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ANALYSIS OF A CARTESIAN PML APPROXIMATION TO THE THREE DIMENSIONAL ELECTROMAGNETIC WAVE SCATTERING PROBLEM

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Abstract. We consider the application of a perfectly matched layer (PML) technique applied in Cartesian geometry to approximate solutions of the electromagnetic wave (Maxwell) scattering problem in the frequency domain. The PML is viewed as a complex coordinate shift ("stretching") and leads to a variable complex coefficient equation for the electric field posed on an infinite domain, the complement of a bounded scatterer. The use of Cartesian geometry leads to a PML operator with simple coefficients, although, still complex symmetric (non-Hermitian). The PML reformulation results in a problem which preserves the original solution inside the PML layer while decaying exponentially outside. The rapid decay of the PML solution suggests truncation to a bounded domain with a convenient outer boundary condition and subsequent finite element approximation (for the truncated problem).

For suitably defined Cartesian PML layers, we prove existence and uniqueness of the solutions to the infinite domain and truncated domain PML equations provided that the truncated domain is sufficiently large. We show that the PML reformulation preserves the solution in the layer while decaying exponentially outside of the layer. Our approach is to develop variational stability for the infinite domain electromagnetic wave scattering PML problem from that for the acoustic wave (Helmholtz) scattering PML problem given in [12]. The stability and exponential convergence of the truncated PML problem is then proved using the decay properties of solutions of the infinite domain problem. Although, we do not provide a complete analysis of the resulting finite element approximation, we believe that such an analysis should be possible using the techniques in [6].

Key words. electromagnetic wave scattering problem, Maxwell scattering, Helmholtz equation, PML layer

1. Introduction

In this paper, we consider the application of PML techniques for approximating the solutions of frequency domain Maxwell scattering problems. These problems are posed on unbounded domains with a far field boundary condition given by the Silver-Müller condition. The PML technique which we shall study is one based on Cartesian geometry where each variable is transformed independently.

In an earlier paper Bérenger [2] introduced a PML method for Maxwell's equations in the time domain. This approach was based on constructing a ficticious absorbing layer designed so that plane waves passed into the layer without reflection. The technique involved the introduction of additional variables and equations in the "ficticious material" region. For more analysis on PML applied to time domain problems see [1, 3, 11] and the included references. PML type techniques were also developed in terms of a formal complex change of variable (or stretching) [10, 18]. This approach was especially well suited for frequency domain problems and led to simpler PML formulations more amendable to analysis. Perhaps the simplest and most widely used of the PML variants for frequency domain problems is one which involves a complex change of the Cartesian coordinates.

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A well designed PML reformulation for a scattering problem has the following properties. First, the PML reformulation and the original problem should have the same solution in the "region of interest", i.e., near the scatterer. Second, the solution of the PML reformulation should decay rapidly (usually exponentially) so that it is feasible to truncate the problem to a bounded computational domain with a convenient outer boundary condition. Third, the variational problem on the truncated domain should be stable and thus amendable to finite element approximation. In this paper, we shall show that the Cartesian PML reformulation of the Maxwell scattering problem satisfies all of these properties.

There has been recent work on the stability of PML equations. For polar or spherical PML, [7, 5] have showed stability of the truncated PML approximations to acoustic, electromagnetic and elastic wave scattering problems for stretching functions $\sigma(r) \in C^2(0,\infty)$ which were constant for $r \geq r_1$ (provided that the size of the computational domain is sufficient large). A key ingredient in these analyses is that the coefficients become constant outside of ball of radius r_1 and hence one can apply compact perturbation techniques. Using the particular form of the acoustic scattering two dimensional polar PML equations, Chen and Liu [8] were able to show stability for a stretching of the form

$$\sigma(r) = \sigma_0 \left(\frac{r - r_0}{\rho - r_0}\right)^m$$

for sufficiently large σ_0. The question of stability of the PML equations in the Cartesian case is a much more invoved matter as compact perturbation arguments do not apply. Recently, Cartesian PML approximations to acoustic scattering problems were successfully analyzed in [12] and [4] for PML functions $\sigma(x)$ which are constant for large x. The first paper, [12], shows that the truncated PML equations are stable if the computational domain is sufficiently large. The second [4] is more general and further proves stability provided that the product of the domain size and σ_0 is sufficiently large. We also mention that stability through the PML layer was provided for two dimensional acoustic PML problems with piecewise constant σ by Chen and Zheng [9].

The goal of this paper is provide stability estimates for the truncated Cartesian PML formulations of the Maxwell scattering problem. To do this, we shall show stability on a sequence of domains, one leading to the next. Specifically, we denote the domain of the (bounded) scatterer by $\Omega \subset \mathbb{R}^3$ and the interior of its complement by Ω^c. We show that:

- PML Cartesian stability for the acoustic probem implies similar stability for the Maxwell problem on all of $\mathbb{R}^3$.
- PML Cartesian stability for the Maxwell problem on $\mathbb{R}^3$ implies similar stability on Ω^c.
- PML Cartesian stability for the Maxwell problem on Ω^c implies similar stability on the computational domain $(-M,M)^3 \setminus \bar{\Omega}$ for sufficiently large M.

The outline of the remainder of the paper is as follows. In Section 2, we formulate the Maxwell scattering problem. The PML operators are defined in Section 3. Sections 4 and 5 prove variational stability of the PML problem in $\mathbb{R}^3$ and Ω^c, respectively. Section 6 shows variational stability on the truncated domain and exponential convergence of the coresponding solution to the solution of the original problem on the region of interest. For simplicity, we only consider the case where the domain (not σ_0) becomes large. Finally, in Section 7, we give the result of computations which illustrate the early theory.

2. Formulation of the Maxwell scattering problem.

Throughout this paper, we shall have to deal with complex valued functions on various Sobolev spaces. For a domain $D \subseteq \mathbb{R}^3$, let $L^2(D)$ be the space of complex valued functions whose absolute values are square integrable on D and let $\boldsymbol{L}^2(D) = (L^2(D))^3$ be the space of vector valued functions whose components are in $L^2(D)$. We shall use $(\cdot,\cdot)_D$ to denote the (vector or scalar) $L^2(D)$-inner product. The scalar and vector Sobolev spaces on D will be denoted $H^s(D)$ and $\boldsymbol{H}^s(D)$ respectively. Sobolev spaces with vanishing boundary conditions are denoted $H^s_0(D)$ or $\boldsymbol{H}^s_0(D)$ and can be characterized as the completion of $C^\infty_0(D)$ or $\boldsymbol{C}^\infty_0(D)$, respectively, under the corresponding Sobolev norms. We shall use bold symbols to denote vector valued functions and operators. When the inner product is on $\mathbb{R}^3$, we will use the simpler notation $(\cdot,\cdot)$.

As we shall later see, it is more natural to set up PML problems with forms that are bilinear (not sesquilinear) even though we are dealing with complex function spaces. For a complex Hilbert space H, we shall denote H^* to be the set of bounded linear functionals on H.

For a domain D, we denote $\boldsymbol{H}(\mathbf{curl}; D)$ to be the functions in $\boldsymbol{L}^2(D)$ whose curls are also in $\boldsymbol{L}^2(D)$. The subset of $\boldsymbol{H}(\mathbf{curl}; D)$ with vanishing tangential trace on ∂D will be denoted $\boldsymbol{H}_0(\mathbf{curl}; D)$. Finally, $\boldsymbol{H}_{loc}(\mathbf{curl}; \overline{\Omega^c})$ denotes functions on Ω^c whose restrictions to $\Omega^c \cap D$ are in $\boldsymbol{H}(\mathbf{curl}; \Omega^c \cap D)$ for any bounded domain D.

In this section, we formulate the Maxwell scattering problem and its far field boundary condition. Let Ω be a bounded domain, with boundary Γ, containing the origin and let Ω^c denote the complement of its closure. We seek a vector valued function $\boldsymbol{u} \in \boldsymbol{H}_{loc}(\mathbf{curl}; \overline{\Omega^c})$ satisfying

$$\boldsymbol{\nabla}\times \mu^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{u} - k^2 \varepsilon \boldsymbol{u} = \mathbf{0} \text{ in } \Omega^c \tag{2.1}$$

and

$$\boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \text{ on } \Gamma. \tag{2.2}$$

Here μ is the magnetic permeability, ε is the electric permittivity, k is positive real number and $\boldsymbol{g}$ is the trace on Γ of a function $\widetilde{\boldsymbol{g}} \in \boldsymbol{H}_{loc}(\mathbf{curl}; \overline{\Omega^c})$. We assume that $\mu = \varepsilon = 1$ outside of a sphere B_{R_0} of radius R_0 centered at the origin.

The formulation is completed by imposing the Silver-Müller radiation condition, i.e.,

$$\lim_{r\to\infty} r(\boldsymbol{\nabla}\times \boldsymbol{u} \times \boldsymbol{n} - ik\boldsymbol{u}) = \mathbf{0} \tag{2.3}$$

(uniformly on the ball of radius r).

We allow piecewise smooth μ and ε inside the sphere satisfying the conditions of Section 10 of [14] (see also, [17]) so that the above problem has a unique solution which is in $\boldsymbol{H}(\mathbf{curl}; D)$ for any bounded domain $D \subset \Omega^c$.

Because the PML equations are only applied in the region where $\mu = \epsilon = 1$, we shall also study the simpler problem with constant coefficients,

$$\boldsymbol{\nabla}\times \boldsymbol{\nabla}\times \boldsymbol{u} - k^2 \boldsymbol{u} = \mathbf{0} \text{ in } \Omega^c. \tag{2.4}$$

The solution of (2.4) is locally smooth and satisfies

$$\int_{\Omega^c} (\boldsymbol{\nabla}\times\boldsymbol{u} \cdot \boldsymbol{\nabla}\times\boldsymbol{\theta} - k^2 \boldsymbol{u}\cdot\boldsymbol{\theta})\, dx = 0 \quad \text{for all } \boldsymbol{\theta} \in \boldsymbol{C}^\infty_0(\Omega^c). \tag{2.5}$$

Note that (2.5) does not give rise to a stable variational formulation nor does it incorporate the far field boundary condition.

