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STABILITY CRITERIA AND MULTIPLE BIFURCATION ANALYSIS FOR SOME NONLINEAR CONTINUOUS-TIME COUPLED SYSTEMS WITH MULTIPLE DELAYS

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Abstract. A coupled system, which consists of multiple delayed neural network loops, is proposed and a detailed analysis of the asymptotic behavior of the zero solution is included. The stable regions and all possible bifurcations, which depend on multiple parameters, are given in a geometrical way for several specific cases.

Key Words. stability, multiple bifurcations, neural network loops, delay.

1. Introduction

Consider a continuous-time Hopfield neural network with m identical subsystems which have n nonidentical neurons with n delays and no self-connection of the following form:

(1.1)

$$
\begin{array}{ll}
subsystem\ 1 & \left\{\begin{array}{rcl}
x'_{11}(t) &=& -a_1x_{11}(t)+\beta_nf_n(x_{1n}(t-\tau_n))+\epsilon_1g_m(x_{mn}(t-\tau_n))\\
x'_{12}(t) &=& -a_2x_{12}(t)+\beta_1f_1(x_{11}(t-\tau_1))\\
&\vdots&\\
x'_{1n}(t) &=& -a_nx_{1n}(t)+\beta_{n-1}f_{n-1}(x_{1(n-1)}(t-\tau_{n-1}))
\end{array}\right.\\
subsystem\ 2 & \left\{\begin{array}{rcl}
x'_{21}(t) &=& -a_1x_{21}(t)+\beta_nf_n(x_{2n}(t-\tau_n))+\epsilon_2g_1(x_{1n}(t-\tau_n))\\
x'_{22}(t) &=& -a_2x_{22}(t)+\beta_1f_1(x_{21}(t-\tau_1))\\
&\vdots&\\
x'_{2n}(t) &=& -a_nx_{2n}(t)+\beta_{n-1}f_{n-1}(x_{2(n-1)}(t-\tau_{n-1}))
\end{array}\right.\\
\vdots & \\
subsystem\ m & \left\{\begin{array}{rcl}
x'_{m1}(t) &=& -a_1x_{m1}(t)+\beta_nf_n(x_{mn}(t-\tau_n))\\
&& \qquad +\epsilon_mg_{m-1}(x_{n(m-1)}(t-\tau_n))\\
x'_{m2}(t) &=& -a_2x_{m2}(t)+\beta_1f_1(x_{m1}(t-\tau_1))\\
&\vdots&\\
x'_{mn}(t) &=& -a_nx_{mn}(t)+\beta_{n-1}f_{n-1}(x_{m(n-1)}(t-\tau_{n-1}))
\end{array}\right.
\end{array}
$$

where x_{ij} denotes the activation of the jth neuron within the ith subsystem, a_j is the internal decay of the neurons, β_j and ϵ_i are the connection weights, nonnegative integers τ_j denotes the synaptic transmission delays, which corresponds to the time when a signal is emitted by the $(j-1)-$th neuron, and becomes available for the

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$j-$th neuron. For above notations, $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$. $f : \mathbb{R} \to \mathbb{R}$ is the activation function.

In the past twenty years, there have been an increasing interest on the study of the dynamical evolution of nonlinear delayed coupled systems. The attractiveness of nonlinear coupled systems may lie in their possible modeling the interaction dynamics among neurons (such as Hopfield/Cohen-Grossberg neuron networks [7, 14, 15, 16]) or oligopolists (such as Cournot duopoly models [20]) etc. Among the most widely studied phenomena is synchronization, where individual networks oscillate at the same frequency and phase when coupled. According to the learning rules of Hebb [13]: synchronous activation increases the synaptic strength, whereas asynchronous activation decreases the synaptic strength.

It is known that the delay increases the dimensionality, and hence the complexity. It is natural for the inclusion of time delay in the realistic consideration of finite transmission of the interaction, such as the propagation of information through a network node or "synapse". Now great efforts have been made on those domains where delay is not the major factor, or where there occur rich dynamics.

For the study of existence and stability of periodic solutions with spatiotemporal symmetries in delay-coupled neural networks of delay-differential equations, we refer the reader to Refs. [5, 9, 11, 12, 17, 23, 26], where multiple periodic/steady-state solutions can be obtained and observed by equivariant Hopf/fold bifurcations from the trivial zero equilibrium solution. But each of the neurons of the networks is described by a one-dimensional nonlinear differential equation systems.

For the study of existence and stability of periodic solutions in delay-coupled asymmetric neural networks, we refer the reader to Refs. [1, 25], where Hopf/fold bifurcations were discussed and the mechanism of how delay affects neural dynamics and learning is explored[1]. But each of the neurons of the networks is also described by a one-dimensional nonlinear differential equation systems.

Moreover, there is an increasing interest in some nonlinear delayed neural networks coupled by two sub-networks [3, 19, 24]. In [3], the authors discussed the stability and bifurcations in the delayed neural network coupled by a pair of three-neuron sub-networks without internal delays. But they did not deal with the direction and stability of Hopf bifurcation and the possible spatio-temporal patterns of bifurcating periodic oscillations. In [19, 24], a neural network coupled by a pair of two-neuron sub-networks is investigated, which contains the time delay not only in the coupling but also in the internal connection. Yet one can find that all the delays have the same size in [24].

Motivated by proposing a more generalized model than those in [3, 24], we consider model (1.1), which consists of multiple nonlinear delayed neural network loops by delay coupling.

It is well-known that an artificial neural network (ANN) is composed of many artificial neurons that are linked together according to a specific network architecture. The objective of the neural network is to transform the inputs into meaningful outputs. Artificial neural networks are inspired by the learning processes that take place in biological systems, which try to imitate the working mechanisms of their biological counterparts. Since McCulloch and Pitts's first formal model of the elementary computation neuron in 1943 [22], which could perform arithmetical logic operations, a great amount of ANN models have been proposed and developed according to the purposes of the applications or theoretical analysis. The applications of ANNs range from classification (including pattern recognition, feature extraction, detection and clustering, image matching), noise reduction (recognizing patterns in the inputs and produce noiseless outputs), prediction (extrapolation based on historical data, such as stock market prediction), function approximation, control for real-world applications (such as robot control), and optimization etc.

The rest of this paper is organized as follows: In Sec. 2 we give a detailed study of asymptotic behavior of system (1.1) and some properties of the polynomial (2.3) are discussed. As application, the stable regions and all possible bifurcations, which depend on multiple parameters, are given in a geometrical way for some specific cases in Sec. 3. Numerical simulation is included in Sec. 4, and a tendency of partially phase-locking phenomenon is discovered. Finally we draw our conclusions in Sec. 5.

2. local stability analysis of Eq. (1.1)

For notational simplicity, let $f_i(0) = 0, f_i'(0) = 1(i = 1, 2, \ldots, n), g_i(0) = 0, g_i'(0) = 1(i = 1, \ldots, m)$.

