INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 9, Number 1, Pages 56–72



# AN OPTIMAL-ORDER ERROR ESTIMATE FOR A FINITE DIFFERENCE METHOD TO TRANSIENT DEGENERATE ADVECTION-DIFFUSION EQUATIONS

TONGCHAO LU AND JINHONG JIA

**Abstract.** We prove an optimal-order error estimate in a degenerate-diffusion weighted energy norm for implicit Euler and Crank-Nicolson finite difference methods to two-dimensional time-dependent advection-diffusion equations with degenerate diffusion. In the estimate, the generic constants depend only on certain Sobolev norms of the true solution but not on the lower bound of the diffusion. This estimate, combined with a known stability estimate of the true solution of the governing partial differential equations, yields an optimal-order estimate of the finite difference methods, in which the generic constants depend only on the Sobolev norms of the initial and right-hand side data.

**Key Words.** convergence analysis, degenerate advection-diffusion equations, finite difference methods, optimal-order error estimates

## 1. Introduction

In this article we study a classical problem of an optimal-order error estimate for the numerical methods for time-dependent advection-diffusion equations with degenerate diffusion. Time-dependent nonlinear degenerate advection-diffusion equations typically arise in a coupled system of partial differential equations that models immiscible displacement of oil by water in secondary oil recovery processes and the movement of non-aqueous phase liquid in groundwater in subsurface porous media [2, 3, 13, 17, 22]. The time-dependent linear degenerate advection-diffusion equation studied in this paper is a linearized version of the nonlinear problems.

Optimal-order error estimates for Galerkin finite element methods to nondegenerate parabolic or advection-diffusion equations dated back to 1970s via the introduction of Ritz projection [44]. Optimal-order error estimates for Eulerian-Lagrangian finite element or finite difference methods to transient advection-diffusion equations were proved in [10, 11, 26, 30, 39]. Optimal-order error estimates for the coupled systems of advection-diffusion equations with the coupled pressure equation can be found in [8, 15, 27, 36]. The advantage of these estimates is that they are valid for any regular partition. However, because the approximation property of Ritz projection depends on the Peclet number of the problem, so these error estimates also depend on the Peclet number of the problem and could potentially blow up as the lower bound of the diffusion approaches to zero. These estimates do not fully reflect the utility of the methods which were observed computationally [1, 14, 28, 29].

Received by the editors August 18, 2009 and, in revised form, December 21, 2009.
2000 *Mathematics Subject Classification.* 65N06, 65N15.

$\varepsilon$ uniform error estimates were sought. In the context of steady-state advection-diffusion equations, optimal-order error estimates were obtained for finite element or difference methods on Shishkin meshes [16, 23]. In the context of Eulerian-Lagrangian methods for transient advection-diffusion equations, $\varepsilon$ uniform error estimates in [4, 21, 31, 32, 33, 35, 43, 41, 42]. Recently, a uniformly optimal-order error estimate was proved for a Eulerian-Lagrangian method for time-dependent advection-diffusion equations with degenerate diffusion [38, 41]. In the context of finite element methods to time-dependent advection-diffusion equation, an $\varepsilon$-uniform optimal-order error estimate was proved in [20]. The authors proved a uniformly optimal-order error estimate for a finite element method for degenerate convection-diffusion equations [18].

In this paper we prove an optimal-order error estimate for a space-centered finite difference method with implicit Euler or Crank-Nicolson temporal discretization for time-dependent advection-diffusion equations with degenerate diffusion. In the estimate, the generate constants depend only on certain Sobolev norms of the true solution but not on the lower bound of the diffusion. This estimate, combined with a known stability estimate of the true solution of the governing partial differential equations in [41], yields an optimal-order estimate of the finite element method, in which the generic constants depend only on the Sobolev norms of the initial data and right-hand side data. The rest of this article is organized as follows. In section 2, we formulate the problem and recall preliminary results that are to be used in the paper. In section 3, we prove the optimal-order error estimate for the problem. In section 4, we summarize the results and draw concluding remarks. In section 5, we prove auxiliary lemma that are used in the analysis.

## 2. Problem Formulation and Preliminaries

**2.1. Model problem.** We consider a time-dependent convection-diffusion equation in two space-dimensions

$$
\begin{array}{rcll}
u_t + \nabla\cdot\big(\boldsymbol{v}(\boldsymbol{x},t)u - D(\boldsymbol{x},t)\nabla u\big) & = & f(\boldsymbol{x},t), & (\boldsymbol{x},t)\in\Omega\times(0,T], \\
u(\boldsymbol{x},t) & = & 0, & (\boldsymbol{x},t)\in\Gamma\times(0,T], \\
u(\boldsymbol{x},0) & = & u_0(\boldsymbol{x}), & \boldsymbol{x}\in\Omega.
\end{array}
\tag{1}
$$

Here $\Omega = (a,b)\times(c,d)$ is a rectangular domain, $\Gamma := \partial\Omega$ is the spatial boundary of $\Omega$. $\boldsymbol{x} = (x,y)$, $\boldsymbol{v} = (V_1(\boldsymbol{x},t), V_2(\boldsymbol{x},t))$ is a velocity field, $f(\boldsymbol{x},t)$ accounts for external sources and sinks, $u_0(\boldsymbol{x})$ is a prescribed initial data, and $u(\boldsymbol{x},t)$ is the unknown solute concentration of a dissolved function. $D(\boldsymbol{x},t)$ is a diffusion coefficient with $0 \le D(\boldsymbol{x},t) \le D_{max} < +\infty$ for any $(\boldsymbol{x},t)\in\Omega\times[0,T]$.