3. PML operators

We shall denote the cube-shaped domain $\Omega_\alpha = (-\alpha, \alpha)^3$ and Γ_α to be its boundary. When α is large enough so that $\bar{\Omega} \subset \Omega_\alpha$ we further define $\Omega'_\alpha = \Omega_\alpha \setminus \bar{\Omega}$. Without loss of generality, we can assume that $\widetilde{\boldsymbol{g}}$ is supported in $B_{R_0} \setminus \bar{\Omega}$.

The differential operators involved in our PML approximations will be defined in terms of a formal complex change of variables (or stretching). This stretching is defined by means of an even function $\sigma \in C^0(\mathbb{R})$ satisfying

$$\begin{aligned} &\sigma(x) = 0 \quad \text{for } |x| \le a, \\ &\sigma(x): \text{ increasing} \quad \text{for } a < x < b, \\ &\sigma(x) = \sigma_0 \quad \text{for } |x| \ge b. \end{aligned} \tag{3.1}$$

Here $\sigma_0 > 0$ is a parameter (the PML strength) and $R_0 < a < b$. We further define

$$\widetilde{\sigma}(r) = \frac{1}{r} \int_0^r \sigma(s) ds. \tag{3.2}$$

Note that $\sigma(r) = (r\widetilde{\sigma}(r))'$ and σ_0 is the maximum value of σ. A simple example of such a function σ is a piecewise linear function.

For a complex number z, the corresponding "complex stretching" is of the form

$$T\boldsymbol{x} = ((1 + z\widetilde{\sigma}(x_1))x_1, (1 + z\widetilde{\sigma}(x_2))x_2, (1 + z\widetilde{\sigma}(x_3))x_3), \qquad \boldsymbol{x} \in \mathbb{R}^3. \tag{3.3}$$

For PML stretching, we shall take $z = i$ or $z = 1 + i$.

Let $\widetilde{d}(r) = 1 + z\widetilde{\sigma}(r)$ and $d(r) = 1 + z\sigma(r)$. The formal change of variables is $\widehat{w}(T^{-1}x) = w(x)$. Note that DT is the 3x3 diagonal matrix,

$$DT = \operatorname{diag}(d(x_1), d(x_2), d(x_3))$$

and we set $J = J(\boldsymbol{x}) = Det(DT) = d(x_1)d(x_2)d(x_3)$. Following Monk [14], we introduce the 3×3 matrices;

$$\boldsymbol{A} = \operatorname{diag}\left(\frac{1}{d(x_2)d(x_3)}, \frac{1}{d(x_1)d(x_3)}, \frac{1}{d(x_1)d(x_2)}\right),$$

and

$$\boldsymbol{B} = DT = \operatorname{diag}(d(x_1), d(x_2), d(x_3)),$$

and consider the following "stretched operators":

$$\begin{aligned} \widetilde{\boldsymbol{\nabla}} w &= \boldsymbol{B}^{-1} \boldsymbol{\nabla}\, w, \\ \widetilde{\Delta} &= \frac{1}{J} \nabla \cdot (\boldsymbol{A}^{-1} \boldsymbol{B}^{-1} \boldsymbol{\nabla}\,), \\ \widetilde{\boldsymbol{\Delta}} \boldsymbol{w} &= \sum_{j=1}^{3} (\widetilde{\Delta} \boldsymbol{w}_j) \mathbf{e}_j, \\ \widetilde{\boldsymbol{\nabla}} \times \boldsymbol{F} &= \boldsymbol{A}\, \boldsymbol{\nabla} \times (\boldsymbol{B}\boldsymbol{F}), \\ \widetilde{\nabla} \cdot \boldsymbol{F} &= \frac{1}{J} \nabla \cdot (\boldsymbol{A}^{-1} \boldsymbol{F}). \end{aligned} \tag{3.4}$$

Here $\boldsymbol{w}_j$ denotes the j'th component of $\boldsymbol{w}$ and $\mathbf{e}_j$ denotes the unit vector in the j'th direction. The natural domain for the $\widetilde{\boldsymbol{\nabla}}\times$-operator is

$$\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3) \equiv \{\boldsymbol{\theta} \ : \ \boldsymbol{B}\boldsymbol{\theta} \in \boldsymbol{H}(\mathbf{curl}; \mathbb{R}^3)\}.$$

This is a Hilbert space with scalar product

$$(\boldsymbol{U},\boldsymbol{V})_{\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)} = (\boldsymbol{U},\boldsymbol{V}) + (\boldsymbol{\nabla}\times(\boldsymbol{B}\boldsymbol{U}), \boldsymbol{\nabla}\times(\boldsymbol{B}\boldsymbol{V})).$$

We also define $\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)$ and $\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c)$ in the obvious way.

In order to have integration by parts with these operators appear as normal as possible, we introduce the **bilinear** forms

$$[f,g] = \int_{\mathbb{R}^3} Jfg\,dx \qquad \text{and} \qquad [\boldsymbol{f},\boldsymbol{g}] = \int_{\mathbb{R}^3} J\boldsymbol{f}\cdot\boldsymbol{g}\,dx \tag{3.5}$$

which we shall use even for complex functions. With this notation, the following integration by parts identities hold:

$$\begin{aligned}
[\widetilde{\boldsymbol{\nabla}}u,\boldsymbol{\phi}] &= -[u,\widetilde{\nabla}\cdot\boldsymbol{\phi}] \quad \text{for all } u\in H^1(\mathbb{R}^3), \boldsymbol{\phi}\in \boldsymbol{C}_0^\infty(\mathbb{R}^3) \text{ and}\\
[\widetilde{\boldsymbol{\nabla}}\times\boldsymbol{u},\boldsymbol{\phi}] &= [\boldsymbol{u},\widetilde{\boldsymbol{\nabla}}\times\boldsymbol{\phi}] \quad \text{for all } \boldsymbol{u}\in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3), \boldsymbol{\phi}\in \boldsymbol{C}_0^\infty(\mathbb{R}^3).
\end{aligned}$$

These formulas extend by density to $\boldsymbol{\phi}\in\boldsymbol{H}^1(\mathbb{R}^3)$ and $\boldsymbol{\phi}\in\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)$, respectively. Moreover, the stretched differentiation operators are well defined up to second order and commute when applied to suitably smooth functions.

The form in (2.5) formally transforms into

$$\widetilde{A}(\boldsymbol{v},\boldsymbol{w}) = [\widetilde{\boldsymbol{\nabla}}\times\boldsymbol{v},\widetilde{\boldsymbol{\nabla}}\times\boldsymbol{w}]_{\Omega^c} - k^2[\boldsymbol{v},\boldsymbol{w}]_{\Omega^c}. \tag{3.6}$$

Here $[\cdot,\cdot]_{\Omega^c}$ denotes the bilinear form analogous to $[\cdot,\cdot]$ but with integration only on Ω^c.

The analogous PML form coming from the variable coefficient problem (2.1) is given by

$$\widetilde{A}_v(\boldsymbol{v},\boldsymbol{w}) = [\mu^{-1}\widetilde{\boldsymbol{\nabla}}\times\boldsymbol{v},\widetilde{\boldsymbol{\nabla}}\times\boldsymbol{w}]_{\Omega^c} - k^2[\varepsilon\boldsymbol{v},\boldsymbol{w}]_{\Omega^c}. \tag{3.7}$$

A goal of this paper is to prove variational stability for problems involving $\widetilde{A}_v$ on both Ω^c and the truncated computational domain Ω'_M (to be defined more precisely later).

Examining the form of the stretched operators, one finds that $\widetilde{\boldsymbol{\nabla}}\times\widetilde{\boldsymbol{\nabla}}\times$ and $\widetilde{\boldsymbol{\nabla}}\widetilde{\nabla}\cdot$ involve at most one derivative of d and hence they map $\boldsymbol{H}^2_{loc}(\mathbb{R}^3)$ into $\boldsymbol{L}^2_{loc}(\mathbb{R}^3)$. Similar results hold for $\widetilde{\Delta}$ and $\widetilde{\boldsymbol{\Delta}}$. A simple computation shows that

$$\widetilde{\boldsymbol{\Delta}}\boldsymbol{w} = -\widetilde{\boldsymbol{\nabla}}\times\widetilde{\boldsymbol{\nabla}}\times\boldsymbol{w} + \widetilde{\boldsymbol{\nabla}}\widetilde{\nabla}\cdot\boldsymbol{w} \tag{3.8}$$

for $\boldsymbol{w}\in\boldsymbol{H}^2_{loc}(\mathbb{R}^3)$.

Because of the form of $\boldsymbol{B}$, multiplication by $\boldsymbol{B}$ is a bicontinuous map of $\boldsymbol{H}^j(\mathbb{R}^3)$ onto itself, for $j=0,1$. Similarly, multiplication by J is a bicontinuous map of $H^j(\mathbb{R}^3)$ onto itself, for $j=0,1$.

Remark 3.1. To keep the notation from becoming too cumbersome, we have used the same scaling function σ in each direction. For many problems, it may be more natural to use brick like regions allowing different values of the stretching parameters (a_i and b_i, $i=1,2,3$). In this case, the functions $d(x_i)$ are replaced by $d_i(x_i)$. All of the above identities hold in this case as well. Moreover, the analysis in the rest of this paper goes over without essential change.

4. Stability of the constant coefficient PML source problem on $\mathbb{R}^3$.

In this section, we consider the PML source problem on $\mathbb{R}^3$ corresponding to $\mu = \varepsilon = 1$ everywhere. Specifically, given a continuous linear functional $\boldsymbol{F}$ in $\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)^*$, we seek a solution $\widetilde{\boldsymbol{U}} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)$ satisfying

$$\widetilde{A}(\widetilde{\boldsymbol{U}}, \boldsymbol{\theta}) = \langle \boldsymbol{F}, \boldsymbol{\theta}\rangle \quad \text{for all } \boldsymbol{\theta} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3). \tag{4.1}$$

Before investigating the stability (4.1), we first mention stability results for the acoustic Cartesian PML problem described as follows. Let $W_1(\mathbb{R}^3)$ denote the weighted Sobolev space of functions defined on $\mathbb{R}^3$ given by

$$W_1(\mathbb{R}^3) = \{u : \ u(1+r^2)^{-1/2} \in L^2(\mathbb{R}^3) \text{ and } \boldsymbol{\nabla} u \in \boldsymbol{L}^2(\mathbb{R}^3)\}.$$

It follows from Theorem 2.5.13 of [17] that $\|\boldsymbol{\nabla}\, \phi\|_{\boldsymbol{L}^2(\mathbb{R}^3)}$ provides an equivalent norm for $W_1(\mathbb{R}^3)$. Moreover, $C_0^\infty(\mathbb{R}^3)$ is dense in $W_1(\mathbb{R}^3)$.

