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NUMERICAL APPROXIMATION OF OPTION PRICING MODEL UNDER JUMP DIFFUSION USING THE LAPLACE TRANSFORMATION METHOD

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Abstract. We propose a **LT** (*Laplace transformation*) method for solving the PIDE (*partial integro-differential equation*) arising from the financial mathematics. An option model under a jump-diffusion process is given by a PIDE, whose non-local integral term requires huge computational costs. In this work, the PIDE is transformed into a set of complex-valued elliptic problems by taking the Laplace transformation in time variable. Only a small number of Laplace transformed equations are then solved on a suitable choice of contour. Then the time-domain solution can be obtained by taking the Laplace inversion based on the chosen contour. Especially a splitting method is proposed to solve the PIDE, and its solvability and convergence are proved. Numerical results are shown to confirm the efficiency of the proposed method and the parallelizable property.

Key Words. Laplace inversion, Option, Derivative, Jump-diffusion

1. Introduction

The Black-Scholes formula [3], introduced by Black and Scholes in 1973, has been adopted as a standard framework for option pricing, particularly due to its closed form solution. However, the difficulty in capturing a large or sudden movement of an underlying asset has been pointed out as a major drawback of the Black-Scholes formula. Practitioners and theorists have tried to extend the model of the underlying to tackle this phenomena, and among them the implementation of jump-diffusion processes has become one of the most popular tools. In the pioneering work of Merton [23], he modeled the underlying assets using a Brownian motion with drift having jumps arriving accordingly as a compound Poisson process. In [16], Kou tried to explain high peaks and heavy tails in asset return distributions incorporating with the volatility smile by proposing a double exponential jump diffusion model. These jump-diffusion processes are considered as specific examples of Lévy processes with stationary independent increments, and have been applied intensively to improve option modeling. For further details, for instance, readers are referred to the book by Cont and Tankov [6] and the references therein.

To evaluate the value of the option modeled by jump-diffusion process, one needs to solve a PIDE (*partial integro-differential equation*) of parabolic type that contains both partial differential operators and a non-local integral term. Several attempts have been tried to reduce the expensive computational cost of solving the PIDE. Most of such attempts may be roughly classified into two types. One is to try to improve the efficiency in the computation of the non-local integral term, and the

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other is try to reduce the number of time steps. In particular, the discretization of the non-local integral term generates dense matrices that are very expensive to deal with: iterative methods or other splitting techniques should be employed instead of any direct matrix inversion schemes. For example, an implicit-explicit method was developed by Zhang [32], and an ADI (*Alternative Direction Implicit*) method was applied by Andersen and Andersen [2]. A fixed point iteration scheme was introduced by d' Halluin *et al.* [8]. As an effort to reduce the number of time steps, Almendral and Oosterlee [1] applied a second order backward difference formula (BDF2) and Feng and Linetsky [9] used a high order extrapolation method.

In spite of the popularity of time marching methods, which all the above mentioned works employed, they require usually as many time steps as spatial meshes in order to balance the errors arising from the spatial and time discretizations. The schemes in [1, 9] reduce the number of time steps considerably compared to previous works, but they are still of polynomial convergence in time.

In the present paper, we propose a new approach for a parabolic type PIDE based on the **LT** (*Laplace transformation*) method. As proven in [26], the method is of exponential convergence in time, and in addition it can be easily parallelized. In the current paper, the method is coupled with a finite element method to solve the Laplace transformed complex elliptic equations. The solvability of the Laplace transformed equations is proved, and numerical experiments are performed to show the efficiency of the proposed method. The numerical results show an exponential order of convergence in time, and a second order in space. The **LT** method has been already applied to the Black-Scholes equation in [17]. Moreover, in [17], a precise absorbing boundary condition is derived and the solvability of the set of complex-valued elliptic problems that are the Laplace transforms of the Black-Scholes equation. Related with **LT** method there are other approaches; for instance, see [12, 10, 11, 20], and so on. Also, high-dimensional parabolic problems can be solved using sparse grids [13, 18, 19, 25]. Application of our **LT** method using sparse grids to option pricing will also be interesting. Other approaches in the fast time-stepping methods can be found in [29, 22, 21]; any of these methods can be also chosen in the **LT** method that is to be developed in this paper.

The rest of the paper is organized as follows. In the following subsection, a brief explanation about the mathematical formulation of the model under jump diffusion is described in terms of PIDE. §2 introduces the **LT** method and provides a convergence theorem with some remarks. In §3, we describe the Laplace transformed equation and prove its solvability. The finite element method for solving the complex elliptic equations and the technique to accelerate the numerical scheme are described in §4. §5 shows numerical results to confirm the convergence and efficiency results of the proposed method.

1.1. The parabolic integro-differential equation. Let S denote the price of an underlying asset. Following Merton [23], the underlying asset that is governed by a Brownian motion with drift having jumps arriving accordingly as a compound Poisson process is assumed to satisfy the following stochastic differential equation:

$$\mathrm{d}S = \nu S\,\mathrm{d}\tau + \sigma S\,\mathrm{d}W + (\eta - 1)S\,\mathrm{d}q, \tag{1.1}$$

where the terms $\mathrm{d}W$ and $\mathrm{d}q$ represent the increment of a Brownian motion and a Poisson process, respectively. In (1.1), ν and σ are the drift rate and the volatility of the Brownian part. The Poisson process $\mathrm{d}q$ is defined by

$$\mathrm{d}q = \begin{cases} 0 & \text{with probability } 1 - \lambda\,\mathrm{d}\tau, \\ 1 & \text{with probability } \lambda\,\mathrm{d}\tau, \end{cases}$$

where λ is the arrival intensity of the Poisson process. Also, $(\eta - 1)$ is an impulse function producing a finite jump in S to $S\eta$ where the average jump size $\mathbb{E}[\eta - 1]$ is denoted by κ. It is assumed that the Brownian and Poisson processes are uncorrelated.

Let T be the maturity date and set $t = T - \tau$. Denote then by r the risk-free interest rate and $g(\eta)$ the probability density function of the jump amplitude η. Denote by $V(S,t)$ the option value depending on the asset value $S \in \mathbb{R}^+$ and the time to maturity $t \in (0,T]$ with a given initial contract $V_0 = V(S,0)$. Under the above assumptions, V then satisfies the following PIDE ([23]):

$$V_t - \frac{1}{2}\sigma^2 S^2 V_{SS} - (r - \lambda\kappa)SV_S + (r+\lambda)V - \lambda \int_0^\infty V(S\eta, t) g(\eta)\,\mathrm{d}\eta = 0, \tag{1.2}$$

for $(S,t) \in \mathbb{R}^+ \times (0,T]$.

Note that we have restricted our attention to the model presented in [23], where jumps are log-normally distributed with mean μ and variance σ_J, and the log-normal probability density function is defined by

$$g(\eta) = \frac{1}{\sqrt{2\pi}\sigma_J \eta} e^{-(\log(\eta)-\mu)^2/2\sigma_J^2} \chi_{\{\eta>0\}},$$

and in this case,

$$\kappa = e^{\mu + \sigma_J^2/2} - 1.$$

Let K be the strike price of the option. The changes of variables in (1.2)

$$S = Ke^x, \quad \eta = e^y, \quad g(\eta) = f(y), \quad \text{and} \quad V(S,t) = U(x,t),$$

result in

$$U_t - \frac{1}{2}\sigma^2 U_{xx} - (r - \lambda\kappa - \frac{1}{2}\sigma^2)U_x + (r+\lambda)U - \lambda \int_{\mathbb{R}} U(x+y,t)p(y)\,\mathrm{d}y = 0, \tag{1.3}$$

for $(x,t) \in \mathbb{R} \times (0,T]$, where the function $p(y)$ is defined by

$$p(y) = \frac{1}{\sqrt{2\pi}\sigma_J} e^{-(y-\mu)^2/2\sigma_J^2}.$$

In order to develop a numerical method, we truncate the infinite domain $\mathbb{R}$ into a bounded domain $\Omega = (x_L, x_R)$ with $-\infty < x_L \ll 0 \ll x_R < \infty$. In order to cooperate the non-local integral term in (1.3), which requires the information on U outside Ω, we need to impose a suitable condition in the exterior to Ω as well as a boundary condition on $\partial\Omega$. We therefore assume that there exists a suitable function $R : \Omega^c \times (0,T] \to \mathbb{R}$ such that

$$U(x,t) = R(x,t), \qquad (x,t) \in \Omega^c \times (0,T], \tag{1.4}$$

where the choice of $R(x,t)$ depends on the initial data. For instance, an European put option has the initial data

$$U_0(x) = \max\left(K(1-e^x), 0\right),$$

and one can choose the following function under the linearity assumption:

$$R(x,t) = \begin{cases} Ke^{-rt} - Ke^x, & \text{if } x \le x_L, \\ 0, & \text{if } x \ge x_R. \end{cases}$$

For other initial contracts, one can also deduce a boundary condition based on an initial data as mentioned in [8].

