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A STABILIZED NONCONFORMING QUADRILATERAL FINITE ELEMENT METHOD FOR THE GENERALIZED STOKES EQUATIONS

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Abstract. In this paper, we study a local stabilized nonconforming finite element method for the generalized Stokes equations. This nonconforming method is based on two local Gauss integrals, and uses the equal order pairs of mixed finite elements on quadrilaterals. Optimal order error estimates are obtained for velocity and pressure. Numerical experiments performed agree with the theoretical results.

Key words. Generalized Stokes equations, nonconforming quadrilateral finite elements, optimal error estimates, *inf-sup* condition, numerical experiments, stability

1. Introduction

Much attention has recently been attracted to using the equal order finite element pairs (e.g., $P_1 - P_1$–the linear function pair and $Q_1 - Q_1$–the bilinear function pair) for the fluid mechanics equations, particularly for the Stokes and Navier-Stokes equations [1, 10, 11, 12]. While they do not satisfy the *inf-sup* stability condition, these element pairs offer simple and practical uniform data structure and adequate accuracy. Many stabilization techniques have been proposed to stabilize them such as penalty [7, 8], pressure projection [1, 10], and residual [15] stabilization methods. Among these methods, the pressure projection stabilization method is a preferable choice in that it is free of stabilization parameters, does not require any calculation of high-order derivatives or edge-based data structures, and can be implemented at the element level. As formulated in [1, 10, 11, 14], it is based on two local Gauss integrals.

Nonconforming finite elements [4] are popular for the discretization of partial differential equations since they are simple and have small support sets of basis functions. These elements on triangles have been studied in the context of the pressure projection stabilization method [9]. However, due to a technical reason, the nonconforming finite elements on quadrilaterals have not been studied for this stabilization method. In this paper, an argument is introduced to study this class of nonconforming finite elements for the stabilization method of the generalized Stokes equations. As examples, the nonconforming rotated element span$\{1,x,y,x^2 - y^2\}$ [3, 13] and the element span$\{1,x,y,((3x^2 - 5x^4) - (3y^2 - 5y^4))\}$ proposed by Douglas et al. [6] will be analyzed. After a stability condition is proven for the pressure projection stabilization method, optimal order error estimates are obtained for velocity and pressure. Numerical experiments will be performed to check the theoretical results derived.

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An outline of this paper is given as follows: In the second section, we introduce some basic notation and the generalized Stokes equations. Then, in the third section, the nonconforming quadrilateral finite elements and the local stabilization method are given. In the fourth section, a stability result is shown. Optimal order error estimates are derived in the fifth section. Finally, numerical experiments are presented in the sixth section.

2. Preliminaries

We consider the following generalized Stokes problem:

$$\begin{cases} \sigma u - \nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω represents a polygonal convex domain in $\Re^2$ with a Lipschitz-continuous boundary $\partial\Omega$, $u(x) = (u_1(x), u_2(x))$ the velocity vector, $p(x)$ the pressure, $f(x)$ the prescribed force, $\nu > 0$ the viscosity, and $\sigma \geq 0$ a nonnegative real number. For a time dependent problem, for example, σ can represent a time step.

To introduce a weak formulation of (2.1), set

$$X = (H_0^1(\Omega))^2, \quad Y = (L^2(\Omega))^2,$$
$$M = \left\{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \right\}.$$

Below the standard notation is used for the Sobolev space $W^{m,r}(\Omega)$, with the norm $\|\cdot\|_{m,r}$ and the seminorm $|\cdot|_{m,r}$, $m, r \geq 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$ when $r = 2$. The spaces $(L^2(\Omega))^m$, $m = 1, 2, 4$, are endowed with the $L^2(\Omega)$-scalar product $(\cdot,\cdot)$ and $L^2(\Omega)$-norm $\|\cdot\|_0$, respectively, as appropriate. Also, the space X is equipped with the scalar product $(\nabla u, \nabla v)$ and the norm $|u|_1$, $u, v \in X$. Because of the norm equivalence between $\|\cdot\|_1$ and $|\cdot|_1$ on X, we sometimes use the same notation for them.