Given $F \in (W_1(\mathbb{R}^3))^*$, we shall be interested in solutions $v \in W_1(\mathbb{R}^3)$ satisfying

$$[\widetilde{\boldsymbol{\nabla}} v, \widetilde{\boldsymbol{\nabla}} w] = F(w), \qquad \text{for all } w \in W_1(\mathbb{R}^3). \tag{4.2}$$

It is not hard to see that

$$|[\widetilde{\boldsymbol{\nabla}} w, \widetilde{\boldsymbol{\nabla}} w]| \ge c_0 \|\boldsymbol{\nabla}\, w\|^2_{\boldsymbol{L}^2(\mathbb{R}^3)}, \quad \text{for all } w \in W_1(\mathbb{R}^3),$$

provided that either

$$z = 1+i \qquad \text{or} \qquad z = i \text{ and } \arg(1+i\sigma_0) < \pi/3. \tag{4.3}$$

We also need stability of the following problem: For $F \in (H^1(\mathbb{R}^3))^*$, find $u \in H^1(\mathbb{R}^3)$ satisfying

$$[\widetilde{\boldsymbol{\nabla}} u, \widetilde{\boldsymbol{\nabla}} \phi] - k^2[u,\phi] = \langle F, \phi\rangle \quad \text{for all } \phi \in H^1(\mathbb{R}^3). \tag{4.4}$$

The analysis of this problem is quite involved and is essential for this paper. It was shown in [12] and [4] that for $z = i$ or $z = 1+i$, (4.4) has a unique solution satisfying

$$\|u\|_{H^1(\mathbb{R}^3)} \le C\|F\|_{H^{-1}(\mathbb{R}^3)}.$$

Remark 4.1. Because of the symmetry of the form $\widetilde{A}(\cdot,\cdot)$, analysis of (4.1) reduces to showing existence of an appropriately bounded solution. Specifically, we need only show that given a bounded linear functional $\boldsymbol{F}$, there exists a solution $\widetilde{\boldsymbol{U}} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)$ satisfying (4.1) with

$$\|\widetilde{\boldsymbol{U}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)} \le C\|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*}. \tag{4.5}$$

To see this, we argue as follows. Suppose $\widetilde{\boldsymbol{U}}$ is in $\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)$ and satisfies (4.1) with $\boldsymbol{F} = \mathbf{0}$. Let $\boldsymbol{V} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)$ be a solution of

$$\widetilde{A}(\boldsymbol{V}, \boldsymbol{\theta}) = (\boldsymbol{\theta}, \widetilde{\boldsymbol{U}})_{\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)} \quad \text{for all } \boldsymbol{\theta} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3).$$

Then

$$\|\widetilde{\boldsymbol{U}}\|^2_{\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)} = \widetilde{A}(\boldsymbol{V}, \widetilde{\boldsymbol{U}}) = 0 \tag{4.6}$$

and the uniqueness of solutions to (4.1) immediately follows. Now, for any $\widetilde{\boldsymbol{U}} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)$, $\widetilde{\boldsymbol{U}}$ is the unique solution of (4.1) with $\boldsymbol{F}(\cdot) = A(\widetilde{\boldsymbol{U}}, \cdot)$ and (4.5) can be rewritten

$$\|\widetilde{\boldsymbol{U}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)} \le C \sup_{\boldsymbol{V} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)} \frac{|\widetilde{A}(\widetilde{\boldsymbol{U}}, \boldsymbol{V})|}{\|\boldsymbol{V}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)}}.$$

We consider $\boldsymbol{L}^2(\mathbb{R}^3)$ as a subset of $(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*$ using the identification given by

$$\langle \boldsymbol{w},\boldsymbol{\theta}\rangle = [\boldsymbol{w},\boldsymbol{\theta}], \quad \text{for all } \boldsymbol{\theta}\in \widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3). \tag{4.7}$$

The analysis of (4.1) is contained in the following theorem and its corollary. Some of the ideas in this proof come from the proof of Theorem 4.2 of [7].

Theorem 4.1. *Assume that* (4.3) *holds. Suppose that* $\boldsymbol{F}$ *is in* $(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*$ *and satisfies*

$$\langle \boldsymbol{F}, \widetilde{\boldsymbol{\nabla}}\theta\rangle = 0 \quad \textit{for all } \theta\in W_1(\mathbb{R}^3). \tag{4.8}$$

Then (4.1) *has a solution* $\widetilde{\boldsymbol{U}}$ *which is in* $\boldsymbol{H}^1(\mathbb{R}^3)$*. In addition, if* $\boldsymbol{F}$ *is in* $\boldsymbol{L}^2(\mathbb{R}^3)$ *(as in* (4.7)*) then* $\widetilde{\boldsymbol{U}}$ *is in* $\boldsymbol{H}^2(\mathbb{R}^3)$*.*

To prove the theorem, we shall need the following lemma whose proof is given at the end of this section:

Lemma 4.1. *Suppose that* $\boldsymbol{F}$ *is in* $(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*$ *and satisfies* (4.8) *and that* (4.3) *holds. Then there is a sequence of functions,* $\boldsymbol{F}_n\in \boldsymbol{L}^2(\mathbb{R}^3)$*,* $n=1,2,\dots$ *satisfying* (4.8) *and converging to* $\boldsymbol{F}$ *in* $(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*$*.*

Proof of Theorem 4.1. Suppose that $\boldsymbol{F}$ is in $\boldsymbol{L}^2(\mathbb{R}^3)$. We consider the following problem: find $\widetilde{\boldsymbol{V}}\in \boldsymbol{H}^1(\mathbb{R}^3)$ satisfying

$$\widehat{D}(\widetilde{\boldsymbol{V}},\boldsymbol{\theta}) - k^2[\widetilde{\boldsymbol{V}},\boldsymbol{\theta}] = \langle \boldsymbol{F},\boldsymbol{\theta}\rangle \quad \text{for all } \boldsymbol{\theta}\in \boldsymbol{H}^1(\mathbb{R}^3) \tag{4.9}$$

where

$$\widehat{D}(\boldsymbol{V},\boldsymbol{W}) \equiv \sum_{i=1}^{3}[\widetilde{\boldsymbol{\nabla}}\boldsymbol{V}_i, \widetilde{\boldsymbol{\nabla}}\boldsymbol{W}_i].$$

This problem is just three copies of (4.4) and is thus stable. Its unique solution $\widetilde{\boldsymbol{V}}$ satisfies

$$\|\widetilde{\boldsymbol{V}}\|_{\boldsymbol{H}^1(\mathbb{R}^3)} \le C\|\boldsymbol{F}\|_{(\boldsymbol{H}^1(\mathbb{R}^3))^*} \le C\|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*}. \tag{4.10}$$

We shall show that $\widetilde{\boldsymbol{V}}$ is, in fact, a solution to (4.1) provided that (4.8) holds. First, as $\boldsymbol{F}\in \boldsymbol{L}^2(\mathbb{R}^3)$, $\widetilde{\boldsymbol{V}}$ is in $\boldsymbol{H}^2(\mathbb{R}^3)$. Indeed, the argument given in Theorem 4.1 of [5] works for Cartesian PML also.

For $\boldsymbol{\theta}\in \boldsymbol{C}_0^\infty(\mathbb{R}^3)$, integration by parts (applied separately over regions where σ is smooth) and (3.8) gives

$$\begin{aligned}\widehat{D}(\widetilde{\boldsymbol{V}},\boldsymbol{\theta}) - k^2[\widetilde{\boldsymbol{V}},\boldsymbol{\theta}] &= [\widetilde{\nabla}\cdot\widetilde{\boldsymbol{V}}, \widetilde{\nabla}\cdot\boldsymbol{\theta}]\\ &\quad + [\widetilde{\boldsymbol{\nabla}}\times\widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{\nabla}}\times\boldsymbol{\theta}] - k^2[\widetilde{\boldsymbol{V}},\boldsymbol{\theta}]\\ &= \langle \boldsymbol{F},\boldsymbol{\theta}\rangle.\end{aligned} \tag{4.11}$$

Indeed, the smoothness of σ and an examination of (3.4) shows that all of the boundary contributions between regions cancel. As $\boldsymbol{C}_0^\infty(\mathbb{R}^3)$ is dense in $\boldsymbol{H}^1(\mathbb{R}^3)$, we conclude that (4.11) holds for all $\boldsymbol{\theta}\in \boldsymbol{H}^1(\mathbb{R}^3)$.

Let θ be in $C_0^\infty(\mathbb{R}^3)$. Setting $\boldsymbol{\theta}=\widetilde{\boldsymbol{\nabla}}\theta$ in (4.11), as above, we find that

$$\begin{aligned}0 &= [\widetilde{\nabla}\cdot\widetilde{\boldsymbol{V}}, \widetilde{\Delta}\theta] - k^2[\widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{\nabla}}\theta]\\ &= -[\widetilde{\boldsymbol{\nabla}}\widetilde{\nabla}\cdot\widetilde{\boldsymbol{V}}, \widetilde{\boldsymbol{\nabla}}\theta] + k^2[\widetilde{\nabla}\cdot\widetilde{\boldsymbol{V}},\theta]\end{aligned}$$

Up to a sign, the form on the right hand side is the same as that in (4.4) (applied to $u=\widetilde{\nabla}\cdot\widetilde{\boldsymbol{V}}$). The stability of (4.4) and the density of $C_0^\infty(\mathbb{R}^3)$ in $H^1(\mathbb{R}^3)$ imply that $\widetilde{\nabla}\cdot\widetilde{\boldsymbol{V}}=0$.

Putting the above together shows that $\widetilde{\boldsymbol{V}}$ satisfies

$$[\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{V}}, \boldsymbol{\nabla}\times\boldsymbol{\theta}] - k^2[\widetilde{\boldsymbol{V}}, \boldsymbol{\theta}] = \langle \boldsymbol{F}, \boldsymbol{\theta}\rangle \quad \text{for all } \boldsymbol{\theta} \in \boldsymbol{H}^1(\mathbb{R}^3).$$

This implies that $\widetilde{\boldsymbol{V}}$ satisfies (4.1) as $\boldsymbol{H}^1(\mathbb{R}^3)$ is dense in $\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3)$.

