}(s_\varepsilon-\xi)_+ = -\varepsilon\Delta_{\mathbf{x}}\chi_{\{\xi<s_\varepsilon(t,\mathbf{x})\}} = -\varepsilon\Delta_{\mathbf{x}}\chi_\varepsilon,\end{aligned}$$

we differentiate (4.5) with respect to ξ and get

$$\partial_t\chi_\varepsilon + \mathrm{div}_{\mathbf{x}}\left(f'(\xi)\chi_\varepsilon\mathbf{v}_\varepsilon\right) + g'(\xi)\nabla_{\mathbf{x}}\chi_\varepsilon\cdot\mathbf{k} + g(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\partial_\xi\chi_\varepsilon - \varepsilon\Delta_{\mathbf{x}}\chi_\varepsilon = \partial_\xi m_\varepsilon.$$

Finally, we observe that

$$\begin{aligned}&g'(\xi)\nabla_{\mathbf{x}}\chi_\varepsilon\cdot\mathbf{k} + g(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\partial_\xi\chi_\varepsilon\\ &\qquad= \mathrm{div}_{\mathbf{x}}\left(g'(\xi)\chi_\varepsilon\mathbf{k}\right) - g'(\xi)\chi_\varepsilon\mathrm{div}_{\mathbf{x}}(\mathbf{k}) + \partial_\xi\big(g(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\chi_\varepsilon\big) - g'(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})\,\chi_\varepsilon\end{aligned}$$

$$=\mathrm{div}_{\mathbf{x}}\left(g'(\xi)\chi_\varepsilon\mathbf{k}\right)+\partial_\xi\big(g(\xi)\mathrm{div}_{\mathbf{x}}\left(\mathbf{k}\right)\chi_\varepsilon\big)-2g'(\xi)\mathrm{div}_{\mathbf{x}}\left(\mathbf{k}\right)\chi_\varepsilon,$$

therefore we have (4.3). □

Lemma 4.2. *There exist*

$$\chi\in L^\infty((0,\infty)\times\Omega\times\mathbb{R}),\qquad m\in\mathcal{M}_+((0,\infty)\times\Omega\times\mathbb{R}),$$

such that (using the same notation of Lemma 3.7 for the subsequence)

$$\chi_{\varepsilon_k}\overset{\star}{\rightharpoonup}\chi\quad\text{weakly-}*\text{ in }L^\infty((0,\infty)\times\Omega\times\mathbb{R}),\tag{4.6}$$

$$m_{\varepsilon_k}\rightharpoonup m\quad\text{weakly in }\mathcal{M}_+((0,\infty)\times\Omega\times\mathbb{R}),\tag{4.7}$$

$$0\le\chi\le1,\qquad\partial_\xi\chi\le0,\tag{4.8}$$

where $\mathcal{M}_+((0,\infty)\times\Omega\times\mathbb{R})$ is the set of positive Radon measures on $(0,\infty)\times\Omega\times\mathbb{R}$. Moreover, the following identity holds in the sense of distributions

$$\begin{aligned}\partial_t\chi&+\mathrm{div}_{\mathbf{x}}\left(f'(\xi)\chi\mathbf{v}\right)+\mathrm{div}_{\mathbf{x}}\left(g'(\xi)\chi\mathbf{k}\right)\\&+\partial_\xi\big(g(\xi)\mathrm{div}_{\mathbf{x}}\left(\mathbf{k}\right)\chi\big)-2g'(\xi)\mathrm{div}_{\mathbf{x}}\left(\mathbf{k}\right)\chi=\partial_\xi m.\end{aligned}\tag{4.9}$$

Proof. The existence of χ, m, (4.6), and (4.7) follow from Lemmas 3.1 and 3.6. Since, from the definition,

$$0\le\chi_\varepsilon\le1,\qquad\partial_\xi\chi_\varepsilon=-\delta_{\{\xi=s_\varepsilon(t,\mathbf{x})\}}\le0,\tag{4.10}$$

we have (4.8).