We define the continuous bilinear forms:

$$a(u, v) = \sigma(u, v) + \nu(\nabla u, \nabla v) \qquad \forall\ u,\ v \in\ X,$$
$$d(v, p) = (\nabla \cdot v, p) \qquad \forall v \in X,\ p \in\ M.$$

Now, the variational formulation of problem (2.1) is to find a pair $(u, p) \in X \times M$ such that

$$B((u, p), (v, q)) = (f, v) \qquad \forall (v, q) \in X \times M, \tag{2.2}$$

where

$$B((u, p), (v, q)) = a(u, v) - d(v, p) - d(u, q).$$

The bilinear form $d(\cdot,\cdot)$ satisfies the *inf-sup* condition [4]:

$$\sup_{0 \neq v \in X} \frac{|d(v, q)|}{\|v\|_1} \geq \beta \|q\|_0, \quad q \in M,$$

where β is a positive constant depending only on the domain Ω.

3. Nonconforming Quadrilateral Finite Elements

Let K_h be a quasi-regular partition of Ω into convex quadrilaterals. Set

$$\bar{\Omega} = \bigcup_j \bar{K}_j, \quad \Gamma_j = \partial\Omega \cap K_j, \quad \Gamma_{jk} = \Gamma_{kj} = K_j \cap K_k, \qquad K_j \in K_h.$$

We denote the centers of Γ_j and Γ_{jk} by c_j and c_{jk}, respectively. Let $\widehat{K}$ be the reference square $[-1,1] \times [-1,1]$ in the (ξ, η)-plane. On this reference element, we define the nonconforming rotated element [3, 13]

$$X(\widehat{K}) = \mathrm{span}\{1, \xi, \eta, \xi^2 - \eta^2\} \tag{3.1}$$

or the element [6]

$$X(\widehat{K}) = \mathrm{span}\{1, \xi, \eta, ((3\xi^2 - 5\xi^4) - (3\eta^2 - 5\eta^4))\}. \tag{3.2}$$

For every convex quadrilateral $K \in K_h$ with vertices (x_i, y_i), $i = 1, 2, 3, 4$, there is a unique bilinear mapping $F_K : \widehat{K} \to K$:

$$(x, y) = F_K(\xi, \eta) = \left(F_K^1, F_K^2\right) = \left(\sum_{i=1}^{4} x_i \phi_i, \sum_{i=1}^{4} y_i \phi_i\right),$$

where

$$\phi_1 = \tfrac{1}{4}(1-\xi)(1-\eta), \quad \phi_2 = \tfrac{1}{4}(1+\xi)(1-\eta),$$
$$\phi_3 = \tfrac{1}{4}(1+\xi)(1+\eta), \quad \phi_4 = \tfrac{1}{4}(1-\xi)(1+\eta).$$

Then we define the element on the quadrilateral $K \in K_h$:

$$X(K) = \{v : v = \widehat{v} \circ F_K^{-1},\ \widehat{v} \in X(\widehat{K})\},$$

where $X(\widehat{K})$ is defined by either (3.1) or (3.2). With the above notation, we construct the following velocity-pressure finite element spaces:

$$\begin{aligned} X_h &= \left\{ v \in Y : v|_{K_j} \in X(K_j) \times X(K_j),\ \int_{\Gamma_{jk}} [v]\, ds = 0,\ \int_{\Gamma_j} v\, ds = 0 \quad \forall j, k \right\}, \\ M_h &= \left\{ q \in M : q|_{K_j} \in Q_1(K_j) \quad \forall j \right\}, \end{aligned}$$

where $[v] = v_{\Gamma_{jk}} - v_{\Gamma_{kj}}$ denotes the jump of the function v across the interface Γ_{jk}. When $X(\widehat{K})$ is defined by (3.2), the space X_h can equivalently be defined as follows [6]:

$$X_h = \left\{ v \in Y : v|_{K_j} \in X(K_j) \times X(K_j),\ v(c_{jk}) = v(c_{kj}),\ v(c_j) = 0 \quad \forall j, k \right\}.$$

Define the energy norm

$$\|v\|_{1,h} = \left(\sum_j |v|_{1,K_j}^2 + \sum_j \|v\|_{0,K_j}^2 \right)^{1/2}, \quad v \in X_h.$$

The two finite element spaces X_h and M_h satisfy the approximation property: For $(v, q) \in (H^2(\Omega) \cap X) \times (H^1(\Omega) \cap M)$, there are two approximations $v_I \in X_h$ and $q_I \in M_h$ such that

$$\|v - v_I\|_0 + h\left(\|v - v_I\|_{1,h} + \|q - q_I\|_0\right) \le Ch^2\left(\|v\|_2 + \|q\|_1\right), \tag{3.3}$$

where (and below) $C > 0$ is a generic constant independent of the mesh size h.

