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NUMERICAL SOLUTIONS OF NONLINEAR PARABOLIC PROBLEMS BY MONOTONE JACOBI AND GAUSS–SEIDEL METHODS

IGOR BOGLAEV

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Abstract. This paper is concerned with solving nonlinear monotone difference schemes of the parabolic type. The monotone Jacobi and monotone Gauss–Seidel methods are constructed. Convergence rates of the methods are compared and estimated. The proposed methods are applied to solving nonlinear singularly perturbed parabolic problems. Uniform convergence of the monotone methods is proved. Numerical experiments complement the theoretical results.

Key Words. nonlinear parabolic problem, monotone iterative method, singularly perturbed problem, uniform convergence.

1. Introduction

Many reaction-diffusion-convection-type problems in the chemical, physical and engineering sciences are described by nonlinear parabolic equations. The parabolic problem under consideration is in the form

$$\frac{\partial u}{\partial t} - Lu + f(x,t,u) = 0, \quad (x,t) \in \omega \times (0,T], \tag{1}$$

$$u(x,t) = g(x,t), \quad (x,t) \in \partial\omega \times (0,T], \quad u(x,0) = u^0(x), \quad x \in \overline{\omega},$$

where ω is a connected bounded domain in $\mathbb{R}^\kappa$ ($\kappa = 1,2,\ldots$) with boundary $\partial\omega$. Lu is given by

$$Lu = \sum_{\nu=1}^{\kappa} \frac{\partial}{\partial x_\nu}\left(k_\nu(x,t)\frac{\partial u}{\partial x_\nu}\right) + \sum_{\nu=1}^{\kappa} v_\nu(x,t)\frac{\partial u}{\partial x_\nu},$$

where the coefficients of the differential operator are smooth and $k_\nu > 0$, $\nu = 1,\ldots,\kappa$, in $\overline{\omega}$. It is also assumed that the functions f and g are smooth in their respective domains.

In the study of numerical methods for nonlinear parabolic problems, the two major points to be developed are: i) constructing convergent nonlinear difference schemes and ii) computing solutions of nonlinear discrete problems. A major point about the nonlinear difference schemes is to obtain reliable and efficient computational methods for computing the solution. The reliability of iterative techniques for solving nonlinear difference schemes can be essentially improved by using componentwise monotone globally convergent iterations. Such methods can be controlled every time. A fruitful method for the treatment of these nonlinear schemes is the method of upper and lower solutions and its associated monotone iterations [7].

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Since an initial iteration in the monotone iterative method is either an upper or lower solution, which can be constructed directly from the difference equation without any knowledge of the exact solution, this method simplifies the search for the initial iteration as is often required in the Newton method. In the context of solving systems of nonlinear equations, the monotone iterative method belongs to the class of methods based on convergence under partial ordering (see Chapter 13 in [7] for details).

The purpose of this paper is to extend the monotone iterative method from [4] to monotone relaxation methods of Jacobi- and Gauss–Seidel type iterations for solving nonlinear monotone difference schemes in the canonical form and to apply the monotone methods to nonlinear singularly perturbed equations of the parabolic type. Convergence rates of these relaxation methods are compared and estimated.

The structure of the paper is as follows. In Section 2, we present the nonlinear monotone difference scheme in the canonical form and formulate the maximum principle. In Section 3, we construct the monotone Jacobi and monotone Gauss–Seidel methods, prove monotone convergence of the methods and compare their convergence rates. Section 4 is devoted to estimation of convergence rates of the monotone methods. In the final Section 5, the monotone methods are applied to solving nonlinear singularly perturbed parabolic problems. We prove that on layer-adapted meshes the monotone methods converge uniformly in a perturbation parameter. Numerical experiments complement the theoretical results.

2. The nonlinear difference scheme

On $\overline{\omega}$ and $[0,T]$, we introduce meshes $\overline{\omega}^h$ and $\overline{\omega}^\tau$, respectively. For simplicity, we assume that the mesh $\overline{\omega}^\tau$ is uniform with the time step τ. For a mesh function $U(p,t)$, $(p,t) \in \overline{\omega}^h \times \overline{\omega}^\tau$, consider the nonlinear implicit difference scheme in the canonical form [9]

$$\mathcal{L}U(p,t) + f(p,t,U) - \tau^{-1}U(p,t-\tau) = 0, \quad (p,t) \in \omega^h \times (\overline{\omega}^\tau \setminus 0), \tag{2}$$

$$U(p,0) = u^0(p), \quad p \in \overline{\omega}^h, \quad U(p,t) = g(p,t), \quad (p,t) \in \partial\omega^h \times (\overline{\omega}^\tau \setminus 0),$$

where $\partial\omega^h$ is the boundary of $\overline{\omega}^h$, and the difference operator $\mathcal{L}$ is defined by

$$\mathcal{L}U(p,t) \equiv \mathcal{L}^h U(p,t) + \tau^{-1}U(p,t),$$

$$\mathcal{L}^h U(p,t) \equiv d(p,t)U(p,t) - \sum_{p' \in \sigma'(p)} e(p',t)U(p',t).$$

Here $\sigma'(p) = \sigma(p) \setminus \{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p \in \omega^h$.

On each time level t, we make the following assumptions on the coefficients of the spatial operator $\mathcal{L}^h$:

$$d(p,t) > 0, \quad e(p,t) \geq 0, \quad p \in \omega^h, \tag{3}$$

$$d(p,t) - \sum_{p' \in \sigma'(p)} e(p',t) \geq 0, \quad p' \in \sigma'(p).$$

We also assume that the mesh $\overline{\omega}^h$ is connected. It means that for two interior mesh points $\tilde{p}$ and $\hat{p}$, there exists a finite set of interior mesh points $\{p_1, p_2, \ldots, p_s\}$ such that

$$p_1 \in \sigma'(\tilde{p}), \quad p_2 \in \sigma'(p_1), \ldots, \quad p_s \in \sigma'(p_{s-1}), \quad \hat{p} \in \sigma'(p_s). \tag{4}$$

On each time level t, introduce the linear problem

$$(\mathcal{L} + c)W(p,t) = f_0(p,t), \quad p \in \omega^h, \tag{5}$$

$$W(p,t)=g(p,t),\quad p\in\partial\omega^h,\quad c(p,t)\geq c_0=\text{const}\geq 0,\quad p\in\overline{\omega}^h.$$

On each time level t, for a mesh function $W(p,t)$, $p\in\overline{\omega}^h$, define the maximal norms

$$\|W(\cdot,t)\|_{\overline{\omega}^h}=\max_{p\in\overline{\omega}^h}|W(p,t)|,\quad \|W(\cdot,t)\|_{\omega^h}=\max_{p\in\omega^h}|W(p,t)|,$$

$$\|W(\cdot,t)\|_{\partial\omega^h}=\max_{p\in\partial\omega^h}|W(p,t)|.$$

We now formulate the maximum principle for the difference operator $\mathcal{L}+c$ and give an estimate of the solution to (5).

Lemma 1. *Let the coefficients of the difference operator $\mathcal{L}^h$ satisfy (3) and the mesh $\overline{\omega}^h$ be connected (4).*

(i) If a mesh function $W(p,t)$ satisfies the conditions

$$(\mathcal{L}+c)W(p,t)\geq 0\ (\leq 0),\quad p\in\omega^h,\quad W(p,t)\geq 0\ (\leq 0),\quad p\in\partial\omega^h,$$

then $W(p,t)\geq 0$ (≤ 0), $p\in\overline{\omega}^h$.

(ii) The following estimate of the solution to (5) holds true

$$\|W(\cdot,t)\|_{\overline{\omega}^h}\leq\max\{\|g(\cdot,t)\|_{\partial\omega^h},\|f_0(\cdot,t)\|_{\omega^h}/(c_0+\tau^{-1})\}. \tag{6}$$

The proof of the lemma can be found in [9].

