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SUPERCONVERGENCE OF STABILIZED LOW ORDER FINITE VOLUME APPROXIMATION FOR THE THREE-DIMENSIONAL STATIONARY NAVIER-STOKES EQUATIONS

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Abstract. We first analyze a stabilized finite volume method for the three-dimensional stationary Navier-Stokes equations. This method is based on local polynomial pressure projection using low order elements that do not satisfy the inf-sup condition. Then we derive a general superconvergent result for the stabilized finite volume approximation of the stationary Navier-Stokes equations by using a L^2-projection. The method is a postprocessing procedure that constructs a new approximation by using the method of least squares. The superconvergent results have three prominent features. First, they are established for any quasi-uniform mesh. Second, they are derived on the basis of the domain and the solution for the stationary Navier-Stokes problem by solving sparse, symmetric positive definite systems of linear algebraic equations. Third, they are obtained for the finite elements that fail to satisfy the inf-sup condition for incompressible flow. Therefore, this method presented here is of practical importance in scientific computation.

Key words. Navier-Stokes equations, stabilized finite volume method, local polynomial pressure projection, inf-sup condition.

1. Introduction

The development of stable mixed finite element methods is a fundamental component in search for efficient numerical methods for solving the Navier-Stokes equations governing the flow of an incompressible fluid by using a primitive variable formulation. The importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the inf-sup condition is widely understood. It is well known that numerous mixed finite elements satisfying this stable condition have been proposed over years. However, elements that do not satisfy the inf-sup condition can be of practical values; some of them are very attractive and usable in many occasions. In particular, the lower order mixed finite elements are of practical importance in scientific computation because they are computationally convenient. However, the violation of the inf-sup condition for the Navier-Stokes equations often leads to unphysical pressure oscillations.

In order to make fully use of these lower order mixed finite elements, a popular strategy is to use stabilized techniques to circumvent or ameliorate the compatibility condition. Some of these techniques have been studied during the past decades for the lower order finite elements [1, 4, 5, 6, 7, 13, 18, 19]. However, most of the stabilized techniques necessarily introduce stabilization parameters either explicitly or implicitly. In addition, some of them are conditionally stable and achieve suboptimal accuracy depending on the choice of the stabilization parameters with respect to the solution regularity [18, 19]. Thus insensitivity to such parameters is important if a method is to be competitive.

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On the other hand, the finite volume method has been a very popular method in fluid computation. The finite volume method is intuitive in that it is directly based on conservation of physical properties over volumes or dual volumes. It has flexibility similar to that of the finite element method for handling complicated geometries but its theoretical analysis is much more complex than the latter [3, 8, 10, 11, 14, 17, 22, 25, 26].

In this paper, the idea of a stabilized finite volume method based on local polynomial pressure projection is derived from [17, 21] for the three dimensional stationary incompressible flow by using lower order finite elements. The well-posedness and optimal error estimates of this method are stated for the stationary Navier-Stokes equations. The main purpose is to establish a general superconvergent result for the finite volume approximation of the three-dimensional stationary Navier-Stokes equations by using a L^2-projection method proposed recently in [24]. This superconvergent result for the stationary Navier-Stokes equations can be applied to any finite element with regular but nonuniform partitions and is introduced by using the L^2 projection in a solution postprocessing manner. The method is demonstrated to generate a convergent scheme for finite element spaces that fail to satisfy the inf-sup condition especially for the incompressible flow. The post-processing technique of superconvergence has the feature that it can yield the superconvergent result anywhere in the domain and even up to the boundary. Moreover, this method has been developed as multi-scale process by capturing coarse information of given problem with lowest order finite element and then projecting the first finite solution on coarse mesh with high order piecewise polynomial in order to obtained more effective approximation solution. Therefore, this post-processing method is of practical importance in scientific computation.

We emphasize that the analysis requires to take special care of the trilinear term and the lower order convergence order $O(h)$ between the base functions of the finite element method and finite volume method. Here an equivalence between the finite element method and the finite volume method and an additional duality argument are applied to analyze the postprocessing of the stabilized finite volume method for the stationary Navier-Stokes equations based on the finite element theoretic results. The main results are summarized in Theorems 4.1 and 4.2. The error estimates for velocity are superconvergence if $s \geq 2$ in the H^1-norm. The superconvergence for pressure can be made in the case of $t > 0$. However, no improvement has been made for the velocity in the L^2-norm.

The remainder of the paper is organized as follows. In the next section, the stabilized finite element approximation of the Navier-Stokes problem is given with some basic statements. The stabilized finite volume approximations are analyzed and optimal estimates are stated in §3. Error estimates of superconvergence for the stabilized finite volume solution (u_h, p_h) are derived in §4.

2. Stabilized Finite Element Approximation

Let Ω be a bounded domain in R^3, assumed to have a Lipschitz-continuous boundary Γ and to satisfy a further condition stated in (A1) below. The stationary Navier-Stokes equations are considered as follows.

$$-\nu\Delta u + \nabla p + (u \cdot \nabla)u + \frac{1}{2}(\text{div } u)u = f, \ \text{div } u = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$u|_{\partial\Omega} = 0, \tag{2.2}$$

where $u(x) = (u_1(x), u_2(x), u_3(x))$ represents the velocity of a viscous incompressible fluid, $p = p(x)$ the pressure, ν the fluid viscosity, and $f(x) = (f_1(x), f_2(x), f_3(x))$

the prescribed force. Note that the term (div u)$u/2$ is added to ensure the dissipativity of the Navier-Stokes equations [23].