Set $(\cdot,\cdot)_j = (\cdot,\cdot)_{K_j}$, $\langle\cdot,\cdot\rangle_j = (\cdot,\cdot)_{\partial K_j}$, $\|\cdot\|_{0,j} = \|\cdot\|_{0,K_j}$, and $|\cdot|_{0,j} = |\cdot|_{0,K_j}$. Then the discrete bilinear forms are given as follows:

$$a_h(u,v) = \sigma\sum_j (u,v)_j + \nu\sum_j (\nabla u,\nabla v)_j, \qquad u,\ v\in X_h,$$
$$d_h(v,q) = \sum_j (\nabla\cdot v, q)_j. \qquad v\in X_h,\ q\in M_h.$$

For the nonconforming vector space X_h, we define the local operator Π_j : $H^1(K_j)\to X(K_j)$ by

$$\int_{\partial K_j} (v-\Pi_j v)\ ds = 0. \tag{3.4}$$

This local operator satisfies [4]:

$$|v-\Pi_j v|_{1,j} \le Ch^i|v|_{i+1,j}, \qquad v\in H^{i+1}(K_j), \quad i=0,1, \tag{3.5}$$
$$\|\Pi_j v\|_{1,j} \le C\|v\|_{1,j}, \qquad v\in H^1(K_j). \tag{3.6}$$

The global operator $\Pi_h : X\to X_h$ is now defined by

$$\Pi_h v|_j = \Pi_j v \qquad \forall v\in X.$$

The operator Π_h has the following properties:

$$d_h(v-\Pi_h v, q_h) = 0 \quad \forall q_h\in W_h, \qquad \|\Pi_h v\|_{1,h} \le C\|v\|_1 \quad \forall v\in X, \tag{3.7}$$

where $W_h\subset L^2(\Omega)$ denotes the piecewise constant space associated with K_h. As a result, the discrete *inf-sup* condition holds [4]:

$$\sup_{0\ne v\in X_h} \frac{|d_h(v,q)|}{\|v\|_{1,h}} \ge \beta\|q\|_0, \quad q\in W_h,$$

where $\beta>0$ is independent of h.

As noted, the $X_h\times M_h$ pair does not satisfy the *inf-sup* condition. However, following [9], we can add a simple local, effective stabilization term $G_h(\cdot,\cdot)$:

$$G_h(p,q) = \sum_{K_j\in K_h}\left\{\int_{K_{j,2}} pq\ dx - \int_{K_{j,1}} pq\ dx\right\}, \quad p,q\in L^2(\Omega),$$

where $\int_{K_{j,i}} pq\ dx$ indicates an appropriate Gauss integral over K_j that is exact for polynomials of degree i $(i=1,2)$, and pq is a polynomial of degree not greater than two. Thus, for all test functions $q\in M_h$, the trial function $p\in M_h$ must be piecewise constant when $i=1$. Consequently, we define the L^2-projection operator $\pi_h : L^2(\Omega)\to W_h$ by

$$(p,q_h) = (\pi_h p, q_h) \qquad \forall p\in L^2(\Omega),\ q_h\in W_h. \tag{3.8}$$

The projection operator π_h has the following properties [9]:

$$\|\pi_h p\|_0 \le C\|p\|_0 \qquad \forall p\in L^2(\Omega), \tag{3.9}$$

and

$$\|p-\pi_h p\|_0 \le Ch\|p\|_1 \qquad \forall p\in H^1(\Omega). \tag{3.10}$$

Now, using (3.8), we can define the bilinear form $G_h(\cdot,\cdot)$ as follows:

$$G_h(p,q) = (p-\pi_h p, q) = (p-\pi_h p, q-\pi_h q). \tag{3.11}$$

Finally, the nonconforming finite element approximation of problem (2.1) is to find a pair $(u_h, p_h) \in (X_h, M_h)$ such that

$$B_h((u_h, p_h), (v_h, q_h)) = (f, v_h) \qquad \forall (v_h, p_h) \in (X_h, M_h), \tag{3.12}$$

where

$$B_h((u_h, p_h), (v_h, q_h)) = a_h(u_h, v_h) - d_h(v_h, p_h) - d_h(u_h, q_h) - G_h(p_h, q_h)$$

is a bilinear form defined on $(X_h, M_h) \times (X_h, M_h)$. In the subsequent two sections we will establish stability and convergence results for method (3.12).