Now, if $\boldsymbol{F} \in \boldsymbol{L}^2(\mathbb{R}^3)$ the above construction produces the desired function $\boldsymbol{V}$ (which is in $\boldsymbol{H}^2(\mathbb{R}^3)$ as already mentioned). For general $\boldsymbol{F} \in (\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3))^*$ satisfying (4.8), we proceed by density. In this case, let $\boldsymbol{F}_n$, $n = 1, 2, \ldots$ be the sequence of functions of Lemma 4.1. For each n, set $\boldsymbol{V}_n$ to be the function constructed above with $\boldsymbol{F}$ replaced by $\boldsymbol{F}_n$. Then $\boldsymbol{V}_n$ converges in $\boldsymbol{H}^1(\mathbb{R}^3)$ to a function $\boldsymbol{V} \in \boldsymbol{H}^1(\mathbb{R}^3)$ satisfying (4.1). □

Corollary 4.1. *Assume that* (4.3) *holds. For* $\boldsymbol{F} \in (\boldsymbol{H}(\mathbf{curl}; \mathbb{R}^3))^*$*, there exists a unique solution* $\widetilde{\boldsymbol{U}}$ *of* (4.1) *satisfying*

$$\|\widetilde{\boldsymbol{U}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)} \le C\|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*}. \tag{4.12}$$

Proof. Given $\boldsymbol{F} \in (\boldsymbol{H}(\mathbf{curl}; \mathbb{R}^3))^*$, we consider $\theta \in W_1(\mathbb{R}^3)$ satisfying

$$[\widetilde{\boldsymbol{\nabla}}\theta, \widetilde{\boldsymbol{\nabla}}\psi] = \langle \boldsymbol{F}, \widetilde{\boldsymbol{\nabla}}\psi\rangle \quad \text{for all } \psi \in W_1(\mathbb{R}^3). \tag{4.13}$$

This problem is just (4.2) and is uniquely solvable since (4.3) holds. Its solution satisfies

$$\|\widetilde{\boldsymbol{\nabla}}\theta\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)} \le C\|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*}. \tag{4.14}$$

The modified functional

$$\langle \widetilde{\boldsymbol{F}}, \boldsymbol{\phi}\rangle = \langle \boldsymbol{F}, \boldsymbol{\phi}\rangle - [\widetilde{\boldsymbol{\nabla}}\theta, \boldsymbol{\phi}]$$

satisfies (4.8) and we denote by $\widetilde{\boldsymbol{V}}$ the solution of (4.1) constructed in the theorem with right hand side $\widetilde{\boldsymbol{F}}$. We observe that $\widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{V}} - k^{-2}\widetilde{\boldsymbol{\nabla}}\theta$ is in $\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3)$ and is a solution to (4.1). That $\widetilde{\boldsymbol{U}}$ satisfies (4.12) follows from the triangle inequality, (4.10), and (4.14). The remainder of the proof follows from Remark 4.1. □

Proof of Lemma 4.1. As $\boldsymbol{H}^1(\mathbb{R}^3)$ is dense in $\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3)$, it is also dense in $(\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3))^*$. Thus, for $\boldsymbol{F}$ in $(\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3))^*$, there is a sequence $\boldsymbol{G}_n$ in $\boldsymbol{L}^2(\mathbb{R}^3)$ converging to $\boldsymbol{F}$ in $(\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3))^*$. Let $\theta_n \in W_1(\mathbb{R}^3)$ be the solution to

$$[\widetilde{\boldsymbol{\nabla}}\theta_n, \widetilde{\boldsymbol{\nabla}}\phi] = \langle \boldsymbol{G}_n, \widetilde{\boldsymbol{\nabla}}\phi\rangle \quad \text{for all } \phi \in W_1(\mathbb{R}^3).$$

Now if $\boldsymbol{F}$ satisfies (4.8) then $\boldsymbol{G}_n$ above can be replaced by $\boldsymbol{G}_n - \boldsymbol{F}$ and hence

$$\|\widetilde{\boldsymbol{\nabla}}\theta_n\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \le C\|\boldsymbol{G}_n - \boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3))^*}.$$

As the right hand side above goes to zero, the functional $\boldsymbol{F}_n$ given by

$$\langle \boldsymbol{F}_n, \boldsymbol{\phi}\rangle \equiv \langle \boldsymbol{G}_n, \boldsymbol{\phi}\rangle - [\widetilde{\boldsymbol{\nabla}}\theta_n, \boldsymbol{\phi}]$$

converges to $\boldsymbol{F}$ in $(\widehat{\boldsymbol{H}}(\mathbf{curl}; \mathbb{R}^3))^*$ and satisfies (4.8) for each n. □

5. Stability of the PML approximation on Ω^c.

In this section, we investigate the stability of the PML approximation to (2.1)–(2.3) on Ω^c. We start with the PML approximation to (2.2)–(2.4).

Theorem 5.1. *Assume that* (4.3) *holds. Then there is a positive constant* C *satisfying*

$$\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)} \le C \sup_{\boldsymbol{\phi}\in\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c)} \frac{|\widetilde{A}(\boldsymbol{u}, \boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)}} \quad \textit{for all } \boldsymbol{u} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c).$$

Proof. The proof of this result involves the construction of an appropriate solution as in Remark 4.1. Specifically, given $\boldsymbol{F} \in (\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c))^*$, we shall construct a solution $\boldsymbol{u} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c)$ of

$$\widetilde{A}(\boldsymbol{u},\boldsymbol{\phi}) = \langle \boldsymbol{F},\boldsymbol{\phi}\rangle, \quad \text{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c) \tag{5.1}$$

satisfying

$$\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)} \leq C\|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c))^*} \tag{5.2}$$

with C independent of $\boldsymbol{F}$. By the Hahn-Banach Theorem, we can extend $\boldsymbol{F}$ to a bounded linear functional $\widehat{\boldsymbol{F}}$ on $\widehat{\boldsymbol{H}}(\mathbf{curl};\mathbb{R}^3)$ without increasing its norm. Let $\widehat{\boldsymbol{u}}$ denote the corresponding solution of (4.1) (with $\boldsymbol{F}$ replaced by $\widehat{\boldsymbol{F}}$). Of course, $\widehat{\boldsymbol{u}}$ satisfies (5.1) but, in general, fails to have vanishing tangential components on Γ. We are thus led to the construction of a function $\widetilde{\boldsymbol{w}} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)$ satisfying the homogeneous equation with $\widetilde{\boldsymbol{w}} \times \boldsymbol{n} = \widehat{\boldsymbol{u}} \times \boldsymbol{n}$ on Γ.

Let $\boldsymbol{w} \in \boldsymbol{H}_{loc}(\mathbf{curl};\mathbb{R}^3)$ denote the solution of the Maxwell scattering problem,

$$\begin{aligned} \boldsymbol{\nabla}\times \boldsymbol{\nabla}\times \boldsymbol{w} - k^2\boldsymbol{w} = \mathbf{0}, &\text{ on } \Omega^c \\ \boldsymbol{w}\times\boldsymbol{n} = \widehat{\boldsymbol{u}}\times\boldsymbol{n}, &\text{ on } \Gamma, \\ \boldsymbol{w} \text{ satisfies } (2.3). \end{aligned}$$

It easily follows from Section 10 of [14] that $\boldsymbol{w}$ satisfies

$$\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl};\Omega'_a)} \leq C\|\widehat{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega'_a)} \leq C\|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c))^*}. \tag{5.3}$$

Since $\nabla\cdot\boldsymbol{w} = 0$ on Ω^c, each component of $\boldsymbol{w}$ satisfies

$$\Delta\boldsymbol{w}_j + k^2\boldsymbol{w}_j = 0, \text{ on } \Omega^c. \tag{5.4}$$

Then,

$$\boldsymbol{w}_j(x) = \int_{\Gamma_R}\left[\boldsymbol{w}_j(y)\frac{\partial\Phi(r)}{\partial n_y} - \Phi(r)\frac{\partial\boldsymbol{w}_j}{\partial n}(y)\right]\mathrm{d}S_y, \quad \text{for all } x\in\Omega^c_R. \tag{5.5}$$

Here R is in (R_0,a) and $\Phi(r)$ is the fundamental solution of (5.4). We define a complexified distance between $\widetilde{x} = (\widetilde{d}(x_1)x_1, \widetilde{d}(x_2)x_2, \widetilde{d}(x_3)x_3)$ and $y\in\Gamma_R$ by

$$\widetilde{r} \equiv \sqrt{(\widetilde{x}_1-y_1)^2 + (\widetilde{x}_2-y_2)^2 + (\widetilde{x}_3-y_3)^2}$$

where we take the negative real axis as the branch cut for the square root. That the argument stays away from the branch cut is a consequence of Lemma 2.1 of [4] (see also, Lemma 3.1 of [12]).

Let R_1 be in (R,a). We then define $\widetilde{\boldsymbol{w}}_j$, for $j=1,2,3$ by

$$\widetilde{\boldsymbol{w}}_j(x) \equiv \int_{\Gamma_R}\left[\boldsymbol{w}_j(y)\frac{\partial\Phi(\widetilde{r})}{\partial n_y} - \Phi(\widetilde{r})\frac{\partial\boldsymbol{w}_j}{\partial n}(y)\right]\mathrm{d}S_y, \quad \text{for all } x\in\Omega^c_{R_1}. \tag{5.6}$$

We extend $\widetilde{\boldsymbol{w}}_j$ to Ω^c by

$$\widetilde{\boldsymbol{w}}_j(x) = \boldsymbol{w}_j(x), \quad \text{for all } x\in\bar{\Omega}'_{R_1}.$$

We note that $\boldsymbol{w}_j(x)$ coincides with $\widetilde{\boldsymbol{w}}_j(x)$ in $\Omega_a\setminus\bar{\Omega}_R$ since $\widetilde{r}$ and r are the same for these values of x and y and hence the transition is smooth across Γ_{R_1}.