From now on, the function $R : \Omega^c \times (0,T] \to \mathbb{R}$ is assumed to be extended to the whole domain $\mathbb{R} \times (0,T]$, still denoted by $R(\cdot,t)$, for the sake of notational simplicity.

2. The Laplace Transformation method

In order to describe the **LT** method for a class of parabolic-type problems that include (1.3), we will consider an abstract setting. We thus consider

$$\begin{aligned} \frac{\partial u}{\partial t} + Au &= f, \quad t \in (0.T], \\ u(0) &= u_0, \end{aligned} \tag{2.1}$$

where u_0 is given initial data and A is a spatial elliptic operator in $L^2(\Omega)$ with its eigenvalues located in a sector Σ_δ for some $\delta \in (0, \frac{\pi}{2})$, where the sector Σ_δ is defined by $\Sigma_\delta = \{z \in \mathbb{C} : |\arg(z)| < \delta, z \neq 0\}$. Furthermore, we assume that the resolvent $(zI + A)^{-1}$ of A satisfies

$$\|(zI + A)^{-1}\| \le \frac{M}{1 + |z|}, \quad \text{for } z \in \Sigma_{\pi-\delta} \cup B,$$

where B is a small circle at the origin and M is a constant independent of z.

For $z \in \Sigma_{\pi-\delta} \cup B$, the Laplace transform in time of a function $u(\cdot, t)$ is given by

$$\widehat{u}(\cdot, z) := \mathcal{L}[u](z) = \int_0^\infty u(\cdot, t)e^{-zt}\,\mathrm{d}t,$$

and for such z, the Laplace transform of (2.1) is given by

$$z\widehat{u} + A\widehat{u} = u_0 + \widehat{f}(\cdot, z). \tag{2.2}$$

Notice that the solution $\widehat{u}(z) = \widehat{u}(\cdot, z)$ of (2.2) is formally given by

$$\widehat{u}(\cdot, z) = (zI + A)^{-1}(u_0(\cdot) + \widehat{f}(\cdot, z)), \tag{2.3}$$

for each z. The *Laplace inversion formula* is given by the following Bromwich integral [4]

$$u(\cdot, t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \widehat{u}(\cdot, z)e^{zt}\,\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \big(zI + A\big)^{-1}\big(u_0 + \widehat{f}(z)\big)e^{zt}\,\mathrm{d}z, \tag{2.4}$$

where the integral contour Γ, which is contained in $\Sigma_{\pi-\delta} \cup B$, is chosen such that all the singularities of the integrand are enclosed.

Observe that if $z \in \Gamma$ has negative real parts, the discretization error in numerically evaluating the integrand in (2.4) will be reduced for positive t as $|z|$ becomes large. There are several types of deformed contours and quadratures for accurate numerical inversion of Laplace transformation. [12, 10, 11, 20, 30, 28, 31, 14]

The following analysis can be modified without difficulty by choosing any other alternative contour and quadrature mentioned above instead of using the approach of the smooth contour of hyperbola type proposed by Sheen *et al.* [26]. The hyperbola contour in [26] is defined by

$$\Gamma = \{z \in \mathbb{C} : z(\omega) = \zeta(\omega) + \mathrm{i}s\omega, \quad \omega \in \mathbb{R}, \quad \omega \text{ increasing}\},$$

where $\zeta(\omega) = \gamma - \sqrt{\omega^2 + \nu^2}$. In this case, since the contour cuts the real line at $\gamma - \nu$, γ and ν must be selected such that $\gamma - \nu$ is larger than the real part of the negative of the smallest eigenvalue of A and the real parts of singularities of $\widehat{f}(z)$. Also s should be chosen such that all the singularities of $\widehat{u}(\cdot, z)$ and $\widehat{f}(z)$ be to the left of the contour Γ.

Using the above deformed contour, the inversion formula can be written as an indefinite integral as follows:

$$u(\cdot, t) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \widehat{u}(\cdot, \zeta(\omega) + \mathrm{i}s\omega)(\zeta'(\omega) + \mathrm{i}s)e^{(\zeta(\omega)+\mathrm{i}s\omega)t}\,\mathrm{d}\omega.$$

It is convenient to change the infinite range of the variable ω in the above integral into to a finite region by the change of variables $y : (-\infty, \infty) \to (-1, 1)$ defined by

$$y(\omega) = \tanh\big(\frac{\tau\omega}{2}\big) \quad \text{and} \quad \omega(y) = \frac{2}{\tau}\tanh^{-1}(y) = \frac{1}{\tau}\log\frac{1+y}{1-y},$$

for some $\tau > 0$. Consequently, the integral (2.4) reduces to the following form:

$$(2.5)\quad u(\cdot,t) = \frac{1}{2\pi \mathrm{i}}\int_{-1}^{1} \widehat{u}(\cdot,\zeta(\omega(y))+\mathrm{i}s\omega(y))(\zeta'(\omega(y))+\mathrm{i}s)e^{(\zeta(\omega(y))+\mathrm{i}s\omega(y))t}\omega'(y)\,\mathrm{d}y.$$

2.1. Semi-discrete approximation. The integral formula (2.5) can be discretized in time using a quadrature rule. A semi-discrete approximation of $u(\cdot,t)$ is defined by a composite trapezoidal rule as follows:

$$(2.6)\qquad U_{N_z,\tau}(\cdot,t) = \frac{1}{2\pi \mathrm{i}}\frac{1}{N_z}\sum_{j=-N_z+1}^{N_z-1} \widehat{u}(\cdot,z_j)\frac{\mathrm{d}z}{\mathrm{d}\omega}(\omega_j)\frac{\mathrm{d}\omega}{\mathrm{d}y}(y_j)e^{z_j t},$$

where

$$z_j = z(\omega_j), \quad \omega_j = \omega(y_j) \quad \text{and} \quad y_j = \frac{j}{N_z}, \quad \text{for } -N_z < j < N_z.$$

It is proved in [26] that the convergence rate of the quadrature scheme (2.6) is as high as the order of the regularity of the source term, stated as follows:

Theorem 2.1. *Let $u(t)$ be the solution of (2.1) and let $U_{N_z,\tau}(t)$ be its approximation defined by (2.6). Assume that $\widehat{f}(z)$ is analytic to the right of the contour Γ and continuous up to Γ, with $\widehat{f}^{(j)}(z)$ bounded on Γ for $j \le r$ and r an integer ≥ 1, Then, for $t > r\tau$,*

(2.7)

$$\|U_{N_z,\tau}(t)-u(t)\| \le \frac{C_{r,s}}{N_z^r}\Big(1+t^r+\frac{1}{\tau^r}\Big)e^{\gamma t}\Big(1+\log_+\frac{1}{t-r\tau}\Big)(\|u_0\|+\max_{k\le r}\sup_{z\in\Gamma}\|\widehat{f}^{(k)}(z)\|).$$

Three important remarks should be stressed.

Remark 2.2. *The implication of the above theorem is such that the convergence of the proposed scheme* (2.6) *is of order $O(\frac{1}{N_z^r})$ with an arbitrary large $r > 0$ if $\widehat{f} \equiv 0$ or it is analytic. This implies that the time discretization errors using the* **LT** *method are negligible compared to the spatial discretization errors in solving parabolic problems with a homogeneous term.*

Remark 2.3. *In the summand* (2.6)*, an important observation is that*

$$\widehat{u}(\cdot,z_j)\frac{dz}{d\omega}(\omega_j)\frac{d\omega}{dy}(y_j),\ j = 0,\cdots,N_z,$$

are independent of t. Therefore, we only have to approximate $\widehat{u}(\cdot,z_j)$ only once by solving the complex-valued elliptic problem (2.2) *for a set of $z_j, j = -N_z + 1,\cdots,N_z-1$. Then, if we need the option pricing at different time t, the same set of spatial solutions $\widehat{u}(\cdot,z_j), j = -N_z+1,\cdots,N_z-1$, can be used in the evaluation of the summation* (2.6) *with the only change in $e^{z_j t}$, with the desired time t.*

Remark 2.4. *Notice that each elliptic problem* (2.2) *for a z_j from the set of $z_j, j = -N_z+1,\cdots,N_z-1$, is independent of all other elliptic problems for the remaining z_j's. This will minimize communication times in solving the elliptic problems* (2.2) *in parallel by assigning each processor to solve an independent elliptic problem without communicating with other processors during solving its assigned problem.*

3. Laplace transformed equations

In this section, we will apply the **LT** method to the given PIDE (1.3), and analyze the solvability of the transformed equation. Let $\widehat{U} = \widehat{U}(z) = \widehat{U}(\cdot, z)$ denote the Laplace transform of $U = U(t) = U(\cdot, t)$. Taking Laplace transform of (1.3) we have the following equation:

$$z\widehat{U} - \frac{1}{2}\sigma^2 \widehat{U}_{xx} - (r - \lambda\kappa - \frac{1}{2}\sigma^2)\widehat{U}_x + (r+\lambda)\widehat{U} - \lambda \int_{-\infty}^{\infty} \widehat{U}(x+y)p(y)\, \mathrm{d}y = U_0, \tag{3.1}$$

for each $z \in \Gamma$.