4. Stability

The stability result will come from the next theorem.

Theorem 4.1. *The bilinear form $B_h((\cdot,\cdot),(\cdot,\cdot))$ satisfies the continuous property*

$$\begin{aligned} B_h((u_h, p_h), (v_h, q_h)) &\leq C(\|u_h\|_{1,h} + \|p_h\|_0)(\|v_h\|_{1,h} + \|q_h\|_0) \\ &\qquad \forall (u_h, p_h),\ (v_h, q_h) \in (X_h, M_h), \end{aligned} \tag{4.1}$$

and the coercive property

$$\begin{aligned} \sup_{0 \neq (v_h, q_h) \in (X_h, M_h)} \frac{B_h((u_h, p_h), (v_h, q_h))}{\|v_h\|_{1,h} + \|q_h\|_0} &\geq \beta\,(\|u_h\|_{1,h} + \|p_h\|_0) \\ &\qquad \forall (u_h, p_h) \in (X_h, M_h), \end{aligned} \tag{4.2}$$

where β is a positive constant depending only on Ω.

Proof. By the continuous property of the bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$, and $G_h(\cdot,\cdot)$, we can easily obtain the continuous property of $B_h((\cdot,\cdot),(\cdot,\cdot))$ so it suffices to show the coercive property.

For all $p_h \in L^2(K_j)$, there exists $w \in \left(H^1(K_j)\right)^2$ such that [4]

$$(\nabla \cdot w, p_h)_j = \|p_h\|_{0,j}^2, \qquad \|w\|_{1,j} \leq C_0 \|p_h\|_{0,j}.$$

Setting $w_h = \Pi_h w$, we see that

$$\|w_h\|_{1,h} \leq C_1 \|p_h\|_0. \tag{4.3}$$

Choosing $(v_h, q_h) = (u_h - \epsilon w_h, -p_h)$ for some constant $\epsilon > 0$ yet to be determined, we have

$$\begin{aligned} &B_h((u_h, p_h), (u_h - \epsilon w_h, p_h)) \\ &= a_h(u_h, u_h) - \epsilon a_h(u_h, w_h) + \epsilon d_h(w_h, p_h) + G_h(p_h, p_h). \end{aligned} \tag{4.4}$$

From (4.3) it follows that

$$\epsilon a_h(u_h, w_h) \quad \leq \quad \frac{\max\{\sigma, \nu\}}{2\gamma} \epsilon C_2 \|u_h\|_{1,h}^2 + \frac{\max\{\sigma, \nu\}\gamma}{2} \epsilon C_3 \|p_h\|_0^2, \tag{4.5}$$

where $\gamma > 0$ is another constant to be determined. Note that, by (3.7),

$$\begin{aligned} \|p_h\|_0^2 &= (p_h, \nabla \cdot w) \\ &= (p_h - \pi_h p_h, \nabla \cdot w) + (\pi_h p_h, \nabla \cdot w) \\ &= (p_h - \pi_h p_h, \nabla \cdot w) + (\pi_h p_h, \nabla \cdot w_h) \\ &= (p_h - \pi_h p_h, \nabla \cdot (w - w_h)) + (p_h, \nabla \cdot w_h), \end{aligned}$$

and, by (3.7) and (4.3),

$$\begin{aligned} |(p_h - \pi_h p_h, \nabla \cdot (w - w_h))| &\leq \|p_h - \pi_h p_h\|_0 \|\nabla \cdot (w - w_h)\|_0 \\ &\leq C_4 G_h^{1/2}(p_h, p_h) \|p_h\|_0 \\ &\leq \frac{1}{2}\|p_h\|_0^2 + \frac{1}{2} C_4^2 G_h(p_h, p_h). \end{aligned}$$

Consequently, we obtain

$$\frac{1}{2}\|p_h\|_0^2 \le C_5 G_h(p_h,p_h) + (p_h, \nabla\cdot w_h). \tag{4.6}$$