In order to introduce a variational formulation, we set

$$X = \left(H_0^1(\Omega)\right)^3,\ Y = \left(L^2(\Omega)\right)^3,\ M = L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q\ dx = 0\right\}.$$

As mentioned above, a further assumption on Ω is presented:

(A1) Assume that Ω is regular in the sense that the unique solution $(v, q) \in (X, M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g,\ \text{div}\ v = 0 \text{ in } \Omega,\ v|_{\partial\Omega} = 0,$$

for a prescribed $g \in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \le C\|g\|_0,$$

where $C > 0$ is a constant depending on Ω. Here, $\|\cdot\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $\left(H^i(\Omega)\right)^3$ for $i = 0, 1, 2$ [2].

We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ the inner product and norm on $L^2(\Omega)$ or $\left(L^2(\Omega)\right)^3$, respectively. The spaces $H_0^1(\Omega)$ and X are equipped with their usual norm and scalar product $\|u\|_1 = \|\nabla u\|_0$ and $(\nabla u, \nabla v)$. In addition, there holds

$$\|u\|_0 \le C_0\|u\|_1, \quad u \in H_0^1(\Omega) \text{ or } X. \tag{2.3}$$

In order to present the discrete variational problem of (2.1) and (2.2), the continuous bilinear forms $a(\cdot,\cdot)$ on $X \times X$ and $d(\cdot,\cdot)$ on $X \times M$, respectively, are defined by

$$a(u, v) = \nu((u, v))\ \forall u, v \in X,\ d(v, q) = -(v, \nabla q) = (q, \text{div} v)\ \forall v \in X,\ q \in M,$$

and a generalized bilinear form on $(X, M) \times (X, M)$ is defined by

$$\mathcal{B}((u, p); (v, q)) = a(u, v) - d(v, p) + d(u, q).$$

Then there hold the following estimates for the term $\mathcal{B}((\cdot,\cdot); (\cdot,\cdot))$ [15, 23]:

$$|\mathcal{B}((u, p); (u, p)) = \nu\|u\|_1^2, \tag{2.4}$$

$$|\mathcal{B}((u, p); (v, q))| \le C(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0), \tag{2.5}$$

$$\beta_0(\|u\|_1 + \|p\|_0) \le \sup_{(v,q)\in(X,M)} \frac{|\mathcal{B}((u, p); (v, q))|}{\|v\|_1 + \|q\|_0}, \tag{2.6}$$

for all $(u, p),\ (v, q) \in (X, M)$ and constant $\beta_0 > 0$.

A trilinear term is defined by

$$\begin{aligned} b(u; v, w) &= ((u \cdot \nabla)v, w) + \frac{1}{2}((\text{div} u)v, w) \\ &= \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v)\ \forall u, v, w \in X, \end{aligned}$$

and satisfies [15, 23]

$$b(u; v, w) = -b(u; w, v), \tag{2.7}$$

$$|b(u; v, w)| \le C_1\|u\|_1\|v\|_1\|w\|_1, \tag{2.8}$$

for all $u, v, w \in X$ and

$$|b(u; v, w)| + |b(v; u, w)| + |b(w; u, v)| \le C_1\|u\|_1\|v\|_2\|w\|_0, \tag{2.9}$$

for all $u \in X, v \in \left(H_0^2(\Omega)\right)^3, w \in Y$, where $C_0,\ C_1, \ldots$ are positive constants depending only on the domain Ω.

Now, the mixed variational form of (2.1) and (2.2) is to seek $(u,p) \in (X,M)$ such that

$$\mathcal{B}((u,p);(v,q)) + b(u;u,v) = (f,v) \quad \forall (v,q) \in X \times M. \tag{2.10}$$

The existence and uniqueness results can be found in [15, 23].

We introduce a finite dimensional subspace pair $(X_h, M_h) \subset (X, M)$, which is characterized by K_h, a partition of Ω into tetrahedra in R^3, assumed to be regular in the usual sense and satisfy the usual inverse inequality and the approximation property (2.19) [9, 12, 15].

A stable and accurate solution to (2.10) requires that (X_h, M_h) satisfies the discrete inf-sup condition

$$\sup_{v_h \in X_h} \frac{d(v_h, q_h)}{\|v_h\|_1} \geq \beta_1 \|q_h\|_0, \tag{2.11}$$

where $\beta_1 > 0$ is a constant independent of h. However, it is not valid for the following lower order finite element pairs:

$$X_h = \{v \in X : \ v_i \in P_1(K), \ i = 1,2,3, \ \forall K \in K_h\},$$

$$M_h = \{q \in M : \ q \in R_j(K), \ j = 0,1 \ \forall K \in K_h\},$$

where $R_j(K)$, $j = 0,1$, is the set of polynomials of degree j defined on K. Fortunately, the local polynomial pressure projection method efficiently stabilizes the lower order finite element pairs by using the local L^2-projection [5]:

$$\Pi_j = \begin{cases} \Pi_0 : \ L^2 \to R_1, \ j = 0, \\ \Pi_1 : \ L^2 \to R_0, \ j = 1. \end{cases}$$