3.1. Solvability of the Laplace transformed equations. Let $u(x,t) = U(x,t) - R(x,t)$ such that $\mathrm{supp}(u(\cdot,t)) \subset \overline{\Omega}$ for all $t \in (0, T]$. Denote by $\widehat{u}$ and $\widehat{R}$ the Laplace transforms of u and R in time, respectively. Multiplying (3.1) by a test function $v \in H_0^1(\Omega)$ and integrating on Ω, one arrives at the weak problem of (3.1): for each $z \in \Gamma$, find $\widehat{u}(z) \in H_0^1(\Omega)$ such that

$$A_z(\widehat{u}, v) = F(v) \quad \forall v \in H_0^1(\Omega), \tag{3.2}$$

where the sesquilinear form $A_z(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ and the linear functional $F(\cdot) : H_0^1(\Omega) \to \mathbb{C}$ are defined by

$$\begin{aligned} A_z(u,v) &= z(u,v) + B(u,v) - J(u,v) \quad \forall u, v \in H_0^1(\Omega), \\ F(v) &= (U_0, v) + A_z(\widehat{R}, v) \quad \forall v \in H_0^1(\Omega). \end{aligned} \tag{3.3}$$

Here, the two sesquilinear operators B and J on $H^1(\Omega)$ are defined by

$$\begin{aligned} B(u,v) &= \frac{1}{2}\int_\Omega \sigma^2 \frac{\partial u}{\partial x}\frac{\partial \overline{v}}{\partial x}\,\mathrm{d}x + \int_\Omega \Big(-r + \lambda\kappa + \frac{1}{2}\sigma^2 + \sigma\frac{\partial \sigma}{\partial x}\Big)\frac{\partial u}{\partial x}\overline{v}\,\mathrm{d}x \\ &\quad + \int_\Omega (r+\lambda)u\overline{v}\,\mathrm{d}x, \\ J(u,v) &= \lambda \int_\Omega \Big[\int_{-\infty}^{\infty} u(x+y)p(y)\,\mathrm{d}y\Big]\,\overline{v}(x)\,\mathrm{d}x \\ &= \lambda \int_\Omega \Big[\int_\Omega u(y)p(y-x)\,\mathrm{d}y\Big]\,\overline{v}(x)\,\mathrm{d}x \\ &= \lambda \int_\Omega \Big[\int_\Omega u(y)p(x-y)\,\mathrm{d}y\Big]\,\overline{v}(x)\,\mathrm{d}x \\ &= \lambda \int_\Omega (u * p)(x)\,\overline{v}(x)\,\mathrm{d}x, \end{aligned}$$

where $p(x) = p(-x)$.

Assumption 3.1. *Assume that σ, $\frac{\partial\sigma}{\partial x}$ and r belong to $L^\infty(\Omega)$. Moreover, assume that there exist two positive constants σ_L and σ_R such that*

$$0 < \sigma_L \le \sigma(x) \le \sigma_R \quad \textit{for all } x \in \Omega.$$

Due to the Poincaré's inequality, there exists a positive constant α satisfying

$$\|u\|_{L^2(\Omega)} \le \alpha |u|_{H_0^1(\Omega)} \qquad \text{for all } u \in H_0^1(\Omega),$$

where α is dependent only on Ω. For the sake of simplicity in notation, set

$$\delta = \|r\|_{L^\infty(\Omega)} + \lambda|\kappa| + \sigma_R\Big(\frac{1}{2}\sigma_R + \|\frac{\partial\sigma}{\partial x}\|_{L^\infty(\Omega)}\Big). \tag{3.4}$$

We now have the following two lemmas for the continuity and coercivity of $A_z(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$.

Lemma 3.2. *Under Assumption 3.1, the sesquilinear form* $A_z(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ *is continuous.*

Proof. Let $u, v \in H_0^1(\Omega)$. Then, we have the following estimates:

$$
\begin{aligned}
\Big| \int_\Omega \frac{1}{2}\sigma^2(x) \frac{\partial u}{\partial x}\frac{\partial \overline{v}}{\partial x}\,\mathrm{d}x \Big| &\le \frac{1}{2}\sigma_R^2 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \\
\Big| \int_\Omega \Big(-r(x) + \lambda\kappa + \frac{1}{2}\sigma^2(x) + \sigma(x)\frac{\partial \sigma}{\partial x}\Big) \frac{\partial u}{\partial x} \overline{v}\,\mathrm{d}x \Big| & \\
&\le \delta\, \|u\|_{H_0^1(\Omega)} \|v\|_{L^2(\Omega)} \\
&\le \delta\, \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \\
\Big| \int_\Omega \Big(z + r(x) + \lambda \Big) u \overline{v}\,\mathrm{d}x \Big| &\le (|z| + \|r\|_{L^\infty(\Omega)} + \lambda) \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}, \\
\Big| \int_\Omega \Big[\int_{-\infty}^{\infty} \lambda u(x+y) p(y)\,\mathrm{d}y \Big] \overline{v}(x)\,\mathrm{d}x \Big| & \\
&\le \lambda \|v\|_{L^2(\Omega)} \Big[\int_\Omega \Big(\int_{-\infty}^{\infty} u(x+y)p(y)\,\mathrm{d}y \Big)^2 \mathrm{d}x \Big]^{\frac{1}{2}} \\
&\le \lambda \|v\|_{L^2(\Omega)} \Big[\int_\Omega \Big(\int_\Omega u(\xi) p(\xi - x)\,d\xi \Big)^2 \mathrm{d}x \Big]^{\frac{1}{2}} \\
&\le \lambda \|v\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \Big[\int_\Omega \Big(\int_\Omega p(\xi - x)^2\,d\xi \Big) \mathrm{d}x \Big]^{\frac{1}{2}} \\
&\le \lambda \|v\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \Big[\int_\Omega \int_{-\infty}^{\infty} p(y)^2\,\mathrm{d}y\,\mathrm{d}x \Big]^{\frac{1}{2}} \\
&\le \lambda \Big(\frac{|\Omega|}{2\sqrt{\pi}\sigma_J} \Big)^{\frac{1}{2}} \|v\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.
\end{aligned}
$$

Therefore the sesquilinear form $A_z(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ is continuous. □

Lemma 3.3. *Under Assumption 3.1, there is a non-negative constant* C_1*, which is independent of* u *and* z*, such that*

$$
\mathrm{Re}\,(A_z(u,u)) \ge \frac{\sigma_L^2}{4} \Big\| \frac{\partial u}{\partial x} \Big\|_{L^2(\Omega)}^2 - (|z| + C_1)\|u\|_{L^2(\Omega)}^2 \quad \textit{for all } u \in H_0^1(\Omega).
$$