It easily follows by differentiating (5.6) under the integral sign that $\widetilde{\boldsymbol{w}}_j$ is in $H^1(D)$ for any bounded subdomain of Ω_{R_1}. By Theorem 4.6 of [12] (applied on Ω^c_R)

$\widetilde{\boldsymbol{w}}_j$ and its gradient decay exponentially at infinity and hence $\widetilde{\boldsymbol{w}}_j \in H^1(\Omega^c_{R_1})$. Moreover, except at interfaces where $\sigma'(x_j)$ may be discontinuous, $\widetilde{\Delta}_x \Phi(\widetilde{r}) + k^2 \Phi(\widetilde{r}) = 0$. It follows that $\widetilde{\boldsymbol{w}}_j$ satisfies

$$[\widetilde{\boldsymbol{\nabla}} \widetilde{\boldsymbol{w}}_j, \widetilde{\boldsymbol{\nabla}} \phi] - k^2 [\widetilde{\boldsymbol{w}}_j, \phi] = 0 \quad \text{for all } \phi \in H^1_0(\Omega^c_{R_1}).$$

Summing over j implies that

$$\widehat{D}(\widetilde{\boldsymbol{w}}, \boldsymbol{\phi}) - k^2 [\widetilde{\boldsymbol{w}}, \boldsymbol{\phi}] = 0 \quad \text{for all } \boldsymbol{\phi} \in \boldsymbol{H}^1_0(\Omega^c_{R_1}). \tag{5.7}$$

Let χ be a smooth cut-off function which satisfies $\chi(x) = 0$ for $x \in \Omega^c_{R_1}$ and $\chi(x) = 1$ for $x \in \Omega'_R$. Then $(1-\chi)\widetilde{\boldsymbol{w}}_j$ (extended by zero into $\bar{\Omega}$) satisfies

$$\begin{aligned} [\widetilde{\boldsymbol{\nabla}}((1-\chi)\widetilde{\boldsymbol{w}}_j), \widetilde{\boldsymbol{\nabla}} \phi] &- k^2 [(1-\chi)\widetilde{\boldsymbol{w}}_j, \phi] \\ &= \int_{\Omega_{R_1} \setminus \bar{\Omega}_R} J(-\boldsymbol{w}_j \boldsymbol{\nabla}\, \chi \cdot \boldsymbol{\nabla}\, \phi + \phi \boldsymbol{\nabla}\, \boldsymbol{w}_j \cdot \boldsymbol{\nabla}\, \chi)\, dx \\ &\qquad \text{for all } \phi \in H^1(\mathbb{R}^3). \end{aligned}$$

Using integration by parts to move the derivatives off ϕ and using the fact that $\boldsymbol{\nabla}\, \chi = \boldsymbol{0}$ on the boundary of the region of integration implies that the integral on the right hand side can be bounded by

$$C \|\boldsymbol{w}_j\|_{H^1(\Omega_{R_1} \setminus \bar{\Omega}_R)} \|\phi\|_{L^2(\Omega_{R_1} \setminus \bar{\Omega}_R)}.$$

It follows from the stability and regularity of (4.4) that

$$\|(1-\chi)\widetilde{\boldsymbol{w}}_j\|_{H^2(\mathbb{R}^3)} \le C \|\boldsymbol{w}_j\|_{H^1(\Omega_{R_1} \setminus \bar{\Omega}_R)} \le C \|\widehat{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega'_a)}.$$

On the other hand,

$$\|\chi \widetilde{\boldsymbol{w}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega'_{R_1})} \le C \|\boldsymbol{w}_j\|_{\boldsymbol{H}(\mathbf{curl};\Omega'_a)} \le C \|\widehat{\boldsymbol{u}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega'_a)}.$$

Now as in the proof of Theorem 4.1 (below (4.11)), we find that $\widetilde{\nabla} \cdot \widetilde{\boldsymbol{w}}$ satisfies

$$[\widetilde{\boldsymbol{\nabla}}(\widetilde{\nabla} \cdot \widetilde{\boldsymbol{w}}), \widetilde{\boldsymbol{\nabla}} \theta] - k^2 [(\widetilde{\nabla} \cdot \widetilde{\boldsymbol{w}}), \theta] = 0 \quad \text{for all } \theta \in H^1_0(\Omega^c_{R_1}).$$

Moreover, $\widetilde{\nabla} \cdot \widetilde{\boldsymbol{w}} = \widetilde{\nabla} \cdot \boldsymbol{w} = 0$ on Γ_{R_1}. The stability of the above problem implies that $\widetilde{\nabla} \cdot \widetilde{\boldsymbol{w}} = 0$ in $\Omega^c_{R_1}$. This, together with (5.7) and (4.11), implies that $\widetilde{\boldsymbol{w}}$ satisfies

$$[\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{w}}, \widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi}] - k^2 [\widetilde{\boldsymbol{w}}, \boldsymbol{\phi}] = 0 \quad \text{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c_{R_1}).$$

This also holds for $\boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega'_a)$ and hence for all $\boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c)$.

Combining the above shows that $\boldsymbol{u} = \widehat{\boldsymbol{u}} - \widetilde{\boldsymbol{w}}$ is a solution of (5.1). That $\boldsymbol{u}$ satisfies (5.2) follows from the triangle inequality, (4.12), and the preceding two inequalities. This completes the proof of the theorem. $\square$

We next extend the above theorem to PML applied to (2.1)–(2.3). Specifically we have the following:

Theorem 5.2. *Assume that* (4.3) *holds. Then there is a positive constant* C *satisfying*

$$\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)} \le C \sup_{\boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c)} \frac{|\widetilde{A}_v(\boldsymbol{u}, \boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)}} \quad \textit{for all } \boldsymbol{u} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c).$$

Proof. Given a functional $\boldsymbol{F} \in (\widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c))^*$, we first define $\boldsymbol{v} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c_{R_0})$ solving

$$\widetilde{A}(\boldsymbol{v}, \boldsymbol{\phi}) =< \boldsymbol{F}, \boldsymbol{\phi} > \quad \text{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c_{R_0}).$$

This is a stable problem by Theorem 5.1 applied with $\Omega = \Omega_{R_0}$ and we have

$$\|\boldsymbol{v}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega^c_{R_0})} \leq C \|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c))^*}.$$

Let $\bar{\boldsymbol{v}}$ denote $\boldsymbol{v}$ extended by zero to Ω^c.

We next construct a solution $\widetilde{\boldsymbol{w}} \in \widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega^c)$ satisfying

$$\widetilde{A}_v(\widetilde{\boldsymbol{w}}, \boldsymbol{\phi}) =< \boldsymbol{G}, \boldsymbol{\phi} > \quad \text{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c) \tag{5.8}$$

where

$$< \boldsymbol{G}, \boldsymbol{\phi} >=< \boldsymbol{F}, \boldsymbol{\phi} > - \widetilde{A}_v(\bar{\boldsymbol{v}}, \boldsymbol{\phi}) \quad \text{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c).$$

Note that the support of $\boldsymbol{G}$ is contained in Ω'_{R_0}.

Let B_A denote the ball of radius A with $A \geq \sqrt{2}a$ and set $D = B_A \setminus \bar{\Omega}$. Let $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}; D)$ solve

$$\begin{aligned} (\mu^{-1}\, \boldsymbol{\nabla}\times \boldsymbol{w}, \boldsymbol{\nabla}\times\boldsymbol{\phi})_D - k^2(\varepsilon \boldsymbol{w}, \boldsymbol{\phi})_D + &< DTN\boldsymbol{w}, \boldsymbol{\phi} >_{\partial B_A} \\ &=< \boldsymbol{G}, \boldsymbol{\phi} > \quad \text{for all } \boldsymbol{\phi} \in \boldsymbol{H}_0(\mathbf{curl}; D). \end{aligned} \tag{5.9}$$

This is just a source problem associated with (2.1)–(2.3). The operator DTN appearing in the boundary term is the Dirichlet to Neumann map associated with the far field boundary condition (2.3) (see [14]). This is a stable problem whose solution satisfies

$$\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl}; D)} \leq C \|\boldsymbol{G}\|_{(\boldsymbol{H}_0(\mathbf{curl}; \Omega^c))^*} \leq C \|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c))^*}.$$

Let R and R_1 be as in the proof of the previous theorem. We now define $\widetilde{\boldsymbol{w}}(x)$ from $\boldsymbol{w}$ by setting $\widetilde{\boldsymbol{w}}(x) = \boldsymbol{w}(x)$ for $x \in \bar{\Omega}'_{R_1}$ and using (5.6) with for $x \in \Omega^c_{R_1}$. The argument in the proof of the previous theorem shows that $\widetilde{\boldsymbol{w}}$ is in $\widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c)$, satisfies

$$\widetilde{A}_v(\widetilde{\boldsymbol{w}}, \boldsymbol{\theta}) =< \boldsymbol{G}, \boldsymbol{\theta} > \quad \text{for all } \boldsymbol{\theta} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c)$$

and

$$\|\widetilde{\boldsymbol{w}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega^c)} \leq C \|\boldsymbol{w}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega_a)} \leq \|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega^c))^*}.$$

It follows that $\boldsymbol{u} = \widetilde{\boldsymbol{w}} + \bar{\boldsymbol{v}}$ solves

$$\widetilde{A}_v(\boldsymbol{u}, \boldsymbol{\phi}) =< \boldsymbol{F}, \boldsymbol{\phi} > \quad \text{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c)$$

and satisfies

$$\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega^c)} \leq C \|\boldsymbol{F}\|_{(\widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega^c))^*}.$$

This completes the proof of the theorem. □

It is immediate consequence of Theorem 5.2 that the PML scattering problem is well posed.