Under the above notation, the variational formulation of the problem (2.10) reads as follows: Find $(\bar{u}_h, \bar{p}_h) \in (X_h, M_h)$ such that, for all $(v,q) \in (X_h, M_h)$,

$$\mathcal{B}((\bar{u}_h, \bar{p}_h);(v,q)) + b(\bar{u}_h;\bar{u}_h,v) + G(\bar{p}_h, q) = (f,v), \tag{2.12}$$

where the stabilization term $G(\cdot,\cdot)$ is defined by

$$G(\bar{p}_h, q) = (\bar{p}_h - \Pi_j \bar{p}_h, q - \Pi_j q),$$

with the following properties:

$$(p, q_h) = (\Pi_h p, q_h) \qquad \forall p \in M, \ q_h \in R_j, \tag{2.13}$$

$$\|\Pi_j p\|_0 \leq C \|p\|_0 \qquad \forall p \in M, \tag{2.14}$$

$$\|p - \Pi_j p\|_0 \leq Ch \|p\|_1 \qquad \forall p \in H^1(\Omega) \cap M. \tag{2.15}$$

The following stabilization theorem establishes the weak coercivity of (2.12) for the lower order finite element pairs [5, 16].

Theorem 2.1 (Stability and convergence). *Let (X_h, M_h) be the above lower order finite element space pairs. Then there exists a positive constant β_2, independent of h, satisfying*

(2.16)

$$|\mathcal{B}_h((u,p);(v,q))| \leq C(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0) \ \forall (u,p), \ (v,q) \in (X,M),$$

(2.17)

$$\beta_2(\|\bar{u}_h\|_1 + \|\bar{p}_h\|_0) \leq \sup_{(v_h,q_h) \in (X_h,M_h)} \frac{|\mathcal{B}_h((\bar{u}_h,\bar{p}_h);(v_h,q_h))|}{\|v_h\|_1 + \|q_h\|_0} \ \forall (\bar{u}_h, \bar{p}_h) \in (X_h, M_h),$$

$$|G(p,q)| \leq C \|p\|_0 \|q\|_0 \ \forall p, q \in M, \tag{2.18}$$

where

$$\mathcal{B}_h((u,p);(v,q)) = \mathcal{B}((u,p);(v,q)) + G(p,q).$$

By the fixed-point theorem for the classical Galerkin method, there exists the unique finite element solution $(\bar{u}_h, \bar{p}_h)$ for (2.12) such that [16]

$$\|u-\bar{u}_h\|_0 + h(\|u-\bar{u}_h\|_1 + \|p-\bar{p}_h\|_0) \leq Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0). \tag{2.19}$$

3. Stabilized Finite Volume Approximation

This section is concentrated on the stabilized finite volume approximation for the three-dimensional stationary Navier-Stokes equations. Based on the finite element partition K_h, we connect the barycenter in each element $K \in K_h$ and the midpoint on each of the edges of K, and construct the control volumes in R^2 by connecting all these barycenters and midpoints. Likewise, we first choose an arbitrary point Q in the interior of each tetrahedron in K_h and then connect Q with the barycenters Q_{ijk} of its 2D faces $\Delta P_i P_j P_k$ by straight lines (see Fig. 1). On each face $\Delta P_i P_j P_k$, we connect Q_{ijk} by straight lines with the middle points of the segments P_iP_j, P_jP_k, and P_kP_i, respectively. Then the contribution of K_h to a control volume in $\tilde{K}_h$ of a vertex P in K_h is the volume surrounding Q by these straight lines; for example, the contribution from one simplex to a control volume in $\tilde{K}_h$ with the interfaces γ_{12} and γ_{13} is shown in Fig. 1.

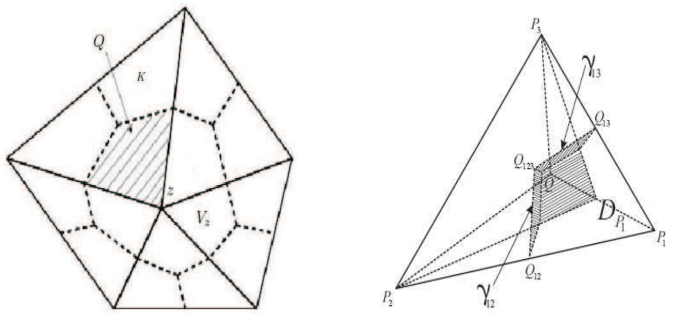

Fig.1. Comparison of the dual volumes in $R^d, d = 2, 3$.

Associated with $\tilde{K}_h$, the dual finite element space is defined by

$$\tilde{X}_h = \left\{ \tilde{v} \in \left(L^2(\Omega)\right)^3 : \tilde{v}|_{\tilde{K}} \in \left(P_0(\tilde{K})\right)^3 \ \forall \tilde{K} \in \tilde{K}_h;\ \tilde{v}|_{\partial\tilde{K}} = 0 \right\}.$$

Obviously, the dimensions of X_h and $\tilde{X}_h$ are the same since they have the same degree of freedoms and vertexes. Furthermore, there exists an invertible linear mapping $\Gamma_h :\ X_h \rightarrow \tilde{X}_h$ such that, for

$$v_h(x) = \sum_{j=1}^{N} v_h(P_j)\varphi_j(x), \qquad x \in \Omega,\ v_h \in X_h, \tag{3.1}$$

we have

$$\Gamma_h v_h(x) = \sum_{j=1}^{N} v_h(P_j)\chi_j(x), \qquad x \in \Omega,\ v_h \in X_h, \tag{3.2}$$

where N denotes the set of all vertices of element $K \in K_h$ except from those on $\partial\Omega$, and $\{\varphi_j\}_{j=0}^{N}$ and $\{\chi_j\}_{j=0}^{N}$ indicate the bases of the finite element space X_h and the finite volume space $\tilde{X}_h$, respectively. The latter are the characteristic functions associated with the dual partition $\tilde{K}_h$:

$$\chi_j(x) = \begin{cases} 1 & \text{if } x \in \tilde{K}_j \in \tilde{K}_h, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the mapping Γ_h has the following properties [22, 26]:

$$\int_K (v_h - \Gamma_h v_h) dx = 0, \tag{3.3}$$

$$\|\Gamma_h v_h\|_0 \le C_2 \|v_h\|_0, \ \|v_h - \Gamma_h v_h\|_{0,r,K} \le C_3 h_K \|v_h\|_{1,r,K}, \tag{3.4}$$

where h_K is the diameter of the element K.