Proof. Let $u \in H^1(\Omega)$ be arbitrary. Then, the following estimates are immediate:

$$\int_\Omega \frac{1}{2}\sigma^2(x)\frac{\partial u}{\partial x}\frac{\partial \overline{u}}{\partial x}\,\mathrm{d}x \geq \frac{\sigma_L^2}{2}\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\Omega)}, \tag{3.5a}$$

$$\begin{aligned}\Big|\mathrm{Re}\Big(\int_\Omega \Big(-r(x)+\lambda\kappa+\frac{\sigma^2(x)}{2}+\sigma(x)\frac{\partial\sigma}{\partial x}\Big)\frac{\partial u}{\partial x}\overline{u}\,\mathrm{d}x\Big)\Big| &\\ \leq \delta\left\|\frac{\partial u}{\partial x}\right\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} &\\ \leq \frac{\sigma_L^2}{4}\left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(\Omega)}+\left(\frac{\delta}{\sigma_L}\right)^2\|u\|^2_{L^2(\Omega)}, &\end{aligned} \tag{3.5b}$$

$$\left|\mathrm{Re}\left(\int_{\mathbb{R}_+}\Big(z+r(x)+\lambda\Big)u\overline{u}\,\mathrm{d}x\right)\right| \leq (|z|+\|r\|_{L^\infty(\Omega)}+\lambda)\|u\|^2_{L^2(\Omega)}, \tag{3.5c}$$

$$\Big|\mathrm{Re}\Big(\int_\Omega\int_{-\infty}^{\infty}\lambda u(x+y)p(y)\,\mathrm{d}y\,\overline{u}(x)\,\mathrm{d}x\Big)\Big| \leq \lambda\Big(\frac{|\Omega|}{2\sqrt{\pi}\sigma_J}\Big)^{\frac{1}{2}}\|u\|^2_{L^2(\Omega)}, \tag{3.5d}$$

where Young's inequality is used in the bound of the second inequality and δ is defined in (3.4). A combination of these inequalities completes the lemma. □

Due to the Gärding's inequality given in the above Lemma 3.3 and the continuity in 3.2, we obtain the following existence and uniqueness theorem.

Theorem 3.4. *Suppose $U_0 \in L^2(\Omega)$. Then, under Assumption 3.1 Problem* (3.2) *has a unique solution $\widehat{u}(\cdot,z)-\widehat{R}(\cdot,z) \in H_0^1(\Omega)$ for each $z \in \mathbb{C}$.*

4. Numerical solution of the transformed equation

4.1. Finite element discretization. Let $(\mathcal{V}_h)_{0<h<1}$ be the set of standard piecewise linear finite-element subspaces of $H_0^1(\Omega)$ associated with uniform grids of mesh size $h = \frac{x_R - x_L}{N_x}$. Let $\{\phi_{h,j}\}_{j=1}^{N_x-1}$ be the space of basis functions defined by

$$\phi_{h,j} = \begin{cases}(x-x_{j-1})/h, & x_{j-1}\leq x\leq x_j,\\ (x_{j+1}-x)/h, & x_j < x \leq x_{j+1},\\ 0, & x\notin[x_{j-1},x_{j+1}],\end{cases}$$

where $x_j = x_L + jh$ and $j = 0,1,2,\cdots,N_x$. Then, for fixed z a finite element solution of (3.2) is represented as a linear combination of $\phi_{h,j}$'s in the following form:

$$\widehat{u}_h(x,z) = \sum_{j=1}^{N_x-1}\widehat{u}_j(z)\phi_{h,j}(x). \tag{4.1}$$

Substituting (4.1) into (3.3) and applying the test functions $\{\phi_{h,k}\}_{k=1}^{N_x-1}$, one obtains the following linear system:

$$z\mathbb{M}\widehat{\mathbf{u}} + \mathbb{B}\widehat{\mathbf{u}} - \mathbb{J}\widehat{\mathbf{u}} = \mathbf{F}, \tag{4.2}$$

where $\widehat{\mathbf{u}} = (\widehat{u}_1,\widehat{u}_2,\cdots,\widehat{u}_{N_x-1})^t$, $\mathbb{M}_{jk} = (\phi_{h,k},\phi_{h,j})$, $\mathbb{B}_{jk} = B(\phi_{h,k},\phi_{h,j})$, $\mathbb{J}_{jk} = J(\phi_{h,k},\phi_{h,j})$ and $\mathbf{F}_j = (U_0,\phi_{h,j}) + A_z(\widehat{R},\phi_{h,j})$.

We follow [9] in order to compute the non-local integral term $\mathbb{J}_{jk}$. First, one gets

$$\begin{aligned}\psi_{h,k}(x) &:= \int_\Omega \phi_{h,k}(y)p(y-x)\,\mathrm{d}y\\ &= \frac{\sigma_J}{\sqrt{2\pi}h}\Big\{e^{-\frac{(x_{k-1}-x-\mu)^2}{2\sigma_J^2}} - 2e^{-\frac{(x_k-x-\mu)^2}{2\sigma_J^2}} + e^{-\frac{(x_{k+1}-x-\mu)^2}{2\sigma_J^2}}\Big\}\\ &\quad -\frac{\sigma_J^2}{2h}\Big\{\Big(\frac{x-x_{k-1}+\mu}{h}\Big)\operatorname{erf}\Big(\frac{x_{k-1}-x-\mu}{\sqrt{2}\sigma_J}\Big)\\ &\quad +\Big(\frac{x-x_{k+1}+\mu}{h}\Big)\operatorname{erf}\Big(\frac{x_{k+1}-x-\mu}{\sqrt{2}\sigma_J}\Big)\\ &\quad +\Big(\frac{2x-x_{k-1}-x_k+2\mu}{h}\Big)\operatorname{erf}\Big(\frac{x_k-x-\mu}{\sqrt{2}\sigma_J}\Big)\Big\},\end{aligned}$$

where $\operatorname{erf}(x)$ represents the error function, $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. Then the approximated value of $\mathbb{J}_{jk}$ is obtained using the finite element interpolation of $\psi_{h,k}(x)$ as follows:

$$\begin{aligned}\mathbb{J}_{jk} &= \lambda\int_\Omega \psi_{h,k}(x)\phi_{h,j}(x)\,\mathrm{d}x\\ &\approx \lambda\int_\Omega \Big(\sum_{l=0}^{N_x}\psi_{h,k}(x_l)\phi_{h,l}(x)\Big)\phi_{h,j}(x)\,\mathrm{d}x\\ &= \lambda h\Big(\frac{1}{6}\psi_{h,k}(x_{j-1}) + \frac{2}{3}\psi_{h,k}(x_j) + \frac{1}{6}\psi_{h,k}(x_{j+1})\Big).\end{aligned} \tag{4.3}$$

Now we turn our attention to the computation of the right hand side $(\mathbb{F})_j = (U_0, \phi_{h,j}) + A_z(\widehat{R}, \phi_{h,j})$. The first term $(U_0, \phi_{h,j})$ is computed by a numerical quadrature rule, and the term $z(\widehat{R}, \phi_{h,j}) + B(\widehat{R}, \phi_{h,j})$ inside $A_z(\widehat{R}, \phi_{h,j})$ is computed analytically. The remaining term $J(\widehat{R}, \phi_{h,j})$ is approximated based on the finite element interpolation again, $\widehat{R}(x) \approx \sum_{k\in\mathbb{Z}} \widehat{R}_k\phi_{h,k}(x)$ where $\widehat{R}_k = \widehat{R}(x_k)$; explicitly we have

$$\begin{aligned}J(\widehat{R}, \phi_{h,j}) &\approx \lambda\int_\Omega\Big[\int_\Omega\sum_{k=-\infty}^{\infty}\widehat{R}_k\phi_{h,k}(y)p(y-x)\,\mathrm{d}y\Big]\phi_{h,j}(x)\,\mathrm{d}x\\ &= \lambda\int_\Omega\sum_{k=-\infty}^{\infty}\widehat{R}_k\psi_{h,k}(x)\phi_{h,j}(x)\,\mathrm{d}x\\ &= \lambda\sum_{k=-\infty}^{\infty}\widehat{R}_k\mathbb{J}_{jk} \approx \lambda\sum_{k=j-N_x+1}^{j+N_x-1}\widehat{R}_k\mathbb{J}_{jk}.\end{aligned}$$

Due to the exponential decay of the jump size probability density $p(y)$, the value of integration is negligibly small if $|j-k| > N_x$, and thus we can replace the infinite summation by a finite one.

In the meanwhile, for the integral part, it is sufficient to use the midpoint rule to keep a second order convergence order in space, that is,

$$\begin{aligned}\mathbb{J}_{jk} &= \lambda\int_\Omega\int_\Omega\phi_{h,k}(y)p(y-x)\,\mathrm{d}y\,\phi_{h,j}(x)\,\mathrm{d}x\\ &\approx \lambda\int_\Omega hp(x_k-x)\phi_{h,j}(x)\,\mathrm{d}x \approx \lambda h^2 p(x_k-x_j).\end{aligned} \tag{4.4}$$

The calculation given in (4.4) gives less accurate approximation than (4.3). However, it still preserves the second order convergence rate, and thus the numerical results are almost indifferent regardless of the choice of the approximation schemes between (4.3) and (4.4). In §5, we will compare these.