**2.2. Preliminaries.** Let $W_p^k(\Omega)$ consist of functions whose weak derivatives up to order-$k$ are $p$-th Lebesgue integrable in $\Omega$, and $H^k(\Omega) := W_2^k(\Omega)$. Let $H_0^1(\Omega) := \{v\in H^1(\Omega) : v(\boldsymbol{x}) = 0, \boldsymbol{x}\in\Gamma\}$. For any Banach space $X$, we introduce Sobolev spaces involving time [12]

$$
\begin{aligned}
W_p^k(t_1,t_2;X) \quad &:= \Big\{ f : \Big\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \Big\|_X \in L^p(t_1,t_2),\ 0 \le l \le k,\ 1 \le p \le \infty \Big\}, \\
\|f\|_{W_p^k(t_1,t_2;X)} \quad &:= \begin{cases} \Big( \displaystyle\sum_{l=0}^{k} \int_{t_1}^{t_2} \Big\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \Big\|_X^p dt \Big)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\max_{0 \le l \le k} \text{ess} \sup_{t \in (t_1,t_2)} \Big\| \frac{\partial^l f}{\partial t^l}(\cdot,t) \Big\|_X, & p = \infty. \end{cases}
\end{aligned}
$$

We define a uniform space-time partition on $\overline{\Omega} \times [0,T]$ by $x_i := a + ih_x$ for $i = 0,1,\ldots,I$ with $h_x := (b-a)/I$; $y_j := c + jh_y$ for $j = 0,1,\ldots,J$ with $h_y := (d-c)/J$; and $t^n := n\Delta t$ for $0 \le n \le N$ with $\Delta t := T/N$. We let $h := (h_x^2 + h_y^2)^{\frac{1}{2}}$ and assume that the partition is quasiuniform. We let function $f(\boldsymbol{x},t)$ is defined only at discrete space-time nodal points $(x_i, y_j, t^n)$. This allows the definition of the following discrete norms

$$
\begin{aligned}
\|f(\cdot,t^n)\|_{\hat{L}^2(\Omega)} \quad &:= \Big( \sum_{j=1}^{J-1} \sum_{i=1}^{I-1} f^2(x_i,y_j,t^n) h_x h_y \Big)^{1/2}, \\
\|f(\cdot,t^n)\|_{\hat{H}_D^1(\Omega)} \quad &:= \Big( \sum_{j=1}^{J} \sum_{i=1}^{I} D(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}, t^n) \big| \nabla f(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}, t^n) \big|^2 h_x h_y \Big)^{1/2}, \\
|||f|||_{\tilde{L}(0,T;\hat{H}_D^1(\Omega))} \quad &:= \max_{0 \le n \le N} \|f(\cdot,t^n)\|_{\hat{L}^2(\Omega)} + \Big( \sum_{n=1}^{N} \|f(\cdot,t^n)\|_{\hat{H}_D^1(\Omega)}^2 \Delta t \Big)^{1/2}, \\
|||f|||_{\hat{L}(0,T;\hat{H}_D^1(\Omega))} \quad &:= \max_{0 \le n \le N} \|f(\cdot,t^n)\|_{\hat{L}^2(\Omega)} + \Big( \sum_{n=1}^{N-1} \Big\| \frac{f(\cdot,t^{n+1}) + f(\cdot,t^n)}{2} \Big\|_{\hat{H}_D^1(\Omega)}^2 \Delta t \Big)^{1/2}.
\end{aligned}
$$

Here, we adopted the notations $x_{i-\frac{1}{2}} := \frac{x_{i-1}+x_i}{2}$ , $y_{j-\frac{1}{2}} := \frac{y_{j-1}+y_j}{2}$.

In this paper we use $C$ to denote a general positive constant which could assume different values at different occurrences.

**2.3. The Finite Difference Scheme.** We introduce the notation of the nodal point values

$$
u_{i,j}^n := u(x_i,y_j,t^n), \quad D_{i\pm\frac{1}{2},j}^n := D(x_{i\pm\frac{1}{2}},y_j,t^n), \quad D_{i,j\pm\frac{1}{2}}^n := D(x_i,y_{j\pm\frac{1}{2}},t^n)
$$

and we present the weighted first- and second-order difference quotients as

$$
\begin{aligned}
\delta_{\hat{x}}(V_1u)_{i,j}^n \quad &:= \frac{(V_1u)_{i+1,j}^n - (V_1u)_{i-1,j}^n}{2h_x}, \\
\delta_{\hat{y}}(V_2u)_{i,j}^n \quad &:= \frac{(V_2u)_{i,j+1}^n - (V_2u)_{i,j-1}^n}{2h_y}, \qquad (2) \\
\nabla_c(\boldsymbol{v}u)_{i,j}^n \quad &:= \delta_{\hat{x}}(V_1u)_{i,j}^n + \delta_{\hat{y}}(V_2u)_{i,j}^n.
\end{aligned}
$$

$$
\begin{aligned}
\delta_{\tilde{x}}(D\delta_x u)^n_{i,j} &:= \frac{D^n_{i+\frac{1}{2},j}(u^n_{i+1,j}-u^n_{i,j}) - D^n_{i-\frac{1}{2},j}(u^n_{i,j}-u^n_{i-1,j})}{h_x^2},\\
\delta_{\tilde{y}}(D\delta_y u)^n_{i,j} &:= \frac{D^n_{i,j+\frac{1}{2}}(u^n_{i,j+1}-u^n_{i,j}) - D^n_{i,j-\frac{1}{2}}(u^n_{i,j}-u^n_{i,j-1})}{h_y^2},\\
\nabla_h(D\nabla_h u)^n_{i,j} &:= \delta_{\tilde{x}}(D\delta_x u)^n_{i,j} + \delta_{\tilde{y}}(D\delta_y u)^n_{i,j}.
\end{aligned}
\tag{3}
$$

We approximate the time and space derivatives by implicit Euler difference quotient and weighted first- and second-order difference quotients defined by (2) and (3), leading to a finite difference formulation for problem (1): For $n = 0, 1, \ldots, N-1$, seek $u(\boldsymbol{x}, t) \in H_0^1$ such that

$$
\begin{aligned}
&\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} + \theta\Big(\nabla_c(\boldsymbol{v}u)^{n+1}_{i,j} - \nabla_h(D\nabla_h u)^{n+1}_{i,j}\Big)\\
&\qquad = -(1-\theta)\Big(\nabla_c(\boldsymbol{v}u)^n_{i,j} - \nabla_h(D\nabla_h u)^n_{i,j}\Big) + f^{n+\theta}_{i,j} + R^{n+\theta}_{i,j},\\
&\qquad 1 \le i \le I-1, \quad 1 \le j \le J-1, \quad 0 \le \theta \le 1.
\end{aligned}
\tag{4}
$$