The above idea of connecting the trial and test spaces in the Petrov-Galerkin method through the mapping Γ_h was first introduced in [26] in the context of elliptic problems. The main procedure is presented as follows: Multiplying equation (2.1) by $\Gamma_h v_h \in \tilde{X}_h$ and integrating over the dual elements $\tilde{K} \in \tilde{K}_h$, multiplying equation (2.2) by $q_h \in M_h$ and integrating over the primal elements $K \in K_h$, and applying Green's formula for both equations, we obtain the following bilinear forms for the finite volume method:

$$A(u_h, \Gamma_h v_h) = -\nu \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} \frac{\partial u_h}{\partial n} ds, \quad u_h, \ v_h \in X_h,$$

$$D(\Gamma_h v_h, p_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} p_h n \ ds, \quad v_h \in X_h, \ p_h \in M_h,$$

$$(f, \Gamma_h v_h) = \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\tilde{K}_j} f \ dx, \quad v_h \in X_h,$$

where n is the unit normal outward to $\partial \tilde{K}_j$. Using a technique similar to that in the trilinear form of the finite element method in the previous section, we define the trilinear form $b(\cdot;\cdot,\cdot): \ X_h \times X_h \times \tilde{X}_h \to \Re$ of the finite volume method [21, 23]:

$$b(u_h; v_h, \Gamma_h w_h) = \left((u_h \cdot \nabla) v_h + \frac{1}{2} (\text{div } u_h) v_h, \Gamma_h w_h \right) \quad \forall u_h, v_h, w_h \in X_h.$$

Now, the new stabilized finite volume method for the Navier-Stokes equations (2.1) and (2.2) is: Find $(u_h, p_h) \in (X_h, M_h)$ such that

$$\mathcal{C}_h((u_h, p_h), (v_h, q_h)) + b(u_h; u_h, \Gamma_h v_h) = (f, \Gamma_h v_h) \quad \forall (v_h, q_h) \in (X_h, M_h), \tag{3.5}$$

where we define the bilinear form $\mathcal{C}_h(\cdot,\cdot)$ on $(X_h, M_h) \times (X_h, M_h)$:

$$\mathcal{C}_h((u_h, p_h), (v_h, q_h)) = A(u_h, \Gamma_h v_h) + D(\Gamma_h v_h, p_h) + d(u_h, q_h) + G_h(p_h, q_h). \tag{3.6}$$

The next results establish a relationship between the finite element and finite volume methods [11, 17, 27] with respect to the linear terms:

$$A(u_h, \Gamma_h v_h) = a(u_h, v_h) \qquad \forall u_h, \ v_h \in X_h, \tag{3.7}$$

$$D(\Gamma_h v_h, q_h) = -\, d(v_h, q_h) \qquad \forall (v_h, q_h) \in (X_h, M_h). \tag{3.8}$$

Thus we will pay more attention to the difference of the trilinear term $b(\cdot;\cdot,\cdot)$ between the two methods. Moreover, the following result on the continuity and weak coercivity of the bilinear form $\mathcal{C}_h(\cdot,\cdot)$ can be directly deduced from the relationship between the two methods and Theorem 2.1.

Lemma 3.1 (Stability). *It holds that*

$$|\mathcal{C}_h((u_h,p_h),(v_h,q_h))| \le C\,(\|u_h\|_1 + \|p_h\|_0)\,(\|v_h\|_1 + \|q_h\|_0) \quad \forall (u_h,p_h),\ (v_h,q_h) \in (X_h,M_h). \tag{3.9}$$

Moreover,

$$\sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{C}_h((u_h,p_h),(v_h,q_h))|}{\|v_h\|_1 + \|q_h\|_0} \ge \beta_3\,(\|u_h\|_1 + \|p_h\|_0) \quad \forall (u_h,p_h) \in (X_h,M_h), \tag{3.10}$$

where $\beta_3 > 0$ is independent of h.

We are now in a position to show the well posedness of system (3.5). For this we define the mesh parameter

$$h_0(h) = \frac{4C_0C_2C_3C_4}{\nu^2}|\log h|^{1/2} h\|f\|_0.$$

Theorem 3.2 (Existence and uniqueness). *For each $h > 0$ such that*

$$0 < h_0 \le 1/2, \tag{3.11}$$

system (3.5) *admits a solution $(u_h,p_h) \in (X_h,M_h)$. Moreover, if the viscosity $\nu > 0$, the body force $f \in Y$, and the mesh size $h > 0$ satisfy*

$$1 - \frac{4C_1C_1C_2}{\nu^2}\|f\|_0 > 0, \tag{3.12}$$

then the solution $(u_h,p_h) \in (X_h,M_h)$ is unique. Furthermore, it satisfies

$$\|u_h\|_1 \le \frac{2C_0C_2}{\nu}\|f\|_0,\ \|p_h\|_0 \le 2\beta_3^{-1}C_0C_2\|f\|_0\left(1 + \frac{2C_0C_1C_2\|f\|_0}{\nu^2}\right). \tag{3.13}$$