3. The monotone iterative methods

Assume that $f(p,t,u)$ from (1) satisfies the two-sided constraint

$$0\leq f_u(p,t,u)\leq c^*,\quad c^*=\text{const},\quad (f_u=\partial f/\partial u). \tag{7}$$

On a time level $t\in\overline{\omega}^\tau\setminus 0$, we say that $\overline{U}(p,t)$ is an upper solution of (2) with respect to $U(p,t-\tau)$ if it satisfies the inequalities

$$\mathcal{L}\overline{U}(p,t)+f(p,t,\overline{U})-\tau^{-1}U(p,t-\tau)\geq 0,\quad p\in\omega^h,$$

$$\overline{U}(p,t)\geq g(p,t)\quad p\in\partial\omega^h.$$

Similarly, $\underline{U}(p,t)$ is called a lower solution if it satisfies all the reversed inequalities. Upper and lower solutions satisfy the inequality

$$\underline{U}(p,t)\leq\overline{U}(p,t),\quad p\in\overline{\omega}^h. \tag{8}$$

Indeed, by the definition of lower and upper solutions and the mean-value theorem, for $\delta U=\overline{U}-\underline{U}$ we have

$$\mathcal{L}\delta U(p,t)+f_u(p,t,W)\delta U(p,t)\geq 0,\quad p\in\omega^h,$$

$$\delta U(p,t)\geq 0,\quad p\in\partial\omega^h,$$

where $W(p,t)$ lies between $\underline{U}(p,t)$ and $\overline{U}(p,t)$. In view of the maximum principle in Lemma 1 and assumption (7), we conclude the required inequality.

We now introduce two monotone iterative methods based on the Jacobi and Gauss–Seidel methods and on the method of upper and lower solutions. On each time level $t\in\overline{\omega}^\tau\setminus 0$, the iterative sequence $\{U^{(n)}\}$, $n\geq 1$, generated by the Jacobi and Gauss–Seidel methods, is defined by the recurrence formulae

$$\mathcal{L}_*Z^{(n)}(p,t)=-\mathcal{R}(p,t,U^{(n-1)}),\quad p\in\omega^h, \tag{9}$$

$$Z^{(1)}(p,t)=g(p,t)-U^{(0)}(p,t),\quad Z^{(n)}(p,t)=0,\quad n\geq 2,\quad p\in\partial\omega^h,$$

$$U^{(n)}(p,t)=U^{(n-1)}(p,t)+Z^{(n)}(p,t),\quad p\in\overline{\omega}^h,$$

$$\mathcal{R}(p,t,U^{(n-1)})=\mathcal{L}U^{(n-1)}(p,t)+f(p,t,U^{(n-1)})-\tau^{-1}U(p,t-\tau),$$

where $\mathcal{R}(p,t,U^{(n-1)})$ is the residual of the difference scheme (2) on $U^{(n-1)}$, and $U(p,t-\tau)$ is given. For the Jacobi method, $\mathcal{L}_*$ is defined by

$$\mathcal{L}_{\text{JAC}} Z^{(n)}(p,t) = (d(p,t) + \tau^{-1} + c^*) Z^{(n)}(p,t),$$

and for the Gauss–Seidel method,

$$\mathcal{L}_{\text{GS}} Z^{(n)}(p,t) = (d(p,t) + \tau^{-1} + c^*) Z^{(n)}(p,t) - \sum_{p' \in \sigma'_L(p)} e(p',t) Z^{(n)}(p',t),$$

where $\sigma'_L(p)$ is a set of stencil points corresponding to a strictly lower triangular part of $\sigma(p)$.

Lemma 2. *Let the coefficients of the difference operator $\mathcal{L}$ from (2) satisfy (3)and $\overline{\omega}^h$ be connected (4). Then for the difference operators $\mathcal{L}_{\text{JAC}}$ and $\mathcal{L}_{\text{GS}}$, the maximum principle in Lemma 1 holds with $\mathcal{L}_{\text{JAC}}$ and $\mathcal{L}_{\text{GS}}$ instead of $\mathcal{L}+c$.*

Proof. The coefficients of the difference operators $\mathcal{L}_{\text{JAC}}$ and $\mathcal{L}_{\text{GS}}$ satisfy the conditions from (3). Indeed, in the case of $\mathcal{L}_{\text{JAC}}$, $d(p,t)+\tau^{-1}+c^* > 0$ is a diagonal entry and $e(p',t) = 0$, $p' \in \sigma'(p)$. In the case of $\mathcal{L}_{\text{GS}}$, $d(p,t)+\tau^{-1}+c^* > 0$ is a diagonal entry, $e(p',t) \geq 0$, $p' \in \sigma'_L(p)$, and

$$d(p,t) + \tau^{-1} + c^* - \sum_{p' \in \sigma'_L(p)} e(p',t) \geq d(p,t) + \tau^{-1} - \sum_{p' \in \sigma'(p)} e(p',t) \geq 0.$$

Thus, from Lemma 1, we conclude the maximum principle for $\mathcal{L}_{\text{JAC}}$ and $\mathcal{L}_{\text{GS}}$. □

3.1. Monotone convergence of the iterative methods. The following theorem gives the monotone property of the iterative methods (9).

Theorem 1. *Assume that the coefficients of the difference operator $\mathcal{L}$ in (2) satisfy (3), $f(p,t,u)$ satisfies (7) and $\overline{\omega}^h$ is connected (4). Let $U(p,t-\tau)$ be given and $\overline{U}^{(0)}(p,t)$, $\underline{U}^{(0)}(p,t)$ be upper and lower solutions of (2) corresponding to $U(p,t-\tau)$. Then the upper sequence $\{\overline{U}^{(n)}(p,t)\}$ generated by (9) converges monotonically from above to the unique solution $U(p,t)$ of the problem*

$$\mathcal{L}U(p,t) + f(p,t,U) - \tau^{-1}U(p,t-\tau) = 0, \quad p \in \partial\omega^h,$$
$$U(p,t) = g(p,t), \quad p \in \partial\omega^h,$$

the lower sequence $\{\underline{U}^{(n)}(p,t)\}$ generated by (9) converges monotonically from below to $U(p,t)$ and the following inequalities hold

$$\underline{U}^{(n-1)}(p,t) \leq \underline{U}^{(n)}(p,t) \leq U(p,t) \leq \overline{U}^{(n)}(p,t) \leq \overline{U}^{(n-1)}(p,t), \quad p \in \overline{\omega}^h.$$

Proof. We consider only the case of the upper sequence for the Gauss–Seidel method. All other cases can be proved in a similar way.

If $\overline{U}^{(0)}_{\text{GS}}$ is an upper solution, then from (9) we conclude that

$$\mathcal{L}_{\text{GS}} Z^{(1)}_{\text{GS}}(p,t) \leq 0, \quad p \in \omega^h, \quad Z^{(1)}_{\text{GS}}(p,t) = g(p,t) - U^{(0)}_{\text{GS}}(p,t) \leq 0, \quad p \in \partial\omega^h.$$

From Lemma 2, it follows that

$$Z^{(1)}_{\text{GS}}(p,t) \leq 0, \quad p \in \overline{\omega}^h, \quad U^{(1)}_{\text{GS}}(p,t) = g(p,t), \quad p \in \partial\omega^h. \tag{10}$$

Using the mean-value theorem and the equation for $Z^{(1)}_{\text{GS}}$ from (9), we represent $\mathcal{R}(p,t,U^{(1)}_{\text{GS}})$ in the form

$$\begin{aligned} \mathcal{R}(p,t,U^{(1)}_{\text{GS}}) &= -(c^* - f^{(1)}_u(p,t)) Z^{(1)}_{\text{GS}}(p,t) - \\ &\quad \sum_{p' \in \sigma'_U(p)} e(p',t) Z^{(1)}_{\text{GS}}(p',t), \quad p \in \omega^h, \end{aligned} \tag{11}$$

where $f_u^{(1)}(p,t) \equiv f_u[p,t,\overline{U}_{\text{GS}}^{(0)}(p) + \theta^{(1)}(p,t)Z_{\text{GS}}^{(1)}(p,t)]$, $0 < \theta^{(1)}(p,t) < 1$, and $\sigma'_U(p)$ is a set of stencil points corresponding to a strictly upper triangular part of $\sigma(p)$. Since the mesh function $Z_{\text{GS}}^{(1)}$ is nonpositive on ω^h and taking into account (3) and (7), we conclude that $U_{\text{GS}}^{(1)}$ is an upper solution to (2). By induction on n, we obtain that $Z_{\text{GS}}^{(n)}(p,t) \leq 0$, $p \in \overline{\omega}^h$, $n \geq 1$, and prove that $\{\overline{U}_{\text{GS}}^{(n)}(p,t)\}$ is a monotonically decreasing sequence of upper solutions.