Here $R^{n+\theta}_{i,j}$ is the local truncation error of the finite difference scheme and can be denoted as $R^{n+\theta}_{i,j} = R^{n+\theta}_{t,i,j} + R^{n+\theta}_{s,i,j}$, $R^{n+\theta}_{s,i,j} = \theta R^{n+1}_{s,i,j} + (1-\theta)R^n_{s,i,j}$, where

$$
R^{n+1}_{t,i,j} = -\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} (t - t^n)\frac{\partial^2 u}{\partial t^2}dt.
$$

$$
\begin{aligned}
R^{n+\frac{1}{2}}_{t,i,j} = &\frac{1}{2\Delta t}\Big(\int_{t^{n+\frac{1}{2}}}^{t^{n+1}} (t^{n+1}-t)^2\frac{\partial^3 u}{\partial t^3}dt + \int_{t^n}^{t^{n+\frac{1}{2}}} (t-t^n)^2\frac{\partial^3 u}{\partial t^3}dt\Big)\\
&+\frac{1}{2}\Big(\int_{t^{n+\frac{1}{2}}}^{t^{n+1}} (t^{n+1}-t)\frac{\partial^2}{\partial t^2}\Big(\nabla\cdot(\boldsymbol{v}(\boldsymbol{x},t)u - D(\boldsymbol{x},t)\nabla u)\Big)dt\\
&\quad+\int_{t^n}^{t^{n+\frac{1}{2}}} (t-t^n)\frac{\partial^2}{\partial t^2}\Big(\nabla\cdot(\boldsymbol{v}(\boldsymbol{x},t)u - D(\boldsymbol{x},t)\nabla u)\Big)dt\Big).
\end{aligned}
$$

$$
R^{n+1}_{s,i,j} = R^{n+1}_{1,i,j} + R^{n+1}_{2,i,j},
$$

$$
\begin{aligned}
R^{n+1}_{1,i,j} = &\frac{1}{4h_x}\Big(\int_{x_{i-1}}^{x_i} (x-x_{i-1})^2\frac{\partial^3(V_1 u)}{\partial x^3}dx + \int_{x_i}^{x_{i+1}} (x_{i+1}-x)^2\frac{\partial^3(V_1 u)}{\partial x^3}dx\Big)\\
&+\frac{1}{4h_y}\Big(\int_{y_{j-1}}^{y_j} (y-y_{j-1})^2\frac{\partial^3(V_2 u)}{\partial y^3}dy + \int_{y_j}^{y_{j+1}} (y_{j+1}-y)^2\frac{\partial^3(V_2 u)}{\partial y^3}dy\Big).
\end{aligned}
$$

The local truncation error $R^{n+1}_{2,i,j}$ can be expressed as $R^{n+1}_{2,i,j} = R^{n+1}_{21,i,j} + R^{n+1}_{22,i,j}$, where

$$
\begin{aligned}
R^{n+1}_{21,i,j} = &-\frac{1}{2h_x}\Big(\int_{x_i}^{x_{i+\frac{1}{2}}}(x_{i+\frac{1}{2}}-x)^2\frac{\partial^3}{\partial x^3}(D\frac{\partial u}{\partial x})dx \\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}(x-x_{i-\frac{1}{2}})^2\frac{\partial^3}{\partial x^3}(D\frac{\partial u}{\partial x})dx\Big) \\
&-\frac{D^{n+1}_{i+\frac{1}{2},j}}{2h_x^2}\Big(\int_{x_{i+\frac{1}{2}}}^{x_{i+1}}(x_{i+1}-x)^2\frac{\partial^3 u}{\partial x^3}dx+\int_{x_i}^{x_{i+\frac{1}{2}}}(x-x_i)^2\frac{\partial^3 u}{\partial x^3}dx\Big) \\
&+\frac{D^{n+1}_{i-\frac{1}{2},j}}{2h_x^2}\Big(\int_{x_{i-\frac{1}{2}}}^{x_i}(x_i-x)^2\frac{\partial^3 u}{\partial x^3}dx+\int_{x_{i-1}}^{x_{i-\frac{1}{2}}}(x-x_{i-1})^2\frac{\partial^3 u}{\partial x^3}dx\Big).
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
R^{n+1}_{22,i,j} = &-\frac{1}{2h_y}\Big(\int_{y_j}^{y_{i+\frac{1}{2}}}(y_{j+\frac{1}{2}}-y)^2\frac{\partial^3}{\partial y^3}(D\frac{\partial u}{\partial y})dy \\
&+\int_{y_{j-\frac{1}{2}}}^{y_j}(y-y_{j-\frac{1}{2}})^2\frac{\partial^3}{\partial y^3}(D\frac{\partial u}{\partial y})dy\Big) \\
&-\frac{D^{n+1}_{i,j+\frac{1}{2}}}{2h_y^2}\Big(\int_{y_{j+\frac{1}{2}}}^{y_{j+1}}(y_{i+1}-y)^2\frac{\partial^3 u}{\partial y^3}dy+\int_{y_j}^{y_{j+\frac{1}{2}}}(y-y_j)^2\frac{\partial^3 u}{\partial y^3}dy\Big) \\
&+\frac{D^{n+1}_{i,j-\frac{1}{2}}}{2h_y^2}\Big(\int_{y_{j-\frac{1}{2}}}^{y_j}(y_j-y)^2\frac{\partial^3 u}{\partial y^3}dy+\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}(y-y_{j-1})^2\frac{\partial^3 u}{\partial y^3}dy\Big).
\end{aligned}
$$

## 3. Error Estimate for Implicit Euler Finite Difference Method

Let $U$ denote a function defined only at discrete space-time nodal points $(x_i, y_j, t^n)$. The implicit Euler finite difference method states as follows: Find $U$, for $n = 0, 1, \cdots, N-1$, such that

$$
\begin{aligned}
&\frac{U^{n+1}_{i,j}-U^n_{i,j}}{\Delta t}+\nabla_c(\boldsymbol{v}U)^{n+1}_{i,j}-\nabla_h(D\nabla_h U)^{n+1}_{i,j}=f^{n+1}_{i,j}, \\
&\qquad 1\le i\le I-1, \quad 1\le j\le J-1.
\end{aligned}
\tag{6}
$$