We have now to prove (4.9). Due to (4.7) we have only to show that

$$\chi_{\varepsilon_k}\mathbf{v}_{\varepsilon_k}\longrightarrow\chi\mathbf{v},\qquad\text{in the sense of distributions on }(0,\infty)\times\Omega\times\mathbb{R}.\tag{4.11}$$

Observe that, from (4.10) and Lemma (3.7), we know

$$\chi_{\varepsilon_k}\rightharpoonup\chi,\quad\mathbf{v}_{\varepsilon_k}\rightharpoonup\mathbf{v},\quad\text{weakly in }L^2((0,\infty)\times\Omega\times\mathbb{R}).\tag{4.12}$$

Since $\mathbf{v}_{\varepsilon_k}$ does not depend on ξ, thanks to Lemma (3.5),

$$\{\mathbf{v}_{\varepsilon_k}\}_k\text{ is uniformly bounded in }L^2((0,\infty);W^{1,2}(\Omega\times\mathbb{R})).\tag{4.13}$$

We claim that

$$\{\partial_t\chi_{\varepsilon_k}\}_k\text{ is uniformly bounded in }L^1((0,T);W^{-1,1}(\Omega\times(-a,a))),\ T,\ a>0.\tag{4.14}$$

Indeed, from (4.3), we know that

$$\begin{aligned}\partial_t\chi_{\varepsilon_k}=&-\mathrm{div}_{\mathbf{x}}\left(f'(\xi)\chi_{\varepsilon_k}\mathbf{v}_{\varepsilon_k}\right)-\mathrm{div}_{\mathbf{x}}\left(g'(\xi)\chi_{\varepsilon_k}\mathbf{k}\right)\\&-\partial_\xi\big(g(\xi)\mathrm{div}_{\mathbf{x}}\left(\mathbf{k}\right)\chi_{\varepsilon_k}\big)+2g'(\xi)\mathrm{div}_{\mathbf{x}}\left(\mathbf{k}\right)\chi_{\varepsilon_k}+\varepsilon_k\Delta_{\mathbf{x}}\chi_{\varepsilon_k}-\partial_\xi m_{\varepsilon_k},\end{aligned}\tag{4.15}$$

Due to (**H.2**), (4.10), and Lemma (3.5),

$$\begin{aligned}&\{\mathrm{div}_{\mathbf{x}}\left(f'(\xi)\chi_{\varepsilon_k}\mathbf{v}_{\varepsilon_k}\right)\}_k\text{ bounded in }L^\infty((0,\infty);W^{-1,1}(\Omega\times(-a,a))),\ a>0,\\&\{\mathrm{div}_{\mathbf{x}}\left(g'(\xi)\chi_{\varepsilon_k}\mathbf{k}\right)\}_k\text{ bounded in }L^\infty((0,\infty);W^{-1,\infty}(\Omega\times(-a,a))),\ a>0,\\&\{\partial_\xi\big(g(\xi)\mathrm{div}_{\mathbf{x}}\left(\mathbf{k}\right)\chi_{\varepsilon_k}\big)\}_k\text{ bounded in }L^\infty((0,\infty);W^{-1,\infty}(\Omega\times(-a,a))),\ a>0,\\&\{g'(\xi)\mathrm{div}_{\mathbf{x}}\left(\mathbf{k}\right)\chi_{\varepsilon_k}\}_k\text{ bounded in }L^\infty((0,\infty)\times\Omega\times(-a,a)),\ a>0.\end{aligned}\tag{4.16}$$

Since

$$\begin{aligned}\varepsilon_k\Delta_{\mathbf{x}}\chi_{\varepsilon_k}-\partial_\xi m_{\varepsilon_k}=&\partial_\xi(\varepsilon_k(s_{\varepsilon_k}-\xi)_+\Delta_{\mathbf{x}}s_{\varepsilon_k})\\=&\partial_\xi(\varepsilon_k\mathrm{div}_{\mathbf{x}}\left((s_{\varepsilon_k}-\xi)_+\nabla_{\mathbf{x}}s_{\varepsilon_k}\right)-\varepsilon_k\chi_{\varepsilon_k}|\nabla_{\mathbf{x}}s_{\varepsilon_k}|^2)\\=&\mathrm{div}_{\mathbf{x}}\left(\varepsilon_k\partial_\xi(s_{\varepsilon_k}-\xi)_+\nabla_{\mathbf{x}}s_{\varepsilon_k}\right)-\partial_\xi(\varepsilon_k\chi_{\varepsilon_k}|\nabla_{\mathbf{x}}s_{\varepsilon_k}|^2)\end{aligned}$$

$$= -\operatorname{div}_{\mathbf{x}}(\varepsilon_k\chi_{\varepsilon_k}\nabla_{\mathbf{x}}s_{\varepsilon_k}) - \partial_\xi(\varepsilon_k\chi_{\varepsilon_k}|\nabla_{\mathbf{x}}s_{\varepsilon_k}|^2),$$

thanks to (3.8) and (4.4), we have that also

$$\{\varepsilon_k\Delta_{\mathbf{x}}\chi_{\varepsilon_k} - \partial_\xi m_{\varepsilon_k}\}_k \text{ bounded in } L^1((0,T);W^{-1,1}(\Omega\times(-a,a))),\ T,\ a>0. \tag{4.17}$$