Combining (4.4)–(4.6) gives

$$\begin{aligned}
& B_h((u_h,p_h),(u_h-\epsilon w_h,p_h)) \\
\ge\ & \min\{\sigma,\nu\}\|u_h\|_{1,h}^2 - \frac{\max\{\sigma,\nu\}}{2\gamma}\epsilon C_2\|u_h\|_{1,h}^2 \\
-\ & \frac{\max\{\sigma,\nu\}\gamma}{2}\epsilon C_3\|p_h\|_0^2 + \frac{1}{2}\epsilon\|p_h\|_0^2 - C_5\epsilon G(p_h,p_h) + G(p_h,p_h) \\
\ge\ & \left(\min\{\sigma,\nu\} - \frac{\max\{\sigma,\nu\}}{2\gamma}\epsilon C_2\right)\|u_h\|_{1,h}^2 \\
+\ & \frac{1}{2}\epsilon\left(1-\max\{\sigma,\nu\}\gamma C_3\right)\|p_h\|_0^2 + (1-C_5\epsilon)G(p_h,p_h).
\end{aligned}$$

Choosing

$$\gamma = \frac{1}{2\max\{\sigma,\nu\}C_3}, \qquad \epsilon = \min\left\{\frac{\min\{\sigma,\nu\}}{2\max\{\sigma,\nu\}^2 C_2 C_3}, \frac{1}{2C_5}\right\},$$

we see that

$$|B_h((u_h,p_h),(u_h-\epsilon w_h,p_h))| \ge C_6(\|u_h\|_{1,h} + \|p_h\|_0)^2. \tag{4.7}$$

Clearly, using (4.3) and the triangle inequality, we have

$$\|u_h-\epsilon w_h\|_{1,h} + \|p_h\|_0 \le C_7(\|u_h\|_{1,h} + \|p_h\|_0). \tag{4.8}$$

Finally, setting $\beta = C_6/C_7$, combining (4.7) and (4.8) yields (4.2). []

5. Error Estimates

The next lemma will be used [2].

Lemma 5.1. *For any $\phi,\ w \in X \cup X_h$,*

$$\left|\sum_j \left\langle \frac{\partial w}{\partial n_j}, \phi\right\rangle_j\right| \le Ch\|w\|_2\|\phi\|_{1,h}, \qquad w \in X\cap (H^2(\Omega))^2, \tag{5.1}$$

$$\left|\sum_j \langle q, \phi\cdot n_j\rangle_j\right| \le Ch\|q\|_1\|\phi\|_{1,h}, \qquad q\in H^1(\Omega). \tag{5.2}$$

Set

$$\widetilde{B}_h((u,p),(v,q)) = B_h((u,p),(v,q)) + G_h(p,q).$$

We introduce the projection operators $(R_h,Q_h) : (X,M) \to (X_h,M_h)$ by

$$\begin{aligned}
& B_h((R_h(v,q),Q_h(v,q)),(v_h,q_h)) \\
&= \widetilde{B}_h((v,q),(v_h,q_h)) \quad \forall (v_h,q_h)\in (X_h,M_h).
\end{aligned} \tag{5.3}$$

which is well defined by Theorem 4.1 and satisfies the next approximation property, whose proof follows [9].

Lemma 5.2. *It holds that, for $(v,q) \in \big(X\cap (H^2(\Omega))^2, M\cap H^1(\Omega)\big)$,*

$$\begin{aligned}
& \|v-R_h(v,q)\|_0 + h(\|v-R_h(v,q)\|_{1,h} + \|q-Q_h(v,q)\|_0) \\
& \le Ch^2(\|v\|_2 + \|q\|_1).
\end{aligned} \tag{5.4}$$

Proof. Using (5.3), we see that

$$B_h((v - R_h(v,q), q - Q_h(v,q)), (v_h, q_h)) \\ = -G_h(q, q_h) \quad \forall (v_h, q_h) \in (X_h, M_h). \tag{5.5}$$

Setting $E = v - \Pi_h v$ and $(w, r) = (\Pi_h v - R_h(v,q), q_I - Q_h(v,q))$, where q_I is the interpolation of q in M_h, we have

$$B_h((w,r),(v_h,q_h)) = -B_h((E, q - q_I)),(v_h,q_h)) - G_h(q,q_h) \\ \forall (v_h,q_h) \in (X_h, M_h). \tag{5.6}$$