4.2. An iterative method to deal with the convolution integral term. As seen in the previous subsection, the discretization of the integral part leads to a full matrix $\mathbb{J}$, whose calculation requires a huge computational cost. In most earlier works, as mentioned in §1, the authors have tried to improve the computational efficiency by avoiding the inversion of the full matrix using various implicit-explicit or fixed point iteration methods. Since these methods are based on the real-valued parabolic problem of the original PIDE, we may not apply these previous approaches to our complex-valued problems directly.

Instead, we employ a regular splitting method to avoid the inversion of the full matrix. In the discretized setting, we can formulate the regular splitting as follows:

$$z\mathbb{M}\widehat{\mathbf{u}}^{n+1} + \mathbb{B}\widehat{\mathbf{u}}^{n+1} - \mathbb{J}\widehat{\mathbf{u}}^{n} = \mathbf{F}. \tag{4.5}$$

To check the convergence region, we examine the regular splitting method for (3.2) in the continuous setting instead of investigating (4.5) directly. We then have to find $\widehat{u}^{n+1} \in H_0^1(\Omega)$ such that

$$z(\widehat{u}^{n+1}, v) + B(\widehat{u}^{n+1}, v) = (\widehat{u}^{n} * p, v) + F(v), \quad \forall v \in H_0^1(\Omega). \tag{4.6}$$

By denoting $\widehat{\mathbf{e}}^n = \widehat{u} - \widehat{u}^n$, where $\widehat{u}$ is the solution of (3.2), we have the problem to find $\widehat{\mathbf{e}}^{n+1}$ such that

$$z(\widehat{\mathbf{e}}^{n+1}, v) + B(\widehat{\mathbf{e}}^{n+1}, v) = (\widehat{\mathbf{e}}^{n} * p, v), \quad \forall v \in H_0^1(\Omega). \tag{4.7}$$

Then we have the following lemma, which is a modification of Lemma 2.1 in [7].

Lemma 4.1. *Let $\widehat{\mathbf{e}}^n \in H_0^1(\Omega)$ satisfy (4.7) for $n = 1, 2, \cdots$. Under Assumption 3.1, for any $\theta \in (\frac{1}{2}\pi, \pi)$ there exist $C \geq 0$ and $\varsigma > 0$, independent of z and $\widehat{\mathbf{e}}^n$, such that*

$$\|\widehat{\mathbf{e}}^{n+1}\|_{L^2(\Omega)} \leq \frac{C}{|z-\varsigma|}\|\widehat{\mathbf{e}}^{n}\|_{L^2(\Omega)}, \qquad \text{for } z \in \sigma_{\varsigma,\theta},\ \widehat{\mathbf{e}}^n \in L^2(\Omega)$$

where $\sigma_{\varsigma,\theta} = \{z \in \mathbb{C} : |\arg(z-\varsigma)| \leq \theta\}$. Explicitly, the coefficients are given by $C = (1 + \frac{1}{2}\varrho)(1+\varrho^2)$ and $\varsigma = \left(1 + \frac{\varrho^2}{2}\right)\left(\frac{\delta}{\sigma_L}\right)^2 + \|r\|_{L^\infty(\Omega)} + \lambda$, where $\varrho = \tan\frac{\theta}{2}$.

Proof. Let $\theta \in \left(\frac{\pi}{2}, \pi\right)$, and $\varsigma > 0$ be arbitrary. For $z \in \sigma_{\varsigma,\theta}$, we write

$$z - \varsigma = (\xi + \mathrm{i}\eta)^2 = \xi^2 - \eta^2 + 2\mathrm{i}\xi\eta \qquad \text{with } \xi + \mathrm{i}\eta \in \sigma_{0,\theta/2},\ \xi, \eta \in \mathbb{R}.$$

Setting $\varrho = \tan\frac{\theta}{2}$, we see that $\xi > 0$, $\varrho > 1$, and $|\eta| \leq \varrho\xi$. Thus the following inequality holds:

$$\xi^2 \leq |z-\varsigma| = \xi^2 + \eta^2 \leq (1+\varrho^2)\xi^2. \tag{4.8}$$

Set

$$F = B(\widehat{\mathbf{e}}^{n+1}, \widehat{\mathbf{e}}^{n+1}) + z\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)}.$$

Taking the real part of F, we obtain

$$\operatorname{Re} B(\widehat{\mathbf{e}}^{n+1}, \widehat{\mathbf{e}}^{n+1}) + (\varsigma + \xi^2 - \eta^2)\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} = \operatorname{Re} F. \tag{4.9}$$

Combining (4.9) with (3.5a)–(3.5c), we have

$$\frac{\sigma_L^2}{4}\left\|\frac{\partial \widehat{\mathbf{e}}^{n+1}}{\partial x}\right\|_{L^2(\Omega)} + (\varsigma + \xi^2 - \eta^2 - \tilde{\mu})\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} \leq |F|, \tag{4.10}$$

where $\tilde{\mu} = (\delta/\sigma_L)^2 + \|r\|_{L^\infty} + \lambda$. By taking the imaginary part of F, we have

$$\operatorname{Im} B(\widehat{\mathbf{e}}^{n+1}, \widehat{\mathbf{e}}^{n+1}) + 2\xi\eta\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} = \operatorname{Im} F,$$

and thus,

$$\begin{aligned}2\xi|\eta|\,\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} &\le |F| + \big|\operatorname{Im} B(\widehat{\mathbf{e}}^{n+1},\widehat{\mathbf{e}}^{n+1})\big| \\ &\le |F| + \left|\operatorname{Im}\int_\Omega \Big(-r(x)+\lambda\kappa+\frac{\sigma^2(x)}{2}+\sigma(x)\frac{\partial\sigma}{\partial x}\Big)\frac{\partial\widehat{\mathbf{e}}^{n+1}}{\partial x}\overline{e^{n+1}}\,\mathrm{d}x\right| \\ &\le |F| + \delta\left\|\frac{\partial\widehat{\mathbf{e}}^{n+1}}{\partial x}\right\|_{L^2(\Omega)}\|\widehat{\mathbf{e}}^{n+1}\|_{L^2(\Omega)}.\end{aligned}$$

Multiplying by $\frac{1}{2}\varrho = \frac{1}{2}\tan(\frac{\theta}{2})$ the last estimate, we have

$$\eta^2\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} \le \varrho\xi|\eta|\,\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} \le \frac{1}{2}\varrho|F| + \frac{\varrho\delta}{2}\left\|\frac{\partial\widehat{\mathbf{e}}^{n+1}}{\partial x}\right\|_{L^2(\Omega)}\|\widehat{\mathbf{e}}^{n+1}\|_{L^2(\Omega)}.$$

Adding this to (4.10), we have

$$\begin{aligned}&\frac{\sigma_L^2}{4}\left\|\frac{\partial\widehat{\mathbf{e}}^{n+1}}{\partial x}\right\|^2_{L^2(\Omega)} + (\varsigma+\xi^2-\tilde{\mu})\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} \\ &\qquad\le (1+\frac{1}{2}\varrho)|F| + \frac{\sigma_L^2}{8}\left\|\frac{\partial\widehat{\mathbf{e}}^{n+1}}{\partial x}\right\|^2_{L^2(\Omega)} + \frac{\varrho^2\delta^2}{2\sigma_L^2}\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)},\end{aligned}$$

from which the choice of

$$\varsigma = \tilde{\mu} + \frac{\varrho^2\delta^2}{2\sigma_L^2},$$

leads to the following estimate:

$$\frac{\sigma_L^2}{8}\left\|\frac{\partial\widehat{\mathbf{e}}^{n+1}}{\partial x}\right\|^2_{L^2(\Omega)} + \xi^2\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} \le (1+\frac{1}{2}\varrho)|F|. \tag{4.11}$$