Therefore (4.14) follows from (4.15), (4.16), and (4.17). Due to (4.12), (4.13), (4.14), and [13, Lemma 5.1] we have (4.11) and the proof is concluded. □

Lemma 4.3. *Let $\eta \in C^2(\mathbb{R})$. The following identity holds in the sense of distributions*

$$\begin{aligned}\partial_t\eta(\chi) &+ \operatorname{div}_{\mathbf{x}}(f'(\xi)\eta(\chi)\mathbf{v}) + \operatorname{div}_{\mathbf{x}}(g'(\xi)\eta(\chi)\mathbf{k})\\ &+ \partial_\xi\big(g(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\chi\eta'(\chi)\big) - 2g'(\xi)\eta(\chi)\operatorname{div}_{\mathbf{x}}(\mathbf{k}) = \eta'(\chi)\partial_\xi m.\end{aligned} \tag{4.18}$$

Proof. Convolving both sides of (4.9) with a family of mollifiers and the arguing as in [9, Lemma II.1] we get

$$\begin{aligned}\partial_t\eta(\chi)&+\operatorname{div}_{\mathbf{x}}(f'(\xi)\chi\mathbf{v})\,\eta'(\chi) + \operatorname{div}_{\mathbf{x}}(g'(\xi)\chi\mathbf{k})\,\eta'(\chi)\\ &+ \partial_\xi\big(g(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\chi\big)\eta'(\chi) - 2g'(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\chi\eta'(\chi) = \eta'(\chi)\partial_\xi m.\end{aligned}$$

Since

$$\begin{aligned}\operatorname{div}_{\mathbf{x}}(f'(\xi)\chi\mathbf{v})\,\eta'(\chi) =& f'(\xi)\nabla_{\mathbf{x}}\chi\cdot\mathbf{v}\eta'(\chi)\\ =& f'(\xi)\nabla_{\mathbf{x}}(\eta(\chi))\cdot\mathbf{v} = \operatorname{div}_{\mathbf{x}}(f'(\xi)\eta(\chi)\mathbf{v}),\\ \operatorname{div}_{\mathbf{x}}(g'(\xi)\chi\mathbf{k})\,\eta'(\chi) =& g'(\xi)\nabla_{\mathbf{x}}\chi\cdot\mathbf{k}\eta'(\chi) + g'(\xi)\chi\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\eta'(\chi)\\ =& g'(\xi)\nabla_{\mathbf{x}}\eta(\chi)\cdot\mathbf{k} + g'(\xi)\chi\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\eta'(\chi)\\ =& \operatorname{div}_{\mathbf{x}}(g'(\xi)\eta(\chi)\mathbf{k}) + g'(\xi)(\chi\eta'(\chi)-\eta(\chi))\operatorname{div}_{\mathbf{x}}(\mathbf{k}),\\ \partial_\xi\big(g(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\chi\big)\eta'(\chi) =& g(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\partial_\xi\chi\eta'(\chi) + g'(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\chi\eta'(\chi)\\ =& g(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\partial_\xi\eta(\chi) + g'(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\chi\eta'(\chi)\\ =& \partial_\xi\big(g(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,\eta(\chi)\big) + g'(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,(\chi\eta'(\chi)-\eta(\chi)),\end{aligned}$$

we have (4.18). □

Lemma 4.4. *The function χ defined in Lemma 4.2 takes only values* 0 *and* 1*. In particular, there exists a function $S \in L^\infty((0,\infty)\times\Omega)$ such that*

$$\chi(t,\mathbf{x},\xi) = \chi_{\{\xi\le S(t,\mathbf{x})\}}, \qquad a.e.\ (t,\mathbf{x},\xi)\in(0,\infty)\times\Omega\times\mathbb{R}. \tag{4.19}$$

Proof. We use the entropy

$$\eta(\chi) = \chi - \chi^2$$

in (4.18) and get

$$\begin{aligned}\partial_t(\chi-\chi^2) &+ \operatorname{div}_{\mathbf{x}}\big(f'(\xi)(\chi-\chi^2)\mathbf{v}\big) + \operatorname{div}_{\mathbf{x}}\big(g'(\xi)(\chi-\chi^2)\mathbf{k}\big)\\ &+ \partial_\xi\big(g(\xi)\operatorname{div}_{\mathbf{x}}(\mathbf{k})\,(\chi-2\chi^2)\big)\\ &- 2g'(\xi)(\chi-\chi^2)\operatorname{div}_{\mathbf{x}}(\mathbf{k}) = (1-2\chi)\partial_\xi m.\end{aligned} \tag{4.20}$$