From Theorem 4.1 and the continuous property of $G_h(\cdot,\cdot)$, we obtain

$$\begin{aligned} \beta(\|w\|_{1,h} + \|r\|_0) &\le \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{B_h((w,r),(v_h,q_h))}{\|v_h\|_{1,h} + \|q_h\|_0}, \\ |G_h(q,q_h)| &\le Ch\|q\|_1(\|v_h\|_{1,h} + \|q_h\|_0). \end{aligned} \tag{5.7}$$

From (3.3), (3.5), and the continuous property of $B_h(\cdot,\cdot)$, it follows that

$$\begin{aligned} B_h((E, q - q_I)),(v_h,q_h)) &\le C(\|E\|_{1,h} + \|q - q_I\|_0)(\|v_h\|_{1,h} + \|q_h\|_0) \\ &\le Ch(\|v\|_2 + \|q\|_1)(\|v_h\|_{1,h} + \|q_h\|_0). \end{aligned} \tag{5.8}$$

Combining (5.6)–(5.8) gives

$$\|w\|_{1,h} + \|r\|_0 \le Ch(\|v\|_2 + \|q\|_1). \tag{5.9}$$

Therefore, we obtain

$$\begin{aligned} &\|v - R_h(v,q)\|_{1,h} + \|q - Q_h(v,q)\|_0 \\ &\le (\|v - \Pi_h v\|_{1,h} + \|w\|_{1,h}) + (\|q - q_I\|_0 + \|r\|_0) \\ &\le Ch(\|v\|_2 + \|q\|_1). \end{aligned} \tag{5.10}$$

Next, we consider the following dual problem, with $(\zeta, \tau) = (v - R_h(v,q), q - Q_h(v,q))$:

$$\begin{cases} \sigma\Phi - \nu\Delta\Phi + \nabla\Psi = \zeta & \text{in } \Omega, \\ \nabla\cdot\Phi = 0 & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

Because of the convexity of the domain Ω, the solution of this problem satisfies the regularity property:

$$\|\Phi\|_2 + \|\Psi\|_1 \le c\|\zeta\|_0. \tag{5.12}$$

Multiplying the first and second equations of (5.11) by ζ and τ, respectively, integrating the resulting equations over Ω, and using (5.5) with $(v_h, q_h) = (\Pi_h\Phi, \Psi_I)$, we see that

$$\begin{aligned} \|\zeta\|_0^2 &= a_h(\zeta,\Phi) - d_h(\zeta,\Psi) - d_h(\Phi,\tau) - \nu\sum_j \left\langle \frac{\partial\Phi}{\partial n_j}, \zeta\right\rangle_j + \sum_j \langle \zeta\cdot n_j, \Psi\rangle_j \\ &= a_h(\zeta, \Phi - \Pi_h\Phi) - d_h(\zeta, \Psi - \Psi_I) - d_h(\Phi - \Pi_h\Phi, \tau) - G_h(\Psi - \Psi_I, \tau) \\ &\quad + G_h(\Psi,\tau) - G_h(q, \Psi_I) - \nu\sum_j \left\langle \frac{\partial\Phi}{\partial n_j}, \zeta\right\rangle_j + \sum_j \langle \zeta\cdot n_j, \Psi\rangle_j. \end{aligned} \tag{5.13}$$

From the continuous property of the bilinear terms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$, and $G_h(\cdot,\cdot)$, the approximation properties (3.3) and (3.5), the estimate (5.10), and the regularity

(5.12), we have

$$|a_h(\zeta,\Phi-\Pi_h\Phi)-d_h(\zeta,\Psi-\Psi_I)-d_h(\Phi-\Pi_h\Phi,\tau)-G_h(\Psi-\Psi_I,\tau)| \tag{5.14}$$
$$\le C(\|\zeta\|_{1,h}+\|\tau\|_0)(\|\Phi-\Pi\Phi\|_{1,h}+\|\Psi-\Psi_I\|_0)$$
$$\le Ch^2(\|v\|_2+\|q\|_1)\|\zeta\|_0,$$

and

$$\begin{aligned}|G_h(\Psi,\tau)-G_h(q,\Psi_I)| &\le Ch\|\Psi\|_1\|\tau\|_0+Ch^2\|q\|_1\|\Psi\|_1\\ &\le Ch^2(\|v\|_2+\|q\|_1)\|\zeta\|_0.\end{aligned} \tag{5.15}$$

In addition, it follows from Lemma 5.1, (5.10), and (5.12) that

$$\left|\nu\sum_j\left\langle\frac{\partial\Phi}{\partial n_j},\zeta\right\rangle_j-\sum_j\langle\zeta\cdot n_j,\Psi\rangle_j\right|\le Ch^2(\|v\|_2+\|q\|_1)\|\zeta\|_0. \tag{5.16}$$

Finally, combining (5.13)–(5.16) gives (5.4). []

We are now in a position to obtain the error estimates for method (3.12).