The linearized system of (1.1) evaluated around the origin (the trivial zero solution) leads to

(2.1)

$$\begin{aligned}
&\textit{subsystem } 1 \left\{\begin{array}{lcl} x_{11}'(t) &=& -a_1x_{11}(t) + \beta_n x_{1n}(t-\tau_n) + \epsilon_1 x_{mn}(t-\tau_n) \\ x_{12}'(t) &=& -a_2x_{12}(t) + \beta_1 x_{11}(t-\tau_1) \\ \quad\vdots && \\ x_{1n}'(t) &=& -a_nx_{1n}(t) + \beta_{n-1} x_{1(n-1)}(t-\tau_{n-1}) \end{array}\right. \\
&\textit{subsystem } 2 \left\{\begin{array}{lcl} x_{21}'(t) &=& -a_1x_{21}(t) + \beta_n x_{2n}(t-\tau_n) + \epsilon_1 x_{1n}(t-\tau_n) \\ x_{22}'(t) &=& -a_2x_{22}(t) + \beta_1 x_{21}(t-\tau_1) \\ \quad\vdots && \\ x_{2n}'(t) &=& -a_nx_{2n}(t) + \beta_{n-1} x_{2(n-1)}(t-\tau_{n-1}) \end{array}\right. \\
&\quad\vdots \\
&\textit{subsystem } m \left\{\begin{array}{lcl} x_{m1}'(t) &=& -a_1x_{m1}(t) + \beta_n x_{mn}(t-\tau_n) + \epsilon_m x_{n(m-1)}(t-\tau_n) \\ x_{m2}'(t) &=& -a_2x_{m2}(t) + \beta_1 x_{m1}(t-\tau_1) \\ \quad\vdots && \\ x_{mn}'(t) &=& -a_nx_{mn}(t) + \beta_{n-1} x_{m(n-1)}(t-\tau_{n-1}) \end{array}\right.
\end{aligned}$$

Then one can derive the characteristic matrix of (2.1)

$$\mathbb{Q}(\lambda) = diag(\lambda + a_1, \lambda + a_2, \ldots, \lambda + a_n, \ldots, \lambda + a_1, \lambda + a_2, \ldots, \lambda + a_n)_{mn} - M,$$

where M is the connection matrix of (2.1), i.e.,

$$M = \begin{pmatrix} A & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B_1 \\ B_2 & A & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_3 & A & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_n & A \end{pmatrix}_{mn\times mn},$$

where

$$A = \begin{pmatrix} 0 & 0 & \cdots & \beta_n e^{-\lambda\tau_n} \\ \beta_1 e^{-\lambda\tau_1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \beta_{n-1}e^{-\lambda\tau_{n-1}} & 0 \end{pmatrix}_{n\times n}$$

and

$$B_i = \begin{pmatrix} 0 & 0 & \cdots & \epsilon_i e^{-\lambda\tau_n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}.$$

Hence, the characteristic equation is

$$\begin{aligned} &\det\mathbb{Q}(\lambda) \\ &= [e^{\lambda\tau}(\lambda+a_1)(\lambda+a_2)\cdots(\lambda+a_n) - \beta_1\beta_2\cdots\beta_n]^m - (\beta_1\beta_2\cdots\beta_{n-1})^m\epsilon_1\cdots\epsilon_m \\ &= \begin{cases} \prod_{s=0}^{m-1}[e^{\lambda\tau}(\lambda+a_1)(\lambda+a_2)\cdots(\lambda+a_n) - (\beta+\eta e^{\frac{2s\pi i}{m}})] \\ \qquad \text{if } (\beta_1\beta_2\cdots\beta_{n-1})^m\epsilon_1\cdots\epsilon_m > 0; \\ \prod_{s=0}^{m-1}[e^{\lambda\tau}(\lambda+a_1)(\lambda+a_2)\cdots(\lambda+a_n) - (\beta+\eta e^{\frac{(2s+1)\pi i}{m}})] \\ \qquad \text{if } (\beta_1\beta_2\cdots\beta_{n-1})^m\epsilon_1\cdots\epsilon_m < 0; \end{cases} \\ &:= \Delta_0\cdots\Delta_{m-1} = 0, \end{aligned} \tag{2.2}$$

where $\beta = \beta_1\beta_2\cdots\beta_n, \eta = |\beta_1\beta_2\cdots\beta_{n-1}|(|\epsilon_1\cdots\epsilon_m|)^{\frac{1}{m}}$.

Now, we give a detailed study of the zero distribution of a polynomial of the type

$$\begin{aligned} \Lambda(\tau,\mathbf{a},\alpha,b) &= e^{\lambda\tau}(\lambda+a_1)\ldots(\lambda+a_n) - be^{i\alpha} \\ &= e^{\lambda\tau}(\lambda+a_1)\ldots(\lambda+a_n) - (c+id), \end{aligned} \tag{2.3}$$

where $a_i > 0 (i \in N(1,n))$, $b \in (0,\infty)$ and $\alpha \in [-\pi,\pi]$.

Define the curve $\sum = \{(u,v)\}$, where u and v are both parameterized by a_i, τ and θ as follows:

$$\mathbf{U}_n = \begin{bmatrix} a_n & -\theta \\ \theta & a_n \end{bmatrix}\mathbf{U}_{n-1} = \begin{bmatrix} a_n & -\theta \\ \theta & a_n \end{bmatrix}\cdots\begin{bmatrix} a_1 & -\theta \\ \theta & a_1 \end{bmatrix}\mathbf{U}_0,$$

where $\mathbf{U}_n = \begin{bmatrix} u_n \\ v_n \end{bmatrix}$ (in what follows, we shall identify (u,v) with (u_n,v_n)), $\mathbf{U}_0 = \begin{bmatrix} \cos\tau\theta \\ \sin\tau\theta \end{bmatrix}$. Then it can be found that

$$\begin{aligned} D_n &:= v_n'u_n - v_nu_n' \\ &= (a_nv_{n-1}' + \theta u_{n-1}')(a_nu_{n-1} - \theta v_{n-1}) - (a_nu_{n-1}' - \theta v_{n-1}')(a_nv_{n-1} + \theta u_{n-1}) \\ &\qquad + u_{n-1}(a_nu_{n-1} - \theta v_{n-1}) + v_{n-1}(a_nv_{n-1} + \theta u_{n-1}) \\ &= a_n(u_{n-1}^2 + v_{n-1}^2) + (a_n^2+\theta^2)(v_{n-1}'u_{n-1} - v_{n-1}u_{n-1}') \\ &= a_n(a_{n-1}^2+\theta^2)\cdots(a_1^2+\theta^2) + (a_n^2+\theta^2)D_{n-1} \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} \delta_n &:= (U_n^T)'U_n = u_n'u_n + v_nv_n' \\ &= \theta U_{n-1}^TU_{n-1} + (a_n^2+\theta^2)(U_{n-1}^T)'U_{n-1} \\ &= \theta(a_{n-1}^2+\theta^2)\cdots(a_1^2+\theta^2) + (a_n^2+\theta^2)\delta_{n-1}, \end{aligned}$$

with $D_1 = a_1$ and $\delta_1 = \theta$. Therefore

$$\begin{aligned} D_n &:= v_n'u_n - v_nu_n' \\ &= \textstyle\sum_{j_1\neq\cdots\neq j_n, j_1,\ldots,j_n\in(1,2,\ldots,n)} a_{j_1}(a_{j_2}^2+\theta^2)\cdots(a_{j_n}^2+\theta^2), \end{aligned}$$

and

$$\begin{aligned} \delta_n &:= u_n'u_n - v_nv_n' \\ &= \textstyle\sum_{j_2\neq\cdots\neq j_n, j_2,\ldots,j_n\in(1,2,\ldots,n)} \theta(a_{j_2}^2+\theta^2)\cdots(a_{j_n}^2+\theta^2). \end{aligned}$$