Since from (4.8) and the definitions of χ and m, we know

$$(1-2\chi)\partial_\xi m = \partial_\xi((1-2\chi)m) + 2m\partial_\xi\chi \le \partial_\xi((1-2\chi)m),$$

$$\chi - \chi^2 \ge 0, \tag{4.21}$$

$$\xi\in(-\infty,0)\cup(1,\infty) \Longrightarrow \chi(t,\mathbf{x},\xi) = m(t,\mathbf{x},\xi) = 0. \tag{4.22}$$

The identity (4.20) gives

$$
\begin{aligned}
&\partial_t(\chi-\chi^2)+\mathrm{div}_{\mathbf{x}}\left(f'(\xi)(\chi-\chi^2)\mathbf{v}+g'(\xi)(\chi-\chi^2)\mathbf{k}\right)\\
&\qquad+\partial_\xi\big(g(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})(\chi-2\chi^2)+(2\chi-1)m\big)\\
&\qquad\le\underbrace{2\left\|g'\right\|_{L^\infty(0,1)}\left\|\mathrm{div}_{\mathbf{x}}(\mathbf{k})\right\|_{L^\infty(\Omega)}}_{\kappa}(\chi-\chi^2).
\end{aligned}
\tag{4.23}
$$

Moreover, in light of (4.22),

$$\int_{\mathbb{R}}\partial_\xi\big(g(\xi)\mathrm{div}_{\mathbf{x}}(\mathbf{k})(\chi-2\chi^2)+(2\chi-1)m\big)d\xi=0,$$

therefore, integrating (4.23) with respect to $\mathbf{x}$ and ξ and using (**H.2**)

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega\times\mathbb{R}}(\chi-\chi^2)d\mathbf{x}d\xi\\
&\quad\le-\int_{\Omega\times\mathbb{R}}\mathrm{div}_{\mathbf{x}}\left(f'(\xi)(\chi-\chi^2)\mathbf{v}+g'(\xi)(\chi-\chi^2)\mathbf{k}\right)d\mathbf{x}d\xi\\
&\qquad+\kappa\int_{\Omega\times\mathbb{R}}(\chi-\chi^2)d\mathbf{x}d\xi\\
&\quad=\int_{\partial\Omega\times\mathbb{R}}\left(f'(\xi)(\chi-\chi^2)\mathbf{v}+g'(\xi)(\chi-\chi^2)\mathbf{k}\right)\cdot\nu d\sigma d\xi\\
&\qquad+\kappa\int_{\Omega\times\mathbb{R}}(\chi-\chi^2)d\mathbf{x}d\xi\\
&\quad=\int_{\partial\Omega\times\mathbb{R}}\underbrace{\left(-\lambda_*f'(\xi)\left(\pi-\frac{\rho_w+\rho_o}{2}\mathbf{k}\cdot\nu\right)+g'(\xi)\mathbf{k}\cdot\nu\right)}_{\le 0\text{ (cf. }(\mathbf{H.2}))}\underbrace{(\chi-\chi^2)}_{\ge 0\text{ (cf. (4.21))}}d\sigma d\xi\\
&\qquad+\kappa\int_{\Omega\times\mathbb{R}}(\chi-\chi^2)d\mathbf{x}d\xi\\
&\quad\le\kappa\int_{\Omega\times\mathbb{R}}(\chi-\chi^2)d\mathbf{x}d\xi.
\end{aligned}
$$

Thanks to the Gronwall's inequality and the fact that $(\chi-\chi^2)\big|_{t=0}=0$, we have

$$\chi-\chi^2=0\qquad\text{a.e. }(0,\infty)\Omega\times\mathbb{R},$$

and then χ takes only values 0 and 1.

Finally, (4.19) follows from [19]. □

Proof of Theorem 2.1. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and s be the one of Lemma 3.7. Due to the the weak convergences Lemmas 3.7 and 4.2, we have $S=s$, where S is the one introduced in Lemma 4.4, we have

$$s_{\varepsilon_k}\to s\quad\text{strongly in }L^r((0,T)\times\Omega),\ T>0,\ 1\le r<\infty,$$

indeed

$$s_\varepsilon(t,\mathbf{x})=\int_{\mathbb{R}}\chi_\varepsilon(t,\mathbf{x},\xi)d\xi,\qquad s(t,\mathbf{x})=\int_{\mathbb{R}}\chi(t,\mathbf{x},\xi)d\xi.$$

□

5. Numerical experiments

In order to test how close the model with $\mu_w > 0$ and $\mu_o > 0$ are to the model with $\mu_w = \mu_o = 0$, we have performed several numerical experiments.