**Theorem 3.1.** *Assume $D, \boldsymbol{v} \in L^\infty(0,T;W^4_\infty)$, $f \in L^2(0,T;H^3)$, and $u_o \in H^3(\Omega)$. Let $u$ be the true solution to problem (1) and $U$ be the corresponding implicit Euler finite difference solution determined by the scheme (6). Then the following optimal-order and superconvergence error estimate holds*

$$
|||U-u|||_{\tilde{L}(0,T;\hat{H}^1_D)} \le C\Delta t\|u\|_{H^2(0,T;L^2)}+Ch^2\|u\|_{L^\infty(0,T;H^4)}.
\tag{7}
$$

*Here the constant $C$ is independent of $u$, $h$, or $\Delta t$.*

*Proof.* We let $e^n_{i,j} = U^n_{i,j} - u^n_{i,j}$, and subtract (4) with $\theta = 1$ from (6) to obtain the relation:

$$
\frac{e^{n+1}_{i,j}-e^n_{i,j}}{\Delta t}+\nabla_c(\boldsymbol{v}e)^{n+1}_{i,j}-\nabla_h(D\nabla_h e)^{n+1}_{i,j}=-R^{n+1}_{i,j},
\tag{8}
$$

We multiply (8) by $e^{n+1}_{i,j}h_xh_y$ and sum the resulting terms for $i = 1, 2, \ldots, I-1$ and $j = 1, 2, \ldots, J-1$, to get an error equation:

$$
\begin{aligned}
&\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}(e_{i,j}^{n+1})^2 h_x h_y - \Delta t \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \nabla_h (D\nabla_h e)_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y \\
&\qquad = \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} e_{i,j}^{n} e_{i,j}^{n+1} h_x h_y - \Delta t \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \nabla_c (\boldsymbol{v}e)_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y \\
&\qquad\qquad - \Delta t \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} R_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y.
\end{aligned}
\tag{9}
$$

The error equation $e_{i,j}^{n+1}$ vanishes on $\Gamma$ for the Dirichlet boundary condition.

We first sum the diffusion term by parts. By symmetry, we need only to consider its $x$ components:

$$
\begin{aligned}
&\sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \delta_{\bar{x}} (D\delta_x e)_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y \\
&\quad = \sum_{j=1}^{J-1} \frac{1}{h_x^2} \Big( \sum_{i=1}^{I} D_{i-\frac{1}{2},j}^{n+1} (e_{i,j}^{n+1} - e_{i-1,j}^{n+1}) e_{i-1,j}^{n+1} \\
&\qquad\qquad - \sum_{i=1}^{I} D_{i-\frac{1}{2},j}^{n+1} (e_{i,j}^{n+1} - e_{i-1,j}^{n+1}) e_{i,j}^{n+1} \Big) h_x h_y \\
&\qquad\qquad\qquad = - \sum_{j=1}^{J-1}\sum_{i=1}^{I} D_{i-\frac{1}{2},j}^{n+1} \Big( \frac{e_{i,j}^{n+1} - e_{i-1,j}^{n+1}}{h_x} \Big)^2 h_x h_y.
\end{aligned}
$$

Thus, the second term on the left-hand side of (9) can be denoted as:

$$
-\Delta t \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \nabla_h (\nabla_h e)_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y = \Delta t \|e^{n+1}\|_{\hat{H}_D^1}^2.
$$

Now we begin to estimate the right-hand side of (9) term by term. The first term on the right-hand side of (9) can be bounded by Cauchy inequality

$$
\Big| \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} e_{i,j}^{n+1} e_{i,j}^{n} h_x h_y \Big| \le \frac{1}{2}\|e^{n+1}\|_{\hat{L}^2}^2 + \frac{1}{2}\|e^{n}\|_{\hat{L}^2}^2.
$$

We decompose the second term on the right side of (9) as follows:

$$
\begin{aligned}
\Big| -\Delta t \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \nabla_c (\boldsymbol{v}e)_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y \Big| &\le \Delta t \Big| \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \delta_{\hat{x}} (V_1 e)_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y \Big| \\
&\quad + \Delta t \Big| \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \delta_{\hat{y}} (V_2 e)_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y \Big|.
\end{aligned}
\tag{10}
$$

Next we need only to concentrate on the first term on the right-hand side of (10) since the second term can be bounded by symmetry. We sum this term by parts to obtain

$$
\begin{aligned}
&\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\delta_{\hat{x}}(V_1e)_{i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| \\
&= \Delta t\Big|\sum_{j=1}^{J-1}\Big(\sum_{i=1}^{I-1}\frac{V_{1i+1,j}^{n+1}e_{i+1,j}^{n+1}e_{i,j}^{n+1}}{2h_x} - \sum_{i=1}^{I-1}\frac{V_{1i-1,j}^{n+1}e_{i-1,j}^{n+1}e_{i,j}^{n+1}}{2h_x}\Big)h_xh_y\Big| \\
&= \Delta t\Big|\sum_{j=1}^{J-1}\Big(\sum_{i=2}^{I-1}\frac{V_{1i,j}^{n+1}e_{i,j}^{n+1}e_{i-1,j}^{n+1}}{2h_x} - \sum_{i=2}^{I-1}\frac{V_{1i-1,j}^{n+1}e_{i,j}^{n+1}e_{i-1,j}^{n+1}}{2h_x}\Big)h_xh_y\Big| \\
&= \Delta t\Big|\sum_{j=1}^{J-1}\Big(\sum_{i=2}^{I-1}\frac{V_{1i,j}^{n+1}-V_{1i-1,j}^{n+1}}{2h_x}e_{i,j}^{n+1}e_{i-1,j}^{n+1}h_x\Big)h_y\Big| \\
&\le C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2
\end{aligned} \tag{11}
$$

Combining this concludes

$$
\Delta t\Big|-\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\nabla_c(\boldsymbol{v}e)_{i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| \le C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2.
$$

To derive an optimal order error estimate, we use Lemma 6.1 to bound the third term on the right-hand side of (9)

$$
\begin{aligned}
\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| \le{}& C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2 + C(\Delta t)^2\|u\|_{H^2(t^n,t^{n+1};L^2)}^2 \\
&+C\Delta t h^4\|u\|_{L^\infty(0,T;H^4)}^2.
\end{aligned} \tag{12}
$$