We now prove that the monotone sequence $\{\overline{U}_{\text{GS}}^{(n)}\}$ converges to the solution of (2). The sequence $\{\overline{U}_{\text{GS}}^{(n)}\}$ is monotonically decreasing and bounded below by $\underline{U}$, where $\underline{U}$ is any lower solution (8). Now by linearity of the operators $\mathcal{L}_{\text{GS}}$, $\mathcal{L}$ and continuity of f, we have also from (9) that the mesh function U defined by

$$U(p,t) = \lim_{n\to\infty} \overline{U}_{\text{GS}}^{(n)}(p,t), \quad p \in \overline{\omega}^h,$$

is the exact solution to (2). If by contradiction, we assume that there exist two solutions U_1 and U_2 to (2), then by the mean-value theorem, the difference $\delta U = U_1 - U_2$ satisfies the problem

$$\mathcal{L}\delta U(p,t) + f_u \delta U(p,t) = 0, \quad p \in \omega^h, \quad \delta U(p,t) = 0, \quad p \in \partial\omega^h.$$

By Lemma 1, $\delta U = 0$ which leads to the uniqueness of the solution to (2). This proves the theorem. □

Remark 1. *Consider the following approach for constructing initial upper and lower solutions $\overline{U}^{(0)}(p,t)$ and $\underline{U}^{(0)}(p,t)$. Suppose that a mesh function $M(p,t)$ is defined on $\overline{\omega}^h$ and satisfies the boundary condition $M(p,t) = g(p,t)$, $p \in \partial\omega^h$. Introduce the difference problems*

$$\mathcal{L}Z_\nu^{(0)}(p,t) = \nu|\mathcal{R}(p,t,M)|, \quad p \in \omega^h, \tag{12}$$

$$Z_\nu^{(0)}(p,t) = 0, \quad p \in \partial\omega^h, \quad \nu = 1, -1.$$

Then the functions $\overline{U}^{(0)} = M + Z_1^{(0)}$, $\underline{U}^{(0)} = M + Z_{-1}^{(0)}$ are upper and lower solutions, respectively. We check only that $\overline{U}^{(0)}$ is an upper solution. From the maximum principle in Lemma 1, it follows that $Z_1^{(0)} \geq 0$ on $\overline{\omega}^h$. Now using the difference equation for $Z_1^{(0)}$ and the mean-value theorem, we have

$$\mathcal{R}(p,t,\overline{U}^{(0)}) = \mathcal{R}(p,t,M) + |\mathcal{R}(p,t,M)| + f_u^{(0)} Z_1^{(0)}.$$

Since $f_u^{(0)} \geq 0$ and $Z_1^{(0)}$ is nonnegative, we conclude that $\overline{U}^{(0)}$ is an upper solution.

Remark 2. *Since the initial iteration in (9) is either an upper or lower solution, which can be constructed directly from the difference equation without any knowledge of the solution as we have suggested in Remark 1, this method simplifies considerably the search for the initial iteration as is often required in Newton's method. This gives a practical advantage in the computation of numerical solutions.*

3.2. Comparisons of the monotone sequences. We now give a comparison result for the monotone sequences obtained by the monotone Jacobi and Gauss–Seidel methods from (9). Let $\overline{U}^{(0)}$ and $\underline{U}^{(0)}$ be upper and lower solutions to (2). Denote the upper and lower sequences from (9) by $\{\overline{U}_{\text{JAC}}^{(n)}\}$, $\{\underline{U}_{\text{JAC}}^{(n)}\}$ and $\{\overline{U}_{\text{GS}}^{(n)}\}$, $\{\underline{U}_{\text{GS}}^{(n)}\}$, respectively. The following theorem gives the comparison result for these sequences.

Theorem 2. *Assume that the coefficients of the difference operator $\mathcal{L}$ in (2) satisfy (3), $f(p,t,u)$ satisfies (7) and $\overline{\omega}^h$ is connected (4). Let $U(p,t-\tau)$ be given and $\overline{U}^{(0)}(p,t)$, $\underline{U}^{(0)}(p,t)$ be upper and lower solutions of (2) corresponding to $U(p,t-\tau)$. If $\overline{U}^{(0)}_{\text{JAC}} = \overline{U}^{(0)}_{\text{GS}} = \overline{U}^{(0)}$ and $\underline{U}^{(0)}_{\text{JAC}} = \underline{U}^{(0)}_{\text{GS}} = \underline{U}^{(0)}$, then for every $n = 1, 2, \ldots,$*

$$\overline{U}^{(n)}_{\text{GS}}(p,t) \le \overline{U}^{(n)}_{\text{JAC}}(p,t), \quad \underline{U}^{(n)}_{\text{GS}}(p,t) \ge \underline{U}^{(n)}_{\text{JAC}}(p,t), \quad p \in \omega^h. \tag{13}$$

Proof. We prove (13) in the case of the upper sequences. The case of the lower sequences can be proved in a similar way.

We use the notation

$$Z^{(n)}_{\text{JAC}} = \overline{U}^{(n)}_{\text{JAC}} - \overline{U}^{(n-1)}_{\text{JAC}}, \quad Z^{(n)}_{\text{GS}} = \overline{U}^{(n)}_{\text{GS}} - \overline{U}^{(n-1)}_{\text{GS}}, \quad W^{(n)} = \overline{U}^{(n)}_{\text{GS}} - \overline{U}^{(n)}_{\text{JAC}}.$$

From (9) with $n = 1$, we conclude

$$(d(p,t) + \tau^{-1} + c^*)W^{(1)}(p,t) = \sum_{p' \in \sigma'_L(p)} e(p',t) Z^{(1)}_{\text{GS}}(p',t).$$

From here, (3) and (10), it follows that

$$W^{(1)}(p,t) \le 0, \quad p \in \overline{\omega}^h. \tag{14}$$

From (9) with $n = 2$, we get

$$\begin{aligned}(d(p,t) + \tau^{-1} + c^*)(W^{(2)}(p,t) - W^{(1)}(p,t)) &= \sum_{p' \in \sigma'_L(p)} e(p',t) Z^{(2)}_{\text{GS}}(p',t) - \\ &\quad [\mathcal{R}(p,t,\overline{U}^{(1)}_{\text{GS}}) - \mathcal{R}(p,t,\overline{U}^{(1)}_{\text{JAC}})].\end{aligned}$$

By the mean-value theorem,

$$\begin{aligned}(d(p,t) + \tau^{-1} + c^*)W^{(2)}(p,t) &= (c^* - f^{(1)}_u(p,t))W^{(1)}(p,t) + \tau^{-1}W^{(1)}(p,t) + \\ &\quad \sum_{p' \in \sigma'(p)} e(p',t) W^{(1)}(p',t) + \\ &\quad \sum_{p' \in \sigma'_L(p)} e(p',t) Z^{(2)}_{\text{GS}}(p',t),\end{aligned}$$

where $f^{(1)}_u(p,t) \equiv f_u[p,t,\overline{U}^{(1)}_{\text{JAC}}(p,t) + \theta^{(1)}(p,t)W^{(1)}(p,t)]$, $0 < \theta^{(1)}(p,t) < 1$. From here, (3), (7), (10) for $Z^{(2)}_{\text{GS}}$ and (14), it follows that

$$W^{(2)}(p,t) \le 0, \quad p \in \overline{\omega}^h.$$

By induction on n, we can prove that

$$W^{(n)}(p,t) \le 0, \quad p \in \overline{\omega}^h, \quad n \ge 1.$$

This proves the theorem. □

Remark 3. *The comparison result (13) shows that with the same initial iteration, which is either an upper or lower solution, the sequence of the Gauss–Seidel iterations converges not slower than the sequence of the Jacobi iterations.*

4. Convergence rates of the monotone methods

Here we analyze convergence rates of the monotone Jacobi and Gauss–Seidel methods from (9).