We have simplified the equations to read

$$\mathbf{v} = \Lambda_T \nabla p + (\rho_w \Lambda_w + \rho_o \Lambda_o)\,\mathbf{k}, \quad t > 0,\, \mathbf{x} \in \Omega, \tag{5.1}$$

$$\operatorname{div}_{\mathbf{x}} (\mathbf{v}_\varepsilon) = q, \quad t > 0,\, \mathbf{x} \in \Omega, \tag{5.2}$$

$$s_t + \operatorname{div}_{\mathbf{x}} \left(\frac{\lambda_w(s)}{\lambda_T(s)} \left(\mathbf{v} + (\rho_w - \rho_o)\,\lambda_o(s)\mathbf{k}\right) \right) = q, \quad t > 0,\, \mathbf{x} \in \Omega, \tag{5.3}$$

$$\Lambda_{w,o} - \mu_{w,o} \Delta \Lambda_{w,o} = \lambda_{w,o}(s), \quad t > 0,\, \mathbf{x} \in \Omega, \tag{5.4}$$

where $\Lambda_T = \Lambda_o + \Lambda_w$, $\lambda_T = \lambda_w + \lambda_o$, $\mathbf{k}$ is the vector $(0,1)$ and the domain Ω is the rectangle $[0,a] \times [0,b]$. The boundary conditions are

$$\left(\frac{\lambda_w(s)}{\kappa_{\mathrm{T}}(s)} \left(\mathbf{v}_\varepsilon + (\rho_w - \rho_o)\,\lambda_o(s)\mathbf{k}\right) \right) \cdot \nu = 0, \quad t > 0,\, \mathbf{x} \in \partial\Omega,$$

$$\mathbf{v} \cdot \nu = 0, \quad t > 0,\, \mathbf{x} \in \partial\Omega, \tag{5.5}$$

$$\Lambda_{r,o} = \lambda_{r,o}(s), \quad t > 0,\, \mathbf{x} \in \partial\Omega \text{ (in the trace sense)}, \tag{5.6}$$

$$s(0,\mathbf{x}) = s_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{5.7}$$

The relative permeabilities of water and oil, λ_w and λ_o, are given by

$$\lambda_{w,o}\left(s_{w,o}\right) = s_{w,o}^2,$$

so that $\lambda_o(s) = (1-s)^2$. The source term q accounts for injection of water and production of oil. Numerically, we model this term as a sum of "delta-functions" located at the relevant wells.

We now specify the numerical scheme used to approximate (5.1)–(5.7). Let

$$D_x^{\pm} k_{ij} = \pm \frac{1}{h_x} \left(k_{i\pm1,j} - k_{ij}\right), \; D_y^{\pm} k_{ij} = \pm \frac{1}{h_y} \left(k_{i,j\pm1,} - k_{ij}\right).$$

We divide Ω into $N \times M$ rectangles, such that $h_x = a/N$, $h_y = b/M$. As an approximation to (5.4), (5.6) we use the scheme

$$\begin{cases} \Lambda_{ij} - \varepsilon\left(D_x^+ D_x^- + D_y^+ D_y^-\right) \Lambda_{ij} = \lambda\left(s_{ij}\right), 1 < i < N, 1 < j < M, \\ \Lambda_{ij} = \lambda\left(s_{ij}\right), i = 1, N,\; 1 \le j \le M \text{ and } j = 1, N,\; 1 \le i \le N. \end{cases} \tag{5.8}$$

This scheme is used both for $\Lambda_{ij}^{\mathrm{w}} \approx \Lambda_w$ and $\Lambda_{ij}^{\mathrm{o}} \approx \Lambda_o$. To solve the "pressure equation", (5.2) with boundary conditions given by (5.5) we use a finite volume scheme similar to (5.8). Set

$$\Lambda_{i+1/2,j} = \begin{cases} 2\frac{\Lambda_{ij}\Lambda_{i+1,j}}{\Lambda_{ij}+\Lambda_{i+1,j}}, & 1 \le i < N,\; 1 \le j \le M, \\ 0 & i = 0 \text{ or } i = N,\; 1 \le j \le M, \end{cases}$$

$$\Lambda_{i,j+1/2} = \begin{cases} 2\frac{\Lambda_{ij}\Lambda_{i,j+1}}{\Lambda_{ij}+\Lambda_{i,j+1}}, & 1 \le j < M,\; 1 \le i \le N, \\ 0 & j = 0 \text{ or } j = M,\; 1 \le i \le N. \end{cases}$$

Then the discrete analogue of (5.2) reads

$$D_x^- \Lambda_{i+1/2,j}^{\mathrm{T}} D_x^+ p_{ij} + D_y^- \Lambda_{i,j+1/2}^{\mathrm{T}} D_y^+ p_{ij} = q_{ij} - D_y^- \gamma_{ij+1/2}, \tag{5.9}$$

for $1 \le i \le N$ and $1 \le j \le M$. Here the gravitational source term is defined by

$$\gamma_{i,j+1/2} = \rho_o \Lambda_{i,j+1/2}^{\mathrm{o}} + \rho_w \Lambda_{i,j+1/2}^{\mathrm{w}}.$$