We incorporate the preceding estimates into (9) to obtain

$$
\begin{aligned}
\|e^{n+1}\|_{\hat{L}^2}^2 + \Delta t\|e^{n+1}\|_{\hat{H}_D^1}^2 \le{}& \frac{1+C\Delta t}{2}(\|e^{n+1}\|_{\hat{L}^2}^2 + \|e^n\|_{\hat{L}^2}^2) \\
&+C(\Delta t)^2\|u\|_{H^2(t^n,t^{n+1};L^2)}^2 + C\Delta t h^4\|u\|_{L^\infty(0,T;H^4)}.
\end{aligned}
$$

We sum the above estimates for $n = 1, 2, \ldots, N_1 - 1 (N_1 \le N)$ and cancel like terms to obtain

$$
\begin{aligned}
\|e^{N_1}\|_{\hat{L}^2}^2 + \Delta t\sum_{n=1}^{N_1}\|e^n\|_{\hat{H}_D^1}^2 \le{}& C\Delta t\sum_{n=1}^{N_1-1}\|e^n\|_{\hat{L}^2}^2 + C(\Delta t)^2\|u\|_{H^2(0,T;L^2)} \\
&+Ch^4\|u\|_{L^\infty(0,T;H^4)}.
\end{aligned}
$$

We apply Gronwall inequality to conclude

$$
|||e|||_{\tilde{L}(0,T;\hat{H}_D^1)} \le C\Delta t\|u\|_{H^2(0,T;L^2)} + Ch^2\|u\|_{L^\infty(0,T;H^4)}.
$$

Above all, we have finished the proof. □

## 4. Error Estimate for Crank-Nicolson Finite Difference Method

In this section, we assume the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ and the diffusion coefficient $D(\boldsymbol{x},t)$ are independent of time. The Crank-Nicolson finite difference method states as follows: Find $U$, for $n = 0, 1, \cdots, N-1$ such that

$$
\begin{aligned}
&\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} - \frac{1}{2}\Big(\nabla_h (D\nabla_h U)_{i,j}^{n+1} + \nabla_h (D\nabla_h U)_{i,j}^{n}\Big) \\
&\qquad\qquad = -\frac{1}{2}\Big(\nabla_c (\boldsymbol{v}U)_{i,j}^{n+1} + \nabla_c (\boldsymbol{v}U)_{i,j}^{n}\Big) + f_{i,j}^{n+\frac{1}{2}}, \\
&\qquad\qquad 1 \le i \le I-1, \quad 1 \le j \le J-1.
\end{aligned} \tag{13}
$$

**Theorem 4.1.** *Assume $D, \boldsymbol{v} \in L^\infty(0,T;W^4_\infty)$, $f \in L^2(0,T;H^3)$, and $u_0 \in H^3(\Omega)$. Let $u$ be the true solution to problem (1) and $U$ be the corresponding Crank-Nicolson finite difference solution determined by the scheme (13). Then the following optimal-order and superconvergence error estimate holds*

$$
|||U - u|||_{\hat{L}(0,T;\hat{H}^1_D)} \le C(\Delta t)^2 (\|u\|_{H^3(0,T;L^2)} + \|u\|_{H^2(0,T;H^2)}) + Ch^2 \|u\|_{L^\infty(0,T;H^4)}.
$$

*Here the constant $C$ is independent of $u$, $h$, or $\Delta t$.*

*Proof.* We let $e_{i,j}^n = U_{i,j}^n - u_{i,j}^n$, subtract (4) with $\theta = 1/2$ form (13), to get the relation

$$
\begin{aligned}
&\frac{e_{i,j}^{n+1} - e_{i,j}^{n}}{\Delta t} - \frac{1}{2}\Big(\nabla_h (D\nabla_h e)_{i,j}^{n+1} + \nabla_h (D\nabla_h e)_{i,j}^{n}\Big) \\
&\qquad\qquad = -\frac{1}{2}\Big(\nabla_c (\boldsymbol{v}e)_{i,j}^{n+1} + \nabla_c (\boldsymbol{v}e)_{i,j}^{n}\Big) - R_{i,j}^{n+\frac{1}{2}}.
\end{aligned} \tag{14}
$$

We multiply (14) by $\dfrac{e_{i,j}^{n+1} + e_{i,j}^{n}}{2} h_x h_y$, and sum the resulting formulation for $i = 1, 2, \ldots, I-1$ and $j = 1, 2, \ldots, J-1$, to obtain:

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1} (e_{i,j}^{n+1})^2 h_x h_y \\
&\quad - \frac{\Delta t}{2}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \Big(\nabla_h (D\nabla_h e)_{i,j}^{n+1} + \nabla_h (D\nabla_h e)_{i,j}^{n}\Big) \frac{e_{i,j}^{n+1} + e_{i,j}^{n}}{2} h_x h_y \\
&= \frac{1}{2}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1} (e_{i,j}^{n})^2 h_x h_y \\
&\qquad - \frac{\Delta t}{2}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \Big(\nabla_c (\boldsymbol{v}e)_{i,j}^{n+1} + \nabla_c (\boldsymbol{v}e)_{i,j}^{n}\Big) \frac{e_{i,j}^{n+1} + e_{i,j}^{n}}{2} h_x h_y \\
&\qquad\qquad - \Delta t \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} R_{i,j}^{n+\frac{1}{2}} \frac{e_{i,j}^{n+1} + e_{i,j}^{n}}{2} h_x h_y.
\end{aligned} \tag{15}
$$

Similarly, we first sum the diffusion term on the left-hand side of (15) by parts. By symmetry, we need only to consider its $x$ components:

$$
\begin{aligned}
&\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\frac{1}{h_x^2}\Big(D_{i+\frac{1}{2},j}\Big(\frac{e_{i+1,j}^{n+1}+e_{i+1,j}^{n}}{2}-\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}\Big)\\
&\quad=\sum_{j=1}^{J-1}\frac{1}{h_x^2}\Big(\sum_{i=1}^{I}D_{i-\frac{1}{2},j}\Big(\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}-\frac{e_{i-1,j}^{n+1}+e_{i-1,j}^{n}}{2}\Big)\frac{e_{i-1,j}^{n+1}+e_{i-1,j}^{n}}{2}h_x\\
&\qquad-\sum_{i=1}^{I}D_{i-\frac{1}{2},j}\Big(\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}-\frac{e_{i-1,j}^{n+1}+e_{i-1,j}^{n}}{2}\Big)\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_x\Big)h_y\\
&\qquad\quad=-\sum_{j=1}^{J-1}\sum_{i=1}^{I}D_{i-\frac{1}{2},j}\Big(\frac{\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}-\frac{e_{i-1,j}^{n+1}+e_{i-1,j}^{n}}{2}}{h_x}\Big)^2h_xh_y.
\end{aligned}
$$