Theorem 3. *Suppose that the coefficients of the difference operator $\mathcal{L}$ in (2) satisfy (3), $f(p,t,u)$ satisfies (7) and $\overline{\omega}^h$ is connected (4). Let $U(p,t-\tau)$ be given and $\overline{U}^{(0)}(p,t)$, $\underline{U}^{(0)}(p,t)$ be upper and lower solutions of (2) corresponding to $U(p,t-\tau)$. Then the monotone Jacobi and Gauss–Seidel methods from (9) converge at the following convergence rates:*

$$\|Z^{(n)}_{\text{JAC}}(\cdot,t)\|_{\overline{\omega}^h} \le q_{\text{JAC}}\|Z^{(n-1)}_{\text{JAC}}(\cdot,t)\|_{\overline{\omega}^h}, \quad q_{\text{JAC}} = 1 - \frac{1}{1+\tau(d^*+c^*)}, \tag{15}$$

$$\|Z^{(n)}_{\text{GS}}(\cdot,t)\|_{\overline{\omega}^h} \le q_{\text{GS}}\|Z^{(n-1)}_{\text{GS}}(\cdot,t)\|_{\overline{\omega}^h}, \quad q_{\text{GS}} = 1 - \frac{1}{1+\tau(e^*+c^*)}, \tag{16}$$

$$\|Z^{(n)}_{\text{JAC}}(\cdot,t)\|_{\overline{\omega}^h} = \max_{p\in\overline{\omega}^h}|Z^{(n)}_{\text{JAC}}(p,t)|, \quad \|Z^{(n)}_{\text{GS}}(\cdot,t)\|_{\overline{\omega}^h} = \max_{p\in\overline{\omega}^h}|Z^{(n)}_{\text{GS}}(p,t)|,$$

where d^* and e^* are defined by

$$d^* = \max_{(p,t)\in\omega^h\times\omega^\tau} d(p,t), \quad e^* = \max_{(p,t)\in\omega^h\times\omega^\tau} \sum_{p'\in\sigma_U(p)} e(p',t).$$

Here $\sigma'_U(p)$ is a set of stencil points corresponding to a strictly upper triangular part of $\sigma(p)$.

Proof. Monotone convergence of upper and lower solutions has been proved in Theorem 1.

We now prove (15). Using the mean-value theorem, we represent the residual $\mathcal{R}(p,t,U^{(n-1)}_{\text{JAC}})$ in the form

$$\mathcal{R}(p,t,U^{(n-1)}_{\text{JAC}}) = \mathcal{L}Z^{(n-1)}_{\text{JAC}}(p,t) + f_u(p,t)Z^{(n-1)}_{\text{JAC}}(p,t) + \mathcal{R}(p,t,U^{(n-2)}_{\text{JAC}})$$

From here and (9) with $Z^{(n-1)}_{\text{JAC}}$, we have

$$-\mathcal{R}(p,t,U^{(n-1)}_{\text{JAC}}) = \mathcal{L}_{\text{JAC}}Z^{(n-1)}_{\text{JAC}}(p,t) - (\mathcal{L}Z^{(n-1)}_{\text{JAC}}(p,t) + f_u(p,t)Z^{(n-1)}_{\text{JAC}}(p)),$$

where

$$\mathcal{L}Z^{(n-1)}_{\text{JAC}}(p,t) = (d(p,t)+\tau^{-1})Z^{(n-1)}_{\text{JAC}}(p,t) - \sum_{p'\in\sigma'(p)} e(p',t)Z^{(n-1)}_{\text{JAC}}(p,t),$$

$$\mathcal{L}_{\text{JAC}}Z^{(n-1)}_{\text{JAC}}(p,t) = (d(p,t)+\tau^{-1}+c^*)Z^{(n-1)}_{\text{JAC}}(p,t).$$

Thus,

$$-\mathcal{R}(p,t,U^{(n-1)}_{\text{JAC}}) = (c^*-f_u(p,t))Z^{(n-1)}_{\text{JAC}}(p,t) + \sum_{p'\in\sigma'(p)} e(p',t)Z^{(n-1)}_{\text{JAC}}(p,t). \tag{17}$$

From here and (9), we conclude that

$$(d(p,t)+\tau^{-1}+c^*)Z^{(n)}_{\text{JAC}}(p,t) = (c^*-f_u(p,t))Z^{(n-1)}_{\text{JAC}}(p,t) + \sum_{p'\in\sigma'(p)} e(p',t)Z^{(n-1)}_{\text{JAC}}(p,t).$$

Let $\max|Z^{(n)}_{\text{JAC}}(p,t)|$ over ω^h attain at point p_*. Then from the last equation, (3) and (7), we have

$$(d(p_*,t)+\tau^{-1}+c^*)\gamma^{(n)}(t) \le c^*\gamma^{(n-1)}(t) + \sum_{p'\in\sigma'(p_*)} e(p',t)\gamma^{(n-1)}(t),$$

where $\gamma^{(n)}(t) = \|Z^{(n)}_{\text{JAC}}(\cdot,t)\|_{\overline{\omega}^h}$. From here and (3), we conclude (15).

Similar to (17), we can obtain

$$-\mathcal{R}(p,t,U_{\rm GS}^{(n-1)}) = (c^* - f_u(p,t))Z_{\rm GS}^{(n-1)}(p,t) + \sum_{p' \in \sigma'_U(p)} e(p',t) Z_{\rm GS}^{(n-1)}(p,t).$$

From here and (9), we conclude that

$$(d(p,t) + \tau^{-1} + c^*)Z_{\rm GS}^{(n)}(p,t) - \sum_{p' \in \sigma'_L(p)} e(p',t) Z_{\rm GS}^{(n)}(p,t) =$$
$$(c^* - f_u(p,t))Z_{\rm GS}^{(n-1)}(p,t) + \sum_{p' \in \sigma'_U(p)} e(p',t) Z_{\rm GS}^{(n-1)}(p,t).$$

Let $\max |Z_{\rm GS}^{(n)}(p,t)|$ over ω^h attain at point p_*. Then from the last equation, (3) and (7), we have

$$(d(p_*,t) + \tau^{-1} + c^*)\delta^{(n)}(t) \quad - \sum_{p' \in \sigma'_L(p_*)} e(p',t)\delta^{(n)}(t) \le c^*\delta^{(n-1)}(t) +$$
$$\sum_{p' \in \sigma'_U(p_*)} e(p',t)\delta^{(n-1)}(t),$$

where $\delta^{(n)}(t) = \|Z_{\rm GS}^{(n)}(\cdot,t)\|_{\overline{\omega}^h}$. From (3), we have

$$d(p,t) - \sum_{p' \in \sigma'_L(p)} e(p',t) \ge \sum_{p' \in \sigma'_U(p)} e(p',t).$$

From here and the previous inequality, we conclude (16). □

Without loss of generality, we assume that the boundary condition $g = 0$. This assumption can always be obtained via a change of variables. On each time level, let $U_\nu^{(0)}(p,t)$ be chosen in the form of (12), that is, $U_\nu^{(0)}(p,t)$ is the solution of the difference problem

$$\mathcal{L}U_\nu^{(0)}(p,t) = \nu \left| f(p,t,0) - \tau^{-1}U(p,t-\tau) \right|, \quad p \in \omega^h, \tag{18}$$
$$U_\nu^{(0)}(p,t) = 0, \ p \in \partial\omega^h, \quad \nu = 1,-1,$$

where $M(p,t) = 0$. Then the functions $U_1^{(0)}(p,t)$, $U_{-1}^{(0)}(p,t)$ are upper and lower solutions.

Theorem 4. *Let initial upper or lower solutions be chosen in the form of (18), and let f satisfy (7). Suppose that on each time level the number of iterates $n_* \ge 2$. Then for the monotone iterative methods (9), the following estimates on convergence rates hold*

$$\max_{1 \le k \le N_\tau} \|U_{\rm JAC}(\cdot,t_k) - U(\cdot,t_k)\|_{\overline{\omega}^h} \le C_{\rm JAC}(c^* + d^*) q_{\rm JAC}^{n_*-1}, \tag{19}$$
$$\max_{1 \le k \le N_\tau} \|U_{\rm GS}(\cdot,t_k) - U(\cdot,t_k)\|_{\overline{\omega}^h} \le C_{\rm GS}(c^* + e^*) q_{\rm GS}^{n_*-1},$$
$$U_{\rm JAC}(p,t_k) = U_{\rm JAC}^{(n_*)}(p,t_k), \quad U_{\rm GS}(p,t_k) = U_{\rm GS}^{(n_*)}(p,t_k),$$

where $U(p,t)$ is the solution to (2), constants $C_{\rm JAC}$, $C_{\rm GS}$ are independent of τ, and $q_{\rm JAC}$, $q_{\rm GS}$, d^* and e^* are defined in Theorem 3.