Then one can find that the following result holds:

Proposition 2.1. *If* $be^{i\alpha} = u_n + iv_n$*, then* $\lambda = e^{i\theta}$ *is one of the zero roots of the polynomial* $\Lambda(\tau, \boldsymbol{a}, \alpha, b)$*, i.e.,*

$$e^{i\theta\tau}(i\theta + a_1)\dots(i\theta + a_n) - (u_n + iv_n) = 0$$

and $\overline{\lambda} = e^{-i\theta}$ *is one of the zero roots of the polynomial* $\Lambda(\tau, \boldsymbol{a}, -\alpha, b)$*.*

Note that

$$re^{i\psi} = be^{i\alpha} = u_n + iv_n,$$

where $u = r\cos\psi, v = r\sin\psi$, which leads to

$$b = r = \sqrt{u_n^2 + v_n^2} = \sqrt{(a_n^2 + \theta^2)(u_{n-1}^2 + v_{n-1}^2)} = \sqrt{\prod_{i=1}^{n}(a_i^2 + \theta^2)}$$

and $\psi = \alpha$.

Then, it follows from the above analysis that

- (i) r is monotonically increasing for $\theta, a_i \in (0, \infty), i = 1, 2, \cdots, n$ and decreasing for $\theta \in (-\infty, 0), a_i \in (0, \infty)$;
- (ii) the curve has the anticlockwise property and the symmetry property about the $u-$axis, i.e., $sign(\psi'(\theta)) = sign(uv' - u'v) > 0$ and $u(-\theta) = u(\theta), v(-\theta) = -v(\theta)$;
- (iii) the curve $\sum^+ = \{(u(\theta), v(\theta)) : \theta \in R^+ := (0, \infty)\}$ is simple, i.e., it cannot intersect with itself.

Let $\{\theta_s\}_{n=0}^{+\infty}$ be the monotonic increasing sequence of the nonnegative zeros of v, and define

$$c_s = |u(\theta_s)|$$

for all $s \in \mathbb{N}_0 := \{0, 1, 2, ...\}$. Then, it follows from

$$v = \Im(e^{i\tau\theta}(i\theta + a_1)\dots(i\theta + a_n)) = 0$$

that

$$\tau\theta + \sum_{j=1}^{n}\arctan(\theta/a_j) = s\pi,$$

which leads to $\theta_0 = 0$ and $\theta_s \in ((2s - n)\pi/(2\tau), s\pi/\tau)$ for all $s \in N_0$ and the curve $\sum^+$ intersects with the u-axis at $(c_s, 0)$, $s \in \mathbb{N}_0$. The anticlockwise property of the curve $\sum^+$ leads to

$$(-1)^s u(\theta_s) > 0, (-1)^s u'(\theta_s) > 0, (-1)^s v'(\theta_s) > 0$$

for all $s \in \mathbb{N}_0$.

For each $n \in \mathbb{N}_0$, define $\sum_s = \{(u(\theta), v(\theta))|\theta \in [-\theta_{s+1}, -\theta_s] \cup [\theta_s, \theta_{s+1}]\}$, which is a closed simple curve with (0, 0) inside. The curve is schematically illustrated in Fig. 2.1.

We need the following lemma about the properties of the distributions of the roots of (2.3), which will play an important role in further study of bifurcation analysis.

Lemma 2.1. *Consider* $\Lambda(\tau, \boldsymbol{a}, \alpha, b)$ *defined in (2.3) with* $be^{i\alpha} \in \mathbb{C}$*. Then the following statements are true:*

- *(i)* $\Lambda(\tau, \boldsymbol{a}, \alpha, b)$ *has purely imaginary zero roots if and only if* $be^{i\alpha} \in \sum_i$*. Moreover, if* $z = u(\theta) + iv(\theta)$ *then the purely imaginary zero is* $i\theta$ *(or* $-i\theta$ *which depends on* $v >$ *(or* $<$*)0), except that* $v = 0$*, where there is a pair of conjugate purely imaginary roots for* $z = (-1)^s c_s$ *for* $s \in \mathbb{N}_0 - \{0\}$ *and zero is one of its root with* $z = a_1a_2\cdots a_n$ *and* $s = 0$*.*

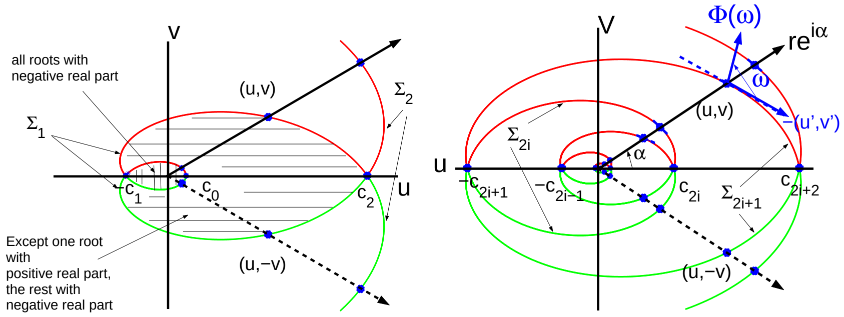


FIGURE 2.1. (a) Stable region and possible Neimark-Sacker bifurcations (NS) near the critical curves $\sum_i$ for model (1.1) and (b) $\sum_i$, c_i and the direction vector $\Phi(\omega)$.

- *(ii) For each fixed $z_0 = u(\theta_0)+iv(\theta_0)$, there exists an open δ-neighborhood of z_0 in the complex plane, denoted by $N(z_0, \delta)$, and an analytical function $\lambda : N(z_0, \delta) \to \mathbb{C}$ such that $\lambda(z_0) = i\theta_0/-i\theta$ and $\lambda(z_0)$ is a zero of $\Lambda(\tau, \boldsymbol{a}, \alpha, b)$ for all $z \in B(z_0, \delta)$.*
- *(iii) Along the vector*

$$\Phi(\omega) = -(u'(\theta), v'(\theta))\Xi(\omega), \tag{2.4}$$

the directional derivative of $\Re\{\lambda(z)\}$ at $z_0 = (u(\theta_0), v(\theta_0))$ is positive, where $\omega \in (0, \pi)$ and

$$\Xi(\omega) = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$

- *(iv) All the roots of $\Lambda(\tau, \boldsymbol{a}, \alpha, b)$ have strictly negative real parts if and only if $z = (u, v)$ is inside the curve $\sum_0$; exactly $j \in N$ roots with positive real parts if z lies in between $\sum_{j-1}$ and $\sum_j$. In particular, if $z \in \sum_0$, either zero is one root of $\Lambda(\tau, \boldsymbol{a}, \alpha, b)$ for $z = c_0$, or a simple purely imaginary root for $\Im(z) \neq 0$, or a pair of simple purely imaginary conjugate roots for $z = -c_1$, except for the rest with strictly negative real parts.*