Thus, the second term on the left-hand side of (15) can be expressed by:

$$
\begin{aligned}
&-\frac{\Delta t}{2}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(\nabla_h(D\nabla_h e)_{i,j}^{n+1}+\nabla_h(D\nabla_h e)_{i,j}^{n}\Big)\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\\
&\quad=\Delta t\|\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}\|_{\hat{H}_D^1}^2.
\end{aligned}
$$

Now we begin to estimate the right-hand side of (15) term by term. The first term can be expressed by

$$
\frac{1}{2}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}(e_{i,j}^{n})^2h_xh_y\Big|=\frac{1}{2}\|e^n\|_{\hat{L}^2}^2.
$$

We decompose the second term on the right-hand side of (16) to obtain:

$$
\begin{aligned}
&-\frac{\Delta t}{2}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(\nabla_c(\boldsymbol{v}e)_{i,j}^{n+1}+\nabla_c(\boldsymbol{v}e)_{i,j}^{n}\Big)\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\\
&\quad=-\frac{\Delta t}{2}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(\delta_{\hat{x}}(V_1e)_{i,j}^{n+1}+\delta_{\hat{x}}(V_1e)_{i,j}^{n}\Big)\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\\
&\qquad-\frac{\Delta t}{2}\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(\delta_{\hat{y}}(V_2e)_{i,j}^{n+1}+\delta_{\hat{y}}(V_2e)_{i,j}^{n}\Big)\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y.
\end{aligned}
$$

We need only to concentrate on the first term on the right-hand side since the second term can be bounded by symmetry. We sum this term by parts

$$\begin{aligned}
&\frac{\Delta t}{2}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(\delta_{\hat{x}}(V_1e)_{i,j}^{n+1}+\delta_{\hat{x}}(V_1e)_{i,j}^{n}\Big)\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big|\\
&=\frac{\Delta t}{2}\Big|\sum_{j=1}^{J-1}\Big(\sum_{i=1}^{I-1}\frac{V_{1i+1,j}}{2h_x}\frac{e_{i+1,j}^{n+1}+e_{i+1,j}^{n}}{2}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}\\
&\quad-\sum_{i=1}^{I-1}\frac{V_{1i-1,j}}{2h_x}\frac{e_{i-1,j}^{n+1}+e_{i-1,j}^{n}}{2}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}\Big)h_xh_y\Big|\\
&=\frac{\Delta t}{2}\Big|\sum_{j=1}^{J-1}\sum_{i=2}^{I-1}\frac{V_{1i,j}-V_{1i-1,j}}{2h_x}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}\frac{e_{i-1,j}^{n+1}+e_{i-1,j}^{n}}{2}h_xh_y\Big|\\
&\le C\Delta t\|\frac{e^{n+1}+e^{n}}{2}\|_{\hat{L}^2}^2\\
&\le C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2+C\Delta t\|e^{n}\|_{\hat{L}^2}^2.
\end{aligned}$$

Thus, the second term on the right-hand side of (15) can be bounded by

$$\begin{aligned}
&\frac{\Delta t}{2}\Big|-\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(\nabla_c(\boldsymbol{v}e)_{i,j}^{n+1}+\nabla_c(\boldsymbol{v}e)_{i,j}^{n}\Big)\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big|\\
&\qquad\qquad\le C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2+C\Delta t\|e^{n}\|_{\hat{L}^2}^2.
\end{aligned}\tag{16}$$

To derive an optimal-order error estimate, we should pay more attention to the third term on the right-hand side of (15)

$$\begin{aligned}
\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{i,j}^{n+\frac{1}{2}}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big| &\le \Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{t,i,j}^{n+\frac{1}{2}}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big|\\
&\quad+\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+\frac{1}{2}}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big|.
\end{aligned}\tag{17}$$

$$\begin{aligned}
&\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{t,i,j}^{n+\frac{1}{2}}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big|\\
&\le C(\Delta t)^{5/2}(\|u\|_{H^3(t^n,t^{n+1};L^2)}+\|u\|_{H^2(t^n,t^{n+1};H^2)})\|e^{n+1}+e^{n}\|_{\hat{L}^2}\\
&\le C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2+C\Delta t\|e^{n}\|_{\hat{L}^2}^2\\
&\qquad+C(\Delta t)^4(\|u\|_{H^3(t^n,t^{n+1};L^2)}^2+\|u\|_{H^2(t^n,t^{n+1};H^2)}^2).
\end{aligned}\tag{18}$$

According to the expression of $R_{s,i,j}^{n+\theta}$, we decompose the second term on the left-hand side of (17) as follows

$$
\begin{aligned}
&\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+\frac{1}{2}}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big| \\
&=\frac{\Delta t}{4}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y+\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+1}e_{i,j}^{n}h_xh_y \\
&\quad+\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n}e_{i,j}^{n+1}h_xh_y+\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n}e_{i,j}^{n}h_xh_y\Big| \\
&\le\frac{\Delta t}{4}\Big(\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big|+\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+1}e_{i,j}^{n}h_xh_y\Big| \\
&\quad+\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n}e_{i,j}^{n+1}h_xh_y\Big|+\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n}e_{i,j}^{n}h_xh_y\Big|\Big).
\end{aligned}
\tag{19}
$$

The first term on the right-hand side of (19) can be estimated in Lemma 6.1

$$\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big|\le C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2+C\Delta th^4\|u\|_{L^\infty(0,T;H^4)}^2.$$

We can use Lemma 6.1 to estimate the second term on the right-hand side of (19)

$$\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+1}e_{i,j}^{n}h_xh_y\Big|\le C\Delta t\|e^{n}\|_{\hat{L}^2}^2+C\Delta th^4\|u\|_{L^\infty(0,T;H^4)}^2.$$