Proof. We prove estimate (19) for the monotone Gauss–Seidel method. Similar to (11), using the mean-value theorem and the equation for $Z_{\rm GS}^{(n)}$ from (9), we have

$$\mathcal{L}U_{\rm GS}^{(n)}(p,t) + f(p,t,U_{\rm GS}^{(n)}) - \tau^{-1}U_{\rm GS}(p,t-\tau) \quad = \quad -[c^* - f_u^{(n)}(p,t)]Z_{\rm GS}^{(n)}(p,t) -$$
$$\sum_{p' \in \sigma'_U(p)} e(p',t) Z_{\rm GS}^{(n)}(p',t), \tag{20}$$

$$f_u^{(n)}(p,t) \equiv f_u[p,t,U_{\mathrm{GS}}^{(n-1)}(p,t)+\theta^{(n)}(p,t)Z_{\mathrm{GS}}^{(n)}(p,t)], \quad 0<\theta^{(n)}(p,t)<1.$$

Introduce the notation

$$W(p,t)=U(p,t)-U_{\mathrm{GS}}(p,t),$$

where $U_{\mathrm{GS}}(p,t)=U_{\mathrm{GS}}^{(n_*)}(p,t)$. Using the mean-value theorem, from (2) and (20), we conclude that $W(p,\tau)$ satisfies the problem

$$\begin{aligned}\mathcal{L}W(p,\tau)+f_u(p,\tau)W(p,\tau) &= (c^*-f_u^{(n_*)}(p,\tau))Z_{\mathrm{GS}}^{(n_*)}(p,\tau)+\\ &\quad \sum_{p'\in\sigma'_U(p)} e(p',t)Z_{\mathrm{GS}}^{(n_*)}(p',\tau), \quad p\in\omega^h,\end{aligned}$$

$$W(p,\tau)=0, \quad p\in\partial\omega^h,$$

where $f_u^{(n_*)}(p,\tau)\equiv f_u[p,\tau,U(p,\tau)+\theta(p,\tau)W(p,\tau)]$, $0<\theta(p,\tau)<1$, and we have taken into account that $U_{\mathrm{GS}}(p,0)=U(p,0)=u^0(p)$. By (6), (7) and (16),

$$\|W(\cdot,\tau)\|_{\overline{\omega}^h}\le(c^*+e^*)\tau q_{\mathrm{GS}}^{n_*-1}\|Z_{\mathrm{GS}}^{(1)}(\cdot,\tau)\|_{\overline{\omega}^h}.$$

Using (7), (18) and the mean-value theorem, estimate $Z_{\mathrm{GS}}^{(1)}(p,\tau)$ from (9) by (6),

$$\begin{aligned}\|Z_{\mathrm{GS}}^{(1)}(\cdot,\tau)\|_{\overline{\omega}^h} &\le \tau\|\mathcal{L}U_{\mathrm{GS}}^{(0)}(\cdot,\tau)\|_{\overline{\omega}^h}+c^*\tau\|U_{\mathrm{GS}}^{(0)}(\cdot,\tau)\|_{\overline{\omega}^h}\\ &\quad +\tau\|f(p,\tau,0)-\tau^{-1}u^0\|_{\overline{\omega}^h}\\ &\le (2\tau+c^*\tau^2)\|f(p,\tau,0)-\tau^{-1}u^0\|_{\overline{\omega}^h}\\ &\le (2+c^*\tau)(\tau\|f(p,\tau,0)\|_{\overline{\omega}^h}+\|u^0\|_{\overline{\omega}^h})\le C_1,\end{aligned}$$

where C_1 is independent of τ ($\tau\le T$). Thus,

$$\|W(\cdot,\tau)\|_{\overline{\omega}^h}\le(c^*+e^*)C_1\tau q_{\mathrm{GS}}^{n_*-1}. \tag{21}$$

Similarly, from (2) and (20), it follows that

$$\begin{aligned}\mathcal{L}W(p,2\tau)+f_u(p,2\tau)W(p,2\tau) &= (c^*-f_u^{(n_*)}(p,2\tau))Z_{\mathrm{GS}}^{(n_*)}(p,2\tau)+\\ &\quad \sum_{p'\in\sigma'_U(p)} e(p',t)Z_{\mathrm{GS}}^{(n_*)}(p',2\tau)+\tau^{-1}W(p,\tau).\end{aligned}$$

Using (16), by (6),

$$\|W(\cdot,2\tau)\|_{\overline{\omega}^h}\le\|W(\cdot,\tau)\|_{\overline{\omega}^h}+(c^*+e^*)\tau q_{\mathrm{GS}}^{n_*-1}\|Z_{\mathrm{GS}}^{(1)}(\cdot,2\tau)\|_{\overline{\omega}^h}. \tag{22}$$

Using (18), estimate $Z_{\mathrm{GS}}^{(1)}(p,2\tau)$ from (9) by (6),

$$\|Z_{\mathrm{GS}}^{(1)}(\cdot,2\tau)\|_{\overline{\omega}^h}\le(2+c^*\tau)(\tau\|f(p,2\tau,0)\|_{\overline{\omega}^h}+\|U_{\mathrm{GS}}(\cdot,\tau)\|_{\overline{\omega}^h})\le C_2,$$

where $U_{\mathrm{GS}}(p,\tau)=U_{\mathrm{GS}}^{(n_*)}(p,\tau)$. As follows from Theorem 1, the monotone sequences $\{\overline{U}_{\mathrm{GS}}^{(n)}(p,\tau)\}$ and $\{\underline{U}_{\mathrm{GS}}^{(n)}(p,\tau)\}$ are bounded from above and below by, respectively, $\overline{U}_{\mathrm{GS}}^{(0)}(p,\tau)$ and $\underline{U}_{\mathrm{GS}}^{(0)}(p,\tau)$. Applying (6) to problem (18) at $t=\tau$, we have

$$\|U_{\mathrm{GS}}^{(0)}(\cdot,\tau)\|_{\overline{\omega}^h}\le\tau\|f(p,\tau,0)-\tau^{-1}u^0(p)\|_{\overline{\omega}^h}\le K_1,$$

where constant K_1 is independent of τ. Thus, we prove that C_2 is independent of τ. From (21) and (22), we conclude

$$\|W(\cdot,2\tau)\|_{\overline{\omega}^h}\le(c^*+e^*)(C_1+C_2)\tau q_{\mathrm{GS}}^{n_*-1}.$$

By induction on k, we prove

$$\|W(\cdot,t_k)\|_{\overline{\omega}^h}\le(c^*+e^*)\left(\sum_{l=1}^{k}C_l\right)\tau q_{\mathrm{GS}}^{n_*-1}, \; k=1,\dots,N_\tau,$$

where all constants C_l are independent of τ. Taking into account that $N_\tau \tau = T$, we prove the estimate (19) with $C_{\text{GS}} = T \max_{1\le l\le N_\tau} C_l$.

Estimate (19) for the monotone Jacobi method can be proved in a similar manner. In this case the summation over $\sigma'_U(p)$ in (20) becomes the summation over $\sigma'(p)$, and e^* must be changed on d^*. $\square$

5. Applications to solving nonlinear singularly perturbed problems

We consider the two dimensional singularly perturbed reaction-diffusion problem

$$(23) \qquad u_t - \mu^2(u_{xx} + u_{yy}) + f(x,y,t) = 0, \quad (x,y,t) \in \omega \times (0,T],$$
$$u(x,y,t) = g(x,t), \quad (x,y,t) \in \partial\omega \times (0,T],$$
$$u(x,y,0) = u^0(x,y), \quad (x,y) \in \overline{\omega}, \quad \omega = (0,1)\times(0,1),$$

where μ is a small positive parameter and f satisfies (7).