Proof. For fixed $f \in Y$, we introduce the set

$$B_M = \Bigg\{(v_h,q_h) \in (X_h,M_h):\ \|u_h\|_1 \le \frac{2C_0C_2}{\nu}\|f\|_0, \\ \|p_h\|_0 \le \frac{2C_0C_2}{\beta_3}\|f\|_0\left(1 + \frac{2C_0C_1C_2\|f\|_0}{\nu^2}\right)\Bigg\}.$$

The mapping $T_h:\ (X_h,M_h) \to (X_h,M_h)$ is defined by

$$T_h(\tilde{v}_h,p_h) = (u_h,p_h)$$

such that

$$\mathcal{C}_h((T_h\tilde{v}_h,p_h),(v_h,q_h)) + b(\tilde{v}_h;T_h\tilde{v}_h,\Gamma_h v_h) = (f,\Gamma_h v_h), \quad (v_h,q_h) \in (X_h,M_h). \tag{3.14}$$

Then, substituting $(v_h,q_h) = (u_h,p_h) \in (X_h,M_h)$ into (3.14) and using (2.7), we see that

$$\mathcal{C}_h((u_h,p_h),(u_h,p_h)) + b(\tilde{v}_h;u_h,\Gamma_h u_h - u_h) = (f,\Gamma_h u_h). \tag{3.15}$$

Obviously,

$$|\mathcal{C}_h((u_h,p_h),(u_h,p_h))| \ge \nu\|u_h\|_1^2. \tag{3.16}$$

Nothing that

$$\|\phi_h\|_{L^\infty} \le C_4|\log h|^{1/2}\|\phi_h\|_1, \tag{3.17}$$

and using the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
|b(\tilde{v_h}; u_h, \Gamma_h u_h - u_h)| &\leq \left(\|\tilde{v}_h\|_{L^\infty} \|u_h\|_1 + \frac{\sqrt{3}}{3} \|\tilde{v}_h\|_1 \|u_h\|_{L^\infty} \right) \|\Gamma_h u_h - u_h\|_0 \\
&\leq 2C_3C_4 |\log h|^{1/2} h \|\tilde{v}_h\|_1 \|u_h\|_1^2 \\
&\leq \frac{4C_0C_2C_3C_4}{\nu} |\log h|^{1/2} h \|f\|_0 \|u_h\|_1^2 \\
&\leq \nu h_0 \|u_h\|_1^2, \qquad (3.18)
\end{aligned}
$$

$$
|(f, \Gamma_h u_h)| \leq \|f\|_0 \|\Gamma_h u_h\|_0 \leq C_0C_2 \|f\|_0 \|u_h\|_1. \qquad (3.19)
$$

Thus we see that

$$
\nu \left(1 - h_0\right) \|u_h\|_1 \leq C_0C_2 \|f\|_0. \qquad (3.20)
$$

By a direct computation,

$$
\|u_h\|_1 \leq \frac{2C_0C_2}{\nu} \|f\|_0. \qquad (3.21)
$$

Now, we deduce from (3.10), and the same approach as for (3.18) and (3.19) that

$$
\begin{aligned}
\|u_h\|_1 + \|p_h\|_0 &\leq \beta_3^{-1} \sup_{(v_h,q_h) \in (X_h, M_h)} \frac{\mathcal{C}_h((u_h, p_h), (v_h, q_h))}{\|v_h\|_1 + \|q_h\|_0} \\
&\leq \beta_3^{-1} \left(\nu h_0 \|u_h\|_1 + C_1 \|\tilde{v}_h\|_1 \|u_h\|_1 + C_0C_2 \|f\|_0 \right) \\
&\leq 2\beta_3^{-1} C_0C_2 \|f\|_0 \left(1 + \frac{2C_0C_1C_2\|f\|_0}{\nu^2}\right). \qquad (3.22)
\end{aligned}
$$

Since the mapping T_h is well defined, there exists a solution to system (3.5) by Brouwer's fixed point theorem.

To prove uniqueness, assume that (u_1, p_1) and (u_2, p_2) are two solutions to (3.5). Then we see that

(3.23)
$$
\mathcal{C}_h((u_1 - u_2, p_1 - p_2), (v_h, q_h)) + b(u_1 - u_2; u_1, \Gamma_h v_h) + b(u_2; u_1 - u_2, \Gamma_h v_h) = 0.
$$

Using the same approach as for (3.18) and (2.8) and setting $(v_h, q_h) = (u_1 - u_2, p_1 - p_2) = (e, \eta)$, we see that

$$
\begin{aligned}
\nu \|e\|_1^2 + G(\eta, \eta) &\leq C_1 \|u_1\|_1 \|e\|_1^2 + 2\nu h_0 \|e\|_1^2 \\
&\leq \left(\frac{2C_0C_1C_2}{\nu} \|f\|_0 + 2\nu h_0 \right) \|e\|_1^2 \\
&\leq \frac{4C_0C_1C_2}{\nu} \|f\|_0 \|e\|_1^2, \qquad (3.24)
\end{aligned}
$$

with $4C_3C_4|\log h|^{1/2} h \leq C_1$. Then it follows that

$$
0 \leq \nu \left(1 - \frac{4C_0C_1C_2}{\nu^2} \|f\|_0\right) \|e\|_1^2 \leq 0. \qquad (3.25)
$$

Thus $u_1 = u_2$. Next, applying (3.10) to (3.23) yields that $p_1 = p_2$. Therefore, it follows that (3.5) has a unique solution. #

We state the error estimates for the finite volume method for the three-dimensional stationary Navier-Stokes equations. A similar proof can be given as in the R^2 case [21].