Since $(\widehat{\mathbf{e}}^n * p) \in L^2(\Omega)$ for $\widehat{\mathbf{e}}^n \in L^2(\Omega)$, by taking $\widehat{\mathbf{e}}^{n+1} \in H^1_0(\Omega)$ in (4.7) instead of v, we have from (4.11) that

$$\begin{aligned}\frac{\sigma_L^2}{8}\left\|\frac{\partial\widehat{\mathbf{e}}^{n+1}}{\partial x}\right\|^2_{L^2(\Omega)} + \xi^2\|\widehat{\mathbf{e}}^{n+1}\|^2_{L^2(\Omega)} &\le (1+\frac{1}{2}\varrho)\Big|\int_\Omega(\widehat{\mathbf{e}}^n * p)v\overline{e^{n+1}}\,\mathrm{d}x\Big| \\ &\le (1+\frac{1}{2}\varrho)\|(\widehat{\mathbf{e}}^n * p)\|_{L^2(\Omega)}\|\widehat{\mathbf{e}}^{n+1}\|_{L^2(\Omega)} \\ &\le (1+\frac{1}{2}\varrho)\|\widehat{\mathbf{e}}^n\|_{L^2(\Omega)}\|\widehat{\mathbf{e}}^{n+1}\|_{L^2(\Omega)},\end{aligned}$$

since $\|\widehat{\mathbf{e}}^n * p\|_{L^2(\Omega)} \le \|\widehat{\mathbf{e}}^n\|_{L^2(\Omega)}\|p\|_{L^1(\Omega)} \le \|\widehat{\mathbf{e}}^n\|_{L^2(\Omega)}$. Therefore, due to (4.8), it follows that

$$\|\widehat{\mathbf{e}}^{n+1}\|_{L^2(\Omega)} \le \frac{1+\frac{1}{2}\varrho}{\xi^2}\|\widehat{\mathbf{e}}^n\|_{L^2(\Omega)} \le \frac{(1+\frac{1}{2}\varrho)(1+\varrho^2)}{|z-\varsigma|}\|\widehat{\mathbf{e}}^n\|_{L^2(\Omega)}.$$

This completes the proof. □

This lemma implies that if one chooses a contour Γ such that the point $z \in \Gamma$ satisfying the condition $|z-\varsigma| > C$, the splitting scheme (4.6) is convergent. In particular, if the parameters in (1.2) are constants, one can calculate ς and C in the closed forms as follows:

$$\begin{aligned}\varsigma &= \Big(\frac{r+\lambda\kappa+\frac{1}{2}\sigma^2}{\sigma}\Big)^2\Big(1+\frac{\tan^2(\frac{1}{2}\arctan(s))}{2}\Big) + r + \lambda, \\ \text{and}\quad C &= \Big(1+\frac{\tan(\frac{1}{2}\arctan(s))}{2}\Big)\Big(1+\tan^2\big(\frac{1}{2}\arctan(s)\big)\Big),\end{aligned}$$

where s denotes the slope of the contour.

The following lemma describes the convergence region of the proposed method (4.5) with respect to z if the parameters in (1.2) are constants.

Lemma 4.2. *The iterative scheme* (4.5) *is convergent to the solution of* (4.2) *if the parameters in* (1.2) *satisfy*

$$\Big||\alpha(z)| - \big(|\beta_-(z)| + |\beta_+(z)|\big)\Big| > \frac{1}{\sqrt{2\pi}\sigma_J}\lambda h^2, \tag{4.12}$$

where $\alpha = \alpha(z), \beta_- = \beta_-(z)$, *and* $\beta_+ = \beta_+(z)$ *are given by*

$$\alpha = \frac{2h(z+r+\lambda)}{3} + \frac{\sigma^2}{h}, \quad \beta_- = -\frac{h(z+r+\lambda)}{6} + \frac{\sigma^2}{2h} + \frac{-2r+2\lambda\kappa+\sigma^2}{4},$$
$$\textit{and} \quad \beta_+ = -\frac{h(z+r+\lambda)}{6} + \frac{\sigma^2}{2h} - \frac{-2r+2\lambda\kappa+\sigma^2}{4}.$$

Proof. Let $\widehat{\mathbf{e}}$ be the eigenvector of the matrix $(z\mathbb{M}+\mathbb{B})^{-1}\mathbb{J}$ associated with an eigenvalue $\lambda_{\widehat{\mathbf{e}}}$ such that $|\widehat{\mathbf{e}}_i| = 1$ and $|\widehat{\mathbf{e}}| \leq 1$. It follows from the definitions of $\mathbb{M}$ and $\mathbb{B}$ that, for $j = 1, \cdots, N_x - 1$,

$$\lambda_{\widehat{\mathbf{e}}}\big(\alpha\widehat{\mathbf{e}}_j - \beta_-\widehat{\mathbf{e}}_{j-1} - \beta_+\widehat{\mathbf{e}}_{j+1}\big) = \sum_{k=1}^{N_x-1}\mathbb{J}_{jk}\widehat{\mathbf{e}}_k,$$

which yields the inequality

$$\big|\lambda_{\widehat{\mathbf{e}}}\big| \leq \frac{\sum_{k=1}^{N_x-1}|\mathbb{J}_{jk}||\widehat{\mathbf{e}}_k|}{\big||\alpha\widehat{\mathbf{e}}_j| - |\beta_-\widehat{\mathbf{e}}_{j-1} + \beta_+\widehat{\mathbf{e}}_{j+1}|\big|}.$$

Due to the inequality $|\mathbb{J}_{jk}| \leq \lambda h^2 \frac{1}{\sqrt{2\pi}\sigma_J}$ from (4.4), we arrive at

$$\big|\lambda_{\widehat{\mathbf{e}}}\big| \leq \frac{1}{\sqrt{2\pi}\sigma_J}\frac{\lambda h^2}{\big||\alpha| - (|\beta_-| + |\beta_+|)\big|},$$

and thus we have the convergent region

$$|\alpha| - \big(|\beta_-| + |\beta_+|\big) > \frac{1}{\sqrt{2\pi}\sigma_J}\lambda h^2.$$

This completes the proof. □

Since $\mathbb{J}$ is a full matrix, a lot of CPU time is required to computing (4.5) even though we do not invert $\mathbb{J}$, To reduce the computation time, we employ the FFT (*Fast Fourier Transform*). Recall that the element $\mathbb{J}_{jk}$ is dependent only on $(k-j)$. Denote by p_{k-j} the (j,k)-component of $\mathbb{J}$. That is,

$$\mathbb{J} = \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_{N_x-3} & p_{N_x-2} \\ p_{-1} & p_0 & p_1 & \cdots & p_{N_x-4} & p_{N_x-3} \\ p_{-2} & p_{-1} & p_0 & \cdots & p_{N_x-5} & p_{N_x-4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ p_{-N_x+2} & p_{-N_x+3} & p_{-N_x+4} & \cdots & p_{-1} & p_0 \end{pmatrix}.$$

Since this is a Toeplitz matrix, $\mathbb{J}\widehat{\mathbf{u}}$ can be computed by FFTs. As described in [5, 15] we embed the matrix $\mathbb{J}$ into a $[2(N_x-1)-1] \times [2(N_x-1)-1]$ circulant matrix, and extend $\widehat{\mathbf{u}}$ to $[2(N_x-1)-1]$-column vector by padding 0 terms to get $(\widehat{\mathbf{u}}, \underbrace{0, 0, \cdots, 0}_{N_x-2})$, and then we apply the FFT. Finally, we retrieve the matrix-vector multiplication by the inverse FFT again.

5. Numerical experiments

In this section we present and analyze some numerical results for pricing options under the jump diffusion process. In the first example, we confirm the convergence of the proposed scheme corresponding to its series solution, and the parallel efficiency is verified by measuring the parallel speed up. In the second example we examine the applicability of the proposed method by applying to an exotic option such as the knock-in option.

In what follows, we define the error reduction rate ρ and speedup by

$$\rho := \log_2 \left| \frac{\text{error of previous step}}{\text{error of current step}} \right|,$$

and

$$\text{speed up} = \frac{\text{time consumption}}{\text{time consumption when using 1-CPU}}.$$

Example 5.1 (European put option under jump diffusion)**.** *We consider an European put option with parameters* $r = 0.03$*,* $\sigma = 0.3$*,* $\mu = 0$*,* $\sigma_J = 0.35$*,* $\lambda = 0.5$*,* $T = 1.0$ *and* $K = 100$ *on a truncated domain* $x_L = -5$ *and* $x_R = 5$.