The solution of problem (23) can be decomposed into two parts $u = S + E$, where S and E are the regular and singular parts of u, respectively. In turn, the singular part can be decomposed in the form

$$E = \Phi + \Psi + (\Upsilon_{00} + \Upsilon_{10} + \Upsilon_{01} + \Upsilon_{11}),$$

where Φ and Ψ are essentially one-dimensional boundary layer functions in some neighborhoods of sides $x=0$, $x=1$ and $y=0$, $y=1$, respectively, and $\Upsilon_{mn}, m,n = 0,1$ are corner layers in the neighborhood of (m,n). The following bounds on the derivatives hold true:

$$\left|\frac{\partial^k S(x,y,t)}{\partial x^{k_x}\partial y^{k_y}\partial t^{k_t}}\right| \le C,$$
$$\left|\frac{\partial^k \Phi(x,y,t)}{\partial x^{k_x}\partial y^{k_y}\partial t^{k_t}}\right| \le C\mu^{-k_x}\Pi(x), \quad \Pi(x) = \Pi_0(x) + \Pi_1(x),$$
$$\left|\frac{\partial^k \Psi(x,y,t)}{\partial x^{k_x}\partial y^{k_y}\partial t^{k_t}}\right| \le C\mu^{-k_y}\hat{\Pi}(y), \quad \hat{\Pi}(y) = \hat{\Pi}_0(y) + \hat{\Pi}_1(y),$$
$$\left|\frac{\partial^k \Upsilon_{mn}(x,y,t)}{\partial x^{k_x}\partial y^{k_y}\partial t^{k_t}}\right| \le C\mu^{-(k_y+k_y)}\Pi_m(x)\hat{\Pi}_n(y), \; m,n = 0,1,$$
$$\Pi_0(x) = \exp(-\kappa_1 x/\mu), \quad \Pi_1(x) = \exp(-\kappa_1(1-x)/\mu),$$
$$\hat{\Pi}_0(y) = \exp(-\kappa_2 y/\mu), \quad \hat{\Pi}_1(y) = \exp(-\kappa_2(1-y)/\mu),$$

where $k = (k_x, k_y, k_t)$, $k_x + k_y + 2k_t \le l$, κ_1 and κ_2 are positive constants, and constant C is independent of μ and the mesh parameters (see [5] for details). For $\mu \ll 1$, problem (23) is singularly perturbed and characterized by boundary layers of width $\mathcal{O}(\mu|\ln\mu|)$ near $\partial\omega$.

On $\overline{\omega}$ introduce nonuniform mesh $\overline{\omega}^h = \overline{\omega}^{hx} \times \overline{\omega}^{hy}$:

$$\overline{\omega}^{hx} = \{x_i,\; 0 \le i \le N_x;\; x_0 = 0,\; x_{N_x} = 1;\; h_{xi} = x_{i+1} - x_i\},$$
$$\overline{\omega}^{hy} = \left\{y_j,\; 0 \le j \le N_y;\; y_0 = 0,\; y_{N_y} = 1;\; h_{yj} = y_{j+1} - y_j\right\}.$$

To approximate (23), we use the classical difference scheme based on the five-point stencil

$$(24) \qquad \mathcal{L}U(p,t) + f(p,t,U) - \tau^{-1}U(p,t-\tau) = 0, \quad (p,t) \in \omega^h \times (\overline{\omega}^\tau \setminus 0),$$

$$\mathcal{L} = \mathcal{L}^h + \tau^{-1}, \quad \mathcal{L}^h U(p,t) = -\mu^2(\mathcal{D}_x^2 + \mathcal{D}_y^2)U(p,t),$$

where $\mathcal{D}_x^2 U$ and $\mathcal{D}_y^2 U$ are the central difference approximations to the second derivatives

$$\mathcal{D}_x^2 U_{ij}^k = (\hbar_{xi})^{-1}[(U_{i+1,j}^k - U_{ij}^k)(h_{xi})^{-1} - (U_{ij}^k - U_{i-1,j}^k)(h_{x,i-1})^{-1}],$$

$$\mathcal{D}_y^2 U_{ij}^k = (\hbar_{yj})^{-1}[(U_{i,j+1}^k - U_{ij}^k)(h_{yj})^{-1} - (U_{ij}^k - U_{i,j-1}^k)(h_{y,j-1})^{-1}],$$
$$\hbar_{xi} = 2^{-1}(h_{x,i-1} + h_{xi}), \quad \hbar_{yj} = 2^{-1}(h_{y,j-1} + h_{yj}),$$

where $p = (x_i, y_j) \in \omega^h$ and $U_{ij}^k = U(x_i, y_j, t_k)$.

The difference scheme (24) satisfies the assumptions in (3). Thus, for the monotone Jacobi and Gauss–Seidel methods (9) applied to the difference scheme (24), Theorems 1–4 hold true.

5.1. Uniform convergence of the monotone iterative methods on layer-adapted meshes. Here we investigate convergence of the monotone iterative methods (9) to the difference scheme (2) defined on meshes of general type introduced in [8].

A mesh of this type is formed in the following manner. We divide each of the intervals $\overline{\omega}^x = [0,1]$ and $\overline{\omega}^y = [0,1]$ into three parts $[0,\varsigma_x]$, $[\varsigma_x, 1-\varsigma_x]$, $[1-\varsigma_x, 1]$, and $[0,\varsigma_y]$, $[\varsigma_y, 1-\varsigma_y]$, $[1-\varsigma_y, 1]$, respectively. Assuming that N_x, N_y are divisible by 4, in the parts $[0,\varsigma_x]$, $[1-\varsigma_x, 1]$ and $[0,\varsigma_y]$, $[1-\varsigma_y, 1]$ we allocate $N_x/4+1$ and $N_y/4+1$ mesh points, respectively, and in the parts $[\varsigma_x, 1-\varsigma_x]$ and $[\varsigma_y, 1-\varsigma_y]$ we allocate $N_x/2+1$ and $N_y/2+1$ mesh points, respectively. Points ς_x, $(1-\varsigma_x)$ and ς_y, $(1-\varsigma_y)$ correspond to transition to the boundary layers. We consider meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ which are equidistant in $\left[x_{N_x/4}, x_{3N_x/4}\right]$ and $\left[y_{N_y/4}, y_{3N_y/4}\right]$ but graded in $\left[0, x_{N_x/4}\right]$, $\left[x_{3N_x/4}, 1\right]$ and $\left[0, y_{N_y/4}\right]$, $\left[y_{3N_y/4}, 1\right]$. On $\left[0, x_{N_x/4}\right]$, $\left[x_{3N_x/4}, 1\right]$ and $\left[0, y_{N_y/4}\right]$, $\left[y_{3N_y/4}, 1\right]$ let the mesh be given by a mesh generating function ϕ with $\phi(0) = 0$ and $\phi(1/4) = 1$ which is supposed to be continuous, monotonically increasing, and piecewise continuously differentiable. Then the mesh is defined by

$$x_i = \begin{cases} \varsigma_x \phi(\xi_i), & \xi_i = i/N_x,\ i = 0,\dots,N_x/4; \\ \varsigma_x + (i - N_x/4)h_x, & i = N_x/4+1,\dots,3N_x/4; \\ 1 - \varsigma_x \phi(\xi_i), & \xi_i = (N_x - i)/N_x,\ i = 3N_x/4+1,\dots,N_x, \end{cases}$$

$$y_j = \begin{cases} \varsigma_y \phi(\xi_j), & \xi_j = j/N_y,\ j = 0,\dots,N_y/4; \\ \varsigma_y + (j - N_y/4)h_y, & j = N_y/4+1,\dots,3N_y/4; \\ 1 - \varsigma_y \phi(\xi_j), & \xi_j = (N_y - j)/N_y,\ j = 3N_y/4+1,\dots,N_y, \end{cases}$$
$$h_x = 2(1-2\varsigma_x)N_x^{-1}, \quad h_y = 2(1-2\varsigma_y)N_y^{-1}.$$