Theorem 3.3 (Convergence). *Let* $(u,p) \in (X,M)$ *and* $(u_h,p_h) \in (X_h,M_h)$ *be the solution of* (2.10) *and* (3.5), *respectively. Then it holds*

$$\|u-u_h\|_1 + \|p-p_h\|_0 \le Ch(\|u\|_2 + \|p\|_1 + \|f\|_0), \tag{3.26}$$

$$\|u-u_h\|_0 \le Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_1). \tag{3.27}$$

4. Superconvergence by L^2-Projection

In this section, the main results are shown about the stabilized finite volume method for the stationary Navier-Stokes equations by using the L^2-projection. The optimal L^2- and H^1-norm error estimates are obtained for the velocity and the optimal L^2-error estimate is derived for the pressure.

The postprocessing technique introduced by Wang and Ye [24] is to project the finite volume solution to another finite element space with a different mesh. The difference in the two mesh sizes can be used to achieve superconvergence after the postprocessing procedure.

Let τ_i, $i=1,2$, be another two finite element partitions with mesh sizes τ_i, respectively, where $h << \tau_i$ $(i=1,2)$. Assume that τ_i and h have the following relation:

$$\tau_i = h^{\alpha_i},\ i=1,2,$$

where $\alpha_1,\ \alpha_2 \in (0,1)$. The parameters α_i will play an important role later in achieving superconvergence for the stabilized finite volume approximation (u_h,p_h). Let X_{τ_1} and M_{τ_2} be any two finite element spaces consisting of piecewise polynomials of degree s and t, respectively, associated with the partitions τ_1 and τ_2.

Next, we introduce the notation of L^2-projections. Let $Q_{\tau_1} : \left(L^2(\Omega)\right)^2 \to X_{\tau_1}$ and $Q_{\tau_2} : L^2(\Omega) \to M_{\tau_2}$ be defined by

$$(Q_{\tau_1}u, v_h) = (u, v_h)\ \forall u \in \left(L^2(\Omega)\right)^2,\ v_h \in X_{\tau_1}, \tag{4.1}$$

$$(Q_{\tau_2}p, q_h) = (p, q_h)\ \forall p \in L^2(\Omega),\ q_h \in M_{\tau_2}. \tag{4.2}$$

We will provide error estimates for $u - Q_{\tau_1}u_h$ and $p - Q_{\tau_2}p_h$ in the following theorems.

In order to analyze the bound of $u - Q_{\tau_1}u_h$, the dual problem to consider for the stationary Navier-Stokes equations is to seek $(\Phi,\Psi) \in X \times M$ such that for any $\phi \in Y$

$$\mathcal{B}((v,q);(\Phi,\Psi)) + b(v;u,\Phi) + b(u;v,\Phi) = (v, Q_{\tau_1}\phi), \tag{4.3}$$

whose solution satisfies the regularity

$$\|\Phi\|_2 + \|\Psi\|_1 \le C\|Q_{\tau_1}\phi\|_0. \tag{4.4}$$

The L^2-norm error estimate for the velocity of the finite volume approximation of the stationary Navier-Stokes equations is more difficult than that of the finite element method since a complicated trilinear term is involved and the test function and the trial function are defined in different finite dimensional spaces with different meshes. Below an additional duality argument is applied to analyze the L^2-norm estimate for the velocity under the Petrov-Galerkin system. Moreover, noting that the postprocessed solution of the velocity and pressure are separately derived without using the so-called inf-sup condition.

Theorem 4.1. *Under the assumptions of Theorems* 3.2 *and* 3.3, *if* τ_1, h, *and* α_1 *are taken as*

$$\tau_1 = O(h^{\alpha_1})\ \textit{with}\ \alpha_1 = \frac{2}{s+1},$$

then the postprocessed solution $Q_{\tau_1}u_h$ *satisfies the following error estimates:*

$$\|u - Q_{\tau_1}u_h\|_i \le Ch^{\frac{2}{1+\theta_i}}(\|u\|_{s+1} + \|p\|_1 + \|f\|_1), \ i = 0, 1, \tag{4.5}$$

with θ_i *given by*

$$\theta_i = \frac{i}{1+s-i}.$$

Proof. Multiplying equation (2.1) by $\Gamma_h v_h \in \tilde{X}_h$ and integrating over the dual elements $\tilde{K} \in \tilde{K}_h$, multiplying (2.2) by $q_h \in M_h$ and integrating over the finite elements $K \in K_h$, and using (3.5), we see that

$$\begin{aligned} &A(u-u_h, \Gamma_h v_h) + D(\Gamma_h v_h, p - p_h) + d(u-u_h, q_h) + G(p-p_h, q_h) \\ &+b(u-u_h; u, \Gamma_h v_h) + b(u; u-u_h, \Gamma_h v_h) - b(u-u_h; u-u_h, \Gamma_h v_h) = G(p, q_h). \end{aligned} \tag{4.6}$$