Corollary 5.1. *There is a unique* $\widetilde{\boldsymbol{u}} \in \widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega^c)$ *satisfying*

$$\begin{aligned} \widetilde{A}_v(\widetilde{\boldsymbol{u}}, \boldsymbol{\phi}) &= 0, && \textit{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega^c), \\ \widetilde{\boldsymbol{u}} \times \boldsymbol{n} &= \boldsymbol{g} \times \boldsymbol{n}, && \textit{on } \Gamma, \end{aligned} \tag{5.10}$$

Moreover, $\widetilde{\boldsymbol{u}}$ *coincides with the solution* $\boldsymbol{u}$ *of* (2.1)-(2.3) *on* Ω'_a *and there are constants* $C > 0$ *and* $\alpha > 0$ *such that for* $M \geq b$,

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma_M)} \leq C e^{-\alpha k M} \|\widetilde{\boldsymbol{g}}\|_{\boldsymbol{H}(\mathbf{curl}; \Omega'_a)}. \tag{5.11}$$

Proof. A solution of (5.10) is constructed exactly as $\widetilde{\boldsymbol{w}}$ was constructed in the proof of the theorem. This solution which we denote by $\widetilde{\boldsymbol{u}}$ coincides with $\boldsymbol{u}$ on Ω_a' by definition and the fact that (5.5) and (5.6) coincide on $\Omega_a \setminus \bar{\Omega}_{R_1}$. That this is the only solution follows immediately from Theorem 5.2 and, moreover,

$$\|\widetilde{\boldsymbol{u}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)} \le C\|\widetilde{\boldsymbol{g}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_a')}.$$

Finally, Theorem 4.6 of [12] implies that, since $\widetilde{\boldsymbol{u}}$ is given by (5.6),

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma_M)} \le Ce^{-\alpha kM}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma_{R_1})}.$$

Since

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\widetilde{\boldsymbol{u}} - k^2\widetilde{\boldsymbol{u}} = \mathbf{0} \text{ in } \Omega_a \setminus \bar{\Omega}_R,$$

interior regularity estimates imply that

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma_R)} \le C\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Omega_a')} \le C\|\widetilde{\boldsymbol{g}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega_a')}.$$

Combining the above estimates completes the proof of the corollary. □

6. Stability of the PML problem on the truncated domain.

The goal of this section is to show the stability of the truncated source problem and the corresponding exponential convergence of the truncated scattering problem. These are posed in terms of the space $\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M') = \{\boldsymbol{\phi} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\Omega_M') : \boldsymbol{\phi}\times\boldsymbol{n} = \mathbf{0} \text{ on } \partial\Omega_M'\}$. Note that the homogeneous boundary condition $\boldsymbol{\phi}\times\boldsymbol{n} = \mathbf{0}$ is equivalent to $(\boldsymbol{B}\boldsymbol{\phi})\times\boldsymbol{n} = \mathbf{0}$ for this domain. The stability is given by the following theorem.

Theorem 6.1. *Assume that* (4.3) *holds. Then there is a positive number M_0 and $C = C(M_0)$ such that for all $M \ge M_0$, $\boldsymbol{u}$ in $\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M')$ and*

$$\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega_M')} \le C \sup_{\boldsymbol{\phi}\in\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M')} \frac{|\widetilde{A}_v(\boldsymbol{u},\boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega_M')}}. \tag{6.1}$$

The bilinear form defines an operator $\mathcal{L}_{\Omega_M'}$ from $\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega_M')$ to $(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M'))^*$ by

$$\langle \mathcal{L}_{\Omega_M'}\boldsymbol{u},\boldsymbol{v}\rangle \equiv \widetilde{A}_v(\boldsymbol{u},\boldsymbol{v}) \quad \text{for all } \boldsymbol{v}\in\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M'). \tag{6.2}$$

Clearly, for $\boldsymbol{u}\in\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M')$,

$$\|\mathcal{L}_{\Omega_M'}\boldsymbol{u}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M'))^*} = \sup_{\boldsymbol{\phi}\in\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M')} \frac{|\widetilde{A}_v(\boldsymbol{u},\boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega_M')}}. \tag{6.3}$$

Before proving the above theorem, we introduce the following lemma whose proof appears after the proof of the theorem.

Lemma 6.1. *Let $\boldsymbol{u}$ be in $\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M')$ with $M \ge 2b$. Then there is an extension $\widetilde{\boldsymbol{u}}$ of $\boldsymbol{u}$ to $\Omega_{3M/2}'$ satisfying*

$$\|\widetilde{\boldsymbol{u}}\|_{\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_{3M/2}')} \le 2\sqrt{2}\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M')}$$

and

$$\|\mathcal{L}_{\Omega_{3M/2}'}\widetilde{\boldsymbol{u}}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_{3M/2}'))^*} \le 2\sqrt{2}\|\mathcal{L}_{\Omega_M'}\boldsymbol{u}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega_M'))^*}. \tag{6.4}$$

Proof of Theorem 6.1. Let $\boldsymbol{u}$ be in $\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_M)$ and $\widetilde{\boldsymbol{u}} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})$ be the extension of $\boldsymbol{u}$ given by Lemma 6.1. Let χ be a smooth function defined on $\Omega_{3/2}$ satisfying $\chi = 1$ on Ω_1 and $\chi = 0$ in a neighborhood of $\partial\Omega_{3/2}$. Set $\chi_M(\boldsymbol{x}) = \chi(\boldsymbol{x}/M)$ for $\boldsymbol{x}$ in $\Omega_{3M/2}$. Finally define $\boldsymbol{v} = \chi_M \widetilde{\boldsymbol{u}}$ on $\Omega'_{3M/2}$ and $\boldsymbol{v} = \mathbf{0}$ on $\Omega^c_{3M/2}$. We note that

$$\widetilde{\boldsymbol{\nabla}} \times (\chi_M \widetilde{\boldsymbol{u}}) = (\widetilde{\boldsymbol{\nabla}}\chi_M) \times \widetilde{\boldsymbol{u}} + \chi_M(\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{u}}). \tag{6.5}$$

This implies that $\boldsymbol{v}$ is in $\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)$ and satisfies

$$\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)} \le \|\boldsymbol{v}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)} \le C\|\widetilde{\boldsymbol{u}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})}.$$

In addition, by Theorem 5.2,

$$\|\boldsymbol{v}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)} \le C\|\mathcal{L}_{\Omega^c}\boldsymbol{v}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c))^*}.$$

Now the support of χ_M is uniformly bounded away from $\Gamma_{3M/2}$ and so there is a stable decomposition of $\boldsymbol{\phi} \in \widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)$ into $\boldsymbol{\phi} = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2$ with the support of $\boldsymbol{\phi}_1$ contained in $\Omega'_{3M/2}$ and the support of $\boldsymbol{\phi}_2$ disjoint with that of χ_M. Indeed if θ is a smooth cutoff function with $\theta = 1$ on the support of χ_M and $\theta = 0$ on $\Omega^c_{3M/2}$ then we may take $\boldsymbol{\phi}_1 = \theta\boldsymbol{\phi}$. The continuity of the decomposition follows from (6.5) with χ_M replaced by θ. We then have

$$\frac{|\widetilde{A}_v(\boldsymbol{v},\boldsymbol{\phi})|}{\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)}} = \frac{|\widetilde{A}_v(\boldsymbol{v},\boldsymbol{\phi}_1)|}{\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega^c)}} \le C\frac{|\widetilde{A}_v(\boldsymbol{v},\boldsymbol{\phi}_1)|}{\|\boldsymbol{\phi}_1\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})}}$$

from which it follows that

$$\|\mathcal{L}_{\Omega^c}\boldsymbol{v}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega^c))^*} \le C\|\mathcal{L}_{\Omega'_{3M/2}}\boldsymbol{v}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_{3M/2}))^*}.$$

We next bound the norm on the right hand side above. By (6.5),

$$\begin{aligned}[]
[\mu^{-1}\widetilde{\boldsymbol{\nabla}} \times (\chi_M\widetilde{\boldsymbol{u}}), \widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi}]_{\Omega'_{3M/2}} - [\mu^{-1}\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\nabla}} \times (\chi_M\boldsymbol{\phi})]|_{\Omega'_{3M/2}} = \\
[\mu^{-1}(\widetilde{\boldsymbol{\nabla}}\chi_M) \times \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi}]_{\Omega'_{3M/2}} - [\mu^{-1}\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{u}}, (\widetilde{\boldsymbol{\nabla}}\chi_M) \times \boldsymbol{\phi}]_{\Omega'_{3M/2}}.
\end{aligned}$$

It is clear that

$$\|\widetilde{\boldsymbol{\nabla}}\chi_M\|_{\boldsymbol{L}^\infty(\Omega'_{3M/2})} \le \frac{c}{M}$$

and hence

$$\begin{aligned}
|[\mu^{-1}\widetilde{\boldsymbol{\nabla}} \times (\chi_M\widetilde{\boldsymbol{u}}), \widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi}]_{\Omega'_{3M/2}} - [\mu^{-1}\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\nabla}} \times (\chi_M\boldsymbol{\phi})]_{\Omega'_{3M/2}}| \\
\le cM^{-1}\|\widetilde{\boldsymbol{u}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})}\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})}.
\end{aligned} \tag{6.6}$$

Now,

$$\begin{aligned}
&\|\mathcal{L}_{\Omega'_{3M/2}}\boldsymbol{v}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_{3M/2}))^*} = \\
&\qquad \sup_{\boldsymbol{\phi}\in\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_{3M/2})} \frac{|[\mu^{-1}\widetilde{\boldsymbol{\nabla}} \times \boldsymbol{v}, \widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi}]_{\Omega'_{3M/2}} - k^2[\varepsilon\boldsymbol{v},\boldsymbol{\phi}]_{\Omega'_{3M/2}}|}{\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})}}.
\end{aligned}$$

The numerator on the right above is equal to

$$|\langle\mathcal{L}_{\Omega'_{3M/2}}\widetilde{\boldsymbol{u}}, \chi_M\boldsymbol{\phi}\rangle - ([\mu^{-1}\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\nabla}} \times (\chi_M\boldsymbol{\phi})]_{\Omega'_{3M/2}} - [\mu^{-1}\widetilde{\boldsymbol{\nabla}} \times (\chi_M\widetilde{\boldsymbol{u}}), \widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi}]_{\Omega'_{3M/2}})|.$$

Thus, Lemma 6.1, (6.5) and (6.6) imply that

$$\begin{aligned}\|\mathcal{L}_{\Omega'_{3M/2}}\boldsymbol{v}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_{3M/2}))^*} &\le \frac{C}{M}\|\widetilde{\boldsymbol{u}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})}\\ &+ C\sup_{\boldsymbol{\phi}\in\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_{3M/2})}\frac{\|\mathcal{L}_{\Omega'_{3M/2}}\widetilde{\boldsymbol{u}}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_{3M/2}))^*}\|\chi_M\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})}}{\|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_{3M/2})}}\\ &\le \frac{C}{M}\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)} + C\|\mathcal{L}_{\Omega'_M}\boldsymbol{u}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_M))^*}.\end{aligned}$$

Combining the above estimates shows that

$$\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)} \le \frac{C_1}{M}\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)} + C\|\mathcal{L}_{\Omega'_M}\boldsymbol{u}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_M))^*}.$$

Taking $M_0 > 1/C_1$ gives

$$\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)} \le (1 - C_1/M_0)^{-1}C\|\mathcal{L}_{\Omega'_M}\boldsymbol{u}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_M))^*}, \quad \text{for all } M \ge M_0.$$

This is the inf-sup condition (6.1) and completes the proof of the theorem. □

Proof of Lemma 6.1. We consider the following situation: Let D be the domain given by

$$D = (-M, M) \times (-l_1, l_1) \times (-l_2, l_2) \setminus \bar{\Omega}$$

where l_i is either M or $3M/2$. Given a function $\boldsymbol{u} \in \widehat{\boldsymbol{H}}(\mathbf{curl}; D)$ whose tangential components vanish on the face $x_1 = M$, we shall define an extension $\widetilde{\boldsymbol{u}} \in \widehat{\boldsymbol{H}}(\mathbf{curl}; \widetilde{D})$ where $\widetilde{D} = (-M, 3M/2) \times (-l_1, l_1) \times (-l_2, l_2) \setminus \bar{\Omega}$. This extension will satisfy:

(a) $\|\widetilde{\boldsymbol{u}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\widetilde{D})} \le \sqrt{2}\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};D)}$.