We also assume that $d(\phi(\xi))/d\xi$ does not decrease. This condition implies that

$$h_{xi} \le h_{x,i+1},\ i = 1,\dots,N_x/4-1, \quad h_{xi} \ge h_{x,i+1},\ i = 3N_x/4+1,\dots,N_x-1,$$
$$h_{yj} \le h_{y,j+1},\ j = 1,\dots,N_y/4-1, \quad h_{yj} \ge h_{y,j+1},\ j = 3N_y/4+1,\dots,N_y-1.$$

5.1.1. Uniform convergence on the piecewise uniform mesh. We choose the transition points ς_x, $(1-\varsigma_x)$ and ς_y, $(1-\varsigma_y)$ as in [10],

$$\varsigma_x = \min\left\{4^{-1}, \upsilon_1 \mu \ln N_x\right\}, \quad \varsigma_y = \min\left\{4^{-1}, \upsilon_2 \mu \ln N_y\right\},$$

where υ_1 and υ_2 are positive constants. If $\varsigma_{x,y} = 1/4$, then $N_{x,y}$ are very large compared to $1/\mu$ which means that the difference scheme (2) can be analyzed using standard techniques. We therefore assume that

$$\varsigma_x = \upsilon_1 \mu \ln N_x, \quad \varsigma_y = \upsilon_2 \mu \ln N_y.$$

Consider the mesh generating function ϕ in the form

$$\phi(\xi) = 4\xi.$$

In this case the meshes $\overline{\omega}^{hx}$ and $\overline{\omega}^{hy}$ are piecewise uniform with the step sizes

$$N_x^{-1} < h_x < 2N_x^{-1}, \quad h_{x\mu} = 4\upsilon_1 \mu N_x^{-1} \ln N_x, \tag{25}$$
$$N_y^{-1} < h_y < 2N_y^{-1}, \quad h_{y\mu} = 4\upsilon_2 \mu N_y^{-1} \ln N_y.$$

The difference scheme (24) on the piecewise uniform mesh (25) converges μ-uniformly to the solution of (23):

$$\max_{t\in\overline{\omega}^\tau} \|U(\cdot,t)-u(\cdot,t)\|_{\overline{\omega}^h} \leq C(N^{-1}\ln N+\tau), \quad N=\min\{N_x,N_y\}, \tag{26}$$

where C denotes a generic constant that is independent of μ, N and τ. The proof of this result can be found in [5].

Lemma 3. *For the monotone Jacobi and Gauss–Seidel methods (9) applied to the difference scheme (24) on the piecewise uniform mesh (25), the convergent factors* q_{JAC} *and* q_{GS} *are defined by (15) and (16), respectively, with*

$$d^* = \frac{N_x^2}{8\upsilon_1^2 \ln^2 N_x} + \frac{N_y^2}{8\upsilon_2^2 \ln^2 N_y}, \quad e^* = \frac{d^*}{2}. \tag{27}$$

Proof. From (23), we have

$$d_{ij} = \frac{\mu^2}{\hbar_{xi}}\left(\frac{1}{h_{x,i-1}}+\frac{1}{h_{xi}}\right) + \frac{\mu^2}{\hbar_{yj}}\left(\frac{1}{h_{y,j-1}}+\frac{1}{h_{yj}}\right).$$

From (25), we obtain

$$\max\{h_{x\mu}^{-1}; h_x^{-1}\} = h_{x\mu}^{-1}, \quad \max\{h_{y\mu}^{-1}; h_y^{-1}\} = h_{y\mu}^{-1}.$$

Thus,

$$d^* = \max_{ij} d_{ij} = 2\mu^2(h_{x\mu}^{-2}+h_{y\mu}^2).$$

From here and (25), we conclude (27) for d^*. The result for e^* follows from the fact that for interior mesh points with step-sizes $h_{x\mu}$, $h_{y\mu}$ and not adjacent to the boundary, we have

$$\sum_{p'\in\sigma_U(p)} e(p,p') = d(p)/2.$$

□

Theorem 5. *Let initial upper or lower solutions be chosen in the form of (18) and* f *satisfy (7). Suppose that on each time level the number of iterates* $n_* \geq 2$. *Then the monotone iterative methods (9), applied to the difference scheme (24) on the piecewise uniform mesh (25), converge* μ*-uniformly to the solution of the difference scheme (24), where the parameters* d^* *and* e^* *in the convergence rates (19) are defined in (27).*

Proof. As follows from the proof of Theorem 4, for the monotone iterative methods (9) on the piecewise uniform mesh (25), constants C_{JAC} and C_{GS} are independent of τ, μ and N. Thus from here, Theorem 4 and Lemma 3, we prove uniform convergence of the monotone iterative methods (9). □

Remark 4. *From (26) and Theorem 5, we conclude that the monotone iterative methods (9), applied to the difference scheme (24) on the piecewise uniform mesh (25), converge* μ*-uniformly to the solution of the differential problem (23).*

5.1.2. Uniform convergence on the log-mesh. Here we assume that $\mu \leq \mu_0 = \text{const} < 1$, and choose the transition points ς_x, $(1-\varsigma_x)$ and ς_y, $(1-\varsigma_y)$ as in [2],

$$\varsigma_x = \upsilon_1\mu\ln(1/\mu), \quad \varsigma_y = \upsilon_2\mu\ln(1/\mu), \tag{28}$$

$$\phi(\xi) = \frac{\ln[1-4(1-\mu)\xi]}{\ln\mu}.$$

The difference scheme (24) on the log-mesh (28) converges μ-uniformly to the solution of (23):

$$\max_{t\in\overline{\omega}^{\tau}} \|U(\cdot,t)-u(\cdot,t)\|_{\overline{\omega}^{h}} \le C(N^{-1}+\tau), \quad N=\min\{N_x,N_y\}, \tag{29}$$

where constant C is independent of μ, N and τ. The proof of this result can be found in [5].

Lemma 4. *For the monotone Jacobi and Gauss–Seidel methods (9) applied to the difference scheme (24) on the log-mesh (28), the convergent factors q_{JAC} and q_{GS} are defined by (15) and (16), respectively, with*

$$d^* = 2(\upsilon_1 \ln(1-4(1-\mu_0)N_x^{-1}))^{-2} + 2(\upsilon_2 \ln(1-4(1-\mu_0)N_y^{-1}))^{-2}, \quad e^* = \frac{d^*}{2}. \tag{30}$$

Proof. From (28), it follows that

$$\max_{i,j} d_{ij} = \frac{\mu^2}{\hbar_{x,1}}\left(\frac{1}{h_{x,0}}+\frac{1}{h_{x,1}}\right) + \frac{\mu^2}{\hbar_{y,1}}\left(\frac{1}{h_{y,0}}+\frac{1}{h_{y,1}}\right) \le \frac{2\mu^2}{h_{x,0}^2}+\frac{2\mu^2}{h_{y,0}^2},$$

$$h_{x,0} = -\upsilon_1\mu\ln(1-4(1-\mu)N_x^{-1}), \quad h_{y,0} = -\upsilon_2\mu\ln(1-4(1-\mu)N_y^{-1}).$$

From here and taking into account that $\mu\le\mu_0$ we prove (30). □

Remark 5. *If N_x and N_y are sufficiently large, then from (30), we conclude the estimate*

$$d^* \approx \frac{N_x^2}{8\upsilon_1^2(1-\mu_0)^2} + \frac{N_y^2}{8\upsilon_2^2(1-\mu_0)^2}.$$

Theorem 6. *Let initial upper or lower solutions be chosen in the form of (18) and f satisfy (7). Suppose that on each time level the number of iterates $n_*\ge 2$. Then the monotone iterative methods (9), applied to the difference scheme (24) on the log-mesh (28), converge μ-uniformly to the solution of the difference scheme (24), where the parameters d^* and e^* in the convergence rates (19) are defined in (30).*

Proof. As follows from the proof of Theorem 4, for the monotone iterative methods (9) on the log-mesh (28), constants C_{JAC} and C_{GS} are independent of τ, μ and N. Thus from here, Theorem 4 and Lemma 4, we conclude uniform convergence of the monotone iterative methods (9). □

Remark 6. *From (29) and Theorem 6, we conclude that the monotone iterative methods (9), applied to the difference scheme (24) on the log-mesh (28), converge μ-uniformly to the solution of the differential problem (23).*

5.2. Numerical experiments. As a test problem for the singularly perturbed problem (23), we use $f(u)=\exp(-1)-\exp(-u)$, $g=0$ and $u^0=0$. This problem gives $c_*=\exp(-1)$, $c^*=1$, and the initial lower and upper solutions are chosen in the form of (18).