We compare the numerical solution with the analytic solution given in [23] as follows:

$$V(S,t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda(1+\kappa)t}(\lambda(1+\kappa)t)^k}{k!} V_{BS}(S, K, t, \sigma_k, r_k),$$

where $\kappa = \mathbb{E}[\eta - 1] = e^{(\mu+\sigma_J^2/2)} - 1$ and V_{BS} if given by the Black-Scholes formula as follows:

$$\begin{aligned} V_{BS}(S, K, t, \sigma_k, r_k) &= Ke^{-rt}\mathbb{N}(-d_2) - S\mathbb{N}(-d_1), \\ \sigma_k^2 = \sigma^2 + \frac{k\sigma_J^2}{t}, &\quad r_k = r - \lambda\kappa + \frac{k\log(1+\kappa)}{t}, \\ d_1 = \frac{\log(S/K) + (r_k + \sigma_k^2/2)t}{\sigma_k\sqrt{t}}, &\quad \text{and} \quad d_2 = d_1 - \sigma_k\sqrt{t}, \end{aligned}$$

where S is the price of an underlying asset and $\mathbb{N}$ is the standard normal cumulative distribution function. When approximating the value of the normal cumulative distribution, we used the subroutine provided in [24], and the infinite series is truncated at $k = 8$.

Tables 1 and 2 show the spatial convergence orders according to the increment in the number of space meshes. In these tables also comparison is shown between the two different approximation schemes (4.3) and (4.4) described in §4.1 on the calculation of the integral part. In both results the reduction rates ρ are almost two. The $L^2(\Omega)$-errors in Table 2 are indifferent from those in Table 1, but the $L^\infty(\Omega)$-errors in Table 2 are slightly smaller than those in Table 1. However, if numerical costs are considered in addition, we conclude that (4.4) seems to be a better choice than (4.3).

Also from Tables 1 and 2, at most 15 N_z-points in a contour are sufficient for the computation of the numerical solution until 2048 spatial meshes. Table 3 shows the convergence order in N_z number of points. Due to the exponential convergence property of the **LT** method, it shows almost a constant reduction rate ρ up to the choice of $N_z = 15$ contour points while N_z increases by a constant number. As N_z exceeds 15, the convergence order becomes almost 0 since the overall errors from this moment are dominated by spatial discretization.

N_z	N_x	$L^2(\Omega)$-Error	ρ	$L^\infty(\Omega)$-Error	ρ
15	16	1.491		0.6955	
15	32	0.4115	1.857	0.1797	1.952
15	64	0.1052	1.968	0.5755E-01	1.643
15	128	0.2645E-01	1.992	0.1460E-01	1.979
15	256	0.6622E-02	1.998	0.3680E-02	1.988
15	512	0.1656E-02	1.999	0.9215E-03	1.998
15	1024	0.4141E-03	2.000	0.2305E-03	1.999
15	2048	0.1036E-03	1.999	0.5760E-04	2.001

TABLE 1. Example 5.1 with increment in N_x using (4.3). N_z, N_x, and ρ represent the numbers of contour points z, the number of subintervals in space, and the reduction rate, respectively.

N_z	N_x	$L^2(\Omega)$-Error	ρ	$L^\infty(\Omega)$-Error	ρ
15	16	2.149		3.602	
15	32	0.4314	2.316	0.7427	2.278
15	64	0.1068	2.014	0.1857	2.000
15	128	0.2665E-01	2.003	0.4648E-01	1.999
15	256	0.6661E-02	2.001	0.1167E-01	1.994
15	512	0.1665E-02	2.000	0.2918E-02	2.000
15	1024	0.4161E-03	2.000	0.7295E-03	2.000
15	2048	0.1040E-03	2.001	0.1823E-03	2.000

TABLE 2. Example 5.1 with increment in N_x using (4.4). N_z, N_x, ρ represent the numbers of contour points z, the number of subintervals in space, and the reduction rate, respectively.

N_z	N_x	$L^2(\Omega)$-Error	ρ	γ	ν	s	τ
3	2048	0.6530		13.47594	12.41899	0.42126	0.16501
6	2048	0.1561E-01	5.387	26.95187	24.83798	0.42126	0.09385
9	2048	0.3636E-03	5.424	40.42781	37.25697	0.42126	0.06809
12	2048	0.1200E-03	1.599	53.90374	49.67596	0.42126	0.05430
15	2048	0.1040E-03	0.207	67.37968	62.09495	0.42126	0.04556
18	2048	0.1041E-03	-0.002	80.85561	74.51394	0.42126	0.03947
21	2048	0.1042E-03	-0.001	94.33155	86.93293	0.42126	0.03494
24	2048	0.1042E-03	0.000	107.80748	99.35192	0.42126	0.03144

TABLE 3. Example 5.1 with increment in N_z. N_z, N_x, and ρ represent the numbers of contour points z, the number of subintervals in space, and the reduction rate, respectively.

Next, we address the efficiency of the scheme (4.5) for the treatment of the Toeplitz matrix part. With $N_x = 1024$ the iteration is set to stop if the difference $\|\widehat{\mathbf{u}}^{n+1} - \widehat{\mathbf{u}}^n\|_{\ell^2} = (\sum_i |\widehat{\mathbf{u}}_i^{n+1} - \widehat{\mathbf{u}}_i^n|^2)^{1/2}$ is less than 10^{-5}, which is sufficient to keep the exact 2nd order convergence as Tables 1 and 2. The iteration numbers required to meet this criterion are shown in Table 4. We observe that, for each z, at most 13 number of iterations are sufficient. We remark that a larger stopping criterion such as 10^{-3} also provides almost same numerical results with the iteration numbers less than 9.

z	N_{iter}	$\|\widehat{\mathbf{u}}^{n+1} - \widehat{\mathbf{u}}^n\|_{\ell^2}$	z	N_{iter}	$\|\widehat{\mathbf{u}}^{n+1} - \widehat{\mathbf{u}}^n\|_{\ell^2}$
(5.28,0.00)	13	0.74E-5	(5.22,1.23)	13	0.70E-5
(5.01,2.38)	13	0.57E-5	(4.65,3.75)	13	0.40E-5
(4.14,5.05)	12	0.94E-5	(3.45,6.41)	12	0.54E-5
(2.56,7.83)	12	0.28E-5	(1.43,9.35)	11	0.58E-5
(-1.77,12.82)	10	0.58E-5	(-4.06,14.88)	10	0.24E-5
(-7.08,17.31)	9	0.58E-5	(-11.24,20.31)	9	0.22E-5
(-17.54,24.40)	8	0.53E-5	(-29.15,31.13)	8	0.13E-5

TABLE 4. The number of iterations (N_{iter}) for the iterative scheme (4.5) in Example 5.1 where $N_z = 15$, $N_x = 1024$.

As mentioned in the remarks in §2, since the linear equations in (4.2) for $z = z_k, k = -N_z + 1, \cdots, N_z - 1$, are independent each other, no communication is required during the computation except for the last summation step in the numerical Laplace inversion. Thus the **LT** method is very well fitted for parallel computation of the option pricing problem. The results in Table 5 are obtained on 2048 uniform spatial meshes with $N_z = 15$ points in a contour using IBM PowerPC97 with 2.2GHz clock speed. The results show almost an ideal speedup because of the minimization of communication time.

Number of CPUs	1	3	5	15
Time (sec)	33.65	12.37	7.48	2.50
Speedup	1.00	2.72	4.50	13.46

TABLE 5. Parallel speed up in Example 5.1

In Figure 1, we plot the analytic solution, the numerical solution, and the option value under a pure diffusion process to show the differences explicitly.

Example 5.2 (Knock-in put option under jump diffusion)**.** *We consider a knock-in put option with parameters* $r = 0.03$, $\sigma = 0.3$, $\mu = 0$, $\sigma_J = 0.35$, $\lambda = 0.5$, $T = 1.0$, $K = 100$ *and knock-in barrier* $B = 80$ *on a truncated domain where* $x_L = -5$ *and* $x_R = 5$.