PROOF. According to Proposition 2.1, we can get the validity of case (i), case (ii) follows from the fact that $\Lambda(\tau, \mathbf{a}, \alpha, b)$ is an analytic function and (iv) is a direct result of case (i) and (iii). Therefore it suffices to verify the validity of case (iii): Consider Eq. (2.3), we find that

$$\frac{\partial\lambda}{\partial c} = \frac{1}{Q'(a,\tau,\lambda)}, \qquad \frac{\partial\lambda}{\partial d} = \frac{i}{Q'(a,\tau,\lambda)},$$
$$\frac{\partial\overline{\lambda}}{\partial c} = \frac{1}{\overline{Q}'(a,\tau,\lambda)}, \qquad \frac{\partial\overline{\lambda}}{\partial d} = \frac{-i}{\overline{Q}'(a,\tau,\lambda)},$$

where $Q(a, \tau, \lambda) = e^{\lambda\tau}(\lambda + a_1)\ldots(\lambda + a_n)$ and its derivative with respect to λ is denoted by $Q'(a, \tau, \lambda)$. Then $\nabla Re\lambda = \left(\frac{\Re(Q'_\lambda(a,\tau,\lambda))}{d_1}, \frac{\Im(Q'_\lambda(a,\tau,\lambda))}{d_1}\right)^T$ where

$$d_1 = Q'_\lambda(a, \tau, \lambda)\overline{Q}'_\lambda(a, \tau, \lambda) > 0,$$

and $Q'_\lambda(a,\tau,\lambda)) = \tau(u+iv)+e^{i\theta\tau}\sum_{j=1}^{n}(a_1+i\theta)\cdots(a_{j-1}+i\theta)(a_{j+1}+i\theta)\cdots(a_n+i\theta)$. Furthermore we have

$$\begin{bmatrix}\Re(Q')\\ \Im(Q')\end{bmatrix} = \begin{bmatrix}\tau u\\ \tau v\end{bmatrix} + \sum_{j=1}^{n}\begin{bmatrix}a_1 & -\theta\\ \theta & a_1\end{bmatrix}\cdots\begin{bmatrix}a_{j-1} & -\theta\\ \theta & a_{j-1}\end{bmatrix}\begin{bmatrix}a_{j+1} & -\theta\\ \theta & a_{j+1}\end{bmatrix}\cdots\begin{bmatrix}a_n & -\theta\\ \theta & a_n\end{bmatrix}\begin{bmatrix}\cos\tau\theta\\ \sin\tau\theta\end{bmatrix}.$$

Then it yields

$$\begin{aligned}\frac{d|\lambda|}{d\Phi}\Big|_{\lambda=\exp(i\theta)} &= -\frac{1}{\sqrt{u'^2+v'^2}}(u',v')\Xi(\omega)\nabla|\lambda|\\ &= -\frac{d_3}{d_1 d_2}\\ &> 0,\end{aligned}$$

where $\omega\in(0,\pi)$, $d_2=\sqrt{u'^2+v'^2}$ and

$$\begin{aligned}d_3 &= [(uu'+vv')\cos\omega-(uv'-vu')\sin\omega]\tau+(u'v')\begin{bmatrix}\cos\omega & \sin\omega\\ -\sin\omega & \cos\omega\end{bmatrix}\begin{bmatrix}\Re(Q')\\ \Im(Q')\end{bmatrix}\\ &= \begin{cases}-\sum_{j_1\neq\cdots\neq j_n, j_1,\ldots,j_n\in(1,2,\ldots,n)}(2a_{j_1}\tau+1)(a_{j_2}^2+\theta^2)\cdots(a_{j_n}^2+\theta^2)\sin\omega, & n\geq 2,\\ -(a_1\tau+1)\sin\omega, & n=1,\end{cases}\\ &< 0.\end{aligned}$$

□

Now, we list the local stability criterion of system (1.1) as follows:

Theorem 2.1. *The zero solution of system (1.1) is locally asymptotically stable if and only if*

$$(\beta,\eta)\in\Omega := \left\{(\beta,\eta)\;\middle|\;\begin{cases}(\beta+\eta\cos\frac{2s\pi}{m},\eta\sin\frac{2s\pi}{m})(s=0,\ldots,m-1)\\ \quad\text{are all lying inside the closed curve } \sum_0,\\ \qquad\qquad\text{if } (\beta_1\beta_2\cdots\beta_{n-1})^m\epsilon_1\cdots\epsilon_m>0;\\ (\beta+\eta\cos\frac{(2s+1)\pi}{m},\eta\sin\frac{(2s+1)\pi}{m})(s=0,\ldots,m-1)\\ \quad\text{are all lying inside the closed curve } \sum_0,\\ \qquad\qquad\text{if } (\beta_1\beta_2\cdots\beta_{n-1})^m\epsilon_1\cdots\epsilon_m<0;\end{cases}\right\}$$

where $\beta=\beta_1\beta_2\cdots\beta_n, \eta=|\beta_1\beta_2\cdots\beta_{n-1}|(|\epsilon_1\cdots\epsilon_m|)^{\frac{1}{m}}$.

PROOF. According to Proposition 2.1 and Lemma 2.1, it is easily to see that all eigenvalues of the linearized system (2.1) have negative real parts, which implies that the zero solution of system (1.1) is locally asymptotically stable. □

3. Applications

Now we consider some specific values of $m=2,3,4$.

3.1. m=2.

The characteristic equation of system (2.1) becomes

(3.1)
$$\begin{aligned}&\det\mathbb{Q}(\lambda)\\ &= [e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-\beta_1\beta_2\cdots\beta_n]^2-(\beta_1\beta_2\cdots\beta_{n-1})^2\epsilon_1\epsilon_2\\ &= \begin{cases}[e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta+\eta)][e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta-\eta)]\\ \qquad\qquad\text{if } \epsilon_1\epsilon_2>0;\\ [e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta+i\eta)][e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta-i\eta)]\\ \qquad\qquad\text{if } \epsilon_1\epsilon_2<0,;\end{cases}\\ &:= \Delta_1\Delta_2=0,\end{aligned}$$

where $\beta=\beta_1\beta_2\cdots\beta_n$ and $\eta=|\beta_1\beta_2\cdots\beta_{n-1}|\sqrt{|\epsilon_1\epsilon_2|}$, which are defined as in Sec.2. Then we divide our discussion into two cases:

Case 1: $\epsilon_1\epsilon_2 > 0$.