Accordingly, we can estimate the remaining two terms on the right-hand side of (19) by Lemma 6.1.
Thus, we get the following estimate

$$
\begin{aligned}
&\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+\frac{1}{2}}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big| \\
&\qquad\le C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2+C\Delta t\|e^{n}\|_{\hat{L}^2}^2+C\Delta th^4\|u\|_{L^\infty(0,T;H^4)}^2.
\end{aligned}
\tag{20}
$$

We incorporate (18) and (20) into the two terms on the right-hand side of (15) to get the following estimate

$$
\begin{aligned}
&\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{i,j}^{n+\frac{1}{2}}\frac{e_{i,j}^{n+1}+e_{i,j}^{n}}{2}h_xh_y\Big| \\
&\le C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2+C\Delta t\|e^{n}\|_{\hat{L}^2}^2+C\Delta th^4\|u\|_{L^\infty(0,T;H^4)}^2 \\
&\qquad+C(\Delta t)^4(\|u\|_{H^3(t^n,t^{n+1};L^2)}^2+\|u\|_{H^2(t^n,t^{n+1};H^2)}^2).
\end{aligned}
$$

By combining all the preceding estimates we get

$$
\begin{aligned}
&\|e^{n+1}\|_{\hat{L}^2}^2 + \Delta t \|\frac{e^{n+1}+e^n}{2}\|_{\hat{H}_D^1}^2 \\
&\le \frac{1+C\Delta t}{2}(\|e^{n+1}\|_{\hat{L}^2}^2 + \|e^n\|_{\hat{L}^2}^2) + C\Delta t h^4 \|u\|_{L^\infty(0,T;H^4)}^2 \\
&\qquad + C(\Delta t)^4(\|u\|_{H^3(t^n,t^{n+1};L^2)}^2 + \|u\|_{H^2(t^n,t^{n+1};H^2)}^2).
\end{aligned}
$$

We sum the above estimates for $n = 1, 2, \ldots, N_1 - 1 (N_1 \le N)$ and cancel like terms to obtain

$$
\begin{aligned}
\|e^{N_1}\|_{\hat{L}^2}^2 + \Delta t \sum_{n=1}^{N_1} \|\frac{e^{n+1}+e^n}{2}\|_{\hat{H}_D^1}^2 &\le C\Delta t \sum_{n=1}^{N_1-1} \|e^n\|_{\hat{L}^2}^2 \\
&+ C(\Delta t)^4(\|u\|_{H^3(0,T;L^2)}^2 + \|u\|_{H^2(0,T;H^2)}^2) + Ch^4\|u\|_{L^\infty(0,T;H^4)}^2.
\end{aligned}
$$

We apply Gronwall inequality to conclude

$$
|||e|||_{\hat{L}(0,T;\hat{H}_D^1)} \le C(\Delta t)^2(\|u\|_{H^3(0,T;L^2)} + \|u\|_{H^2(0,T;H^2)}) + Ch^2\|u\|_{L^\infty(0,T;H^4)}.
$$

This finishes the proof of the theorem. □

## 5. Concluding Remarks

In this article, we proved an optimal-order error estimate for implicit Euler and Crank-Nicolson finite difference methods in a degenerate-diffusion weighted energy norm for degenerate convection-diffusion equations with a Dirichlet boundary condition. The generic constant in the estimate depends only on certain Sobolev norms of the true solution but not on the lower bound of the diffusion. This estimate, combined with the priori stability estimate in [**?**], yields an optimal-order error estimate in the weighted energy norm that holds uniformly with respect to the degenerate diffusion.

## 6. Appendix: Proofs of Auxiliary Lemma

In this section we prove the auxiliary lemma that was used in the proof of the theorems in section 3 and section 4.

**Lemma 6.1.** *Assume $u \in L^\infty(0,T;H^4)$. Then the following superconvergence estimate holds*

$$
\begin{aligned}
\Delta t \Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1} R_{i,j}^{n+1} e_{i,j}^{n+1} h_x h_y\Big| &\le C\Delta t \|e^{n+1}\|_{\hat{L}^2}^2 + C(\Delta t)^2 \|u\|_{H^2(t^n,t^{n+1};L^2)}^2 \\
&+ C\Delta t h^4 \|u\|_{L^\infty(0,T;H^4)}^2.
\end{aligned}
\tag{21}
$$

*Proof.* We decompose the third term on the left side of (9) as follows:

$$
\begin{aligned}
&\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| \\
&\leq \Delta t\Big(\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{t,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| + \Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{s,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big|\Big) \\
&\leq \Delta t\Big(\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{t,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| + \Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{1,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| \\
&\qquad + \Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{2,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big|\Big).
\end{aligned}
\tag{22}
$$

We bound the first term on the right-hand side of (22) as follows

$$
\begin{aligned}
\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R_{t,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| &= \Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\int_{t^n}^{t^{n+1}}(t-t^n)\frac{\partial^2 u}{\partial t^2}dt e_{i,j}^{n+1}h_xh_y\Big| \\
&\leq C(\Delta t)^{\frac{3}{2}}\|u\|_{H^2(t^n,t^{n+1};L^2)}\|e^{n+1}\|_{\hat{L}^2} \\
&\leq C\Delta t\|e^{n+1}\|_{\hat{L}^2}^2 + C(\Delta t)^2\|u\|_{H^2(t^n,t^{n+1};L^2)}^2.
\end{aligned}
\tag{23}
$$

We now decompose the second term on the right-hand side of (22) as follows:

$$
\begin{aligned}
\Big|R_{1,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| \leq{}& \frac{1}{4h_x}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2\frac{\partial^3(V_1u)}{\partial x^3}dx e_{i,j}^{n+1}h_xh_y\Big| \\
&+\frac{1}{4h_x}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)^2\frac{\partial^3(V_1u)}{\partial x^3}dx e_{i,j}^{n+1}h_xh_y\Big| \\
&+\frac{1}{4h_y}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\int_{y_{j-1}}^{y_j}(y-y_{j-1})^2\frac{\partial^3(V_2u)}{\partial y^3}dy e_{i,j}^{n+1}h_xh_y\Big| \\
&+\frac{1}{4h_y}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\int_{y_j}^{y_{j+1}}(y_{j+1}-y)^2\frac{\partial^3(V_2u)}{\partial y^3}dy e_{i,j}^{n+1}h_xh_y\Big|.
\end{aligned}
\tag{24}
$$