(b) $\|\mathcal{L}_{\widetilde{D}}\widetilde{\boldsymbol{u}}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};\widetilde{D}))^*} \le \sqrt{2}\|\mathcal{L}_D\boldsymbol{u}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl};D))^*}$.

(c) If the tangential component of $\boldsymbol{u}$ vanishes on any of the outer boundary faces of D excluding $x_1 = M$, then the tangential component of $\widetilde{\boldsymbol{u}}$ vanishes on the corresponding boundary face of $\widetilde{D}$.

The extension is defined by a reflection involving the two domains $D_1 = (M/2, M) \times (-l_1, l_1) \times (-l_2, l_2)$ and $D_2 = (M, 3M/2) \times (-l_1, l_1) \times (-l_2, l_2)$. Clearly, D_2 is the reflection $\tilde{\boldsymbol{x}} = (2M - x_1, x_2, x_3)^t$ of D_1 across the plane $x_1 = M$. Moreover, since $M \ge 2b$, if $\boldsymbol{x} \in D_1$ reflects into $\tilde{\boldsymbol{x}}$ then $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}(\tilde{\boldsymbol{x}})$ and $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{B}(\tilde{\boldsymbol{x}})$.

Let F denote the common boundary between D_1 and D_2. For $\boldsymbol{u} \in \widehat{\boldsymbol{H}}(\mathbf{curl}; D)$ with $\boldsymbol{u} \times (1, 0, 0)^t = \mathbf{0}$ on F, define $\widetilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})$ for $\boldsymbol{x} \in D$ and

$$\widetilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}) = (\boldsymbol{u}_1(\boldsymbol{x}), -\boldsymbol{u}_2(\boldsymbol{x}), -\boldsymbol{u}_3(\boldsymbol{x}))^t \quad \text{for } \tilde{\boldsymbol{x}} \in D_2. \tag{6.7}$$

It is clear that (c) holds for this extension. Moreover, a simple computation shows that

$$\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{u}}(\tilde{\boldsymbol{x}}) = (-(\widetilde{\boldsymbol{\nabla}} \times \boldsymbol{u})_1(\boldsymbol{x}), (\widetilde{\boldsymbol{\nabla}} \times \boldsymbol{u})_2(\boldsymbol{x}), (\widetilde{\boldsymbol{\nabla}} \times \boldsymbol{u})_3(\boldsymbol{x}))^t \quad \text{for } \tilde{\boldsymbol{x}} \in D_2. \tag{6.8}$$

Here the differentiation on the left denotes differentiation with respect to the $\tilde{x}$ variables while that on the right denotes differentiation with respect to the x variables. It follows that $\widetilde{\boldsymbol{u}}|_{D_2}$ is in $\widehat{\boldsymbol{H}}(\mathbf{curl}; D_2)$. Because $(\boldsymbol{B}\boldsymbol{u}) \times (1, 0, 0)^t = \mathbf{0}$ on F, the tangential trace of $\boldsymbol{B}\widetilde{\boldsymbol{u}}$ from either side is also zero and hence $\widetilde{\boldsymbol{u}}$ is in $\widehat{\boldsymbol{H}}(\mathbf{curl}; \widetilde{D})$. We clearly have

$$\|\widetilde{\boldsymbol{u}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\widetilde{D})} \le \sqrt{2}\|\boldsymbol{u}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};D)}.$$

We next verify (b). Let $\boldsymbol{\phi}$ be in $\widehat{\boldsymbol{H}}_0(\mathbf{curl};\widetilde{D})$ and define $\widetilde{\boldsymbol{\phi}}$ on D by

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{x}) = \begin{cases} (\boldsymbol{\phi}_1(\tilde{\boldsymbol{x}}), -\boldsymbol{\phi}_2(\tilde{\boldsymbol{x}}), -\boldsymbol{\phi}_3(\tilde{\boldsymbol{x}}))^t : & \text{if } \boldsymbol{x} \in D_1, \\ \mathbf{0} : & \text{otherwise.} \end{cases}$$

Clearly $\widetilde{\boldsymbol{\phi}}$ is in $\widehat{\boldsymbol{H}}(\mathbf{curl}; D)$ and has vanishing tangential components on $\partial D \setminus F$. We note that $\boldsymbol{\phi} + \widetilde{\boldsymbol{\phi}}$ (restricted to D) has vanishing tangential components on F and hence is in $\widehat{\boldsymbol{H}}_0(\mathbf{curl}; D)$. Similar to (6.8),

$$\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{\phi}}(\boldsymbol{x}) = (-(\widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi})_1(\tilde{\boldsymbol{x}}), (\widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi})_2(\tilde{\boldsymbol{x}}), (\widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi})_3(\tilde{\boldsymbol{x}}))^t \quad \text{for } \boldsymbol{x} \in D_1.$$

It easily follows that

$$[\widetilde{\boldsymbol{u}}, \boldsymbol{\phi}]_{D_2} = [\boldsymbol{u}, \widetilde{\boldsymbol{\phi}}]_{D_1}$$

and

$$[\widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\nabla}} \times \boldsymbol{\phi}]_{D_2} = [\widetilde{\boldsymbol{\nabla}} \times \boldsymbol{u}, \widetilde{\boldsymbol{\nabla}} \times \widetilde{\boldsymbol{\phi}}]_{D_1}.$$

Combining the above gives

$$< \mathcal{L}_{\widetilde{D}} \widetilde{\boldsymbol{u}}, \boldsymbol{\phi} >= \widetilde{A}_v(\widetilde{\boldsymbol{u}}, \boldsymbol{\phi}) = \widetilde{A}_v(\boldsymbol{u}, \boldsymbol{\phi} + \widetilde{\boldsymbol{\phi}}) =< \mathcal{L}_D \boldsymbol{u}, \boldsymbol{\phi} + \widetilde{\boldsymbol{\phi}} > .$$

Thus,

$$\begin{aligned} | < \mathcal{L}_{\widetilde{D}} \widetilde{\boldsymbol{u}}, \boldsymbol{\phi} > | &= | < \mathcal{L}_D \boldsymbol{u}, \boldsymbol{\phi} + \widetilde{\boldsymbol{\phi}} > | \\ &\le \|\mathcal{L}_D \boldsymbol{u}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl}; D))^*} \|\boldsymbol{\phi} + \widetilde{\boldsymbol{\phi}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl}; D)} \\ &\le \sqrt{2} \|\mathcal{L}_D \boldsymbol{u}\|_{(\widehat{\boldsymbol{H}}_0(\mathbf{curl}; D))^*} \|\boldsymbol{\phi}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl}; \widetilde{D})} \end{aligned}$$

from which (b) immediately follows.

It is clear that the above construction could simultaneously be applied to the faces $x_1 = \pm M$ (we reflect $(-M, -1/2M) \times (-l_1, l_1) \times (-l_2, l_2)$ across the face $x_1 = -M$). To prove the lemma, we start with $\boldsymbol{u} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega'_M)$ and reflect across the $x_1 = \pm M$ faces using $D = \Omega'_M$ and $\widetilde{D} = (-3M/2, 3M/2) \times (-M, M)^2$. Note that (a) and (b) still hold for the corresponding extension and the extended function has vanishing tangential components on the boundary faces $x_2 = \pm M$ and $x_3 = \pm M$. The above argument can be applied with respect to the faces $x_2 = \pm M$ and $x_3 = \pm M$ in succession. In this case, we use the result of the previous extension to get the extension on the larger domain. The result of the $x_1 = \pm M$ extension on $D = (-3M/2, 3M/2) \times (-M, M)^2 \setminus \bar{\Omega}$ is extended across $x_2 = \pm M$ to define the extension on $\widetilde{D} = (-3M/2, 3M/2)^2 \times (-M, M) \setminus \bar{\Omega}$. The result of this extension is the further extended to define the desired extension on all of $\Omega'_{3M/2}$. The norm bounds for the final extension follow trivially completing the proof of the lemma. $\square$

We now consider the truncated PML problem corresponding to Theorem 6.1, i.e., $\widetilde{\boldsymbol{u}}_M \in \widehat{\boldsymbol{H}}(\mathbf{curl}; \Omega'_M)$ satisfying

$$\begin{aligned} &\widetilde{A}_v(\widetilde{\boldsymbol{u}}_M, \boldsymbol{\phi}) = 0, \qquad \text{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl}; \Omega'_M), \\ &\widetilde{\boldsymbol{u}}_M \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n}, \qquad \text{on } \Gamma, \\ &\widetilde{\boldsymbol{u}}_M \times \boldsymbol{n} = \mathbf{0}, \qquad \text{on } \Gamma_M. \end{aligned} \tag{6.9}$$

Theorem 6.1 implies that this problem has a unique solution provided that $M \ge M_0$. The following theorem shows that the solution of the truncated PML problem converges exponentially to the PML solution on Ω'_M.