- (i) The zero solution of system (1.1) is locally asymptotically stable if and only if

$$\begin{aligned}(\beta,\eta)\in\Omega &:= \{(\beta,\eta)\,|{-c_1} < \beta\pm\eta < a_1\cdots a_n\,\}\\ &:= \{(\beta,\eta)\,|{-c_1} < \beta < a_1\cdots a_n, < \eta < \eta^+\,\}\,,\end{aligned}$$

where $\eta^+ = \min\{a_1\cdots a_n - \beta, c_1+\beta\}$.

Furthermore, as $\beta\pm\eta$ increases through a series of critical values c_0, c_2, …, respectively, there may occur subsequently (two types of) fold bifurcations (such as transcritical/pitch-fork bifurcations) and Neimark-Sacker bifurcations, whereas β increases through a series of critical values $-c_1$, $-c_3$, …, there may occur subsequently (two types of) Hopf bifurcations (there occur subsequently *two* pairs of complex conjugate eigenvalues) (please see Fig. 3.1).

- (ii) Except for the co-dimension-one (Fold, Hopf) bifurcations, there exists the following types of co-dimension-two bifurcations:
  (a) cusp bifurcations;
  (b) Bautin (generalized Hopf) bifurcations;
  (c) Bogdanov-Takens(BT) bifurcations: two zero eigenvalues at the critical point;
  (d) double Hopf bifurcations (H^2): two pairs of purely imaginary conjugate eigenvalues;
  (e) Hopf bifurcation and fold bifurcation (HF): A zero eigenvalue and one pair of purely imaginary conjugate eigenvalues.

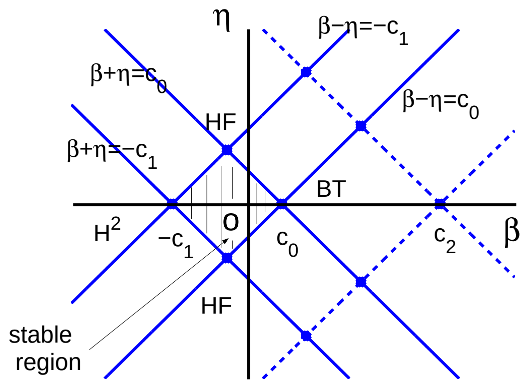


Figure 3.1. The stable region and possible bifurcations near the critical lines $\beta\pm\eta = (-1)^s c_s$ for model (1.1) with $\tau+2 = 2m+1$.

Case 2: $\epsilon_1\epsilon_2 < 0$.

- (i) the zero solution of system (1.1) is locally asymptotically stable if and only if

$$\begin{aligned}(\beta,\eta)\in\Omega &:= \{(\beta,\eta)|(\beta,\eta) \text{ lying inside the closed curve } \textstyle\sum_0\}\\ &:= \{(\beta,\eta)\,|{-c_0} < \beta < c_1, 0 < \eta < \eta^+,\,\}\,,\end{aligned}$$

where $(\beta,\eta^+)\in\sum_0$, i.e., there exists a $\theta^*\in(0,\theta_1)$ such that $u(\theta^*) = \beta$, and $\eta^+ := v(\theta^*)$.

Furthermore, as (β,η) passes through a series of critical curves $\sum_i (i = 1, 2, \ldots,$, respectively, there may occur subsequently a series of Hopf bifurcations (please see Fig. 2.1).

3.2. m=3.
The characteristic equation of system (2.1) is
(3.2)

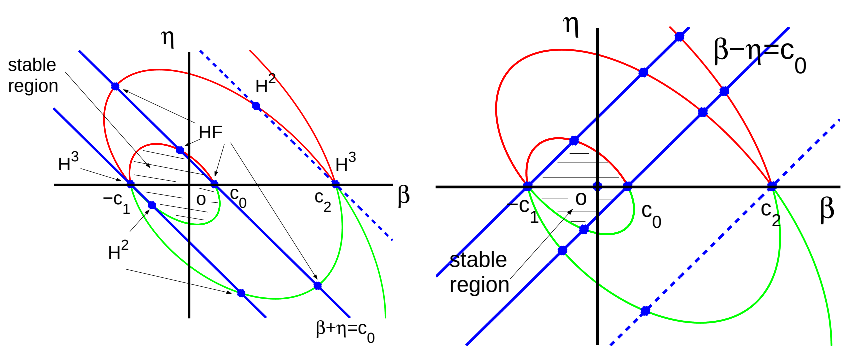
$$
\begin{aligned}
&\det\mathbb{Q}(\lambda)\\
&= [e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-\beta_1\beta_2\cdots\beta_n]^3-(\beta_1\beta_2\cdots\beta_{n-1})^3\epsilon_1\epsilon_2\epsilon_3\\
&= \begin{cases}
[e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta+\eta)]\ \ [e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta+\eta e^{\frac{2i\pi}{3}}))] \\
\qquad [e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta+\eta e^{\frac{-2i\pi}{3}})], \\
\qquad \text{if } (\beta_1\beta_2\cdots\beta_{n-1})^3\epsilon_1\epsilon_2\epsilon_3>0; \\
[e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta-\eta)]\ \ [e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta+\eta e^{\frac{i\pi}{3}}))] \\
\qquad [e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n)-(\beta+\eta e^{\frac{-i\pi}{3}}))] \\
\qquad \text{if } (\beta_1\beta_2\cdots\beta_{n-1})^3\epsilon_1\epsilon_2\epsilon_3<0,
\end{cases}\\
&:= \Delta_0\Delta_1\Delta_2=0,
\end{aligned}
$$

where $\beta=\beta_1\beta_2\cdots\beta_n$ and $\eta=|\beta_1\beta_2\cdots\beta_{n-1}|(|\epsilon_1\epsilon_2\epsilon_3|)^{1/3}$. Then we have:

- (i) The zero solution of system (1.1) is locally asymptotically stable if and only if

$$
(\beta,\eta)\in\Omega:=\begin{cases}
\left\{(\beta,\eta)\,\middle|\, \begin{array}{c} -c_1<\beta+\eta<a_1\cdots a_n \\ (\beta-\eta/2,\sqrt{3}\eta/2) \text{ lying inside the closed curve } \sum_0 \end{array}\right\} \\
\qquad \text{for } (\beta_1\beta_2\cdots\beta_{n-1})^3\epsilon_1\epsilon_2\epsilon_3>0; \\
\left\{(\beta,\eta)\,\middle|\, \begin{array}{c} -c_1<\beta-\eta<a_1\cdots a_n \\ (\beta+\eta/2,\sqrt{3}\eta/2) \text{ lying inside the closed curve } \sum_0 \end{array}\right\} \\
\qquad \text{for } (\beta_1\beta_2\cdots\beta_{n-1})^3\epsilon_1\epsilon_2\epsilon_3<0.
\end{cases}
$$

- (ii) Except for the co-dimension-one (Fold, Hopf) bifurcations, there exists the following types of co-dimension-two bifurcations (see Fig. 3.2):
  (a) cusp bifurcations;
  (b) Bautin (generalized Hopf) bifurcations;
  (c) double Hopf bifurcations (H^2);
  (d) triplicate Hopf bifurcations (H^3): three pairs of purely imaginary conjugate eigenvalues;
  (e) Hopf bifurcation and fold bifurcation (HF).