Next, we estimate the first and second term on the right-hand side of (24) as follows

$$
\begin{aligned}
&\frac{1}{4h_x}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2\frac{\partial^3(V_1u)}{\partial x^3}dx e_{i,j}^{n+1}h_xh_y\Big| \\
&\qquad\qquad \leq Ch^2\|u\|_{L^\infty(0,T;H^3)}\|e^{n+1}\|_{\hat{L}^2}.
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&\frac{1}{4h_x}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\int_{x_i}^{x_{i+1}}(x-x_{i-1})^2\frac{\partial^3(V_1u)}{\partial x^3}dx e_{i,j}^{n+1}h_xh_y\Big| \\
&\qquad\qquad \leq Ch^2\|u\|_{L^\infty(0,T;H^3)}\|e^{n+1}\|_{\hat{L}^2}.
\end{aligned}
\tag{26}
$$

Similarly, we can estimate the remaining two terms right-hand side of (24).

Combing (24) - (26) concludes

$$
\begin{aligned}
\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1} R_{1,i,j}^{n+1}e_{i,j}^{n+1}h_xh_y\Big| &\leq C\Delta t h^2\|u\|_{L^\infty(0,T;H^3)}\|e^{n+1}\|_{\hat{L}^2} \\
&\leq C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^3)} + C\Delta t\|e^{n+1}\|^2_{\hat{L}^2}.
\end{aligned} \tag{27}
$$

Combining the expression of the truncation error (13), we estimate the third term on the right-side of (22) term by term

$$
\begin{aligned}
\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\frac{1}{2h_x}\int_{x_i}^{x_{i+\frac12}}(x_{i+\frac12}-x)^2\frac{\partial^3}{\partial x^3}(D\frac{\partial u}{\partial x})dx e_{i,j}^{n+1}h_xh_y\Big| \\
\leq C\Delta t h^2\|u\|_{L^\infty(0,T;H^4)}\|e^{n+1}\|_{\hat{L}^2} \\
\leq C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^4)} + C\Delta t\|e^{n+1}\|^2_{\hat{L}^2}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{\Delta t}{h_x^2}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(D_{i+\frac12,j}^{n+1}\int_{x_{i+\frac12}}^{x_{i+1}}(x_{i+1}-x)^2\frac{\partial^3 u}{\partial x^3}dx \\
&\qquad -D_{i-\frac12,j}^{n+1}\int_{x_{i-\frac12}}^{x_i}(x_i-x)^2\frac{\partial^3 u}{\partial x^3}dx\Big)e_{i,j}^{n+1}h_xh_y\Big| \\
&=\frac{\Delta t}{h_x^2}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(D_{i+\frac12,j}^{n+1}\int_{x_{i+\frac12}}^{x_{i+1}}(x_{i+1}-x)^2\frac{\partial^3 u}{\partial x^3}dx \\
&\qquad -D_{i-\frac12,j}^{n+1}\int_{x_{i+\frac12}}^{x_{i+1}}(x_{i+1}-x)^2\frac{\partial^3 u(x-h_x)}{\partial x^3}dx\Big)e_{i,j}^{n+1}h_xh_y\Big| \\
&=\frac{\Delta t}{h_x}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(\frac{D_{i+\frac12,j}^{n+1}-D_{i-\frac12,j}^{n+1}}{h_x}\int_{x_{i+\frac12}}^{x_{i+1}}(x_{i+1}-x)^2\frac{\partial^3 u}{\partial x^3}dx \\
&\qquad +D_{i-\frac12,j}^{n+1}\int_{x_{i+\frac12}}^{x_{i+1}}(x_{i+1}-x)^2\frac{\frac{\partial^3 u(x)}{\partial x^3}-\frac{\partial^3 u(x-h_x)}{\partial x^3}}{h_x}\Big)e_{i,j}^{n+1}h_xh_y\Big| \\
&\leq C\Delta t h^2(\|u\|_{L^\infty(0.T;H^3)}+\|u\|_{L^\infty(0,T;H^4)})\|e^{n+1}\|_{\hat{L}^2} \\
&\leq C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^4)} + C\Delta t\|e^{n+1}\|^2_{\hat{L}^2}.
\end{aligned} \tag{28}
$$

Accordingly, we have

$$
\begin{aligned}
&\frac{\Delta t}{h_x^2}\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Big(D^{n+1}_{i+\frac12,j}\int_{x_i}^{x_{i+\frac12}}(x-x_i)^2\frac{\partial^3 u}{\partial x^3}dx\\
&\qquad -D^{n+1}_{i-\frac12,j}\int_{x_{i-1}}^{x_{i-\frac12}}(x-x_{i-1})^2\frac{\partial^3 u}{\partial x^3}dx\Big)e^{n+1}_{i,j}h_xh_y\Big| \qquad (29)\\
&\qquad\qquad \le C\Delta t h^2\|u\|_{L^\infty(0.T;H^4)}\|e^{n+1}\|_{\hat L^2}\\
&\qquad\qquad \le C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^4)}+C\Delta t\|e^{n+1}\|^2_{\hat L^2}.
\end{aligned}
$$

Thus, we conclude

$$\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R^{n+1}_{21,i,j}e^{n+1}_{i,j}h_xh_y\Big|\le C\Delta t h^4\|u\|_{L^\infty(0,T;H^4)}+C\Delta t\|e^{n+1}\|^2_{\hat L^2}.$$

Inserting these estimates to the third term on the right-hand side of (22) to obtain

$$\Delta t\Big|\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}R^{n+1}_{2,i,j}e^{n+1}_{i,j}h_xh_y\Big|\le C\Delta t\|e^{n+1}\|^2_{\hat L^2}+C\Delta t h^4\|u\|_{L^\infty(0,T;H^4)}. \qquad (30)$$

Combining all the preceding estimates we have proved the lemma. □

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School of Mathematics, Shandong University, Jinan, Shandong 250100, China
*E-mail*: `lutc@sdu.edu.cn`
*E-mail*: `jhj.jia@mail.sdu.edu.cn`