Given s_{ij}, the total velocity is then found by solving (5.8) and (5.9), and defined by

$$v^x_{i+1/2,j} = \Lambda^{\mathrm{T}}_{i+1/2,j} D^+_x p_{ij}, \quad v^y_{i+1/2,j} = \Lambda^{\mathrm{T}}_{i,j+1/2} D^+_y p_{ij} + g_{i,j+1/2}.$$

Once we have a total velocity on the cell edges, we can use a finite volume scheme to advance the saturation in time

$$s^{n+1}_{ij} = s^n_{ij} - h\left(D^-_x F^x_{i+1/2,j} + D^-_y\left(F^y_{i,j+1/2} + F^{y,\mathrm{grav}}_{i,j+1/2}\right)\right) + hq_{ij}. \tag{5.10}$$

We have split the numerical flux into the part multiplied by the total velocity, and the gravitational part. Set $f(s) = \lambda_w(s)/\lambda_T(s)$ and $g(s) = \lambda_o(s)f(s)$. Then the inter cell fluxes read

$$\begin{aligned}
F^x_{i+1/2,j}\left(s_{ij}, s_{i+1,j}\right) &= \begin{cases} v^x_{i+1/2,j} f\left(s_{ij}\right) & v^x_{i+1/2,j} > 0, \\ v^x_{i+1/2,j} f\left(s_{i+1,j}\right) & \text{otherwise}, \end{cases} \\
F^y_{i,j+1/2}\left(s_{ij}, s_{i,j+1}\right) &= \begin{cases} v^y_{i,j+1/2} f\left(s_{ij}\right) & v^y_{i,j+1/2} > 0, \\ v^y_{i,j+1/2} f\left(s_{i,j+1}\right) & \text{otherwise}, \end{cases} \\
F^{y,\mathrm{grav}}_{i,j+1/2}\left(s_{ij}, s_{i,j+1}\right) &= \left(\rho_w - \rho_o\right)\left(g\left(\min\left(s_{i,j+1}, 1/2\right)\right) + g\left(\max\left(s_{i,j}, 1/2\right)\right)\right).
\end{aligned}$$

Having numerical methods for (5.1) – (5.4), we can formulate an "IMPES" method to find the pressure and saturation as functions of time. As is commonly believed in the reservoir simulation community, it is sufficient to update the total velocity less frequently than the saturation. Hence, an algorithm for solving (5.1) – (5.4) reads as follows:

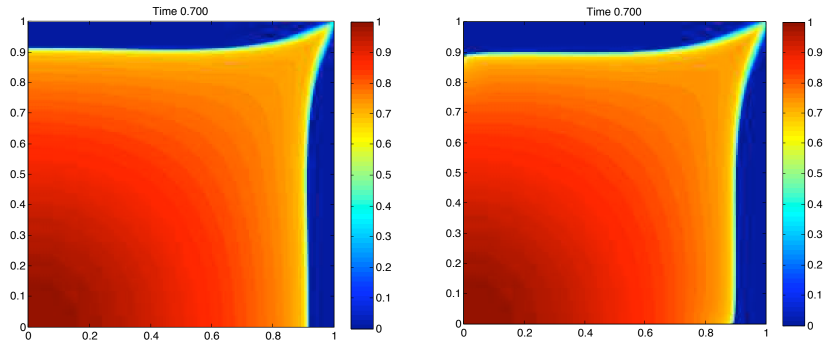
Algorithm 1 Simple reservoir simulation

given s_{ij}, q_{ij}, T, N, ε
$dt \leftarrow T/N$
for $k = 1$ to N **do**
  solve (5.8) to get $\Lambda^{\mathrm{w}}_{ij}$ and $\Lambda^{\mathrm{o}}_{ij}$
  solve (5.9) to get $v^x_{i+1/2,j}$ and $v^y_{i,j+1/2}$
  $t \leftarrow 0$
  while $t < dt$ **do**
    determine h by a CFL-condition
    $t \leftarrow t + h$
    update s_{ij} by (5.10)
  end while
end for

Experiment 1. Our first test is a so-called "quarter five spot", modeling injection of water into a oil filled homogeneous horizontal reservoir, with one injection well in the lower left corner, and one production well in the upper right hand corner. In this case

$$\rho_o = \rho_w = 0, \quad q_{ij} = \begin{cases} 1 & \text{if } i = j = 1, \\ -1 & \text{if } i = N \text{ and } j = M, \\ 0 & \text{otherwise}, \end{cases}$$
$$T = 0.7, \quad N = 25, \quad s_{ij} = 0, \quad \text{for all } i,\ j.$$

We set $a = b = 1$ and $M = N$.