(*a*) $(\beta_1\beta_2\cdots\beta_{n-1})^3\epsilon_1\epsilon_2\epsilon_3>0$ (*b*) $(\beta_1\beta_2\cdots\beta_{n-1})^3\epsilon_1\epsilon_2\epsilon_3<0$

FIGURE 3.2. Stable region and possible higher-codimensional bifurcations near the critical values for model (1.1) with $m=3$.

3.3. m=4.
The characteristic equation of system (2.1) is

$$
\begin{aligned}
&\det\mathbb{Q}(\lambda) \\
&= [e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - \beta_1\beta_2\cdots\beta_n]^4 - (\beta_1\beta_2\cdots\beta_{n-1})^4\epsilon_1\epsilon_2\epsilon_3\epsilon_4 \\
&= \begin{cases}
[e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - (\beta+\eta)][e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - (\beta-\eta)] \\
[e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - (\beta+i\eta][e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - (\beta-i\eta], \\
\qquad \text{if } \epsilon_1\epsilon_2\epsilon_3\epsilon_4 > 0; \\
[e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - (\beta+\eta e^{i\pi/4})][e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - (\beta+\eta e^{-i\pi/4})] \\
[e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - (\beta+\eta e^{i3\pi/4})][e^{\lambda\tau}(\lambda+a_1)\cdots(\lambda+a_n) - (\beta+\eta e^{-i3\pi/4})], \\
\qquad \text{if } \epsilon_1\epsilon_2\epsilon_3\epsilon_4 < 0;
\end{cases} \\
&:= \Delta_0\Delta_1\Delta_2\Delta_3 = 0,
\end{aligned}
\tag{3.3}
$$

where $\beta = \beta_1\beta_2\cdots\beta_n$ and $\eta = |\beta_1\beta_2\cdots\beta_{n-1}|(|\epsilon_1\epsilon_2\epsilon_3\epsilon_4|)^{1/4} > 0$. Then there are two cases to consider:

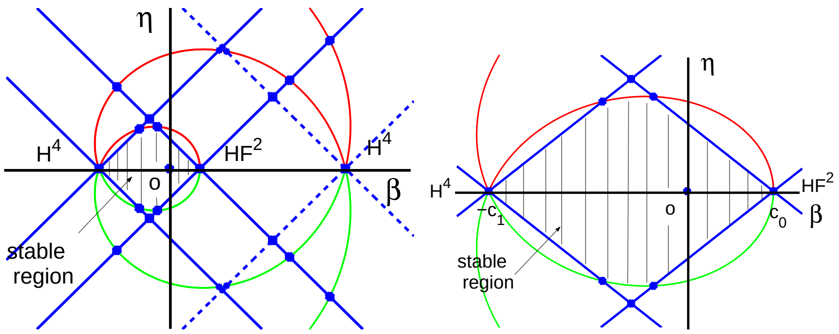
Case 1: $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 > 0$.

- (i) The zero solution of system (1.1) is locally asymptotically stable if and only if

$$
\begin{aligned}
(\beta,\eta)\in\Omega &:= \left\{(\beta,\eta)\left|\begin{array}{c} -c_1 < \beta\pm\eta < a_1\cdots a_n \\ (\beta,\eta) \text{ lying inside the closed curve } \sum_0 \end{array}\right.\right\} \\
&:= \{(\beta,\eta)\,|-c_1 < \beta < a_1\cdots a_n, 0 < \eta < \eta^+\},
\end{aligned}
$$

  where $\eta^+ = \min\{a_1\cdots a_n - \beta, c_1+\beta, \eta^*\}$, and (β,η^*) lies in $\sum_0^+$, i.e., there exists a $\theta^* \in (0,\theta_1)$ such that $u(\theta^*) = \beta$, and $\eta^* := v(\theta^*)$.
- (ii) Except for the co-dimension-one (Fold, Hopf) bifurcations, there exists the following types of co-dimension-two bifurcations:
  (a) cusp bifurcations;
  (b) Bautin (generalized Hopf) bifurcations;
  (c) Bogdanov-Takens(BT) bifurcations;
  (d) double Hopf bifurcations (H^2): two pairs of purely imaginary conjugate eigenvalues;
  (e) Hopf bifurcation and double fold bifurcations (HF^2): two zero eigenvalues and one pair of purely imaginary conjugate eigenvalues;
  (f) duplicate Hopf bifurcations (H^4): four pairs of purely imaginary conjugate eigenvalues (please see Fig. 3.3).

. Case 2: $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 < 0$.



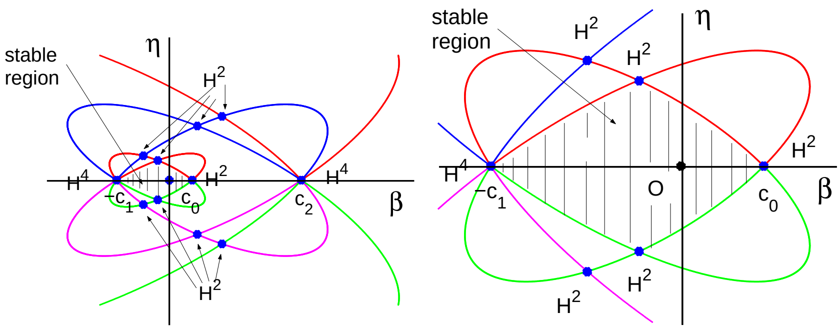
(*a*) *a larger region* (*b*) *a smaller region*

FIGURE 3.3. Stable region and possible higher-codimensional bifurcations near the critical values for model (1.1) with $m = 4$.

- (i) The zero solution of system (1.1) is locally asymptotically stable if and only if

$$(\beta,\eta)\in\Omega := \left\{(\beta,\eta)\,\middle|\,(\beta\pm\tfrac{\sqrt{2}\eta}{2},\tfrac{\sqrt{2}\eta}{2}) \text{ lying inside the closed curve } \textstyle\sum_0.\right\}$$

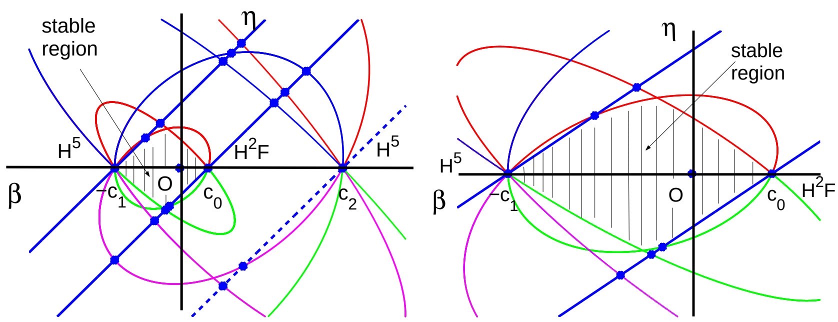
- (ii) Except for the co-dimension-one (Fold, Hopf) bifurcations, there exists the following types of co-dimension-two bifurcations:
  (a) cusp bifurcations;
  (b) Bautin (generalized Hopf) bifurcations;
  (c) double Hopf bifurcations (H^2): two pairs of purely imaginary conjugate eigenvalues;
  (d) quadruplicate Hopf bifurcations (H^4): four pairs of purely imaginary conjugate eigenvalues (please see Fig. 3.4).