Theorem 5.3. *Let* (u,p) *and* (u_h,p_h) *be the respective solutions of* (2.1) *and* (3.12). *Then*

$$\|u-u_h\|_{1,h}+\|p-p_h\|_0\le Ch(\|u\|_2+\|p\|_1). \tag{5.17}$$

Proof: Let $(\zeta,\tau)=(R_h(u,p)-u_h,Q_h(u,p)-p_h)$. Using (5.3) and a similar argument as for (5.15) and (5.16), we see that

$$\begin{aligned}B_h((\zeta,\tau),(v_h,q_h)) &= \widetilde{B}_h((u,p),(v_h,q_h))-B_h((u_h,p_h),(v_h,q_h))\\ &= \nu\sum_j\left\langle\frac{\partial u}{\partial n_j},v_h\right\rangle_j-\sum_j\langle v_h\cdot n_j,p\rangle_j\\ &\le Ch(\|u\|_2+\|p\|_1)\|v_h\|_{1,h}.\end{aligned}$$

Consequently, by the triangle inequality, (4.1), (4.2), and Lemma 5.2, we obtain

$$\begin{aligned}\|u-u_h\|_{1,h}+\|p-p_h\|_0 &\le C\Bigg\{\|u-R_h(u,p)\|_{1,h}+\|p-Q_h(u,p)\|_0\\ &\quad+\frac{1}{\beta}\sup_{(v_h,q_h)\in(X_h,M_h)}\frac{|B_h((\zeta,\tau),(v_h,q_h))|}{\|v_h\|_{1,h}+\|q_h\|_0}\Bigg\}\\ &\le Ch(\|u_h\|_2+\|p_h\|_0),\end{aligned} \tag{5.18}$$

which implies the desired result (5.17). []

Theorem 5.4. *Let* (u,p) *and* (u_h,p_h) *be the respective solutions of* (2.1) *and* (3.12). *Then*

$$\|u-u_h\|_0\le Ch^2(\|u\|_2+\|p\|_1). \tag{5.19}$$

Proof. As in the proof of Lemma 5.2, setting $(\zeta,\tau)=(u-u_h,p-p_h)$, multiplying the first and second equations of (5.11) by ζ and τ, respectively, and integrating the resulting equations over Ω, we see that

$$\begin{aligned}\|\zeta\|_0^2 &= a_h(\zeta,\Phi)-d_h(\zeta,\Psi)-d_h(\Phi,\tau)\\ &\quad-\nu\sum_j\left\langle\frac{\partial\Phi}{\partial n_j},\zeta\right\rangle_j+\sum_j\left\langle\zeta\cdot n_j,\Psi\right\rangle_j.\end{aligned} \tag{5.20}$$

Using (2.1) and (3.12) with $(v_h, q_h) = (\Pi_h\Phi, \Psi_I)$, it follows that

$$\begin{aligned}\|\zeta\|_0^2 &= a_h(\zeta, \Phi - \Pi_h\Phi) - d_h(\zeta, \Psi - \Psi_I) - d_h(\Phi - \Pi_h\Phi, \tau) - G_h(\Psi - \Psi_I, \tau)\\ &\quad + G_h(\Psi, \tau) - G_h(p, \Psi_I) - \nu\sum_j \left\langle \frac{\partial\Phi}{\partial n_j}, \zeta\right\rangle_j + \sum_j \langle \zeta\cdot n_j, \Psi\rangle_j\\ &\quad + \nu\sum_j \left\langle \frac{\partial u}{\partial n_j}, \Pi_h\Phi\right\rangle_j - \sum_j \langle \Pi_h\Phi\cdot n_j, p\rangle_j.\end{aligned} \tag{5.21}$$