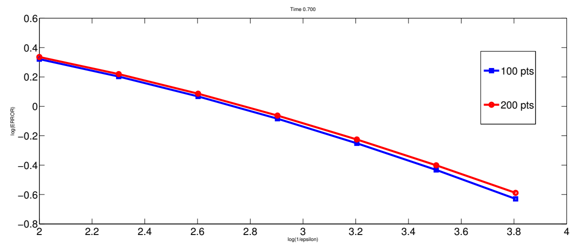
(a) $\mu = 0$ (b) $\mu = 0.01$

FIGURE 2. Solutions to the unregularized and regularized versions of the two-phase flow equations (2.1) without gravity. The plotted quantity is the saturation of water. Left: $\mu = 0$ and Right: $\mu = 0.01$ with μ being the regularization parameter in (2.1). Both solutions are computed with the IMPES scheme on a 200×200 mesh.

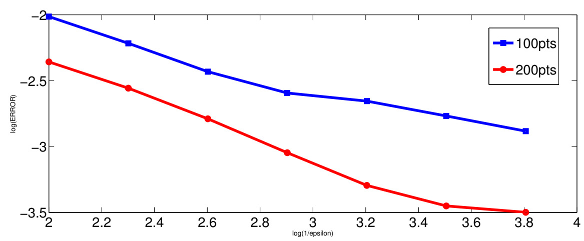
We compute the approximate solutions for the unregularized version of the two-phase flow equations (1.6) and the regularized version (2.1) by varying the regularization parameter $\mu_o = \mu_w = \mu$ in (2.1). We consider two different values of μ namely $\mu = 0$ (unregularized problem) and $\mu = 0.01$ and show the water saturation, computed on a 200×200 mesh at time $T = 0.7$ in figure 2. The figure shows that the saturation consists of sharp fronts and including a finger at the upper right hand corner near the production well. Furthermore, there are very few visible differences between the saturations for the unregularized and regularized problems.

In order to obtain quantitative comparison of the two models, we vary the regularization parameter μ by orders of magnitude and compute the L^1 norm of the difference between the regularized and unregularized saturations and pressure. The relative errors are shown in figure 3. The figure shows that the errors in pressure are very small and converge to zero as the regularization parameter goes to zero. Furthermore, the pressure errors are lower on the finer 200×200 mesh than the 100×100 mesh. The saturation errors are larger than the pressure errors. The saturation errors do converge to zero for both the coarse and the fine meshes as the regularization parameter tends to zero. However, the saturation errors are slightly larger on the finer mesh than on the coarser mesh due to interaction between the discretization error and the regularization error. The figure illustrates that the regularized model (2.1) is quite close to the original two-phase flow equations (1.7) with the difference being very small when the regularization parameter is close to zero.

Experiment 2. The second test case is intended to test the possible effects of gravitation. The setup is supposed to model injection of water into a vertical reservoir initially filled with oil. The injection point is in the top left corner and the production is in the top right one. Since water is heavier than oil, water will tend to sink to the bottom of the reservoir, where it will pile up since we impose



(a) Saturation



(b) Pressure

FIGURE 3. The difference in L^1 between unregularized and regularized versions of the two-phase flow equations (2.1) without gravity. The plotted quantity is the log of $\frac{1}{\mu}$ vs. the log of the relative error. Left: Saturation error and Right: Pressure. We consider solutions computed with the IMPES scheme on a 100×100 and 200×200 mesh.

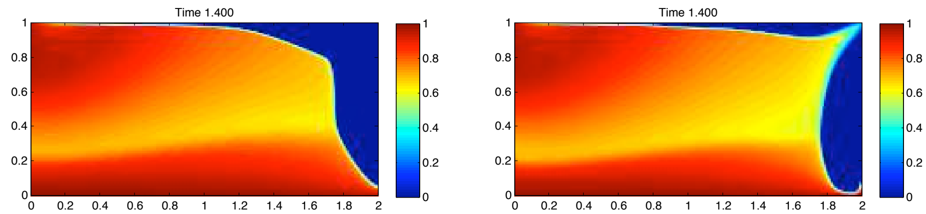
no-flow boundary conditions. The parameters for this set up is

$$\rho_o = 5.5, \quad \rho_w = 7, \quad q_{ij} = \begin{cases} 1 & \text{if } i = 1, j = M, \\ -1 & \text{if } i = N \text{ and } j = M, \\ 0 & \text{otherwise,} \end{cases}$$
$$T = 1.4, \quad N = 50, \quad s_{ij} = 0, \quad \text{for all } i,\ j.$$

We have used $a = 2$, $b = 1$, and $M = N/2$.