(*a*) *a larger region* (*b*) *a smaller region*

FIGURE 3.4. Stable region and possible higher-codimensional bifurcations near the critical values for model (1.1) with $m = 4$.

3.4. m=5.
As to $m = 5$, much richer dynamics can be observed, including quintuplicate hopf bifurcatoins (H^5), and Hopf-double-Fold (HF^2) bifurcations etc. (please see Fig. 3.5). But the detail is omitted.



(*a*) *a larger region* (*b*) *a smaller region*

FIGURE 3.5. Stable region and possible higher-codimensional bifurcations near the critical values for model (1.1) with $m = 5$ for $\beta_1\beta_2\cdots\beta_{n-1})^5\epsilon_1\cdots\epsilon_5 < 0$.

4. Numerical simulation

Consider Eq. (1.1) with $n = 2$, $m = 5$, $f = g = \sin(x)$, $\tau_1 = 1$, $\tau_2 = 2$, $a_1 = 2.5, a_2 = 0.3$, $\beta_1 = 1$, $\epsilon_s = c(s = 2, 3, 4, 5)$, $\epsilon_1 = a$ and $\beta_2 = b$, i.e., system (1.1) becomes

$$
\begin{aligned}
&\textit{subsystem } 1 \begin{cases} y'_{11}(t) = -2.5y_{11}(t) + b\sin(y_{12}(t-1)) + a\sin(y_{52}(t-1)) \\ y'_{12}(t) = -0.3y_{12}(t) + \sin(y_{11}(t-2)) \end{cases} \\
&\textit{subsystem } 2 \begin{cases} y'_{21}(t) = -2.5y_{21}(t) + b\sin(y_{22}(t-1)) + c\sin(y_{12}(t-1)) \\ y'_{22}(t) = -0.3y_{22}(t) + \sin(y_{21}(t-2)) \end{cases} \\
&\textit{subsystem } 3 \begin{cases} y'_{31}(t) = -2.5y_{31}(t) + b\sin(y_{32}(t-1)) + c\sin(y_{22}(t-1)) \\ y'_{32}(t) = -0.3y_{32}(t) + \sin(y_{31}(t-2)) \end{cases} \\
&\textit{subsystem } 4 \begin{cases} y'_{41}(t) = -2.5y_{41}(t) + b\sin(y_{42}(t-1)) + c\sin(y_{32}(t-1)) \\ y'_{42}(t) = -0.3y_{42}(t) + \sin(y_{41}(t-2)) \end{cases} \\
&\textit{subsystem } 5 \begin{cases} y'_{51}(t) = -2.5y_{51}(t) + b\sin(y_{52}(t-1)) + c\sin(y_{42}(t-1)) \\ y'_{52}(t) = -0.3y_{52}(t) + \sin(y_{51}(t-2)). \end{cases}
\end{aligned}
\tag{4.1}
$$

Our numerical result is shown in Figs. 4.1, 4.2 and 4.3 with the initial condition $y_{s1}(t) = -\cos(2(s-1)\pi/5)(-1 \le t \le 0)$ and $y_{s2}(t) = -\sin(2(s-1)\pi/5)(-2 \le t \le 0, s = 1, 2 \ldots, 5)$: In Figs. 4.1 and 4.2, periodic motions and a tendency of partially phase-locking phenomenon can be observed. Moreover, as the parameter a is varying, no new oscillation modes occur. But in Fig. 4.3, different oscillation modes can be observed as the sign of a is varying: the stability of the zero solution, the occurrence of nontrivial steady states or periodic waves, which gives a solid verification of our theoretical analysis.

A much richer dynamic of Eq. (1.1) can be observed in Fig. 4.4 with $n = 3$, $m = 5$, $f = g = \sin(x)$, $\tau_1 = 1$, $\tau_2 = 2$, $\tau_3 = 3$, $a_i = 0.3(i = 1, 2, 3)$, $\beta_1 = \beta_2 = 1$, $\epsilon_s = 0.1(s = 2, 3, 4, 5)$, $\epsilon_1 = 0.009$ and $\beta_3 = b$: there exists a periodic doubling bifurcation from Fig. 4.4(i) to (ii), then a complicated regular/chaotic oscillatory behavior in 4.4(iii)-(iv).

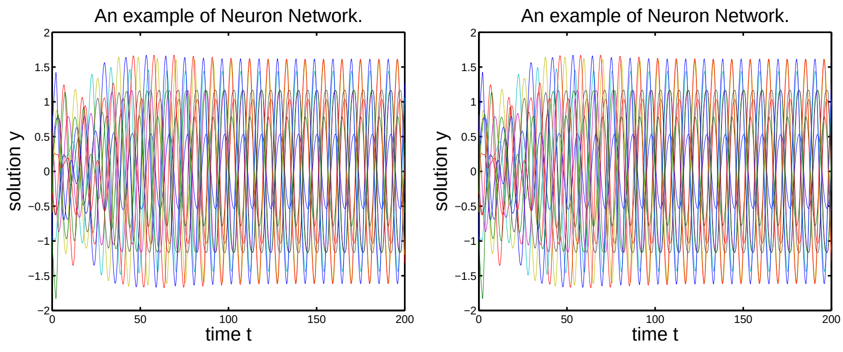
5. Conclusions

In this paper, we propose a generalized type of coupled systems with unidirectional coupling. The zero distribution in a special polynomial of the form $e^{\lambda\tau}(\lambda+a_1)\ldots(\lambda+a_n)-(c+id)$ are discussed, which generalize and extend those obtained in [1, 10, 24, 25]. As its applications, new criteria for local stability of coupled systems with different topological structures are established and the geometrical structure of the stable region is drawn for some specific cases $n = 1, 2, 3, 4, 5$.

All possible bifurcations are also concerned, including higher-codimensional bifurcations, such as quintuplicate/quadruplicate/triplicate hopf bifurcations ($\mathrm{H}^{5,4,3}$), fold-double-Hopf ($\mathrm{H}^2\mathrm{F}$), and Hopf-double-Fold (HF^2) bifurcations etc. As to the lower-codimensional bifurcation analysis in delayed systems, we refer the reader to Refs. [2, 6, 4, 8, 10] and dynamical systems without delay, please see Ref. [21].

It may be very interesting and complicated for the detailed analysis of higher-codimensional bifurcations, the interaction of multiple oscillation patterns, and the mechanism of how delay plays its dominant role in converting a simple system to be complex/chaotic, which needs a further discussion.

Some improvements can be made on Eq. (1.1), such as adding the spatio-temporal symmetrical structure so as to study synchronization phenomenon and the mechanism of processing information among subsystems .



$(a)\ a = -0.0009$ $(b)\ a = 0.0009$

Time Serials

phase portrait

FIGURE 4.1. Time serials and phase portraits near the critical values for model (4.1) with $b = -1.93, c = 1$ as a is varying.