Applying an analogous argument as for (5.14)–(5.16), we have

$$\begin{aligned}&|a_h(\zeta, \Phi - \Pi_h\Phi) - d_h(\zeta, \Psi - \Psi_I) - d_h(\Phi - \Pi_h\Phi, \tau) - G_h(\Psi - \Psi_I, \tau)|\\ &\quad\le Ch^2(\|u\|_2 + \|p\|_1)\|\zeta\|_0,\\ &|G_h(\Psi, \tau) - G_h(p, \Psi_I)| \le Ch^2(\|u\|_2 + \|p\|_1)\|\zeta\|_0,\\ &\left|\nu\sum_j \left\langle \frac{\partial\Phi}{\partial n_j}, \zeta\right\rangle_j - \sum_j \langle \zeta\cdot n_j, \Psi\rangle_j\right| \le Ch^2(\|u\|_2 + \|p\|_1)\|\zeta\|_0.\end{aligned} \tag{5.22}$$

Note that, by the regularity of the solution u and p,

$$\nu\sum_j \left\langle \frac{\partial u}{\partial n_j}, \Pi_h\Phi\right\rangle_j - \sum_j \langle \Pi_h\Phi\cdot n_j, p\rangle_j = \nu\sum_j \left\langle \frac{\partial u}{\partial n_j}, \Pi_h\Phi - \Phi\right\rangle_j - \sum_j \langle (\Pi_h\Phi - \Phi)\cdot n_j, p\rangle_j.$$

As a consequence, it follows from Lemma 5.1, (3.5), and (5.12) that

$$\begin{aligned}\left|\nu\sum_j \left\langle \frac{\partial u}{\partial n_j}, \Pi_h\Phi\right\rangle_j - \sum_j \langle \Pi_h\Phi\cdot n_j, p\rangle_j\right| &\le Ch(\|u\|_2 + \|p\|_1)\|\Pi_h\Phi - \Phi\|_{1,h}\\ &\le Ch^2(\|u\|_2 + \|p\|_1)\|\zeta\|_0.\end{aligned} \tag{5.23}$$

Finally, combining (5.20)–(5.23) yields (5.19). []

6. Numerical Experiments

In this section we present numerical experiments to check the numerical theory developed in the previous sections. In all the experiments, $\Omega = (0,1)^2$ is the unit square in $\Re^2$, the exact solution for the velocity $u = (u_1, u_2)$ and the pressure p is given as follows:

$$\begin{aligned} p &= \cos(\pi x_1)\cos(\pi x_2),\\ u_1 &= 2\pi\sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_1),\\ u_2 &= -2\pi\sin(\pi x_1)\sin^2(\pi x_2)\cos(\pi x_1),\end{aligned} \tag{6.1}$$

and the right-hand side $f(x)$ is determined by (2.1) through this exact solution. Furthermore, the nonconforming finite element given in (3.2) is used.

When $\sigma = 0$ and $\nu = 0.1$, the convergence results are given in Table 1. We now fix $\nu = 0.1$ and vary $\sigma = 0.1, 1, 10, 100$; the results are given in Table 2. These convergence results fully agree with the theoretical results we have obtained in the previous two sections.

Table 1. Convergence results.

$1/h$	$\frac{\|u-u_h\|_0}{\|u\|_0}$	$\frac{\|u-u_h\|_1}{\|u\|_1}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	$u_{L^2}rate$	$u_{H^1}rate$	$p_{L^2}rate$
8	0.0461	0.2981	0.1308			
12	0.0205	0.2000	0.0602	1.9944	0.9845	1.9130
16	0.0116	0.1503	0.0352	1.9973	0.9929	1.8629
20	0.0074	0.1203	0.0234	1.9984	0.9960	1.8271
24	0.0051	0.1003	0.0168	1.9989	0.9975	1.6892

Table 2. Convergence results for different σ.

	16×16			24×24		
σ	$u_{L^2}rate$	$u_{H^1}rate$	$p_{L^2}rate$	$u_{L^2}rate$	$u_{H^1}rate$	$p_{L^2}rate$
0.1	1.9954	0.9881	1.8934	1.9985	0.9967	1.8159
1	1.9941	0.9887	1.9032	1.9980	0.9969	1.8288
10	1.9884	0.9912	1.9546	1.9963	0.9977	1.9134
100	1.9596	0.9939	1.8649	1.9869	0.9987	1.9611

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