Substituting $(v_h, q_h) = (\Phi_h, \Psi_h)$ into (4.6) and using (4.3) with $(v, q) = (\Phi, \Psi)$, we find that

$$\begin{aligned} (e, Q_{\tau_1}\phi) =& a(e, \Phi - \Phi_h) - d(\Phi - \Phi_h, \eta) + d(e, \Psi - \Psi_h) - G(\eta, \Psi_h) + G(p, \Psi_h) \\ &+ a(e, \Phi_h) - A(e, \Gamma_h) + d(\Phi_h, \eta) - D(\Gamma_h, \eta) \\ &+ b(e; u, \Phi - \Gamma_h\Phi_h) + b(u; e, \Phi - I_h\Phi) - b(e; e, \Gamma_h\Phi_h). \end{aligned} \tag{4.7}$$

Due to the definition of the L^2-projection Q_{τ_1}, the Cauchy-Schwartz inequality, and the property of the finite element approximation, we have

$$(e, Q_{\tau_1}\phi) = (Q_{\tau_1}e, \phi), \tag{4.8}$$

$$\begin{aligned} &|a(e, \Phi - \Phi_h) - d(\Phi - \Phi_h, \eta) + d(e, \Psi - \Psi_h) - G(\eta, \Psi_h) + G(p, \Psi_h)| \\ \le& C(\|e\|_1 + \|\eta\|_0)(\|\Phi - \Phi_h\|_0 + \|\Psi - \Psi_h\|_0 + \|\Psi_h - \Pi_j\Psi_h\|_0) \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)(\|\Phi\|_2 + \|\Psi\|_1) \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)\|Q_{\tau_1}\phi\|_0 \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)\|\phi\|_0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &|b(e; u, \Phi - \Gamma_h\Phi) + b(u; e, \Phi - \Gamma_h\Phi)| \\ \le& C\|e\|_1\|u\|_2\|\Phi - \Gamma_h\Phi_h\|_0 \\ \le& C\|e\|_1\|u\|_2(\|\Phi - \Phi_h\|_0 + \|\Phi_h - \Gamma_h\Phi_h\|_0) \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)\|\Phi\|_2 \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)\|Q_{\tau_1}\phi\|_0 \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)\|\phi\|_0. \end{aligned} \tag{4.10}$$

Also, noting that

$$\|\phi_h\|_{L^4} \le \sqrt{2}\|\phi_h\|_0^{3/4}\|\phi_h\|_1^{1/4} \tag{4.11}$$

and using the Cauchy-Schwartz inequality and the inverse inequality, it follows that [21]

$$\begin{aligned} b(e; e, \Gamma_h\Phi_h) =& |b(e; e, \Phi - \Gamma_h\Phi) + b(e; e, \Phi)| \\ \le& C\|e\|_{L^4}\|\nabla e\|_0\|\Phi - \Gamma_h\Phi\|_{L^4} + c_0\|e\|_1\|e\|_1\|\Phi\|_1 \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)\|\Phi\|_2 \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)\|Q_{\tau_1}\phi\|_0 \\ \le& Ch^2(\|u\|_2 + \|p\|_1 + \|f\|_0)\|\phi\|_0. \end{aligned} \tag{4.12}$$

In addition, we deduce from (2.1) and (3.3) that

$$\begin{aligned} &|a(e,\Phi_h)-A(e,\Gamma_h\Phi_h)+d(\Phi_h,\eta)-D(\Gamma_h\Phi_h,\eta)| \\ =&|(f-(u\cdot\nabla)u-\Pi_j(f-(u\cdot\nabla)u),\Phi_h-\Gamma_h\Phi_h)| \\ \le& Ch^2\|f\|_1\|Q_{\tau_1}\phi\|_1\le Ch^2\|f\|_1\|\phi\|_0. \end{aligned} \tag{4.13}$$

Combining (4.8)–(4.13) with (4.7), we obtain

$$\|Q_{\tau_1}e\|_0\le Ch^2(\|u\|_2+\|p\|_1+\|f\|_1). \tag{4.14}$$

The gradient term can be estimated by the inverse inequality as follows:

$$\|\nabla_{\tau_1}(Q_{\tau_1}e)\|_0\le C\tau_1^{-1}\|Q_{\tau_1}e\|_0\le C\tau_1^{-1}h^2(\|u\|_2+\|p\|_1+\|f\|_1). \tag{4.15}$$

The definition of Q_{τ_1} gives

$$\|u-Q_{\tau_1}u\|_0+\tau_1\|u-Q_{\tau_1}u\|_1\le C\tau_1^{s+1}\|u\|_{s+1}, \tag{4.16}$$

which, together with (4.15), yields (4.5). #

In order to analyze the error $p-Q_{\tau_2}p_h$ in the L^2-norm, the same duality argument as for Theorem 4.1 is used: For some given $\psi\in L^2(\Omega)$, we define

$$\mathcal{B}((v,q);(\omega,\xi))+b(v;u,\omega)+b(u;v,\omega)=(q,Q_{\tau_2}\psi), \tag{4.17}$$

for all $(v,q)\in X\times M$. It satisfies the regularity

$$\|\omega\|_2+\|\xi\|_1\le C\|Q_{\tau_2}\psi\|_1. \tag{4.18}$$

Then a similar result can be derived for the pressure component.