Theorem 6.2. *Assume that* (4.3) *holds. Let M be greater than or equal to M_0, $\widetilde{\boldsymbol{u}}$ denote the solution of* (5.10) *and $\widetilde{\boldsymbol{u}}_M$ denote the solution of* (6.9). *Then*

$$\|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_M\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)} \le Ce^{-\alpha k M}\|\widetilde{\boldsymbol{g}}\|_{\boldsymbol{H}(\mathbf{curl};\Omega'_a)}.$$

Here $C = C(M_0)$ can be chosen to be independent of M.

Proof. Note that $\widetilde{\boldsymbol{v}} = \widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_M$ is in $\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)$ and satisfies

$$\begin{aligned} \widetilde{\boldsymbol{v}} \times \boldsymbol{n} &= \mathbf{0} && \text{on } \Gamma,\\ \widetilde{\boldsymbol{v}} \times \boldsymbol{n} &= \widetilde{\boldsymbol{u}} \times \boldsymbol{n} && \text{on } \Gamma_M,\\ \widetilde{A}_v(\widetilde{\boldsymbol{v}}, \boldsymbol{\phi}) &= 0 && \text{for all } \boldsymbol{\phi} \in \widehat{\boldsymbol{H}}_0(\mathbf{curl};\Omega'_M). \end{aligned}$$

It is a consequence of Theorem 6.1 that the above problem has a unique solution satisfying

$$\|\widetilde{\boldsymbol{v}}\|_{\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)} \le C\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2}(\Gamma_M)}.$$

The theorem now follows from (5.11). □

Remark 6.1. For numerical computation, it is more convenient to approximate functions in $\boldsymbol{H}(\mathbf{curl};\Omega'_M)$ instead of $\widehat{\boldsymbol{H}}(\mathbf{curl};\Omega'_M)$. Setting $\boldsymbol{v} = \boldsymbol{B}\boldsymbol{u}_M$ and $\boldsymbol{\psi} = \boldsymbol{B}\boldsymbol{\phi}$ we find that $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl};\Omega'_M)$ satisfies the boundary conditions of (6.9) and

$$\mathcal{C}_M(\boldsymbol{v}, \boldsymbol{\psi}) = 0, \qquad \text{for all } \boldsymbol{\psi} \in \boldsymbol{H}_0(\mathbf{curl};\Omega'_M). \tag{6.10}$$

Here

$$\mathcal{C}_M(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Omega'_M} \mu^{-1}(\boldsymbol{A}\boldsymbol{B}\,\boldsymbol{\nabla}\times \boldsymbol{v})\cdot(\boldsymbol{\nabla}\times\boldsymbol{w})\,dx - k^2\int_{\Omega'_M} \varepsilon((\boldsymbol{A}\boldsymbol{B})^{-1}\boldsymbol{v})\cdot\boldsymbol{w}\,dx. \tag{6.11}$$

It is immediate from Theorem 6.1 that the inf-sup conditions for the source problem corresponding to (6.10) hold as well.

7. Numerical results

In this section, we report the results of numerical calculations which illustrate the theory of the previous sections. In particular, we consider a model scattering problem in two spatial dimension given by

$$\boldsymbol{\nabla}\times \nabla\times\boldsymbol{u} - k^2\boldsymbol{u} = \mathbf{0} \text{ in } \Omega^c \tag{7.1}$$

Here $\boldsymbol{\nabla}\times f = (f_y, -f_x)$ and $\nabla\times\boldsymbol{u} = \boldsymbol{u}_{2,x} - \boldsymbol{u}_{1,y}$. The boundary conditions (2.2) and (2.3) are replaced by

$$\boldsymbol{u}\cdot\boldsymbol{\tau} = \boldsymbol{g}\cdot\boldsymbol{\tau} \text{ on } \Gamma, \tag{7.2}$$

and

$$\lim_{r\to\infty} r^{1/2}(\nabla\times\boldsymbol{u} - ik\boldsymbol{u}\cdot\boldsymbol{\tau}) = 0, \tag{7.3}$$

respectively.

We choose a two dimensional problem because meaningful results can be obtained with far fewer elements in the finite element discretization. In addition, it is somewhat simpler to code. The analysis of PML given in the earlier sections is valid in the two dimensional case with simple modification (in this case, the condition on $\arg(1 + i\sigma_0)$ is no longer required).

For our numerical experiments, we take Ω to be the square $[-1, 1]^2$ and $k = 1$. For discretization, we use lowest order Nedelec elements on a regular partitioning of the computational domain $\Omega'_M = [-4, 4]^2 \setminus [-1, 1]^2$ into small squares of side h. For the subsequent discussion, we denote by V_h the corresponding discrete functions with vanishing tangential components on the outer boundary of Ω'_M while

V_h^0 denotes those functions of V_h whose tangential components also vanish on the inner boundary.

For a PML function, we use

$$\sigma(x) = \begin{cases} 0: & \text{for } |x| \le 2, \\ \sigma_0(x-2): & \text{for } 2 < x < 3, \\ \sigma_0: & \text{for } |x| \ge 3. \end{cases}$$

The two dimensional analog of (6.11) is given by

$$\mathcal{C}_M(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Omega'_M} (d(x_1)d(x_2))^{-1}(\nabla\times\boldsymbol{v})(\nabla\times\boldsymbol{w})\,dx - k^2 \int_{\Omega'_M} (\boldsymbol{\mu}\boldsymbol{v}) \cdot \boldsymbol{w}\,dx$$

where $\boldsymbol{\mu} = diag(d(x_2)/d(x_1), d(x_1)/d(x_2))$.

We consider approximation of the function

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{\nabla}\times[H^1(r)e^{i\theta}]$$

from its tangential values on Γ. Note that $\boldsymbol{u}(\boldsymbol{x})$ satisfies (7.1) and (7.3). We compute the finite element approximation on the truncated domain, i.e., the function $\boldsymbol{u}_h \in V_h$ which coincides $\boldsymbol{u}$ on the nodes of V_h on Γ and satisfies

$$\mathcal{C}_M(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = 0, \quad \text{for all } \boldsymbol{\phi}_h \in V_h^0.$$

We should note that stability and convergence of the finite element approximation is not completely trivial even after the stability properties of the continuous problem have been verified. Even the stability of the finite element approximation of constant coefficient curl-curl problem on a bounded domain requires some analysis (see, [13, 15, 14, 16]). The variable coefficient case is more difficult as it requires an analysis based on "stretched" Helmholtz decompositions analogous to those used in the proof of Corollary 4.1. Such an analysis for the case of spherical PML was given in [6] and we believe that those techniques should carry over here. However, the extension of the analysis of [6] to the Cartesian PML case is beyond the scope of this paper.

The PML approximation is a good approximation to the original problem only in the region inside the PML zone, i.e., on $\Omega'_a = [-2,2]^2 \setminus [-1,1]^2$. Accordingly, we report the $\boldsymbol{L}^2(\Omega'_a)$-error.

The first runs that we made were with $\sigma_0 = 1$. The results for different values of h are given in the first column of Table 7.1. The behavior as h becomes smaller clearly indicates that the PML is not adequate. There are two possible remedies, either increase σ_0 or increase the size of the computational domain (i.e., increase M). The results of the first strategy, although not supported by our theory, are reported in the remaining columns of Table 7.1. An increase of σ_0 from one to two gives better results for modest values of h while the increase of σ_0 to four gave better results for all of the reported values of h. These calculations illustrate the necessity of modifying the PML parameters as a function of the desired accuracy.

The second strategy for decreasing the PML induced error involves increasing the size of the computational domain as suggested by the theory of this paper. There are several ways that this can be achieved. First, one could remesh a larger domain Ω'_M. A uniform mesh on the larger domain, of course, would be quite expensive. A second option would be to use a graded mesh on Ω'_M with roughly the same number of elements as the original mesh. This is somewhat more complicated but keeps the number of elements under control. A third option is to perform a change of variables on the region $[-4,4]^2 \setminus [-3,3]^2$ mapping it to the larger domain $[-M,M]^2 \setminus [-3,3]^2$. This means that one still computes on $[-4,4]^2 \setminus [-1,1]^2$ with

TABLE 7.1. Errors as a function of the PML strength σ_0

h	# dofs	$\sigma_0 = 1$	$\sigma_0 = 2$	$\sigma_0 = 4$
$\frac{1}{16}$	880	0.148	0.104	0.111
$\frac{1}{32}$	3680	0.092	0.033	0.035
$\frac{1}{64}$	15040	0.082	0.011	0.0094
$\frac{1}{128}$	60800	0.082	0.0054	0.0024
$\frac{1}{256}$	244480	0.082	0.0044	0.00061
$\frac{1}{512}$	980480	0.082	0.0042	0.00016

a uniform mesh while getting the PML decay corresponding to Ω'_M with larger M. The stability of this new problem is a direct consequence of that on Ω'_M. The integrals defining the resulting quadratic form remain unchanged inside $[-3, 3]^2$ while on $[-4, 4]^2 \setminus [-3, 3]]^2$ become

$$\int (d_r(x_1)d_r(x_2)d(x_1)d(x_2))^{-1}(\nabla\times\boldsymbol{v})(\nabla\times\boldsymbol{w})\,dx - k^2\int(\boldsymbol{\mu}_r\boldsymbol{\mu}\boldsymbol{v})\cdot\boldsymbol{w}\,dx$$

where $\boldsymbol{\mu}_r = diag(d_r(x_2)/d_r(x_1), d_r(x_1)/d_r(x_2))$ and $d_r(x)$ is the derivative of the real stretching function.

We used the stretching approach for the results of Table 7.2. Specifically, we used the transformation (real stretching)

$$\hat{x} = 3 + \int_3^x (1 + \beta(t-3)^2)\,dx$$

with β chosen so that the right hand side is equal to M when $x = 4$. Table 7.2 gives the error as a function of h and M for $M = 4$ (no stretch), $M = 6$ and $M = 8$. The table clearly demonstrates that smaller PML errors can be obtained by an increase in the stretching factor (computational domain size) as the mesh size is decreased.

TABLE 7.2. Errors for real stretching as a function of the domain size M

h	# dofs	$M = 4$	$M = 6$	$M = 8$
$\frac{1}{16}$	880	0.148	0.099	0.128
$\frac{1}{32}$	3680	0.092	0.030	0.031
$\frac{1}{64}$	15040	0.082	0.0078	0.0083
$\frac{1}{128}$	60800	0.082	0.0020	0.0021
$\frac{1}{256}$	244480	0.082	0.00044	0.00053
$\frac{1}{512}$	980480	0.082	0.0012	0.00013

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