The stopping criterion for the monotone methods (9) is

$$\|U^{(n)}(\cdot,t)-U^{(n-1)}(\cdot,t)\|_{\overline{\omega}^{h}} \le \delta,$$

where $\delta=10^{-5}$ is the prescribed accuracy. We denote by $\underline{n}$ and $\overline{n}$ numbers of iterative steps required to get the required accuracy on each time level over 10 time steps using lower and upper sequences, respectively.

It is found that in all numerical experiments the basic feature of monotone convergence of the upper and lower sequences is observed. In fact, the monotone property of the sequences holds at every mesh point in the domain. This is, of course, to be expected from the analytical consideration.

We mention here that our numerical results on the piecewise uniform mesh (25) and log-mesh (28) are almost the same, thus we present the results only on the piecewise uniform mesh. In (25), we use $N_x = N_y = N$ and $\upsilon_1 = \upsilon_2 = 1$.

In Tables 1 and 2, for various values of N, μ and $\tau = 10^{-2}, 10^{-1}$, we present convergence iteration counts for the monotone Jacobi and Gauss–Seidel methods (9), respectively. From the data, we conclude that for $\mu \leq 10^{-2}$, the numbers of iterations are independent of the perturbation parameter μ. These numerical results confirm our theoretical results stated in Theorem 5.

$\mu\backslash N$	16	32	64	128	256
	$\underline{n}_{\text{JAC}}$; $\overline{n}_{\text{JAC}}$ ($\tau = 10^{-2}$)				
10^{-1}	42; 40	74; 70	166; 159	463; 442	1410; 1337
$\mu \leq 10^{-2}$	32; 30	42; 40	61; 59	94; 90	188; 180
	$\underline{n}_{\text{JAC}}$; $\overline{n}_{\text{JAC}}$ ($\tau = 10^{-1}$)				
10^{-1}	122; 116	308; 294	910; 864	2738; 2574	7791; 7215
$\mu \leq 10^{-2}$	67; 65	106; 102	195; 187	431; 411	1049; 994

TABLE 1. Convergence iteration counts over ten time steps for the monotone Jacobi method.

$\mu\backslash N$	16	32	64	128	256
	$\underline{n}_{\text{GS}}$; $\overline{n}_{\text{GS}}$ ($\tau = 10^{-2}$)				
10^{-1}	44; 40	62; 60	116; 113	281; 272	817; 782
$\mu \leq 10^{-2}$	31; 30	41; 40	52; 50	72; 71	128; 124
	$\underline{n}_{\text{GS}}$; $\overline{n}_{\text{GS}}$ ($\tau = 10^{-1}$)				
10^{-1}	90; 88	195; 189	534; 513	1598; 157	4695; 4397
$\mu \leq 10^{-2}$	59; 57	81; 79	133; 129	266; 257	617; 592

TABLE 2. Convergence iteration counts over ten time steps for the monotone Gauss–Seidel method.

Convergence rates of the monotone Jacobi and Gauss–Seidel methods, λ_{JAC} and λ_{GS}, respectively, can be estimated using the formulae

$$\lambda_{\text{JAC}}(2N) = \frac{n_{\text{JAC}}(2N)}{n_{\text{JAC}}(N)}, \quad \lambda_{\text{GS}}(2N) = \frac{n_{\text{GS}}(2N)}{n_{\text{GS}}(N)}.$$

For $\tau = 10^{-2}, 10^{-1}$, Tables 3 and 4 present uniform convergence rates of the monotone Jacobi and Gauss–Seidel methods, respectively. From (15) and (16), we can estimate theoretical convergence rates of the monotone methods in the form

$$\lambda^{th}_{\text{JAC}}(2N) \approx \frac{\ln(q_{\text{JAC}}(N))}{\ln(q_{\text{JAC}}(2N))}, \quad \lambda^{th}_{\text{GS}}(2N) \approx \frac{\ln(q_{\text{GS}}(N))}{\ln(q_{\text{GS}}(2N))},$$

where q_{JAC}, q_{GS} are defined in (15) and (16), respectively, and d^*, e^* are given in (27). In Tables 3 and 4, we give the theoretical convergence rates as well. From the data in these tables, we can conclude that the numerical convergence rates are close to the theoretical convergence rates.

We next compare the monotone iterative methods (9) with the monotone iterative method from [3]. The last method is defined by the recurrence formulae

$$(\mathcal{L} + c^*)Z^{(n)}(p,t) = -\mathcal{R}(p,t,U^{(n-1)}), \quad p \in \omega^h,$$

τ	$\underline{\lambda}_{\text{JAC}};\ \overline{\lambda}_{\text{JAC}};\ \lambda^{th}_{\text{JAC}}$			
10^{-2}	1.31; 1.33; 1.18	1.45; 1.48; 1.31	1.54; 1.53; 1.53	2.00; 2.00; 1.90
10^{-1}	1.58; 1.57; 1.37	1.84; 1.83; 1.71	2.21; 2.20; 2.18	2.43; 2.42; 2.67
$2N$	32	64	128	256

TABLE 3. Convergence rates for the monotone Jacobi method.

τ	$\underline{\lambda}_{\text{GS}};\ \overline{\lambda}_{\text{GS}};\ \lambda^{th}_{\text{GS}}$			
10^{-2}	1.32; 1.33; 1.12	1.27; 1.25; 1.23	1.39; 1.42; 1.40	1.78; 1.75; 1.67
10^{-1}	1.37; 1.39; 1.23	1.64; 1.63; 1.48	2.01; 1.93; 1.90	2.32; 2.30; 2.42
$2N$	32	64	128	256

TABLE 4. Convergence rates for the monotone Gauss–Seidel method.

$$Z^{(1)}(p,t) = g(p,t) - U^{(0)}(p,t), \quad Z^{(n)}(p,t) = 0, \quad n \geq 2, \quad p \in \partial\omega^h,$$
$$U^{(n)}(p,t) = U^{(n-1)}(p,t) + Z^{(n)}(p,t), \quad p \in \overline{\omega}^h,$$
$$\mathcal{R}(p,t,U^{(n-1)}) \equiv \mathcal{L}U^{(n-1)}(p,t) + f(p,t,U^{(n-1)}) - \tau^{-1}U(p,t-\tau),$$

where $\mathcal{L}$ is defined in (2). As a linear solver at each iterative step, we employ the conjugate gradient method with the preconditioner based on the incomplete LU factorization (ILUCG) (see [1] for details).

$\mu\backslash N$	128	256	512
	$t_{\text{JAC}};\ t_{\text{GS}};\ t_{\text{ILUCG}}\ (\tau = 10^{-2})$		
10^{-1}	20; 12; 15	231; 136; 116	2581; 1653; 873
$\mu \leq 10^{-2}$	4; 3; 9	31; 24; 50	342; 196; 273
	$t_{\text{JAC}};\ t_{\text{GS}};\ t_{\text{ILUCG}}\ (\tau = 10^{-1})$		
10^{-1}	111; 62; 48	1105; 659; 394	8362; 6209; 3386
$\mu \leq 10^{-2}$	17; 11; 20	176; 104; 127	1447; 943; 1006

TABLE 5. Execution times for the monotone Jacobi, monotone Gauss–Seidel and ILUCG methods over twenty time levels.

Table 5 displays the execution times of the monotone Jacobi, monotone Gauss–Seidel and ILUCG methods over 20 time levels. All execution times are rounded up to the nearest second. The data show that for $\mu \leq 10^{-2}$ the monotone Gauss–Seidel method executes faster than the monotone Jacobi and ILUCG methods. For $\tau = 10^{-2}$, $\mu \leq 10^{-2}$, $N \leq 256$ and $\tau = 10^{-1}$, $\mu \leq 10^{-2}$, $N \leq 128$ the monotone Jacobi method executes faster than ILUCG method.

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Institute of Fundamental Sciences, Massey University, Private Bag 11-222, Palmerston North, New Zealand

E-mail: `I.Boglaev@massey.ac.nz`

URL: `http://ifs.massey.ac.nz/staff/boglaev.htm`