Theorem 4.2. *Assume that τ_2, h, and α_2 are taken as*

$$\tau_2=O(h^{\alpha_2})\ \textit{with}\ \alpha_2=\frac{2}{t+2},$$

then the postprocessed solution $Q_{\tau_2}p_h$ satisfies the following error estimate:

$$\|p-Q_{\tau_2}p_h\|_0\le Ch^{\frac{2(t+1)}{t+2}}(\|u\|_2+\|p\|_{t+1}+\|f\|_1). \tag{4.19}$$

Proof. Subtracting (4.6) from (4.17) by taking $(v,q)=(e,\eta)$ in (4.17) and $(v,q)=(I_h\omega,J_h\xi)$ in (4.6), we obtain

$$\begin{aligned} (\eta,Q_{\tau_2}\psi)=&\mathcal{B}((e,\eta);(\omega-\omega_h,\xi-\xi_h))+b(e;u,\omega-\Gamma_h\omega_h)+b(u;e,\omega-\Gamma_h\omega_h) \\ &+a(u,\omega_h)-A(u,\Gamma_h\omega_h)-d(\omega_h,p)-D(\Gamma_h\omega_h,p) \\ &+b(e;e,\omega_h-\Gamma_h\omega_h)-b(e;e,\omega_h)+G(p,\xi_h). \end{aligned} \tag{4.20}$$

Similar to the proof in Theorem 4.1, we have

$$(\eta,Q_{\tau_2}\psi)=(Q_{\tau_2}\eta,\psi), \tag{4.21}$$

$$\begin{aligned} \mathcal{B}((e,\eta);(\omega-\omega_h,\xi-\xi_h))\le& C(\|e\|_1+\|\eta\|_0)(\|\omega-\omega_h\|_1+\|\xi-\xi_h\|_0) \\ \le& Ch^2(\|u\|_2+\|p\|_1+\|f\|_0)(\|\omega\|_2+\|\xi\|_1) \\ \le& Ch^2(\|u\|_2+\|p\|_1+\|f\|_0)\|Q_{\tau_2}\psi\|_1 \\ \le& Ch^2\tau_2^{-1}(\|u\|_2+\|p\|_1+\|f\|_0)\|\psi\|_0, \end{aligned} \tag{4.22}$$

$$\begin{aligned} |G(p,\xi_h)|=&G(p,\xi_h-\xi)+G(p,\xi) \\ \le& C\|p-\Pi_jp\|_0(\|\xi_h-\xi\|_0+\|\xi-\Pi_j\xi\|_0) \\ \le& Ch^2\|p\|_1\|\xi\|_1\le Ch^2\|p\|_1\|Q_{\tau_2}\psi\|_1 \\ \le& Ch^2\tau_2^{-1}\|p\|_1\|\psi\|_0. \end{aligned} \tag{4.23}$$

As for the trilinear terms, it follows by the same approach as for (4.12) that

$$
\begin{aligned}
&|b(e;u,\omega-\Gamma_h\omega_h)+b(u;e,\omega-\Gamma_h\omega_h)|\\
\le& C\|e\|_1\|u\|_2\|\omega-\Gamma_h\omega_h\|_0\\
\le& C\|e\|_1\|u\|_2(\|\omega-\omega_h\|_0+\|\omega_h-\Gamma_h\omega_h\|_0)\\
\le& Ch^2(\|u\|_2+\|p\|_1+\|f\|_0)\|\omega\|_2\\
\le& Ch^2(\|u\|_2+\|p\|_1+\|f\|_0)\|Q_{\tau_2}\psi\|_1\\
\le& Ch^2(\|u\|_2+\|p\|_1+\|f\|_0)\tau_2^{-1}\|\psi\|_0,
\end{aligned} \tag{4.24}
$$

$$
\begin{aligned}
&b(e;e,\Gamma_h\omega_h)\\
=&|b(e;e,\omega_h-\Gamma_h\omega_h)+b(e;e,\omega_h)|\\
\le& C\|e\|_{L^4}\|\nabla e\|_0\|\omega-\Gamma_h\omega_h\|_{L^4}+c\|e\|_1\|e\|_1\|\omega_h\|_1\\
\le& Ch^2(\|u\|_2+\|p\|_1+\|f\|_0)\|\omega\|_2\\
\le& Ch^2(\|u\|_2+\|p\|_1+\|f\|_0)\|Q_{\tau_1}\psi\|_1\\
\le& C\tau_2^{-1}h^2(\|u\|_2+\|p\|_1+\|f\|_0)\|\psi\|_0.
\end{aligned} \tag{4.25}
$$

Similarly,

$$
\begin{aligned}
&|a(u,\omega_h)-A(u,\Gamma_h\omega_h)-d(\omega_h,p)-D(\Gamma_h\omega_h,p)|\\
=&|(f-(u\cdot\nabla)u-\Pi_j(f-(u\cdot\nabla)u),\omega_h-\Gamma_h\omega_h)|\\
\le& Ch^2\|f\|_1\|Q_{\tau_1}\psi\|_1\le Ch^2\|f\|_1\|\psi\|_0.
\end{aligned} \tag{4.26}
$$

The definition of Q_{τ_2} gives

$$
\|p-Q_{\tau_2}p\|\le C\tau_2^{t+1}\|p\|_{t+1}. \tag{4.27}
$$

Finally, combing (4.21)–(4.27) with (4.20) yields the desired result. #

Observed from Theorems 4.1 and 4.2, there is a superconvergence result for the velocity in the H^1-norm if $s\ge 2$ and one for the pressure in the L^2-norm in the case $t\ge 1$. However, no improvement can be made for the velocity in the L^2-norm. Furthermore, the postprocessed solutions of the stabilized finite volume method provided in Theorems 4.1 and 4.2 achieve the same supconvergence results as the stabilized finite element method in the R^2 case [20] for the three-dimensional Navier-Stokes equations studied in the present paper.

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