A knock-in put option has no value until the underlying asset price touches the barrier B, but from this moment it starts to act as an ordinary put option. As described in [27], the barrier B being monitored continuously, it is considered as a boundary condition

$$V_{ki}(S,t) = V_{sp}(S,t), \quad \text{for } S \leq B,$$

where $V_{ki}(S,t)$ is the value of the knock-in option and $V_{sp}(S,t)$ is the value of the standard put option. If a time marching scheme is employed, $V_{sp}(\cdot, t_n)$ should be computed for every discretized time step t_n, in order to be used as the boundary condition in the computation of $V_{ki}(\cdot, t_n)$. Instead, if the **LT** method is used, the boundary condition is also transformed into

$$\widehat{V}_{ki}(B,z) = \widehat{V}_{sp}(B,z) \quad \text{for } S \leq B,$$

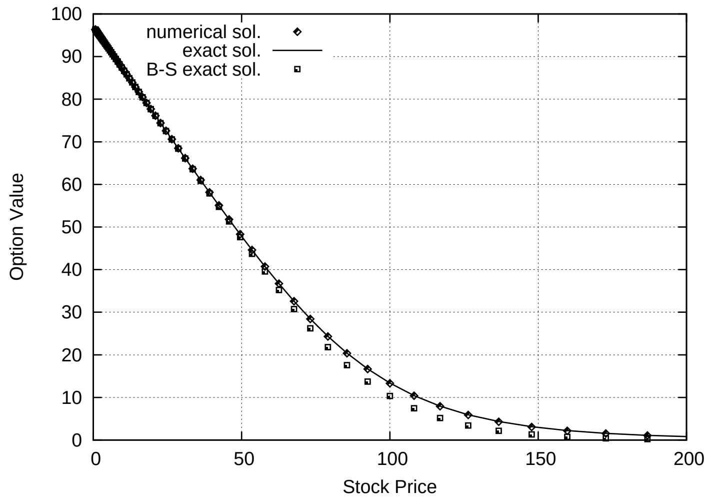

FIGURE 1. Comparison of option values in Example 5.1. The solid line and ∘'s represent the numerical and analytic solutions, and the □'s the option value under a pure diffusion process.

and thus, for each z on a contour, the Laplace transformed value $\widehat{V}_{sp}(S, z)$ of the ordinary put option, is imposed as the boundary value in computing $\widehat{V}_{ki}$.

We evaluate the option value on 1024 uniform spatial meshes with $N_z = 15$ points in a contour. Figure 2 depicts the option values and the delta of the standard put option and the knock-in put option. As the stock price becomes close to the knock-in barrier, due to the higher probability to hit the barrier, the more expensive value of the knock-in option is observed. After hitting the barrier, the option value approaches the standard option value.

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References

[1] A. Almendral and C. W. Oosterlee. Numerical valuation of options with jumps in the underlying. *Applied Numerical Mathematics*, 53(1):1–18, 2005.

[2] L. Andersen and J. Andreasen. Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing. *Review of Derivatives Research*, 4(3):231–262, 2000.

[3] F. Black and M. S. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–54, 1973.

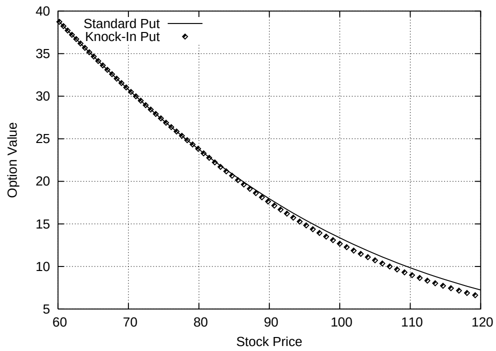
[4] T. J. I'a. Bromwich. Normal coordinates in dynamical systems. *Proc. Lond. Math. Soc.*, 15(Ser. 2):401–448, 1916.

[5] R. H.-F. Chan and X.-Q. Jin. *An Introduction to Iterative Toeplitz Solvers (Fundamentals of Algorithms)*. SIAM, Philadelphia, 2007.

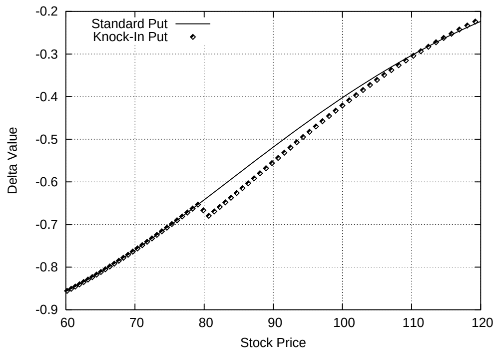
[6] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman and Hall, Boca Raton, FL, 2003.

[7] M. Crouzeix, S. Larsson, and V. Thomée. Resolvent estimates for elliptic finite element operators in one dimension. *Math. Comp.*, 63:121–140, 1994.

[8] Y. d'Halluin, P. A. Forsyth, and K. R. Vetzal. Robust numerical methods for contingent claims under jump diffusion processes. *IMA Journal of Numerical Analysis*, 25(1):87, 2005.



(a) Option values



(b) Delta values

FIGURE 2. Comparison of standard and Knock-in put option

[9] L. Feng and V. Linetsky. Pricing options in jump-diffusion models: An extrapolation approach. *Operations Research-Baltimore*, 56(2):304–325, 2008.
[10] I. P. Gavrilyuk, W. Hackbusch, and B. N. Khoromskij. $\mathcal{H}$-matrix approximation for the operator exponential with applications. *Numer. Math.*, 92:83–111, 2002.
[11] I. P. Gavrilyuk, W. Hackbusch, and B. N. Khoromskij. Data-sparse approximation to a class of operator-valued functions. *Math. Comp.*, 74(250):681–708 (electronic), 2005.
[12] I. P. Gavrilyuk and V. L. Makarov. Exponentially convergent parallel discretization method for the first order evolution equations. *Comput. Methods Appl. Math.*, 1:333–355, 2001.
[13] M. Griebel. A domain decomposition method using sparse grids. In *Domain Decomposition Methods in Science and Engineering: The Sixth International Conference on Domain Decomposition*, volume 157 of *Contemporary Mathematics*, pages 255–261, Providence, Rhode Island, 1994. American Mathematical Society.
[14] K. J. in 't Hout and J. A. C. Weideman. Appraisal of a contour integral method for the Black-Scholes and Heston equations. *arXiv*, (0912.0434), 2009.
[15] X.-Q. Jin. *Developments and Applications of Block Toeplitz Iterative Solvers*. Combinatorics and Computer Science. Springer, New York, 2003.
[16] S. G. Kou. A jump-diffusion model for option pricing. *Management Science*, 48(8):1086–1101, 2002.
[17] H. Lee and D. Sheen. Laplace transformation method for the Black-Scholes equation. *Int. J. Numer. Anal. Model.*, 6(4):642–658, 2009.
[18] C. C. W. Leentvaar and C. W. Oosterlee. Pricing multi-asset options with sparse grids and fourth order finite differences. In *Numerical mathematics and advanced applications*, pages 975–983. Springer, Berlin, 2006.
[19] C. C. W. Leentvaar and C. W. Oosterlee. On coordinate transformation and grid stretching for sparse grid pricing of basket options. *J. Comput. Appl. Math.*, 222(1):193–209, 2008.

[20] M. López-Fernández and C. Palencia. On the numerical inversion of the Laplace transform of certain holomorphic mappings. *Appl. Numer. Math.*, 51:289–303, 2004.

[21] A.-M. Matache, C. Schwab, and T. P. Wihler. Fast numerical solution of parabolic integrodifferential equations with applications in finance. *SIAM J. Sci. Comput.*, 27(2):369–393, 2005.

[22] A.-M. Matache, T. von Petersforff, and C. Schwab. Fast deterministic pricing of options on Lévy driven assets. *M2AN Math. Model. Numer. Anal.*, 38(1):37–71, 2004.

[23] R. C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.

[24] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical recipes in Fortran 90*, volume 2 of *Fortran Numerical Recipes*. Cambridge University Press, Cambridge, second edition, 1996.

[25] C. Reisinger and G. Wittum. Efficient hierarchical approximation of high-dimensional option pricing problems. *SIAM J. Sci. Comput.*, 29(1):440–458 (electronic), 2007.

[26] D. Sheen, I. H. Sloan, and V. Thomée. A parallel method for time-discretization of parabolic equations based on Laplace transformation and quadrature. *IMA J. Numer. Anal.*, 23(2):269–299, 2003.

[27] D. Tavella and C. Randall. *Pricing financial instruments: The finite difference method*. John Wiley & Sons New York, 2000.

[28] L. N. Trefethen, J. A. C. Weideman, and T. Schmelzer. Talbot quadratures and rational approximations. *BIT*, 46(3):653–670, 2006.

[29] T. von Petersdorff and C. Schwab. Numerical solution of parabolic equations in high dimensions. *M2AN Math. Model. Numer. Anal.*, 38(1):93–127, 2004.

[30] J. A. C. Weideman. Optimizing Talbot's contours for the inversion of the Laplace transform. *SIAM J. Numer. Anal.*, 44(6):2342–2362 (electronic), 2006.

[31] J. A. C. Weideman and L. N. Trefethen. Prabolic and hyperbolic contours for computing the Bromwich integral. *Math. Comp.*, 76(259):1341–1356, Mar 2007.

[32] X. L. Zhang. Numerical analysis of American option pricing in a jump-diffusion model. *Mathematics of Operations Research*, pages 668–690, 1997.

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