We compute the approximate solutions for the unregularized version of the two-phase flow equations (1.6) and the regularized version (2.1) by varying the regularization parameter μ in (2.1). We consider two different values of μ namely $\mu = 0$ (unregularized problem) and $\mu = 0.01$ and show the water saturation, computed on a 200×200 mesh at time $T = 0.7$ in figure 4. In contrast to the previous experiment, there are qualitative differences between the unregularized and regularized versions. These differences are visible near the production well. The regularized version seems to include a finger whereas the unregularized version is yet to form a finger near the production well. There is a roll-up at the bottom right corner in the regularized version that is not visible in the unregularized version.

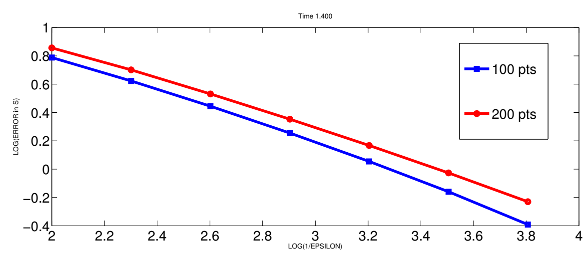
A quantitative comparison in terms of the L^1 norm of the differences between the unregularized and regularized versions is shown in figure 5. The differences in pressure are very small and tend to zero as μ is reduced. As expected from the saturation plots, the difference in saturation is greater than in the non-gravitational case (compare with figure 3). However, the errors tend to zero as $\mu \to 0$ showing



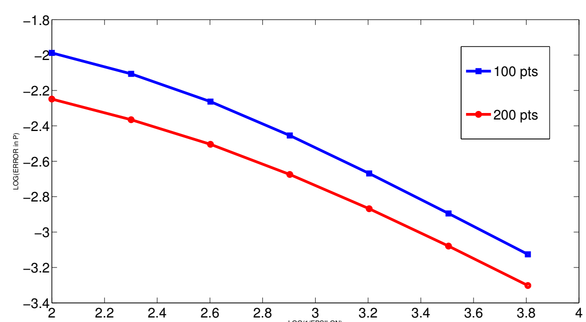
(a) $\mu = 0$ (b) $\mu = 0.01$

FIGURE 4. Solutions to the unregularized and regularized versions of the two-phase flow equations (2.1) with gravity. The plotted quantity is the saturation of water. Left: $\mu = 0$ and Right: $\mu = 0.01$ with μ being the regularization parameter in (2.1). Both solutions are computed with the IMPES scheme on a 200×200 mesh.

that the regularized model leads to a solution very close to the two-phase flow equations (1.7), even when gravity effects are included.



(a) Saturation



(b) Pressure

FIGURE 5. The difference in L^1 between unregularized and regularized versions of the two-phase flow equations (2.1) with gravity. The plotted quantity is the log of $\frac{1}{\mu}$ vs. the log of the relative error. Left: Saturation error and Right: Pressure. We consider solutions computed with the IMPES scheme on a 100×100 and 200×200 mesh.

6. Conclusion

We consider the flow of two-phases, say oil and water, in a porous medium. The classical model of this flow involves a elliptic-hyperbolic system, based on the Darcy's law. The saturation is governed by a hyperbolic conservation law and pressure obeys an elliptic equation. The problem of existence of global weak solutions for this model is still open. The main difficulty being the lack of regularity of the velocity field.

We propose a modified version of the Darcy's law via a Helmholtz regularization of the phase velocities. The resulting model is a hyperbolic-elliptic system with more regular velocity field. The kinetic formulation of scalar conservation laws is modified to show compactness of approximating solutions. The limit is shown to be weak solution of the modified system.

We perform numerical experiments and show that the solutions resulting from the modified version of the Darcy's law are very close to those obtained from a classical two-phase flow system. Furthermore, these approximations converge to the corresponding classical two-phase flow solution as the regularization parameter tends to zero. Thus, the numerical experiments provide an *a posteriori* justification for the proposed model.

Since the problem of proving existence for the classical version of the two-phase flow equations presents formidable difficulties, we propose that this modified version of the Darcy's law be used. It can be motivated by scaling arguments and we are able to provide rigorous proof of existence. The main advantage of the proposed model over non-zero capillary pressure models lies in the fact that the saturation fronts are not diffused by this model. We plan to conduct more intensive numerical study of the proposed model and compare it with the classical model in a future paper.

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