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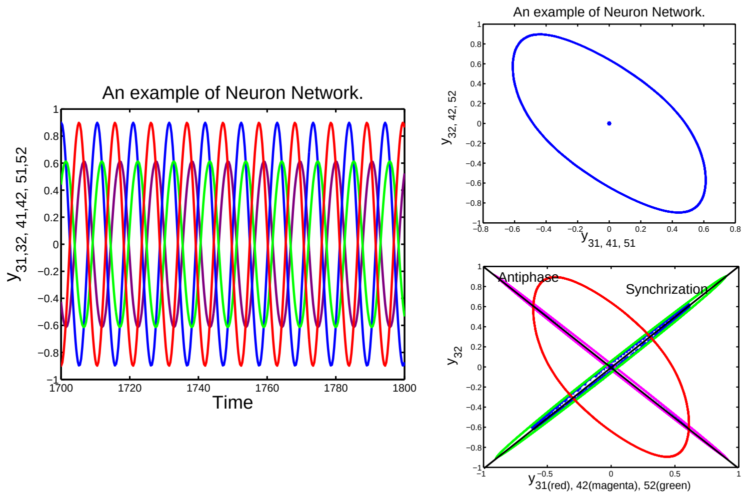
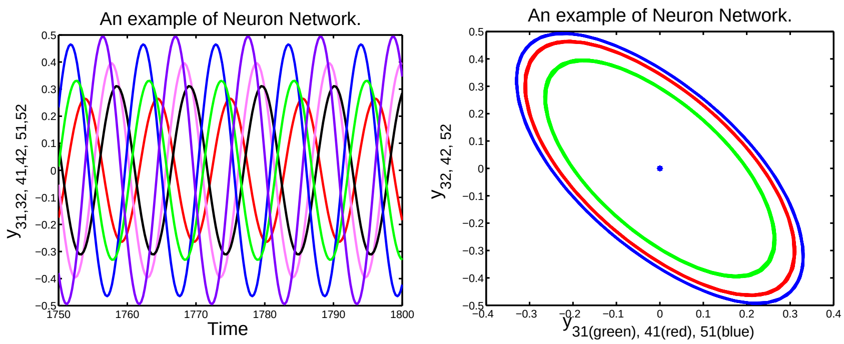


Time Serials *phase portrait*

$b = -1.90$



$b = -1.70$

FIGURE 4.2. A tendency of partially phase-locking phenomenon in model (4.1) with $a = 0.009$, $c = 0.1$, as b is varying.

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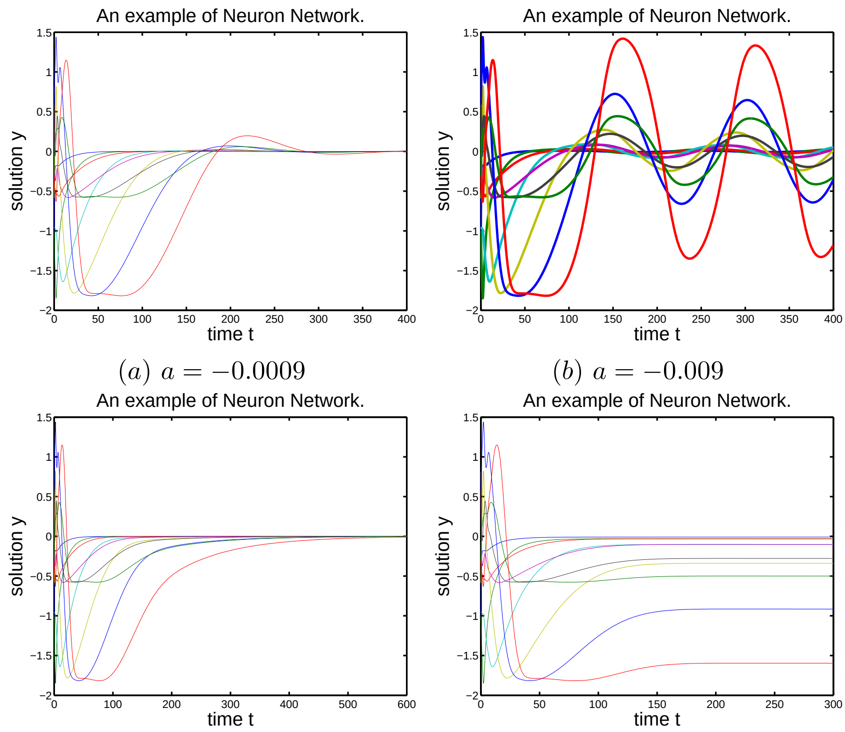

$(a)\ a = -0.0009$ $(b)\ a = -0.009$ $(c)\ a = 0.0009$ $(d)\ a = 0.009$

FIGURE 4.3. Time serials for model (4.1) near the critical values with $b = 0.456, c = 1$ as a is varied.

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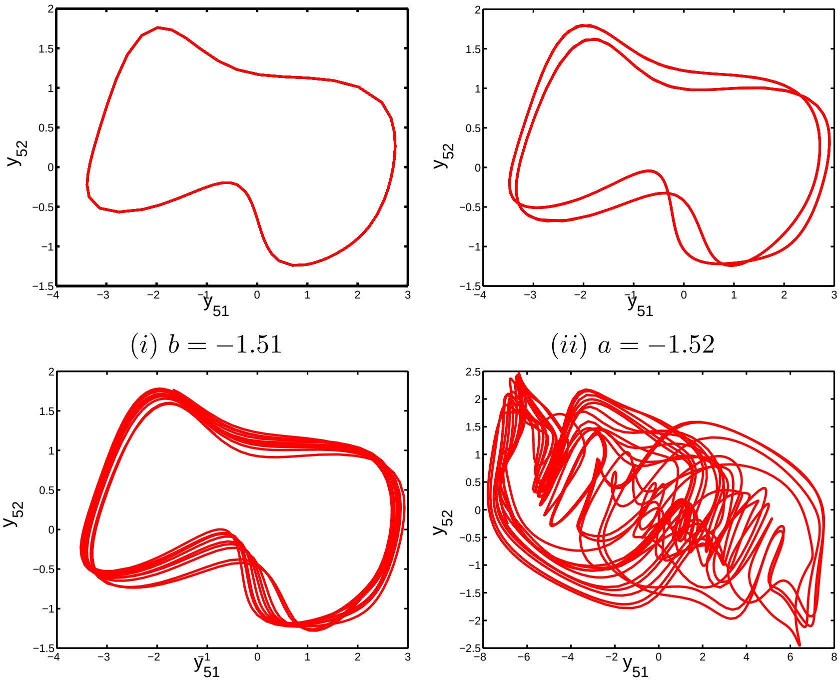

(i) $b = -1.51$ (ii) $a = -1.52$

(iii) $b = -1.55$ (iv) $b = -2.51$

FIGURE 4.4. Rich dynamic for model (4.1) near the critical values with $a = 0.009, c = 0.1$ as b is varied.

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